THE COMPUTATION OF OPTIMUM LINEAR TAXATION

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PREFACE

The equitable sharing of the benefits arising from planned development is a subject of lively contemporary debate. One of the tasks being carried out by the System and Decision Sciences Area of the International Institute for Applied Systems Analysis (IIASA) concerns the treatment of planning and redistribution problems in ways that can provide some guidance to decision makers in the formulation of economic policy. This report examines the first part of a study undertaken to assess the redistributive leverage provided by different instruments of planning. It is devoted specifically to the analysis and computation of optimal redistributive policies in small, general equilibrium models of economic planning.

CONTENTS

1 INTRODUCTION

The recent interest in redistributive welfare economics has prompted its practitioners to inquire how governments might further their distributional goals in economics characterized by the unavailability of lump-sum transfers varying from one household to another. It is not surprising that commodity and income taxes and subsidies have featured prominently in that discussion; an exploration of their redistributive aspects was initiated by Diamond and Mirrlees (1971) and Mirrlees (1971), and was continued by Atkinson (1973), Feldstein (1972a; 1972b; 1973), Mirrlees (1975; 1976), Diamond (1975), Atkinson and Stiglitz (1975), and Stern (1976). These contributions to the literature on linear taxation illustrate that, in general, it is impossible to arrive at explicit expressions for optimal linear tax rates and, therefore, to derive a convenient mnemonic for linear tax policy. The substantial extension of policy options allowed by nonlinear taxation makes more explicit statements about marginal tax rates possible. See Mirrlees (1976).

This report takes as its point of departure the observation that significant analytical advances with the tax equations are unlikely. It would therefore seem worthwhile to calculate optimum tax structures under alternative assumptions about individual preferences, abilities, and social attitudes towards inequality. Such an approach could provide an indication of tax rates and accounting prices for project evaluation. It can also be used to calculate optimal deviations from marginal cost pricing in public utilities or to compute optimum trade taxes and subsidies in an international model. It allows us to dispense with the assumption of constant shadow prices which has been used recently in the literature on this subject. It can also throw some light on the important choice of direct versus indirect tax instruments and, more generally, on the efficiency of alternative fiscal packages in securing specific redistributive ends. Clearly, a representation of an economic system as abstract as the Walrasian general equilibrium model cannot be used directly to generate precise recommendations for this or that government regarding tax rates and shadow prices. The present analysis should instead be viewed as a systematic attempt to isolate some of the more important determinants of redistributive policies and to focus on their consequences in a way untutored intuition alone cannot hope to imitate.

The standard optimum redistributive tax problem is presented in this report as a nonlinear programming problem. It is solved by successively combining a parametric linear program with a recent computational algorithm developed by Scarf. (The actual version of Scarf's algorithm used in this report is the variant devised by Merrill (1972) and expounded by Kuhn and MacKinnon (1975). Shoven and Whalley (1973) have used Scarf's algorithm to compute equilibrium in the presence of fixed taxes. The analysis presented in this report differs considerably from theirs because the choice of tax structure is endogenous to our model.) Scarf's algorithm has been used by Hansen (1969), Scarf (1973), and Hansen and Koopmans (1972) to solve concave programming problems. It is therefore worth underlining those respects in which the procedure adopted here represents a formal advance over earlier work in this area.

To begin with, optimization exercises in the welfare economics of the second best, of which our problem is a special case, are formulated normally as nonconcave, nonlinear programs. Since we cannot rely on concavity arguments and their associated regularity conditions to construct a proof of convergence, we have introduced a constraint qualification which ensures that Scarf's algorithm identifies solutions to the first-order necessary conditions for optimality and which, most conveniently, lends itself to an intuitive economic explanation. Secondly, the labeling rule for a vector that fails to satisfy the constraints of the nonlinear program is not the standard Hansen-Scarf rule. The reason for this will be explained in Chapter 4; but to anticipate that discussion, the procedure uses a single column label that is a suitable convex combination of all the constraint gradient vectors rather than a number of constraint gradient column labels equal to the number of constraints in the programming problem. Thirdly, our formulation of the optimum tax problem allows the government to manipulate two sets of control variables: consumer prices, and public production levels. Since the computational burden of Scarf's algorithm increases considerably with the dimensionality of the space on which the search for a solution is executed, we have applied that algorithm to the unit simplex of consumer prices alone. Production levels and shadow prices corresponding to any consumer price vector are generated by the use of a parametric linear program. Although dimensionality-reducing linear programs were introduced in Scarf (1973) to help solve certain types of general equilibrium models in conjunction with the Scarf algorithm, they have not, to the authors' knowledge, been employed in the solution of nonlinear programming problems using that algorithm. Furthermore, the linear program used in this report is amenable to economic interpretation. Finally, labeling rules for nonlinear programs in the spirit of Scarf and Hansen do not need to be well defined when applied to optimum tax-price problems at the frontier of the unit simplex of consumer prices. This report therefore introduces and discusses economically meaningful boundary conditions that explore this difficulty.

Chapter 2 sets out the basic general equilibrium model used in this work. Chapter 3 describes the computational algorithm for the optimal tax problem. In Chapter 4 we demonstrate that, under certain regularity conditions, the algorithm locates approximate solutions to the first-order conditions characterizing a tax optimum. Two numerical examples are presented in Chapter 5; the economies they illustrate are characterized respectively by (a) households differing in the efficiency of labor supplied, identical Stone-Geary utility functions, linear production, and no income taxation; and by (b) households differing as in (a) above, identical nonhomothetic CES utility functions, and linear production. The parameters chosen in the first example for the linear expenditure system are those estimated for the United Kingdom by Lluch and Powell (1975); the numbers in the nonhomothetic CES example are, however, not based on data for any real economy. These exercises provide an indication of the performance of the algorithm and of the tax rates implied by alternative social welfare functions and efficiency distributions. Chapter 6 discusses possible extensions of the basic model. Finally, an appendix to the report treats boundary problems, discussing in detail specific technical features and providing economic interpretations.

2 THE MODEL

This chapter outlines the basic general equilibrium model and the principal assumptions to be used in the sequel. The presentation parallels that of Diamond and Mirrlees (1971).

Diamond and Mirrlees (1971) effectively assume that every household is endowed with every commodity. Later in this report we shall assume likewise for expository convenience. The assumption is, however, not satisfied by the numerical examples given later in Chapter 5; the appendix demonstrates how the labeling rules may be modified in its absence.

Vector ordering notation is as follows: $a \geqq b$ implies $a_i \geqq b_i$ for all i; $a \geqslant b$ implies $a \geqq b$ but $a \neq b$; $a > b$ implies $a_i > b_i$ for all i.

2.1 HOUSEHOLDS

The economy comprises H households, indexed by h. There are n commodities. Let

$\tilde{x}_h$: n-dimensional vector of "full" endowments of household h
$\tilde{q}^T$: $(q_1, \ldots, q_n)$ a vector of consumer prices
q_{n+1} : poll subsidy unchanging across households

All vectors are column vectors and T denotes transpose operations. Then the full income of household h, M_h, defined as the value of its endowment plus transfer income, may be written as

$$M_h = q^T \cdot \tilde{x}_h + q_{n+1}. \tag{2.1}$$

Let x_h^0 denote the vector of actual consumptions (of goods and leisure) of household h. The budget constraint facing household h may then be written as

$$\tilde{q}^T x_h^0 = \tilde{q}^T \cdot \tilde{x}_h + q_{n+1} = M_h .$$

The government, however, finds it possible to levy taxes and subsidies on market transactions rather than on endowments. It will therefore prove convenient for certain purposes to focus on a household's net trades, $x_h = x_h^0 - \tilde{x}_h$; whence the above reduces to

$$\tilde{q}^T x_h = q_{n+1} . \qquad (2.2)$$

The problem is one of choosing the most preferred consumption vector, x_h^0, from the admissible consumption set, C_h, subject to (2.2).

