OPTIMAL MIGRATION POLICIES: AN ANALYTICAL APPROACH

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FOREWORD

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. During the past three years this interest has given rise to a concentrated research activity focusing on migration dynamics and settlement patterns. Four sub-tasks have formed the core of this research effort:

- the study of spatial population dynamics;
- the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;
- the analysis and design of migration and settlement policy;
- a comparative study of national migration and settlement patterns and policies.

This paper focuses on normative population modeling. It suggests an extension of demographic projection models to the policy domain by introducing concepts and findings of mathematical system theory.

Reports, summarizing previous work on migration and settlement at IIASA, are listed at the end of this report.

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OPTIMAL MIGRATION POLICIES

An Analytical Approach – Part I*

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This paper explores the analytical features of population distribution or human settlement policies. It proposes a methodology for quantitative policy analysis and policy design based on optimal control and system theory. The paper consists of two parts. This part shows how policy models may be derived from demographic and demoeconomic or demometric models by adding a new dimension: the goals–means relationship of population distribution policy. It examines a large class of relevant policy models and demonstrates their relationship to the original Tinbergen *Theory of Policy*, which provides a paradigm for static and dynamic policy analysis. Problems of *existence* and of *design* of optimal population distribution policies are studied analytically. In designing optimal policies, use may be made of the minimizing properties of generalized inverses.

1. Introduction

In recent years there has been an increasing interest in the dynamics of spatial demographic growth. Models for multiregional population growth have been developed to describe the growth process and to analyze its impact on future population characteristics [Rogers (1975)]. The various economic, social, climatological and cultural forces influencing spatial population growth have been brought together in explanatory demoeconomic or demometric models [Greenwood (1975)]. The mathematical demographic models and the demometric models have a common feature. They are designed to describe and to explain the dynamics of the spatial population growth.

Once the dynamics of a phenomenon are understood, human nature comes up with the ultimate question: can we control it, and how? The models associated with this third concern are population policy models. The subject of migration policy models has been introduced by Rogers (1966; 1968, ch. 6;

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1971, pp. 98–108), and more recently a number of authors devoted attention to the design of optimal-seeking migration policy models [MacKinnon (1975), Mehra (1975), Stern (1974), Evtushenko and MacKinnon (1975), Willekens and Rogers (1977), Willekens (1978), and Propoi and Willekens (1978)].

This paper is devoted to a methodological analysis of migration policy models. We assume that a demometric or a demographic model, consisting of a system of linear simultaneous equations, has been successfully specified and estimated. Therefore, we do not deal with identification and estimation procedures. The main thread of the analysis is provided by the *Tinbergen paradigm*, to which we will refer frequently. The paper consists of two highly interrelated parts. Part I focuses on the Tinbergen framework itself and on its application to population distribution policy making. Part II, which will be published in a later issue, addresses the generalization of Tinbergen's ideas to dynamic problems, derives consistent formulations of state-space and optimal control models, and applies them to the quantitative study of dynamic population distribution policies.

Part I is divided into two major sections, sections 2 and 3. Section 2 presents a conceptual survey of various possible policy models. Each model is related back to the original Tinbergen framework. The matrix of impact multipliers, well known in economic analysis, is seen to be of crucial importance to the classification scheme. After this section has set the scene, we devote our attention to the two central issues in the theory of policy: the concepts of *existence* and of *design* (section 3). The existence problem deals with the question whether the system is controllable, i.e., whether a set of arbitrary targets can be achieved at all, given the internal dynamics of the system and given the set of available instruments. The answer to the controllability problem provides input information for the design problem. In designing optimal policies, the policy maker may apply a wide range of mathematical programming techniques, assuming that he has a clear idea of his preferences. Attention is focused, however, on policy models for which solutions may be obtained analytically. The concept of generalized inverse has an interesting potential here.

2. Optimal migration policies: A conceptual framework

There are several analytical differences between a policy model and a conventional demographic or demometric model. The most basic classification of variables in any model consists of two categories: *endogenous variables*, which are determined within the model, and *exogenous variables*, which are predetermined. Suppose the population system is linear and may be modeled as

$$A\{y\}=E\{z\}, \tag{2.1}$$

where $\{y\}$ is a $M\times 1$ vector of endogenous variables, $\{z\}$ is a $L\times 1$ vector of exogenous variables, A is a $M\times M$ matrix of coefficients, and E is a $M\times L$ matrix of coefficients.

Eq. (2.1) is the reduced form of a population model. The endogenous and the exogenous variables are separated. Assuming that A is non-singular, we obtain

$$\{y\}=A^{-1}E\{z\}=C\{z\}, \tag{2.2}$$

where C is the matrix of multipliers, i.e., the reduced form matrix. The elements of C represent the impact on $\{y\}$ of a unit change in $\{z\}$.

The policy models treated here, will be discussed with reference to (2.2). Tinbergen (1963) proposed a classification of the variables of (2.2) better suited for policy analysis. His ideas are general enough to encompass the whole range of policy models. Starting from the Tinbergen paradigm, we try to present a unified treatment of various classes of models, which are relevant for population distribution policy.

2.1. The Tinbergen paradigm

Tinbergen (1963) distinguished two categories of variables in both the endogenous and the exogenous variables. The endogenous variables consist of *target variables*, which are of direct interest for policy purposes, and other variables which are not. The latter are labeled by Tinbergen as *irrelevant variables*. However, they may be of indirect interest for policy planning, since their values may in turn influence various target variables. The exogenous variables are divided according to their controllability. *Instrument variables* are subject to direct control by the policy authorities. *Data variables* are beyond their control. The latter include exogenously predetermined and uncontrollable variables, as well as lagged endogenous variables. They define the environment in which the levels of instrument variables have to be set. Applying this approach, eq. (2.2) may be partitioned to give

$$\begin{bmatrix}\{y_1\}\\\{y_2\}\end{bmatrix}=\begin{bmatrix}R & S\\P & Q\end{bmatrix}\begin{bmatrix}\{z_1\}\\\{z_2\}\end{bmatrix},$$

where $\{y_1\}$ is the $N\times 1$ vector of target variables, $\{y_2\}$ is the $(M-N)\times 1$ vector of other endogenous variables, $\{z_1\}$ is the $K\times 1$ vector of instrument variables, $\{z_2\}$ is the $(L-K)\times 1$ vector of uncontrollable exogenous variables and lagged endogenous variables, and $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are conformable partitions of the model's reduced form matrix.