The following assumptions are made about the typical household:

(a.1) C_h is closed, convex and bounded below.
(a.2) The preference ordering is continuous.
(a.3) The preference ordering is strictly convex, i.e., if x_h^1 is preferred to or indifferent with x_h^2, $x_h^1 \neq x_h^2$ and $0 \leqslant \alpha < 1$, then $(1 - \alpha)x_h^1 + \alpha x_h^2$ is preferred to x_h^2.
(a.4) There is no satiation consumption in C_h.

Under assumptions (a.1) and (a.2) a continuous utility function, $u_h(x_h)$, is known to exist (cf. Debreu, 1959). Household h can therefore be taken to maximize $u_h(x_h)$ subject to (2.2); assumptions (a.1)–(a.3) ensure that when the solution to that maximization problem, $x_h(q)$, is defined, it is defined uniquely where

$$q = \begin{pmatrix} \tilde{q} \\ q_{n+1} \end{pmatrix},$$

as a $(n + 1)$-dimensional vector of consumer prices augmented by a poll subsidy. Since net trades are homogeneous of degree zero in consumer prices and the poll subsidy, we can normalize these variables to lie on the $(n + 1)$-dimensional unit simplex, S_{n+1}, defined by

$$q^T e_{n+1} = 1$$
$$q \geqslant 0$$ [6]

where e_{n+1} is a $(n + 1)$-dimensional vector containing "1" everywhere. Next we define an indirect utility function:

$$v_h(q) = u_h[x_h(q)] .$$

On noticing that "full" income defined in (2.1) is a function of q, $M_h(q)$, we can define the set

$$D_h = \{q \in S_{n+1} | M_h(q) \geqslant \eta\} \tag{2.3}$$

where η is an arbitrarily small positive number. Let

$$D = \bigcap_{h=1}^{H} D_h. \tag{2.4}$$

Assumptions (a.1)–(a.3) guarantee that, in circumstances where C_h is bounded, $x_h(q)$ and therefore $v_h(q)$ are defined and continuous on D_h for all $\eta > 0$ (cf. Debreu, 1959). It is next assumed that

(a.5) $v_h(q)$ and $x_h(q)$ are differentiable continuously on D_h for all $\eta > 0$ whenever C_h is bounded.

2.2 PRODUCTION

The technology is described by a list of m constant returns to scale activities; it is assumed for convenience that all production is controlled by the government. Let

A: $(n \times m)$-dimensional activity matrix
y: m-dimensional nonnegative vector of activity levels

The production possibility set is then given by

$$G = \{g = Ay | y \geqq 0\}.$$

G is thus a convex polyhedral cone with the origin as vertex. It is taken to satisfy the following assumptions:

(b.1) $-\Omega \subset G$, where Ω is the n-dimensional nonnegative orthant.
(b.2) There exists no vector $y \geqq 0$ such that $g = Ay \geqslant 0$.

Assumption (b.2) ensures the existence of a strictly positive vector $r > 0$ such that $r^T A \leqq 0$ (cf. Gale, 1951). Let

$$X(q) = \sum_{h=1}^{H} x_h(q),$$

the vector of aggregate net trades of households.

Define

$$F = \{X \in G | \exists q \in S_{n+1} \quad \text{such that} \quad X = X(q)\}. \tag{2.5}$$

$$E = \{q \in D | X(q) \leqq Ay \quad \text{for some} \quad y \geqq 0\}. \tag{2.6}$$

We can now state the following:

LEMMA 2.1. *Under assumptions (a.1)–(a.3), (b.1) and (b.2)*

(i) F *is bounded.*†
(ii) E *is closed.*

Proof: Boundedness of F is proved in Diamond and Mirrlees (1971). Closure of E can be proved, as in Diamond and Mirrlees (1971), once it has been observed that $X(q)$ is continuous on E.

2.3 GOVERNMENT

The government's planning problem is one of choosing consumer prices, a poll subsidy, and public production to maximize social welfare, subject to the constraint that aggregate excess demand be nonpositive in every market. If, as will be seen, such an optimum is realized as a price-taking equilibrium, the (weak) assumption that not all consumer prices are zero at equilibrium allows us, by Walras' Law, to omit any reference to the government's budget constraint in the statement of the planning problem. It is natural to think of the difference between consumer prices and the producer prices, which will be associated with optimum production as the vector of commodity taxes-cum-subsidies; it is worth noting that the inability of the government to levy lump-sum taxes and subsidies varying across households causes this to be an exercise in "second best" welfare economics.

The government is assumed to possess a social welfare function

$$V(q) = W[v_1(q), \ldots, v_H(q)]$$

where

(c.1) W is differentiable in $(v_1, \ldots, v_H)$,
(c.2) W is increasing strictly in some q_k $(k = 1, \ldots, n + 1)$.

Thus, if W is individualistic (i.e., $(\partial W/\partial v_h) > 0$ for all h), the nonsatiation assumption (a.4) ensures that assumption (c.2) is satisfied for $k = n + 1$,

†Assumptions (a.2) and (a.3) are not needed for part (i) of Lemma 2.1.

i.e., increasing the poll subsidy must increase social welfare for an individualistic social welfare function.

Formally, the government has to select q and y in order to maximize $V(q)$ subject to

$$Ay - X(q) \geqq 0$$
$$q \in S_{n+1}$$
$$y \geqslant 0.$$

If Lagrange's method of undetermined multipliers can be used to solve this problem, we may write the Lagrangian as

$$L = V(q) + vp^T[Ay - X(q)] \tag{2.7}$$

where p is the n-dimensional vector of shadow prices and v is the Lagrange multiplier. This formulation leads to the following first-order conditions:

$$V'(q)^T - v[X'(q)]^T p \leqq 0 \qquad (q \geqslant 0) \tag{2.8}$$
$$p^T A \leqq 0 \qquad (y \geqslant 0) \tag{2.9}$$
$$Ay - X(q) \geqq 0 \qquad (p \geqslant 0) \tag{2.10}$$

where primes denote derivatives; thus

$V'(q)$: gradient vector of V
$X'(q)$: Jacobian matrix of X†

and each inequality bears the relation of complementary slackness with the corresponding variable appearing in brackets on the right.

Clearly, under certain regularity conditions, every solution to the planning problem satisfies assumptions (2.8)–(2.10). The converse, however, fails to hold because the Langrangian (2.7) is not concave in q. Thus there is no guarantee that a computational procedure locating solutions to the necessary conditions for optimality can solve the government's planning problem.

The first-order conditions indicate the possibilities of decentralization in this economy. First, equation (2.9) is a standard no-positive-profit-at-shadow-prices condition for a linear technology: p is a vector of producer prices facing the public sector. Second, the earlier discussion leading to the derivation of the household demand functions $x_h(q)$ shows that, poll subsidy apart, the government may deal with households through the consumer price system alone. Third, equation (2.10) is a market clearance condition. Finally, equation (2.8) provides first-order necessary conditions for an optimum tax structure.

†The typical element in $V'(q)$ is $V_i'(q)$, the derivative of V with respect to the ith consumer price. The element in the ith row and the jth column in $X'(q)$ is denoted by $X_{ij}(q)$ and represents the derivative of the aggregate net trade in good i with respect to the consumer price of good j.

3 THE COMPUTATIONAL PROCEDURE

A recent computational algorithm developed by Scarf is used to execute a structured search on the unit simplex of consumer prices. In order to apply Scarf's algorithm, a specific $(n + 1)$-dimensional column vector b must be associated with each element of a fine grid of vectors $q^1, \ldots, q^k$ on the unit simplex. The labeling rules of the Hansen-Scarf type, which are aimed at locating solutions to the first-order necessary conditions for optimality, may not be well defined for vectors $q^1, \ldots, q^k$ or for the boundary of the unit simplex. This matter, to which every general equilibrium treatment needs to pay some technical attention, is explored in the appendix. The exposition of the main argument in the text is considerably simplified by assuming that

(d.1) C_h is bounded above for all h,
(d.2) $\tilde{x}_h > 0$ for all h.

It is easy to construct examples of economies not satisfying (d.1) when certain consumer prices go to zero. An excess demand function may well exhibit discontinuities as parts of the boundary of the consumer price simplex, if (d.2) is not satisfied. Thus (d.1) and (d.2) are considerably weakened in the appendix, but at the cost of a more elaborate treatment.

Meanwhile, we concentrate on the implications of the assumptions made above. Assumptions (a.1) and (d.1) imply that C_h is bounded. Since the set of feasible allocations, F, is bounded, the bound on C_h can be chosen such that it contains all feasible allocations. Assumption (d.2) ensures that there exists an $\eta > 0$ such that $q^T \times x_h^0 \geqslant \eta$ for all h and for all $q \in S_{n+1}$. With that choice of η, $D_h = S_{n+1}$ for all h. It then follows from the discussion in Section 2.1 and from assumption (a.5) that $v_h(q)$

and $x_h(q)$ are differentiable continuously on the entire unit simplex, S_{n+1}. So, therefore, are $V(q)$ and $X(q)$.

We are now in a position to describe the rules used to associate each element of the grid $q^1, \ldots, q^k$ with a $(n + 1)$-dimensional column vector b.

RULE 1. *For the vectors* q *in which at least one element is zero,* b *is given by* δ^j*, where* δ^j *contains a 1 in the* j*th place and zeroes elsewhere and where* j *is the index of the first zero element in* q.

RULE 2. *For all* $q \in S_{n+1}$, *we calculate* X(q), *the vector of aggregate net trades. Two cases need to be distinguished:*

(i) *If* X(q) *satisfies the feasibility requirement* $Ay - X(q) \geqq 0$, *then* b *is defined as* $[e_{n+1} + V'(q)]$.
(ii) *If* X(q) *fails to satisfy one of the market clearance inequalities, say the* j*th, then* b *is defined as* $[e_{n+1} - X'(q)^T p]$.

The test of feasibility and the producer prices required by such a rule of association is furnished by the linear programs: choose p to maximize $p^T X(q)$ subject to

$$\left.\begin{aligned} p^T A &\leqq 0 && (3.3) \\ p^T e_n &= 1 && (3.4) \\ p &\geqslant 0. && \end{aligned}\right\} \text{(P)}$$

The dual program is: choose (y, z) to minimize z subject to

$$\left.\begin{aligned} Ay + ze_n &\geqq X(q) && \\ y &\geqslant 0. && (3.5) \end{aligned}\right\} \text{(D)}$$

The statement following assumption (b.2) in Section 2.2 ensures that the set $\{p \in S_n | p^T A \leqq 0\}$ is nonempty. Since $X(q)$ is continuous on the compact set S_{n+1}, solutions to the above programs exist. The solution to the primal program yields a producer price vector; equation (3.4) is, again, a normalization permitted by homogeneity of degree zero of the supply response in producer prices. The dual program seeks to minimize the largest deviation between net trades arising at prices q and the supply response Ay which, in turn, is supported by p. Inequalities (3.5) provide the test of feasibility demanded by the rules of association given above.

We now establish a useful preliminary result which shows that the labeling under Rule (2) (ii) is continuous. Towards this end, we define

$$L = \{p \in S_n \mid p^T A \leqq 0\}.$$

$$M(q) = \max_p \{pX(q) \mid p \in L\}; \qquad q \in S_{n+1}.$$

$$T(q) = \{p \in L \mid p^T X(q) = M(q)\}; \qquad q \in S_{n+1}.$$

LEMMA 3.1. $T(q)$ *is an upper semicontinuous mapping from* S_{n+1} *to* L.

Proof. Since L, being a subset of the n-dimensional unit simplex, is compact, upper semicontinuity of the mapping T is equivalent to closedness of its graph. The latter is easily proved as follows.† Consider a sequence (q^v, p^v) in the graph of the correspondence T tending to (q^*, p^*). Clearly

$$(p^v)^T X(q^v) \geqq p^T X(q^v) \qquad \text{for all } p \text{ in } L.$$

Since $X(q)$ is continuous on S_{n+1}, the above inequality may be written in the limit as

$$(p^*)^T X(q^*) \geqq p^T X(q^*) \qquad \text{for all } p \text{ in } L.$$

Thus (q^*, p^*) is in the graph of T, showing that the graph is closed.

The rules of association lead to a matrix B whose columns correspond to the grid vectors $q^1, \ldots, q^k$. That is,

$$\begin{array}{cccccc} & q^1 & q^2 \;\ldots & q^{n+1} & q^{n+2} & \ldots\; q^k \end{array}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & b_{1,n+2} & b_{1k} \\ 0 & 1 & 0 & b_{2,n+2} & b_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & b_{n+1,n+2} & b_{n+1,k} \end{bmatrix}.$$

The theorem underlying Scarf's algorithm may now be stated as follows (Scarf, 1973).

†We are grateful to an anonymous referee for pointing out that this result follows from the more general Maximum Theorem; see, for example, Berge (1959).

THEOREM 3.1. *Let* q^j *be associated with the* j*th column of the matrix* B. *Assume that the set of nonnegative solutions to the equations* $Bw = e_{n+1}$ *is bounded. Then there exists a primitive set* $q^{j_1}, \ldots, q^{j_{n+1}}$ *such that the columns* $j_1, \ldots, j_{n+1}$ *form a feasible basis for* $Bw = e_{n+1}$ *(i.e., these equations have a nonnegative solution, where* $w_j = 0$ *for* $j \neq j_1, \ldots, j_{n+1}$*).*

This theorem may be used to solve our problem if it can be shown that the boundedness condition on which it relies is satisfied here. Toward this end, recall assumption (c.2) of Section 2.3. Consider the kth row of the B matrix. The nonslack entries in this row contain either $[1 + V_k(q)]$ elements or

$$\left[1 - \sum_{i=1}^{n} p_i X_{ik}(q)\right]$$

elements. $V_k(q)$ is, of course, positive. Since the p_i are bounded and the X_{ik} are continuous in $q\,(i = 1, \ldots, n)$, clearly the sums

$$\sum_{i=1}^{n} p_i X_{ik}(q)$$

are bounded as q varies over the unit simplex. We therefore choose the units in which the goods are measured such that

$$\left[\sup_{q} \sum_{i=1}^{n} p_i X_{ik}(q)\right] < 1,$$

where the supremum is taken over S_{n+1}. This ensures that all nonslack entries in the kth row of the B matrix are positive, guaranteeing that the set of nonnegative solutions to $Bw = e_{n+1}$ is bounded.