The value of the target vector is

$$\{y_1\}=\boldsymbol{R}\{z_1\}+\boldsymbol{S}\{z_2\}. \tag{2.3}$$

The policy problem, as formulated by Tinbergen, is to choose an appropriate value of the instrument vector $\{z_1\}$ so as to render the value of the target vector $\{y_1\}$ equal to some previously established desired value $\{\bar{y}_1\}$. The choice of the level of the instrument variables depends on the levels of the uncontrollable variables, represented by $\{z_2\}$, and on how much they affect the targets.

It is important to keep in mind that the policy model (2.3) is derived from the explanatory model (2.2) by adding a new dimension to (2.2). This new dimension is the goals–means relationship of population policy. The explanatory model may be a pure demographic model, relating population growth and distribution to demographic factors such as fertility, mortality and migration. It may also be a demometric model, which statistically relates spatial population growth to socioeconomic variables. *Any model may be converted into a policy model if and only if all the target variables of the policy model are part of the set of endogenous variables of the explanatory model and if at least one of the exogenous variables is controllable.* Most migration models found in the literature are single-equation models with gross or net migration as the dependent variable. They serve only a restricted category of policy models, namely those with targets that consist of migration levels and instruments which are socioeconomic in nature. Various regional economic models include migration as an exogenous variable. Therefore, they are not suited to become migration policy models if population distribution is the goal. Simultaneous equation models, such as the ones developed by Greenwood (1973, 1975) and Olvey (1972), are relevant to model population policy problems of all types, because they include demographic and socioeconomic variables in both the set of endogenous and the set of exogenous variables. Thus they may be applied in situations where the goals–means relationship consists of demographic, as well as of socioeconomic measures. Finally, the multiregional population growth models of Rogers (1975) may be converted to policy models to study purely demographic policy problems, i.e., both targets and instruments are demographic in nature.

Before proceding, it may be important to stress that the analytical solution of Tinbergen's formulation of the policy problem is restricted to linear policy models. If the model is non-linear, one can only solve it numerically through simulation. In this paper, we only deal with linear models and their analytical solutions and do not discuss the simulation approach.

2.2. Survey of policy models

Conceptually, any policy model may be related to (2.3). For convenience, we drop the subscript of the target vector,

$$\{y\}=\boldsymbol{R}\{z_1\}+\boldsymbol{S}\{z_2\}.$$

Throughout our discussion of policy models, it will be assumed that both the targets and the instruments are linearly independent. The matrix $\boldsymbol{R}$ then plays a crucial role in policy analysis. The *existence* of an optimal policy, i.e., a solution to (2.3), depends on the rank of $\boldsymbol{R}$. The *design* of an optimal policy, i.e., the assignment of values to the instrument variables, depends on the structure of $\boldsymbol{R}$, and on the values of its entries. In economic literature, the matrix $\boldsymbol{R}$ is known as the *matrix of impact multipliers*. The name refers to the fact that an element r_{ij} gives the change in the value of the target variable i when the instrument variable j is varied by one unit. The ratio $-r_{ij}/r_{ik}$ is the amount by which the jth instrument may be cut down without changing the level of the ith target, if the value of the kth instrument is increased with one unit. It is, therefore, the marginal rate of substitution between two instruments [Fromm and Taubman (1968, p. 109)].

It is the purpose of this section to classify relevant policy models without going into technical detail. Detailed treatment will be given later. The survey revolves around the matrix multiplier $\boldsymbol{R}$ and its characteristics. A first classification scheme is based on the rank of $\boldsymbol{R}$, or alternatively on the relation between the number of targets and the number of instruments. A second classification scheme relates to the structure of $\boldsymbol{R}$. The structure of $\boldsymbol{R}$ also provides us with a link between the reduced form models and the models of optimal control.

2.2.1. Classification of policy models according to the rank of the matrix multiplier

We may distinguish between three categories of policy models: $\boldsymbol{R}$ is non-singular and of rank N, $\boldsymbol{R}$ is singular and of rank K, $\boldsymbol{R}$ is singular and of rank N. The parameters K and N are respectively, the number of instruments and the number of targets. An illustration is given by a typical policy model, namely the Theil (1964) model.

(*a*) *The matrix multiplier is non-singular and of rank N.* If $\boldsymbol{R}$ is non-singular, i.e., if there are as many instruments as there are targets, then there exists a unique combination of instruments leading to the set of desired targets. Once the targets are specified, the unique vector is given by

$$\{z_1\} = \boldsymbol{R}^{-1}[\{\bar{y}\} - \boldsymbol{S}\{z_2\}]. \tag{2.4}$$

The solution to (2.3) is unique, and there is no need for the policy maker to provide any other information than the set of target values.

(*b*) *The matrix multiplier is singular and of rank $K<N$.* If the number of instruments is less than the number of targets, however, the system (2.3) is inconsistent and there is no way that all the target values can be reached. This poses an additional decision problem for the policy maker. Does he give up some targets in order to reach others, or does he want to achieve all the targets as closely as possible with the limited resources? In the latter case, the policy maker may also wish to weight the targets differently. If the first alternative is chosen, some targets are deleted, and the instrument vector is given by (2.4). The second alternative often leads to the formulation of a quadratic programming model. If $\{\bar{y}\}$ is the vector of desired target values, and $\{\hat{y}\}$ is the vector of realized values, then the problem is to minimize the squared deviation between $\{\bar{y}\}$ and $\{\hat{y}\}$ subject to (2.3), which describes the behavior of the population system. That is,

$$\min[\{\bar{y}\} - \{\hat{y}\}]'\boldsymbol{A}[\{\bar{y}\} - \{\hat{y}\}], \tag{2.5}$$

subject to

$$\{\hat{y}\} = \boldsymbol{R}\{z_1\} + \boldsymbol{S}\{z_2\}. \tag{2.6}$$

The weight matrix $\boldsymbol{A}$ represents the policy maker's differential preferences towards the targets. The target variables with the highest weights will be forced very close to their desired values. Those with the lowest weights will not.

(*c*) *The matrix multiplier is singular and of rank N.* If the number of instrument variables exceeds the number of targets, then there is an infinite number of solutions to (2.3) and, therefore, an infinite number of instrument vectors. To get a unique solution, the policy maker may force the number of instruments to be equal to the number of targets, by deleting some instruments. On the other hand, he may put some constraints on the instruments.

By restricting the value instruments may take, the freedom of policy action is reduced in a way such that only one strategy is available to achieve the targets. Some alternative restrictions will be introduced later in the paper.