4 A PROOF OF CONVERGENCE

This chapter demonstrates that, provided a certain regularity condition is satisfied, the final primitive set where columns form a feasible basis for $Bw = e_{n+1}$ defines an approximate solution to the first-order necessary conditions for a linear tax optimum. We begin by imposing the following condition.

Regularity condition (R). There does not exist a pair of price vectors (p, q) where p satisfies the program (P) such that $p^T X \geqslant 0$ and

$$X'(q)^T p \geqslant 0 \qquad (\hat{q} \geqslant 0)\dagger \tag{4.1}$$

where each inequality in (4.1) bears the relations of complementary slackness with the corresponding variable appearing in brackets on the right.

The economic implication of this condition, which will be seen to play a role analogous to that of the constraint qualification in nonlinear programming, will be examined later in this chapter. Its introduction at this stage allows us to state and prove the following theorem.

THEOREM 4.1. *The computational procedure of Chapter 3 converges to an approximate solution to the first-order necessary conditions (2.8)–(2.10) for a linear tax optimum, provided that the regularity condition (R) is satisfied.*

Proof. The rules of association allow the equation $Bw = e_{n+1}$ to be written as

†It should be noted that homogeneity implies $q^T \times X'(q)^T p = 0$. The inequality in (R) can therefore be strict only for an element corresponding to a zero consumer price.

$$\sum_j \lambda_j [e_{n+1} + V'(q^j)] + \sum_j \mu_j [e_{n+1} - X'(q^j)^T p^j] \leqslant e_{n+1} \qquad (q \geqslant 0) \tag{4.2}$$

where positive w_j elements corresponding to columns arrived at using rule (3.1) (respectively (3.2)) have been renamed λ_j (respectively μ_j).

Imagine now the consequences of employing an increasingly finer sequence of grids. As the grid size approaches infinity in the limit, the q vectors become everywhere dense on the $(n + 1)$-dimensional unit simplex and all vectors of the final primitive set approach the vector $\hat{q}$. A more formal presentation of this type of argument is given in Hansen and Koopmans (1972) and is therefore not repeated here. Since all partial derivatives have been assumed continuous, all functions of q in (4.2) approach subsequential limits. Furthermore, the p vectors are constrained to lie in the unit simplex. Hence corresponding to each (p^j) subsequence tending to $\hat{q}$, there will exist a (p^j) subsequence tending to $\hat{p}^j$, where the superscript j on the subsequential limit indicates that the limit approached depends on the path followed by the producer price sequence. Finally, it should be noticed that the weights λ_j and μ_j, being elements of a closed bounded set, also tend to $\hat{\lambda}_j$ and $\hat{\mu}_j$ respectively.

With this fine grid covering the simplex, (4.2) becomes

$$\hat{\lambda}[e_{n+1} + V'(\hat{q})] + \sum_j \hat{\mu}_j [e_{n+1} - X'(q)^T p^j] \leqslant e_{n+1} \qquad (q \geqslant 0) \tag{4.3}$$

where $\hat{\lambda} = \Sigma_j \hat{\lambda}_j$ and where upper semicontinuity of the $T(q)$ mapping defined in Lemma 3.1 ensures that the $\hat{p}^j$ may be interpreted as producer price vectors.

It needs to be shown that the subsequential limits to which the computational procedure converges satisfy the first-order conditions (2.8)–(2.10) for the government's planning problem. Accordingly, the proof presented below argues (a) that all the unit entries in (4.3) may be eliminated; (b) that it is possible to generate an "average" producer price vector, i.e., one without the superscript j; and (c) that columns of the type defined by (3.1) are included in (4.3), i.e., $\hat{\lambda} \neq 0$. This last argument allows us to claim not only that limiting values of the variables are feasible solutions but that they satisfy the first-order conditions (2.8) for optimum linear taxation as well.

Step 1. Premultiplying (4.3) by q^T, we have

$$\hat{\lambda}[\hat{q}^T e_{n+1} + \hat{q}^T V'(\hat{q})] + \sum_j \hat{\mu}_j [\hat{q}^T e_{n+1} - (\hat{q}^T X'(\hat{q})^T) \hat{p}^j] = \hat{q}^T e_{n+1}.$$

Because of the homogeneity of degree zero of the indirect social welfare function and all aggregate net trades in the q variables and because these variables lie on the unit simplex, the above equation may be reduced to

$$\hat{\lambda} + \sum_j \hat{\mu}_j = 1. \tag{4.4}$$

The expression (4.3) may now be simplified to

$$\hat{\lambda} V'(\hat{q}) - \sum_j \hat{\mu}_j X'(\hat{q})^T \hat{p}^j \leqslant 0 \qquad (q \geqslant 0). \tag{4.5}$$

Step 2. We next construct a weighted average of all the limiting $\hat{p}^j$ producer price vectors appearing in (4.5), the weight being that associated with the column in which that $\hat{p}^j$ vector appears, i.e., $\hat{\mu}_j$. Such an average is well defined only if the proposed weights are not all zero. Suppose, therefore, that $\hat{\mu}_j = 0$ for all j. Then, from (4.4) and (4.5),

$$V'(\hat{q}) \leqslant 0.$$

But this contradicts assumption (c.2) that V is increasing strictly in at least one of its arguments; the supposition that $\hat{\mu}_j = 0$ for all j is therefore false.

Consider the vector

$$\hat{p} = \frac{\sum_j \hat{\mu}_j \hat{p}^j}{\hat{\mu}} \tag{4.6}$$

where $\hat{\mu} = \Sigma_j \hat{\mu}_j$. Clearly $\hat{p}$ is a convex average of the subsequential limits $\hat{p}^j$; since each $\hat{p}^j$ supports the production vector $A\hat{y}$ generated by the parametric program (P), so must $\hat{p}$. One is therefore entitled to regard $\hat{p}$ as a vector of producer prices.

The equations (4.5) may now be written as

$$\hat{\lambda} V'(\hat{q}) - \hat{\mu} X'(\hat{q})^T \hat{p} \leqslant 0 \tag{4.7}$$

where

(i) the kth row above is an equality if $\hat{q}_k > 0$ $(k = 1, \ldots, n+1)$.
(ii) $\hat{\lambda} > 0$ implies $A\hat{y} \geqq X(q)$.
(iii) $\hat{\mu} > 0$ implies that there exists a j such that

$A_j \hat{y} \leqq X_j(\hat{q})$ where A_j is the jth row of A (4.8)

and where

$$\hat{\lambda} + \hat{\mu} = 1. \tag{4.9}$$

The statements in (4.8) follow directly from the rules of association (3.1) and (3.2) that have been applied to the system (4.7).

The existence of the average price vector $\hat{p}$ of (4.6) is dependent on our demonstrating that $\hat{\mu}_j \neq 0$ for all j and, therefore, that $\hat{\mu} = \Sigma_j \hat{\mu}_j$ is positive. Before going on to the final step of the proof, it is worth noting an important implication of this result. On the one hand, statement (iii) of (4.8) asserts that $\hat{\mu} > 0$ implies $A_j \hat{y} \leqq X_j(\hat{q})$ for some j. Inspection of the parametric program (P), on the other hand, shows that $\hat{z} < 0$ implies $A\hat{y} > X(\hat{q})$ or, alternatively, that $A_j \hat{y} \leqq X_j(\hat{q})$ for some j implies $\hat{z} \geqq 0$. Since by the duality theorem of linear programming applied to (P), $\hat{z} = \hat{p}^T X(\hat{q})$, the statements (4.8.ii) and (4.8.iii) allow us to deduce that $\hat{p}^T X(\hat{q}) \geqslant 0$, a fact that plays an important role in the following step.

Step 3. This part of the proof ensures that $\hat{\lambda} \neq 0$. For, in that case, statement (ii) of (4.8) and the complementary slackness relations of linear programming applied to the program (P) imply that (2.10) is satisfied. Second, statement (i) of (4.8) shows that (2.8) is satisfied if one defines $v = \hat{\mu}/\hat{\lambda}$. Finally, the very construction of the program (P) ensures that (2.9) is also satisfied.

The argument $\hat{\lambda} \neq 0$, like the proof of the Kuhn-Tucker theorem of nonlinear programming, fails to hold in the absence of certain regularity conditions. Suppose, to the contrary, that $\hat{\lambda} = 0$. Then from (4.7) and (4.9)

$$[X'(\hat{q})]^T \hat{p} \geqslant 0 \qquad (\hat{q} \geqslant 0). \tag{4.10}$$

But this violates the regularity condition (R) when one remembers that it has already been proved that $\hat{p}^T X(\hat{q}) \geqslant 0$. This establishes that $\hat{\lambda} > 0$ and proves Theorem 4.1.

The rest of this chapter discusses the regularity condition (R) used in the proof of the convergence theorem. The regularity condition (R) states that whenever the value of aggregate net trades is nonnegative at producer prices, there exists a direction of movement in either consumer prices or the poll subsidy; this direction decreases the value of aggregate net trades measured at the producer prices prevailing before that movement.

The meaning of the regularity condition (R) might become clearer on examining its implications for tax revenue. Summation over the household budget constraints $\tilde{q}^T x_h = q_{n+1}$ yields $\tilde{q}^T X = Hq_{n+1}$. Define the vector of commodity tax rates, $\tilde{t} = \tilde{q} - p$, the first n components of $t = q - p$. Since $p^T X = (\tilde{q} - \tilde{t})^T X = (Hq_{n+1} - \tilde{t}^T X)$, the budget deficit, (R), states that there does not exist a pair of price vectors (p, q) such that $[X'(q)]^T p \geqq 0 \quad (\hat{q} \geqslant 0)$ whenever the budget deficit is nonnegative.

A somewhat sharper implication of (R) becomes available if, in keeping with many recent contributions to optimum tax theory, one assumes that producer prices are constant, so that changes in tax rates are reflected completely in consumer prices. In that case

$$[X'(q)]^T p = \frac{\partial}{\partial t}[p^T X] = \frac{\partial}{\partial t}[Hq_{n+1} - \tilde{t}^T X]$$

and (R) postulates that there does not exist a pair (p, q) such that $(\partial/\partial t)[Hq_{n+1} - \tilde{t}^T X] \geqq 0 \quad (\hat{q} \geqslant 0)$ whenever $[Hq_{n+1} - \tilde{t}^T X] \geqq 0 \quad (q \geqslant 0)$. The regularity condition (R) may therefore be interpreted as follows: whenever the government is not running a net budget surplus, it can always use the tax controls at its disposal to reduce the size of its budget deficit. To assume that such an option is available to the government is to assume away Edgeworth's so-called tax paradox (where no tax change can increase revenue); but we do not impose the condition except when the budget deficit is nonnegative. Analogous "constraint qualifications" have been employed by previous authors, e.g., Diamond and Mirrlees (1971) and Green (1975). We should like to emphasize that the assumption of constant producer prices does not underlie our statement of the regularity condition (R) or, indeed, the analysis in the rest of this report. It is made in this paragraph merely to allow a convenient interpretation of (R) by focusing on its implications for tax revenue.

Finally, the algorithm is able to locate points satisfying only the first-order necessary conditions for an optimum. Our analysis therefore represents a generalization of standard treatments of nonlinear programming using Scarf's algorithm insofar as those treatments appeal to concavity properties. The first-order necessary conditions do not, however, guarantee even a local maximum to our planning problem. Consequently, we terminate the algorithm by checking the second-order conditions. (This checking involves "smoothing" of the production set.)

The Scarf algorithm will accordingly be started from different corners of the unit simplex in the numerical examples of Chapter 5 in order to help locate a number of local maxima. The values of social welfare will be noted in each of these cases in order to arrive at the best solution by means of the computational procedure. But, there is no guarantee that the procedure will find the global maximum; furthermore, it is *theoretically* possible for the algorithm to fail to discover any local maxima at all.

5 NUMERICAL EXAMPLES

This section presents two numerical examples which indicate the algorithm's performance while, at the same time, point to the magnitudes of tax rates implied by the recent literature. The examples differ with respect to the utility function assumed for households as well as in the fact that the first example draws on data relating to demand patterns in the United Kingdom. The choice of functional forms in each case is motivated by a desire to avoid a large number of parameters.

5.1 STONE-GEARY UTILITY FUNCTION

A typical household maximizes

$$\log u_h = \sum_{i=1}^{n} \beta_i \log (x_{hi} - \gamma_i); \qquad \sum_{i=1}^{n} \beta_i = 1 \tag{5.1}$$

subject to

$$\sum_{i=1}^{n-1} q_i x_{hi} + q_n h x_{hn} = 0 \tag{5.2}$$

where

β_i: marginal budget share for commodity i
γ_i: "subsistence" consumption on commodity i $\Big\}$ $(i = 1, \ldots, n)$

The budget constraint (5.2) requires comment. Since (5.1) is the utility function common to all households (identical tastes), it is assumed that households differ only with respect to the efficiency of labor hours put in. Thus

$-x_{hn}$:	number of hours worked by household h	$(h = 1, \ldots, H)$
$-hx_{hn}$:	number of efficiency hours worked by h	
q_i:	consumer price of commodity i	$(i = 1, \ldots, n-1)$
q_n:	efficiency wage rate	

The reader will have noticed that no lump-sum subsidy is allowed in (5.2). The example therefore excludes the possibility of progressive income taxation which would be characterized by an exemption level and a proportional rate of tax both above and below this level. This is because a linear income tax renders commodity superfluous as an instrument of redistribution in an economy where households have identical Stone-Geary utility functions and differ only in wages received per physical hour of effort. (Such a result is to be found in Atkinson (1977).) The example is therefore interesting for economies where, for administrative reasons, the government may resort only to commodity taxes and subsidies.

The production side of the economy was kept as simple as possible in this example. There are eight consumer goods, each produced by the one factor of production-labor at constant returns to scale. Thus relative producer prices were constant.

The parameters of the utility function were selected to reflect demand patterns in the United Kingdom. The units were chosen to each represent one U.S. dollar's worth of output or labor input at 1970 prices. The parameters that refer to consumer goods are taken from the estimates of Lluch and Powell (1975), with modifications designed to ensure that the demands never become negative. Parameters pertaining to the labor-leisure choice were chosen so as to produce a labor supply elasticity of -0.19, which was the central value used by Stern (1976), and a labor supply of 1,725, which was the approximate per capita gross national product in 1959. The average labor endowment in efficiency units (some of which would be consumer as leisure) implied by these requirements was 3,070. This inclusion of the labor-leisure choice illustrates the additional flexibility that can be gained by using numerical methods. By way of contrast, analytical methods employed by Deaton (1975) required the simplicity that is provided by assuming a fixed labor supply. The parameters are shown in Table 1.

The calculations were carried out using a social welfare function of the form

$$W = \frac{1}{\rho} \sum_{h=1}^{H} (u_h)^{\rho} \qquad \text{when } \rho \neq 0$$

$$= \sum_{h=1}^{H} \log (u_h) \qquad \text{when } \rho = 0.$$

TABLE 1 Parameters of the linear expenditure system.