(*d*) *Illustration: The Theil quadratic programming model.* We have described how policy models are related to the rank of the matrix of impact multipliers or, equivalently, to the number of targets and instruments. Only some alternative policy models have been indicated. A wider variety is possible. For example, the targets and the instruments may be constrained at the same time, and these constraints need not be linear. The objective function (2.5) may not be quadratic, and (2.6) can be supplemented with both equality and inequality constraints. The reader is referred to the mathematical programming literature for such illustrations. The quadratic objective function with linear constraints, however, is common in economic policy analysis. It is based on two assumptions. The first is that the policy maker's preferences are quadratic in targets and controls. The second assumption is that each of the targets depends linearly on all the instruments, the coefficients of these linear relations being fixed and known. The basic structure of this linear quadratic model is due to Theil (1964, pp. 34–35), and may be expressed as

$$\begin{aligned} \min W = {} & \{\boldsymbol{a}\}'\{z_1\} + \{\boldsymbol{b}\}'\{\hat{y}\} \\ & + \tfrac{1}{2}[\{z_1\}'\boldsymbol{A}\{z_1\} + \{\hat{y}\}'\boldsymbol{Q}\{\hat{y}\} + \{z_1\}'\boldsymbol{C}\{\hat{y}\} + \{\hat{y}\}\boldsymbol{C}'\{z_1\}], \end{aligned} \tag{2.7}$$

subject to (2.3),

$$\{\hat{y}\} = \boldsymbol{R}\{z_1\} + \boldsymbol{S}\{z_2\},$$

where $\{\hat{y}\}$ is the vector of realized values of the target variables, $\{z_1\}$ is the vector of instrument variables, $\{z_2\}$ is the vector of exogenous variables, $\boldsymbol{A}$, $\boldsymbol{Q}$, $\boldsymbol{C}$ are weight matrices, and $\boldsymbol{R}$, $\boldsymbol{S}$ are matrices of multipliers.

Applications of the Theil model in economic policy literature may be found in Fox, Sengupta and Thorbecke (1972, p. 215), and in Friedman (1975, pp. 158–160). To simplify matters we may suppose that $\{\boldsymbol{a}\} = \{\boldsymbol{b}\} = \{\boldsymbol{0}\}$ and $\boldsymbol{C} = \boldsymbol{0}$. The problem then reduces to

$$\min \tfrac{1}{2}[\{\hat{y}\}'\boldsymbol{Q}\{\hat{y}\} + \{z_1\}'\boldsymbol{A}\{z_1\}], \tag{2.8}$$

subject to (2.3),

$$\{\hat{y}\} = \boldsymbol{R}\{z_1\} + \boldsymbol{S}\{z_2\},$$

where $\boldsymbol{Q}$ and $\boldsymbol{A}$ are weights attached to the target vector and to the instrument vector respectively.

To illustrate the application of the Theil model in migration policy analysis, consider the following problem. The costs of public services are held to be too high because some regions are over-urbanized and are subject to diseconomies of scale, while other areas have insufficient people to reach the threshold needed for an efficient public service system. The high costs in the public sector can, therefore, be related to the inefficient population distribution. To reduce the costs, a migration policy is needed. However, there is a cost associated with the redistribution of people over space. Assume that the cost function of public services is a quadratic function of the population distribution $\{\hat{y}\}$, i.e.,

$$C_p = \{\boldsymbol{b}\}'\{\hat{y}\} + \{\hat{y}\}'\boldsymbol{E}\{\hat{y}\}. \tag{2.9}$$

Assume also that the cost associated with population distribution is quadratic in the vector of the number of people relocated by the policy program, $\{z_1\}$, i.e.,

$$C_m = \{z_1\}'\boldsymbol{F}\{z_1\}. \tag{2.10}$$

An element z_{1i} of $\{z_1\}$ is positive if the program attracts people to region i. It is negative if the program has an out-migration effect. On comparing the cost functions with the preference function (2.7), we see that

$$\{\boldsymbol{a}\} = \{\boldsymbol{0}\}, \qquad \boldsymbol{C} = \boldsymbol{0}, \qquad \boldsymbol{Q} = 2\boldsymbol{E},$$

and

$$\boldsymbol{A} = 2\boldsymbol{F}.$$

Since $\{z_1\}$ represents the additional migration, $\boldsymbol{R} = \boldsymbol{I}$ in the constraint. The vector of uncontrollable variables is the population distribution in the previous time period, and $\boldsymbol{S}$ is the multiregional population growth matrix.

2.2.2. *Classification of policy models according to the structure of the matrix multiplier*

We now turn to the question of how policy models may be related to the structure of the matrix $\boldsymbol{R}$. The structure determines the nature of the dependence of $\{z_1\}$ upon $\{y\}$. Several assumptions may be adopted to simplify the form of $\boldsymbol{R}$. They have been studied by Tinbergen (1963, ch. 4), by Fox, Sengupta and Thorbecke (1972, pp. 24–25) and by Friedman (1975, pp. 149–153) among others. We consider four different structures of $\boldsymbol{R}$: diagonal, triangular, block-diagonal and block-triangular. Our illustration considers the block-triangular multiperiod policy model.

(*a*) *The matrix multiplier is diagonal.* If $\boldsymbol{R}$ is diagonal, then each target variable can be associated with one and only one instrument variable and vice versa. Since $\boldsymbol{R}^{-1}$ is also diagonal, eq. (2.4) implies a series of expressions,

$$\bar{z}_{1i} = \frac{1}{r_{ii}} \left[\bar{y}_i - \sum_k s_{ik} z_{2k} \right], \qquad i = 1, \ldots, N,$$

each of which may be solved independently. The practical implication of this is that the policy maker can, in such an instance, pursue each target with a single specific instrument, and no coordination between the various policies is required.

(*b*) *The matrix multiplier is triangular.* Eq. (2.3) is recursive. The two-way simultaneity between the vectors $\{y\}$ and $\{z_1\}$, i.e., $\{z_1\}$ affecting $\{y\}$ and $\{y\}$ affecting $\{z_1\}$, can be reduced to a unilateral dependence or a unidirectional causality. Suppose $\boldsymbol{R}$ is lower triangular, then $\boldsymbol{R}^{-1}$ is also lower triangular, and the decision making procedure is recursive,

$$\bar{z}_{11} = \frac{1}{r_{11}} \left[\bar{y}_1 - \sum_k s_{1k} z_{2k} \right],$$

$$\bar{z}_{12} = \frac{1}{r_{22}} \left[\bar{y}_2 - \sum_k s_{2k} z_{2k} - r_{21} \bar{z}_{11} \right],$$

$$\vdots$$

$$\bar{z}_{1i} = \frac{1}{r_{ii}} \left[\bar{y}_i - \sum_k s_{ik} z_{2k} - \sum_{j=1}^{i-1} r_{ij} \bar{z}_{1j} \right].$$

These expressions may be solved in sequence, and the model has a simple policy interpretation. If each equation were assigned to a different policy maker, the system of equations would specify a hierarchy. In order to make an optimal decision, each policy maker would not need to look at the instruments selected by those who were below his position in the hierarchy.

(*c*) *The matrix multiplier is block-diagonal.* In the case of a block-diagonal policy model, the overall model can be decomposed into several independent parts. This would occur if a policy can be decentralized into independent subpolicies, each having a goals–means relationship unrelated to the goals and the instruments of the other subpolicies. This would permit efficient decentralized decision making.

(*d*) *The matrix multiplier is block-triangular.* Here, as in the case of a triangular $\boldsymbol{R}$, the set of instruments corresponding to any given block can be solved for without any knowledge of the instruments belonging to blocks which are lower in the hierarchy. The overall policy could be decomposed into a hierarchical system of policies.

(*e*) *Illustration: The multiperiod policy problem.* An important application of the block-triangular form of $\boldsymbol{R}$ is found in dynamic policy analysis. The models presented thus far have been static, but they are general enough to handle dynamic policy problems as well. If the entries of the target vector and of the instrument vector belong to different time periods, we clearly have a dynamic or multiperiod policy model. Suppose, for example, that a target vector is given for a sequence of time periods from 1 to T, say. Then $\{y\}$ is itself composed of vectors, one for each time period. Suppose, moreover, that there exists an instrument vector for each time period. The reduced form model (2.3) now may be expressed as

$$\{y\}=\boldsymbol{R}\{z_1\}+\boldsymbol{S}\{z_2\}, \tag{2.11}$$

where

$$\{y\}=\begin{bmatrix}\{y^{(1)}\}\\ \{y^{(2)}\}\\ \vdots\\ \{y^{(t)}\}\\ \vdots\\ \{y^{(T)}\}\end{bmatrix},\qquad \{z_1\}=\begin{bmatrix}\{z_1^{(1)}\}\\ \{z_1^{(2)}\}\\ \vdots\\ \{z_1^{(t)}\}\\ \vdots\\ \{z_1^{(T)}\}\end{bmatrix},\qquad \{z_2\}=\begin{bmatrix}\{z_2^{(1)}\}\\ \{z_2^{(2)}\}\\ \vdots\\ \{z_2^{(t)}\}\\ \vdots\\ \{z_2^{(T)}\}\end{bmatrix},$$

$$\boldsymbol{R}=\begin{bmatrix}\boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \cdots & \boldsymbol{R}_{1T}\\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} & & \\ \vdots & & \ddots & \vdots\\ \boldsymbol{R}_{T1} & \cdots & & \boldsymbol{R}_{TT}\end{bmatrix},$$

$$\boldsymbol{S}=\begin{bmatrix}\boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \cdots & \boldsymbol{S}_{1T}\\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & & \\ \vdots & & \ddots & \vdots\\ \boldsymbol{S}_{T1} & \cdots & & \boldsymbol{S}_{TT}\end{bmatrix}.$$

Vector $\{z_1\}$ is of order KT, and $\{z_2\}$ and $\{y\}$ are of order NT. The submatrix $\boldsymbol{R}_{ij}$ is $N\times K$ and its elements are dynamic policy multipliers which express the impact on the target vector $\{y^{(t)}\}$ in time period $t=i$ of changes in the instrument vector $\{z_1^{(t)}\}$ in time period $t=j$. $\boldsymbol{R}$ is $NT\times KT$; $\boldsymbol{S}$ is $NT\times NT$ and the submatrices $\boldsymbol{S}_{ij}$ are of order $N\times N$. $\boldsymbol{S}$ shows the dynamic effects of predetermined variables on the target variables.

Most policy models assume that policy actions do not influence events which precede them in time and, therefore, generally ignore expectational effects or advance announcement effects. This assumption of *unidirectional* *causality* yields a block-triangular $\boldsymbol{R}$ matrix,

$$\boldsymbol{R}=\begin{bmatrix} \boldsymbol{R}_0 & 0 & 0 & \cdots & 0 \\ \boldsymbol{R}_1 & \boldsymbol{R}_0 & 0 & & \\ \boldsymbol{R}_2 & \boldsymbol{R}_1 & \cdot & & \vdots \\ \vdots & \vdots & & \ddots & \\ \boldsymbol{R}_{T-1} & \boldsymbol{R}_{T-2} & & & \boldsymbol{R}_0 \end{bmatrix}, \tag{2.12}$$

where the elements of $\boldsymbol{R}_i$ are dynamic policy multipliers. A triangular $\boldsymbol{R}$ matrix leads to a sequential decision making procedure analogous to that of the static model. The key distinction is that here the sequence is across time, rather than across individual instrument and target variables.

By way of illustration, consider the application of the Theil model in population policy. Assume that there is a time sequence of target population distributions, and a time sequence of vectors of induced migration. Suppose that no tough policy actions are expected by the potential migrants, therefore the population distribution at time t does not depend on the migration policies beyond t. Eq. (2.11) may, therefore, be written with $\boldsymbol{R}$ being lower block-triangular.

We may reduce the form of this policy model even further. Suppose that the migration policy at time t only affects the population distribution at $t+1$ directly. The impact on the population distributions at a later time is indirect in the sense that the population distribution at $t+1$ affects the distribution beyond $t+1$. This implies the recurrence equation

$$\{\boldsymbol{y}(t+1)\}=\boldsymbol{R}_0\{z_1(t)\}+\boldsymbol{S}_{t+1,t}\{\boldsymbol{y}(t)\}. \tag{2.13}$$

The submatrix $\boldsymbol{S}_{t+1,t}$ is the growth matrix of the population between t and $t+1$. If we assume the growth matrix to be time-independent, i.e., $\boldsymbol{G}=\boldsymbol{S}_{t+1,t}$ for all t, we may write

$$\{\boldsymbol{y}(t+1)\}=\boldsymbol{R}_0\{z_1(t)\}+\boldsymbol{G}\{\boldsymbol{y}(t)\}. \tag{2.14}$$