	β	γ
Leisure	0.547	0
Food	0.054	405
Clothing	0.031	86
Housing	0.116	41
Durables	0.034	27
Personal care	0.014	6
Transport	0.126	0
Recreation	0.030	46
Other services	0.048	0

Different degrees of aversion to inequality can be generated by altering the value of ρ from utilitarianism when $\rho = 1$ to Rawlsiam "maximum" as ρ tends to minus infinity.

The distribution of the efficiency index was assumed to be log-normal and different degrees of skill dispersion were obtained by varying the value of the standard deviation of the logarithms (σ).

Tables 2 and 3 show the results yielded by this model. It is always possible, by appropriate normalization, to choose a zero tax rate on one commodity. In Tables 2 and 3 there is no tax on labor. Table 2 shows the effect of varying the degree of inequality aversion (ρ) while keeping the skill dispersion constant (at $\sigma = 0.39$). As expected, both the tax rates on luxuries and the subsidies on necessities increase with the degree of inequality aversion.

Table 3 shows the effect of varying the skill dispersion (σ) while keeping the degree of inequality aversion constant (at $\rho = 0$).

As one might expect, the rates of tax and subsidy increase with the degree of skill dispersion.

TABLE 2 Tax rates for different degrees of inequality aversion (percentage).

	$\rho = 1$	$\rho = 0$	$\rho = -1$	$\rho = -5$
Food	−8.7	−18.4	−25.4	−43.8
Clothing	−1.6	− 3.8	− 7.2	−18.7
Housing	5.4	11.6	17.9	39.3
Durables	2.8	6.8	10.8	17.1
Personal care	1.5	6.8	12.7	29.6
Transport	6.4	14.4	24.1	73.7
Recreation	1.1	2.2	3.0	− 1.8
Other services	5.5	13.3	22.9	70.7

TABLE 3 Tax rates for different degrees of skill dispersion (percentage).

	$\sigma = 0.3$	$\sigma = 0.39$	$\sigma = 0.5$
Food	−11.9	−18.4	−27.3
Clothing	− 2.2	− 3.8	− 7.3
Housing	7.0	11.6	19.7
Durables	3.9	6.8	11.5
Personal care	3.2	6.8	13.9
Transport	8.3	14.4	26.6
Recreation	1.6	2.2	2.8
Other services	7.6	13.3	25.4

5.2 NONHOMOTHETIC CES UTILITY FUNCTION

In this section we assume that households possess nonhomothetic CES utility functions, that a second factor of production is introduced, and that there are only two consumer goods. The data used in this example are not based on estimates for any real economy.

The nonhomothetic utility function is of the form†

$$F(x,u) = \begin{cases} \sum_{i=1}^{n} D_i u^{-de_i} x_i^d \equiv 1 & \text{for } d \neq 0 \\ \sum_{i=1}^{n} D_i (\log(x_i) - e_i \log(u)) \equiv 1 & \text{for } d = 0 \end{cases}$$

where $D_i, e_i > 0$, and $d < 1$.

The advantage of this utility function is that it provides a role for both income and commodity taxes. In our example, d was given the value 0.5, implying an elasticity of substitution of 0.5. The values of D and e are given in Table 4.

The second factor of production can be regarded as a capital good. It

TABLE 4 Values for D and e.

	D	e
Luxury	0.3	0.8
Necessity	0.7	1.2
Leisure	0.5	1.0

†A convenient account of the properties of nonhomothetic CES functions can be found in Hanoch (1975). We are grateful to Nick Rau for bringing this paper to our attention.

TABLE 5 Production activities.

	1	2	3	4	5
Luxury	0.0	0.0	6.0	8.0	7.0
Necessity	4.0	3.5	0.0	0.0	0.0
Labor	−9.0	−10.0	−11.0	−17.0	−12.0
Capital	−5.3	− 5.0	− 2.0	− 2.0	− 2.0

was assumed that the capital good was owned by the government and that the ratio of the quantity of capital to total labor time (work plus leisure) was 1:12. Thus the mean of the log-normal efficiency distribution in this case was 12. Production was represented by five activities, as shown in Table 5.

The results from this model are summarized in Tables 6 and 7, which correspond to Tables 2 and 3. In this case the necessity was chosen as the good on which there would be no tax. The lump-sum is given in terms of the quantity of the necessity that it can buy. It is worth noting that a positive income tax (tax on labor) implies a smaller consumer price than a producer price, while a positive commodity tax implies a larger consumer price than a producer price.

The results in Table 6 are interesting in that, although they show the expected pattern of increased luxury tax with increased aversion to inequality, the income tax rate is actually negative. This superficially paradoxical result appears because, although the consumer price of labor is lower after the imposition of the taxes, the producer price of labor has fallen considerably more as a result of the shift in the pattern of demand.

This fact is confirmed by Table 7, where it can be seen that an increase in skill dispersion has reduced the tax on labor – an apparently paradoxical result. However, the consumer price of labor was virtually unchanged. The reason for the differences between movements in taxes and consumer prices is the change in producer prices that results from the change in demand patterns. This example, therefore, underlines the importance of allowing for producer price variations when calculating optimum tax rates, even though the sparseness of the set of activities might be responsible for the particularly large changes in producer prices that are observed here.

The behavior of producer prices is summarized in Table 8, where the necessity is chosen as numeraire.

The examples given here were started from all the corners of the

TABLE 6 Tax rates for different degrees of inequality aversion.

	$\rho = 1$	$\rho = 0$	$\rho = -1$	$\rho = -5$
Labor	−263%	−261%	−259%	−261%
Luxury	74%	79%	80%	90%
Lump-sum	0.43	0.43	0.43	0.44

TABLE 7 Tax rates for different degrees of skill dispersion.

	$\sigma = 0.3$	$\sigma = 0.39$	$\sigma = 0.5$
Labor	54%	−261%	−248%
Luxury	−35.5%	79%	74%
Lump-sum	0.41	0.43	0.43

TABLE 8 Producer prices.

	$\sigma = 0.3$	$\sigma = 0.39$
Capital	0.000	0.658
Labor	0.444	0.058
Luxury	0.762	0.286
Necessity	1.000	1.000

simplex and, in each case, the solutions obtained were identical. Thus, the underlying nonconvexities do not appear to have caused any serious problems in these cases.

An indication of the algorithm's efficiency can be obtained from the fact that the examples in Section 5.1, with a final grid size of 1,000, terminated in approximately 4 minutes while those in Section 5.2 terminated in approximately 20 seconds, both of which were run on the IBM 370/158 computer. Other experience with the Scarf algorithm and with linear programs suggests that the computation time will increase in proportion to the number of activities and in proportion to the fourth power of the number of commodities.

These calculations illustrate the sensitivity of tax rates to alternative combinations of the various parameters. No particular significance attaches to the actual numbers presented here.

6 CONCLUSIONS

This report has presented a technique for computing optimum linear redistributive policies in a general equilibrium framework. Its performance in the case of linear taxation has been illustrated by two numerical examples. The basic model employed here is rich enough to permit numerous extensions. Three such extensions, which are being pursued at length in a sequel to this report, are restricted taxation, government expenditure, and the computation of shadow prices for public projects. But it is interesting to sketch the approach used.