Therefore, (2.11) may be reduced to a set of recurrence equations,

$$\begin{aligned}
\{\boldsymbol{y}(1)\} &= \boldsymbol{R}_0\{z_1(0)\}+\boldsymbol{G}\{\boldsymbol{y}(0)\},\\
\{\boldsymbol{y}(2)\} &= \boldsymbol{R}_0\{z_1(1)\}+\boldsymbol{G}\{\boldsymbol{y}(1)\}\\
&= \boldsymbol{R}_0\{z_1(1)\}+\boldsymbol{G}\boldsymbol{R}_0\{z_1(0)\}+\boldsymbol{G}^2\{\boldsymbol{y}(0)\},\\
&\vdots\\
\{\boldsymbol{y}(t)\} &= \boldsymbol{R}_0\{z_1(t-1)\}+\boldsymbol{G}\{\boldsymbol{y}(t-1)\}\\
&= \boldsymbol{G}^t\{\boldsymbol{y}(0)\}+\sum_{i=0}^{t-1}\boldsymbol{G}^{t-1-i}\boldsymbol{R}_0\{z_1(i)\}.
\end{aligned}$$

In matrix form, we have that

$$\begin{bmatrix} \{y(0)\} \\ \{y(1)\} \\ \{y(2)\} \\ \vdots \\ \{y(T)\} \end{bmatrix} = \begin{bmatrix} I & & & & \\ G & R_0 & & & \\ G^2 & GR_0 & R_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ G^T & G^{T-1}R_0 & G^{T-2}R_0 & \cdots & R_0 \end{bmatrix} \begin{bmatrix} \{y(0)\} \\ \{z_1(0)\} \\ \{z_1(1)\} \\ \vdots \\ \{z_1(T-1)\} \end{bmatrix}. \tag{2.15}$$

Eq. (2.14) has received much attention in system theory. It is called the discrete state equation and forms the central component of the discrete version of the state-space model. Stimulated by recent work in system theory and optimal control, an increasing number of authors have adopted the state-space approach to describe dynamic models in the social sciences.[1] We have shown how the state-space model may be derived conceptually from the reduced form model.

By introducing the assumption of unidirectional causality of the population system, we may write the Theil model [(2.8), (2.3)] as

$$\min J = \tfrac{1}{2}[\{\hat{y}\}'Q\{\hat{y}\} + \{z_1\}'A\{z_1\}],$$

subject to (2.14),

$$\{y(t+1)\} = R_0\{z_1(t)\} + G\{y(t)\}.$$

Recall that Q is a $NT \times NT$ matrix, where T is the planning horizon and N the number of target variables at each period, and A is a $KT \times KT$ matrix, where K is the number of instrument variables.

2.2.3. From the Tinbergen model to the optimal control model

In this section, we started out with the Tinbergen paradigm. The original model, based on this paradigm, was simple in the sense that the number of instruments was equal to the number of targets and that the optimal policy was the unique solution to a system of linear equations. When the number of instruments and targets differs, the policy maker is confronted with an additional decision problem. He needs to specify his preferences in order to get a unique policy which is optimal. This led us to the Theil model and to the broad application of mathematical programming. When policy problems become large, there is a need for simplification. The simplifying assumptions, discussed up to now, are related to the structure of the multiplier matrix $\boldsymbol{R}$.

[1]See, for example, Pindyck (1973), Kenkel (1974) and Chow (1975).

The assumption of unidirectional causality of the system, represented by the block-triangularity of $\boldsymbol{R}$ is crucial to the further development of dynamic policy models. Now we extend the simplification of the policy models to the objective function.

Assume that the effect of the target and control vector at time t on the value of the objective function, is independent of the target and control vectors at other time periods. This implies that the matrices $\boldsymbol{Q}$ and $\boldsymbol{A}$ are block-diagonal. The large multiperiod problem may then be decomposed into a sequence of smaller single-period problems. The objective function becomes

$$\min \tfrac{1}{2} \sum_{t=0}^{T} \{\hat{y}(t)\}'\boldsymbol{Q}(t)\{\hat{y}(t)\} + \{z_1(t)\}'\boldsymbol{A}(t)\{z_1(t)\}. \tag{2.16}$$

It is assumed that $\{z_1(T)\} = \{\boldsymbol{0}\}$. In most practical applications, it is assumed that $\boldsymbol{Q}(t) = \boldsymbol{Q}$ is equal for all time periods up to $T-1$. This assumption is only valid if the preference system and tastes do not change over time. It also implies that the contribution of a certain set of target and control values is independent of when they appear on the time path, since no discounting measure has been introduced. Denote the matrix $\boldsymbol{Q}(T)$ by $\boldsymbol{F}$. The weight matrices $\boldsymbol{A}(t)$ associated with the instruments or controls are usually also assumed to be time independent, $\boldsymbol{A}(t) = \boldsymbol{A}$ for all t.

Therefore, we have

$$\min \tfrac{1}{2}\{\hat{y}(T)\}'\boldsymbol{F}\{\hat{y}(T)\} + \tfrac{1}{2} \sum_{t=0}^{T-1} [\{\hat{y}(t)\}'\boldsymbol{Q}\{\hat{y}(t)\} + \{z_1(t)\}'\boldsymbol{A}\{z_1(t)\}]. \tag{2.17}$$

The block-diagonal structure of $\boldsymbol{Q}$ and $\boldsymbol{A}$ implies that the values of the target variables at time t are independent of their values at previous and at later time periods. This is denoted as the assumption of *inter-temporal separability of the objectives.* The combination of (2.17) with (2.14) is known as the linear-quadratic (LQ) control problem. *This shows that the multiperiod Theil problem may be reduced to a linear quadratic control problem by assuming intertemporal separability of the objectives and unidirectional causality of the population system.* The advantage of describing a dynamic policy problem by an LQ control model is not only a reduction in scale, but also that an analytical solution to the problem can be obtained. The optimal time path of the policy measures may be represented by a simple feedback control law. The calculation of the optimal trajectory will be discussed in Part II of this paper. If the conditions of intertemporal separability of the objectives and unidirectional causality of the population system are not met, the dynamic generalization of the Theil problem must be solved using a quadratic programming algorithm [Theil (1964, ch. 4)].

3. Existence and design of optimal migration policies in the Tinbergen framework

It is argued that there are two central issues in the theory of policy. These are the concepts of existence and of design. Existence of policy refers to the controllability of the system or the ability to design any policy at all; design refers to the techniques for designing optimal policies once existence is assured. Although both issues have been recognized for a long time in system theory, policy analysis in the social sciences, led by the theory of economic policy, has focused almost entirely on the design problem.

Only Tinbergen (1963) has given considerable attention to both issues. His work is reviewed in this section and reformulated to provide a direct link with the study of existence and design in the dynamic state-space framework in Part II of the paper. The purpose of integrating the Tinbergen and the state-space frameworks is to show that the Tinbergen policy problem is a particular case of a more general class of policy problems.

3.1 Existence theorem

Recall the Tinbergen model (2.3):

$$\{y\}=\boldsymbol{R}\{z_1\}+\boldsymbol{S}\{z_2\}.$$

In the original formulation, (2.3) represented a static policy problem, i.e., the targets and the instruments belonged to the same time period. The model, however, may include lagged variables in the vector of uncontrollable variables $\{z_2\}$. Contrary to Preston's (1974, p. 65) claim, the Tinbergen model also fits dynamic situations, where the targets and instruments belong to different time periods. This is shown in (2.11). The cornerstone of Tinbergen's theory of policy is the condition for which there exists for any $\{\bar{y}\}$ a corresponding unique policy vector $\{\bar{z}_1\}$ such that (2.3) is satisfied. In other words, under what conditions has (2.3) a unique solution for $\{z_1\}$? The necessary and sufficient condition is that $\boldsymbol{R}$ is of full rank, and that the number of targets is equal to the number of instruments. This statement is general enough to encounter dynamic policy problems where time series of targets are given and where time series of instruments are sought. The uniqueness of the instrument vector is an unnecessarily restrictive condition. An infinite number of policy vectors may exist which lead to the same target vector. The controllability theorem for the Tinbergen model is, therefore, stated as follows:

Theorem 1. Tinbergen Controllability Theorem. The policy model (2.3), $\{y\}=R\{z_1\}+S\{z_2\}$, *is controllable for all* $\{y\}=\{\bar{y}\}$ *if and only if the matrix multiplier* R *satisfies the condition* $\operatorname{rank}(R)=N$, *where* N *is the number of targets. The control vector* $\{z_1\}$ *is unique if* R *is* $N\times N$. *This condition is a reformulation of Tinbergen's proposition that there exist as many instruments as there are targets.*

To prove Theorem 1, recall that R is a $N\times K$ matrix, where N is the number of targets and K is the number of instruments. In the previous section, we made the assumption that both the targets and the instruments are linearly independent. This implies that the equations of (2.3) are independent. The system (2.3) is consistent, i.e., has a solution if and only if the number of unknowns K is equal to the number of equations N. But this implies that the rank of R is N. If K is less than N, the rank of R is $K<N$, and the system is inconsistent. The general solution to a consistent system is [Rogers (1971, p. 258)]

$$\{\bar{z}_1\}=R^{(1)}[\{\bar{y}\}-S\{z_2\}]+[I-R^{(1)}R]\{c\},$$

where $R^{(1)}$ is a generalized inverse of R, satisfying

$$RR^{(1)}R=R,$$

and $\{c\}$ is an arbitrary vector.

If $K>N$, there exists an infinite number of instrument vectors associated with $\{\bar{y}\}$. However, in most cases, there is only one instrument vector which is most suited to the policy maker's preferences. The design of such a policy vector will be discussed in the next section. If on the other hand $K=N$, then R is non-singular and (2.3) has a unique solution,

$$\{\bar{z}_1\}=R^{-1}[\{\bar{y}\}-S\{z_2\}].$$

3.2. Design of optimal migration policies

Any design of optimal policies should begin with a statement of objectives. In the previous section, the question was answered: Under what conditions is it possible to specify certain objectives or targets and to achieve them by the instruments at hand? It was shown that under very specific conditions, there is a unique instrument vector assuring the achievement of the targets. The optimal levels of the instrument variables then follow directly. Under other conditions, however, there is an infinite number of combinations of the instruments that lead to the desired targets. In this case, the policy maker is confronted with an additional decision problem: which alternative set of instruments to choose. This requires the set-up of a cost function or welfare loss function which aggregates the relative costs incurred by the implementation of each instrument. The feasible set of instruments may also be limited by imposing constraints on them. A further possibility is that the objectives are overstated, i.e., that no combination of instruments can be found that realizes all the targets. The system is uncontrollable and again the policy maker has an additional decision to make: where should he modify his preference system? Is he willing to give up some targets completely in order to achieve the others, or is he satisfied with approximating all the targets without reaching them exactly? This amounts to specifying a welfare function of the target variables of interest. The coefficients of the welfare function are the trade-offs between the target variables. The specification of the cost and the welfare function is the most difficult and the most socially sensitive task in the policy design process. In this paper, we make the assumption that these functions are given by the policy maker.

This section discusses the design of optimal policies in the Tinbergen framework. It will be shown that in some instances implicit objective functions may be used to derive the optimal policy. The unifying feature of this section is the notion of the *generalized inverse* [see also Russell and Smith (1975)]. The importance of the minimizing properties of generalized inverses for policy analysis will be illustrated.

From section 3.1, we know that an optimal policy exists if the rank of the impact multiplier matrix $\boldsymbol{R}$ is equal to the number of targets. Following Tinbergen, we consider three cases according to the relationship between the number of targets (N) and the number of instruments (K) or, equivalently, to the rank of the multiplier matrix and its singularity property.

3.2.1. The matrix multiplier is non-singular and of rank N

Recall eq. (2.3):

$$\{y\} = \boldsymbol{R}\{z_1\} + \boldsymbol{S}\{z_2\}.$$

If $\boldsymbol{R}$ is non-singular, then the optimal policy is unique and given by (2.4),

$$\{\bar{z}_1\} = \boldsymbol{R}^{-1}[\{\bar{y}\} - \boldsymbol{S}\{z_2\}].$$

It is clear from (2.4) that the policy depends not only on the target vector, but also on the uncontrollable variables. If $\{z_2\}$ has some lagged endogenous variables, then the effects of past policies will be felt in the current policy.

The nature of the dependence of $\{\bar{z}_1\}$ upon $\{\bar{y}\}$ is associated with different types of structures of the matrix $\boldsymbol{R}$. They were discussed in section 2. Since there is only one possible set of instruments leading to the target vector $\{\bar{y}\}$, no cost or welfare function is needed to distinguish between alternatives.