The assumption that the government can tax every commodity in the economy ignores political and administrative considerations. Thus, to quote one example, it might be impractical to suggest differential taxation on different types of labor. We shall therefore assume that the set of all commodities is partitioned into preselected groups, that all elements of a group must be taxed at an *ad valorem* rate common to that group, and that the government may choose group tax rates optimally. The introduction of these constraints alters the planning problem in certain ways. Since consumer prices are restricted to bearing a particular relationship with producer prices, the government can no longer use commodity taxation to mimic the effects of quantitative controls on private production. This usually implies the desirability of aggregate production inefficiency and calls for a distinction between private and public production on the one hand and consumer prices, shadow prices, and private producer prices on the other. These considerations lead to modifications in the rules of association described in Chapter 3.

A second extension we should like to explore consists in placing greater emphasis on the consequences of government expenditure. The analysis so far has considered a purely redistributive government. It would therefore be instructive to examine the effects on the optimum redistributive tax structure and on the lump-sum grant of the existence of alternative vectors of government requirements which are fixed *a priori* and which represent a prior charge on the revenue. Such an extension is easy to do: the feasibility condition for such an economy requires that production be sufficient to meet both public and aggregate private net demand for goods and services. Similar exercises in optimum income tax models have been carried out by Atkinson (1973), Feldstein (1972c), and Stern (1976).

Finally, the algorithm developed in this report can be used to compute shadow prices for public sector projects. We should therefore like to compare the results of applying this general equilibrium procedure with the partial equilibrium methods that are used typically in this area. This comparison will make use of data from a less developed country and will highlight the effect of the government's redistributive values on the system of accounting prices.

In conclusion, two facts should be borne in mind during the course of further work with the tax algorithm. First, the lack of concavity of the tax program in the control variables prevents our computational procedure from being certain of finding a global optimum. Second, the applicability of optimum tax computations depends on both the adequacy of available specifications of economywide general equilibrium models, and the reliability of data for the degree of heterogeneity within the population. Parallel research on specification of optimum income tax models suggests that the difficulties to be overcome in these areas are not inconsiderable. (See Stern, 1976).

APPENDIX†

It was assumed in the text for expository convenience that

(d.1) The consumption set, C_h, is bounded above for all h.
(d.2) The vector of full endowments, $\tilde{x}_h$, is strictly positive for all h.

Neither of the assumptions is satisfied by the numerical examples of Chapter 5. This appendix therefore dispenses with (d.1) and (d.2) and modifies the rule of association for vectors $q^1, \ldots, q^k$ on the boundary of the unit simplex. Certain boundary conditions are introduced to ensure that the final primitive set does not include any of the boundary labels; the discussion is concluded with an examination of their economic significance.

The rules of association used in the text are well defined for all q in S_{n+1} such that $q_i \geqq \epsilon \quad (i = 1, \ldots, n + 1)$ where ϵ is a small positive number. This guarantees that "full" income, M_h, is above the minimum possible for all h and ensures that net trades are bounded above. Hence, by assumption (a.5), $v_h(q)$ and $x_h(q)$ are differentiable continuously for all h whenever $q_i \geqq \epsilon \quad (i = 1, \ldots, n + 1)$; so also are $V(q)$ and $X(q)$. We enlarge the above set by removing any restriction on the domain of variation of q_{n+1}, the poll subsidy, and define

$$S_n^* = \{q \in S_{n+1} \mid q_i \geqq \epsilon \quad (i = 1, \ldots, n)\} \tag{A.1}$$

where ϵ is an arbitrarily small positive number. We now make the following assumption:

†We should like to thank an anonymous referee for emphasizing to us the importance of boundary problems.

Assumption 1. $\tilde{x}_h \geqslant 0$ for all h.

In other words, no household has a zero vector of "full" endowments. This assumption, which is much weaker than (d.2) is satisfied by the numerical examples given in this report. It implies that every household's "full" income is above the minimum possible in the set S_n^*. $V(q)$ and $X(q)$ are therefore differentiable continuously in S_n^* and the rules of association in the text are well defined there. Difficulties can therefore arise only within the set defined by

$$S_{\epsilon,n} = \{q \in S_{n+1} | q_i \leqq \epsilon \quad \text{for some } i; \quad (i = 1, \ldots, n)\}. \quad \text{(A.2)}$$

In order to anticipate subsequent discussion, elements in $S_{\epsilon,n}$ will be given labels corresponding to elements to be specified in S_n^*; we should then like to exclude these boundary labels from the final primitive set. Such a procedure is justified provided that a solution to the redistributive tax problem does not actually lie in $S_{\epsilon,n}$. This requires a more careful examination of that set. Towards that end, let us recall the set E defined in (2.6),

$$E = \{q \in D | X(q) \leqq Ay \qquad \text{for some } y \geqq 0\}$$

where D is given by (2.4); let us then define

$$Z = \{q \in S_{n+1} | X(q) \leqq Ay \qquad \text{for some } y \geqq 0\}$$

as the set of feasible price vectors, so that $E = Z \cap D$.

Clearly, $S_{\epsilon,n} = [S_{\epsilon,n} \cap E] \cup [S_{\epsilon,n} \cap E^c]$. We may then postulate the following assumption.

Assumption 2. $S_{\epsilon,n} \cap E = \phi$.

On noting that $E = Z \cap D$, we may paraphrase the above as follows. Price vectors in $S_{\epsilon,n}$ lead to the realization of, at most, one of two possibilities: (i) net trades are producible, or (ii) "full" incomes of all households exceed a small positive number. The assumption is satisfied by the numerical examples given in the text for low enough values of ϵ, since $S_{\epsilon,n} \cap Z = \phi$ in both cases. The economies treated in this appendix (by way of contrast with those analyzed in the text) therefore call for modifications to the Scarf-Hansen labeling procedure at the boundaries of the unit simplex.

Assumption 2 implies that $S_{\epsilon,n} \subset E^c$ where E^c is the complement of E in S_{n+1}. Furthermore, Lemma (2.1) asserts that E is closed in S_{n+1}; when E^c is open and for any point in $S_{\epsilon,n}$ there exists a nontrivial neighborhood in E^c $(= Z^c \cup D^c)$.

We are now in a position to modify, where necessary, the rules of association (3.1) and (3.2) of the text to take account of boundary problems. No changes are made for $q \in S_n^*$. For $q \bar{\in} S_n^*$, we find the element

q^* in S_n^* that is closest in Euclidean distance to q with $q_{n+1}^* = q_{n+1}$. We assign to q the label q^*, which is obtained by means of the rules (3.1) and (3.2) of the text. We can easily check that the extended labeling both preserves boundedness of solutions to $Bw = e_{n+1}$ and satisfies the upper semicontinuity argued in Lemma 3.1.

It remains to show that the final primitive set does not include any of the boundary labels introduced above. Toward this end, we introduce two boundary conditions whose economic significance is explored later in the appendix.

DEFINITION. *An indirect social welfare function is said to exhibit minimal consideration (MC) toward all households provided that there exist small positive numbers* η, ϵ *such that whenever a vector* q *leads to the full income of some household being no greater than* $\eta (0 < M_h(q) \leq \eta$ *for some* h) *and* $0 < q_i \leq \epsilon$ *for some* i (i = 1, . . . , n), *any small price change* δq (*where* $\delta q^T e_{n+1} = 0$) *with* $\delta q_i = \alpha q_i (\alpha > 0)$ *for all* i = 1, . . . , n *for which* $q_i \leqq \epsilon$ *and with* $\delta q_i = -\beta q_i (\beta > 0)$ *for all* i = 1, . . . , n *for which* $q_i > \epsilon$ *and* i = n + 1 *will have the property that* $\delta q^T V'(q) > 0$.