3.2.2. *The matrix multiplier is singular and of rank N*

If $N<K$, there exists an infinite number of instrument vectors which lead to the achievement of a preassigned value of the target vector. The solution set to (2.3) may be represented by

$$\boldsymbol{R}\{\bar{z}_1\}=\{\bar{y}\}-\boldsymbol{S}\{z_2\}, \tag{3.1}$$
$$\{\bar{z}_1\}=\boldsymbol{R}^{(1)}[\{\bar{y}\}-\boldsymbol{S}\{z_2\}]+[\boldsymbol{I}-\boldsymbol{R}^{(1)}\boldsymbol{R}]\{\boldsymbol{c}\},$$

where $\boldsymbol{R}^{(1)}$ is a generalized inverse of $\boldsymbol{R}$, satisfying

$$\boldsymbol{R}\boldsymbol{R}^{(1)}\boldsymbol{R}=\boldsymbol{R},$$

and $\{\boldsymbol{c}\}$ is an arbitrary vector.

In order to get a unique instrument vector, one must impose additional conditions on $\{\bar{z}_1\}$. Two illustrations are given of how this may be done. Both minimize a function of $\{z_1\}$ over a constrained set. The first illustration is the formulation of a general mathematical programming problem. The second makes use of the minimizing properties of some types of generalized inverses.

Illustration (*a*). Suppose a cost or welfare loss function $f(\{z_1\})$ has been defined. One wants to minimize this function subject to the dynamic behavior of the system and to some other constraints imposed upon the instrument vector and represented by the vector-valued inequality $g(\{z_1\}) \geqq 0$. The problem then may be formulated as a mathematical programming problem,

$$\min f(\{z_1\}),$$

subject to

$$\{y\}=\boldsymbol{R}\{z_1\}+\boldsymbol{S}\{z_2\}, \tag{3.2}$$
$$g(\{z_1\})\geqq 0.$$

If $g(\{z_2\})$ and $f(\{z_1\})$ are both linear, the problem is a linear programming problem and can be solved by the simplex technique.

Illustration (*b*). This illustration is a special case of the problem (3.2). We delete the constraint $g(\{z_1\}) \geqq 0$, and we let $f(\{z_1\})$ be the Euclidean norm defined on $\{z_1\}$, i.e.,

$$f(\{z_1\}) = [\{z_1\}'\{z_1\}]^{\frac{1}{2}}. \tag{3.3}$$

Ben-Israel and Greville (1974, p. 114) prove that the unique solution to this problem is given by

$$\{z_1\} = \boldsymbol{R}^{(1,4)}[\{\bar{y}\} - \boldsymbol{S}\{z_2\}], \tag{3.4}$$

where $\boldsymbol{R}^{(1,4)}$ is a generalized inverse satisfying

$$\boldsymbol{R}\boldsymbol{R}^{(1,4)}\boldsymbol{R} = \boldsymbol{R},$$

and

$$[\boldsymbol{R}^{(1,4)}\boldsymbol{R}]' = [\boldsymbol{R}^{(1,4)}\boldsymbol{R}].$$

Because $\boldsymbol{R}^{(1,4)}$ defines a minimum norm solution to (2.3), it is often called the '*minimum-norm inverse*'.

There may be other norms defined on the instrument vector. Suppose the policy maker lists some most acceptable values of the instrument variables $\{\bar{z}_1\}$, and wants to minimize the squared deviation between the optimal values and these preassigned values. The policy model is then

$$\begin{aligned}\min f(\{z_1\}) &= \|\{z_1\} - \{\bar{z}_1\}\| \\ &= [[\{z_1\} - \{\bar{z}_1\}]'[\{z_1\} - \{\bar{z}_1\}]]^{\frac{1}{2}},\end{aligned} \tag{3.5}$$

subject to

$$\{\bar{y}\} = \boldsymbol{R}\{z_1\} + \boldsymbol{S}\{z_2\}.$$

The optimal solution is given by

$$\{z_1\} = \boldsymbol{R}^{(1,4)}[\{\bar{y}\} - \boldsymbol{S}\{z_2\}] + [\boldsymbol{I} - \boldsymbol{R}^{(1,4)}\boldsymbol{R}]\{\bar{z}_1\}. \tag{3.6}$$

The matrix $\boldsymbol{R}^{(1,4)}$ has a special meaning for policy analysis. An element $r_{ij}^{(1,4)}$ indicates the change in the ith instrument variable required for a unit change in the jth target variable, assuming that $\{z_2\}$, and, in the second case, also $\{\bar{z}_1\}$ remain unchanged. It is, therefore, a multiplier in the economic sense, measuring the relative effectiveness of the ith instrument.

3.2.3. *The matrix multiplier is singular and of rank K*

If $N>K$, the system (2.3) is inconsistent and no solution exists, i.e., the residual vector $\{r\}$ is non-zero, where

$$\{r\}=[\{\bar{y}\}-S\{z_2\}]-R\{z_1\}=\{\bar{y}\}-\{\hat{y}\},$$

where $\{\hat{y}\}$ is the realized value of the target vector.

In this case, it is common to search for an approximate solution of (2.3), which makes $\{r\}$ closest to zero in some sense. Again two illustrations will be given. As before, the first is a mathematical programming model, namely, a quadratic programming model, and the second applies the minimizing properties of some generalized inverses.

Illustration (*a*). Theil (1964, p. 159) was the first to assume that a policy maker, confronted with an overstatement of his goals set, i.e., $N>K$, formulates his preferences as a quadratic function of the target and control variables. The Theil model has been given in section 2 without proposing a solution to it. Recall the model (2.7):

$$\begin{aligned}\min W(\{z_1\})=\{a\}'\{z_1\}+\{b\}'\{\hat{y}\}+\tfrac{1}{2}[\{z_1\}'A\{z_1\}\\ +\{\hat{y}\}'B\{\hat{y}\}+\{z_1\}'C\{\hat{y}\}+\{\hat{y}\}'C'\{z_1\}],\end{aligned}$$

subject to (2.3),

$$\{\hat{y}\}=R\{z_1\}+S\{z_2\},$$

where A, B, C, are symmetric positive definite weight matrices. This optimization problem may be solved by means of the Lagrangean technique. An alternative method of deriving the optimum consists of using the constraints to eliminate the target vector in the objective function and then minimizing this function unconditionally with respect to the instruments [Theil (1964, pp. 40–41)]. This solution procedure is also followed by Friedman (1975, p. 159). Substituting the constraint in the objective function gives