Boundary condition (B). For all ϵ sufficiently small, there does not exist a pair of price vectors (p, q) where $q \in (S_{\epsilon,n} \cap S_n^*)$ and p satisfies the program (P) such that $p^T X(q) \geqq 0$ and

$$\delta q^T X'(q)^T(p) \geqq 0$$

where δq $(\delta q^T e_{n+1} = 0)$ is a vector with the property $\delta q_i = \alpha q_i$ $(\alpha > 0)$ for all $i = 1, \ldots, n$ for which $q_i \leqq \epsilon$ and $\delta q_i = -\beta q_i$ $(\beta > 0)$ for all $i = 1, \ldots, n$ for which $q_i > \epsilon$ and $i = n + 1$.

We can now state and prove the following theorem

THEOREM. *Assume*

(i) *that the indirect social welfare function* V(q) *extends minimal consideration (MC) to all households*

(ii) *that the boundary condition (B) is satisfied. Then the final primitive set whose columns form a feasible basis for* $Bw = e_{n+1}$ *does not include any of the labels introduced by the extended labeling rule of this appendix.*

Proof. Choose ϵ and η to suit those that are defined in (MC) and (B) and suppose to the contrary that the final primitive set includes labels introduced for $S_{\epsilon,n}$. Equation (4.2) then becomes

$$\lambda^*[e_{n+1} + V'(q^*)] + \sum_j \mu_j^*[e_{n+1} - X'(q^*)^T p^{*j}] \leqslant e_{n+1} \qquad (\hat{q} \geqslant 0) \tag{A.4}$$

where λ^* and μ_j^* are subsequential limits, q^* is the nearest point in S_n^* to $\hat{q}$ with $q_{n+1}^* = q_{n+1}$ and the p^{*j} solve the parametric program (P) with $X(q) = X(q^*)$. Premultiplying (A.4) by q^{*T} and exploiting, as before, homogeneity of degree zero of the indirect social welfare function and aggregate net trades, we obtain

$$\lambda^* + \sum_j \mu_j^* \leqslant 1;$$

whence (A.4) becomes

$$\lambda^* V'(q^*) - X'(q^*)^T \left[\sum_j \mu_j^* p^{*j}\right] \leqq \left[1 - \left(\lambda^* + \sum_j \mu_j^*\right)\right] e_{n+1} \qquad (\hat{q} \geqslant 0) \tag{A.5a}$$

$$\left[1 - \left(\lambda^* + \sum_j \mu_j^*\right)\right] e_{n+1} \geqq 0. \tag{A.5b}$$

Assume first that $\lambda^* > 0$. This implies, from statement (4.8.ii) of the text, that $X(q^*)$ is feasible, i.e., $q^* \in Z$. Since $q^* \in Z^c \cup D^c$, then $q^* \in D^c$, i.e., $M_h(q^*) \leqslant \eta$ for some h. The (MC) condition may then be invoked. Premultiply (A.5a) by the δq^T introduced in the definition of (MC). Let L be the set of indices for which the inequalities in (A.5a) are strict, where L is a proper subset of $(1, \ldots, n+1)$. Then $\hat{q}_l = 0$ for $l \in L$ and the extended labeling rule ensures that $q_l^* = \epsilon$ for all $l \in L$ from 1 to n and $\delta q_l^* = \alpha q_l^* > 0$. The multiplication therefore preserves the sense of the inequalities in (A.5a), and we have

$$\lambda^* \delta q^T V'(q^*) - \delta q^T X'(q^*)^T \left[\sum_j \mu_j^* p^{*j}\right] \leqq 0 \tag{A.6}$$

since $\delta q^T e_{n+1} = 0$. Since $X(q^*)$ is feasible, it follows from the parametric program (P) that $E_j = (p^{*j})^T X(q^*)$ is nonpositive for all j. If $E_j < 0$, statement (4.8.iii) of the text implies that $\mu_j^* = 0$ for all j. (A.6) then reduces to

$$\lambda^* \delta q^T V'(q^*) \leqq 0,$$

which contradicts (MC). Hence $E_j = 0$ for all j. Condition (B) may therefore be used and δq^T in (A.6) chosen to be that which satisfies both (MC) and (B). But (MC) implies that $\lambda^* \delta q^T V'(q^*) > 0$; whence (A.6) shows that

$$\delta q^T X'(q^*)^T \left[\sum_j \mu_j p^{*j}\right] > 0.$$

But this contradicts (B). Hence the supposition that $\lambda^* > 0$ is false.

Since $\lambda^* = 0$, $\mu_j^* > 0$ (for some j), then equations (A.5) become

$$X'(q^*)^T \left[\sum_j \mu_j p^{*j}\right] \geqq -\left[1 - \sum_j \mu_j^*\right] e_{n+1} \qquad (\hat{q} \geqslant 0) \qquad \text{(A.7a)}$$

$$\left[1 - \sum_j \mu_j^*\right] e_{n+1} \geqq 0. \qquad \text{(A.7b)}$$

Since $\mu_j^* > 0$ for some j, then statement (4.8.iii) of the text and the parametric program (P) together imply that $E_j = (p^{*j})^T X(q^*) \geqq 0$ for all j. Now premultiply (A.5a) by a δq^T introduced in (B). As before, the inequalities are preserved and we obtain

$$\delta q^T X'(q^*)^T \left[\sum_j \mu_j^* p^{*j}\right] \geqq -\left(1 - \sum_j \mu_j^*\right) \delta q^T e_{n+1} = 0.$$

But this once again contradicts (B). Thus, the final primitive set cannot contain any of the boundary labels introduced in this appendix and the theorem is proved.

The rest of this appendix is devoted to a discussion of the boundary conditions introduced above. We begin with the notion of minimal consideration. The condition has force when certain households are at or are near starvation level, presumably because the post tax returns to their endowments are very low. Minimal consideration asserts that any proportional increase in the consumer prices of goods with a very low price, together with a proportional decrease in other consumer prices and in the poll subsidy, improves social welfare. An increase in the prices of low-priced goods and services can be expected to benefit improvident households because Assumption 2 guarantees that no household has zero endowments. The condition is a weak one and we imagine that a society that cares about redistribution would find it appealing.

The boundary condition (B) requires that any price change of the above kind made when certain prices are very low and the value of aggregate net trades are positive at producer prices must decrease the value of those trades at those same producer prices. This interpretation follows from the observation that, for small changes δq,

$$\delta q^T X'(q)^T p = [X(q + \delta q) - X(q)]^T p.$$

The boundary condition (B) is therefore related in spirit to the regularity condition (R). It applies only for q belonging to $S_{\epsilon,n} \cap S_n^*$. But within that area it is not possible to postulate the existence of a direction of movement which reduces the value of aggregate net trades at pre-change producer prices. Any change leading to a proportional increase in all consumer prices which are equal to ϵ and to a proportional decrease in all other prices and in the poll subsidy must indicate the existence of a direction. If, pursuing analogous interpretations as in the case of (R), producer prices are now assumed constant, the condition (B) postulates that, starting from $S_{\epsilon,n} \cap S_n^*$ proportionate changes in consumer prices of the kind described must be accompanied by an improvement in the government's budget deficit whenever that deficit is nonnegative. The changes contemplated here might typically include reduction of a subsidy on a final good whose price is very low. Or, to choose another example, they might involve a reduction in the rate of a high income tax to improve an after-tax wage which has become very low. Under these circumstances, the claim that the budget deficit will improve is not an unreasonable one.

Finally, the poll subsidy, q_{n+1}, has received special treatment throughout this appendix because the condition (B) could not be regarded as reasonable if the increases in components of q considered included increases in the poll subsidy.

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