$$W(\{z_1\})=K_0+\{k\}'\{z_1\}+\tfrac{1}{2}\{z_1\}'K\{z_1\}, \tag{3.7}$$

where

$$K_0=\{b\}'S\{z_2\}+\tfrac{1}{2}[[S\{z_2\}]'B[S\{z_2\}]],$$

$$\{k\}=\{a\}+R'\{b\}+[C+R'B][S\{z_2\}],$$

$$K=A+R'BR+CR+R'C'.$$

The first order condition for minimizing $W(\{z_1\})$ with respect to the instrument vector $\{z_1\}$ is

$$\mathrm{d}W(\{z_1\})/\mathrm{d}\{z_1\}=\{0\}=\{k\}+K\{z_1\}.$$

The optimal solution follows immediately,

$$\{z_1\}=-K^{-1}\{k\}, \tag{3.8}$$

where K and $\{k\}$ are as defined in (3.7). The second order condition for the minimization of $W(\{z_1\})$ with respect to $\{z_1\}$ is that K is positive definite. The corresponding value of the target vector is

$$\{\hat{y}\}=RK^{-1}\{k\}+S\{z_2\}. \tag{3.9}$$

It should be noted that a non-trivial solution to (3.9) exists only if $\{k\}$ is non-zero.

Illustration (*b*). Suppose the policy maker only wants to minimize $\{r\}$. The model may be considered as a variant of the Theil model,

$$\min[[\{\bar{y}\}-\{\hat{y}\}]'[\{\bar{y}\}-\{\hat{y}\}]]^{\frac{1}{2}}, \tag{3.10}$$

subject to

$$\{\hat{y}\}=R\{z_1\}+S\{z_2\}.$$

The objective function defines the Euclidean norm of $\{r\}$. Ben-Israel and Greville (1974, p. 104) show that the optimal solution to this problem is given by

$$\{z_1\}=R^{(1,3)}[\{\bar{y}\}-S\{z_2\}], \tag{3.11}$$

where $R^{(1,3)}$ is the generalized inverse of R satisfying

$$RR^{(1,3)}R=R, \qquad [RR^{(1,3)}]'=RR^{(1,3)}.$$

Because of the property that $R^{(1,3)}$ minimizes the Euclidean norm of the residual vector, i.e., the sum of squares of the residuals, it is called the '*least-squares inverse*'. An element $r_{ij}^{(1,3)}$ indicates how much the ith instrument has to change for a unit change in the jth target variable, in order to maintain the smallest sum of squared deviations between the realized and the preassigned values of the target variables. The general least-squares solution is

$$\{z_1\} = R^{(1,3)}[\{\bar{y}\} - S\{z_2\}] + [I - R^{(1,3)}R]\{c\}, \tag{3.12}$$

where $\{c\}$ is an arbitrary $K \times 1$ vector.

Ben-Israel and Greville note that the least-squares solution is unique only when R is of full column rank. This condition is always satisfied in policy models discussed here, since we have assumed initially that the instruments are linearly independent.

This illustration shows that the least-squares generalized inverse is the solution to a special variant of the Theil model. A similar observation was made by Russell and Smith (1975, p. 143).

4. Conclusion

This was the first part of a paper on a comprehensive analytical framework for the study of optimal migration and population distribution policies. It presented the conceptutal framework, namely, the Tinbergen paradigm, and demonstrated how it may encompass a wide variety of policy models. The Tinbergen paradigm is particularly useful to isolate the policy-relevant part of a system described by a demometric or demographic-economic model, and to transform this model into a policy model. It was shown that any linear descriptive or explanatory model may be converted to a policy model if and only if all the target variables of the policy model belong to the set of endogenous variables of the descriptive or explanatory model, and if at least one of the exogenous variables is controllable.

The general formulation of a policy model is given in (2.3),

$$\{y\} = R\{z_1\} + S\{z_2\},$$

with $\{y\}$ the vector of target variables, $\{z_1\}$ the vector of instrument variables and $\{z_2\}$ the vector of uncontrollable exogenous and lagged endogenous variables. *An important role in policy analysis is played by the matrix multiplier* $\boldsymbol{R}$. Our discussion of policy models centered around this multiplier. This is consistent with the economic literature on policy models. To present an overview of policy models, a classification scheme was set up that is based on the rank and the structure of R. This scheme enabled us to relate seemingly unrelated models to each other. For example, it was shown that the linear-quadratic control problem may be derived from the Tinbergen and Theil model by assuming intertemporal separability of the objectives and unidirectional causality of the population system. The state-space model also may be derived from the Tinbergen model, and from the reduced form model in general.

The fundamental questions of quantitative migration policy may be expressed in terms of existence and design. In section 3 we dealt with these two topics. The discussion revolved around the matrix multiplier. Whether arbitrarily specified levels of target variables can be reached by the existing set of instruments depends on the rank of $\boldsymbol{R}$. The condition that must be satisfied for a population system, described by a Tinbergen model, to be controllable, was formulated as an existence theorem. In Part II of the paper, some existence theorems will be formulated for dynamic populations described by a state-space model. The introduction of dynamics into the study of the existence of optimal policies will require to change the way of thinking engendered by Tinbergen's *Theory of Policy*. The reformulation of Tinbergen's findings in this part of the paper aimed at providing the necessary connections.

The design procedure of optimal policies is dictated by the structure and the rank of the matrix multiplier $\boldsymbol{R}$. If $\boldsymbol{R}$ is non-singular, then the unique solution to (2.3) for $\{z_1\}$ is found by simply inverting $\boldsymbol{R}$. When $\boldsymbol{R}$ is singular, there may be no instrument vector leading to the desired target values, or there may be an infinite number of them. To find a unique optimal solution, an objective function reflecting the policy maker's preferences is introduced, and mathematical programming techniques may be applied. There is a wide variety of algorithms available in the literature. The common characteristic of most of them is that they determine the optimal solution numerically. In this study, we have directed our attention to cases where solutions to policy problems can be found analytically.

In this regard, there is the applicability of the notion of generalized inverse. We have shown how the minimizing properties of generalized inverses may be relevant in solutions of policy models with a singular multiplier matrix. For example, no matter what the rank of the $N \times K$ matrix $\boldsymbol{R}$ is, a unique solution to (2.3) is given by

$$\{z_1\} = \boldsymbol{R}^p[\{\bar{y}\} - \boldsymbol{S}\{z_2\}],$$

where $\boldsymbol{R}^p$ is the Moore–Penrose inverse [Ben-Israel and Greville (1974, p. 7)]. If $\boldsymbol{R}$ is non-singular, then $\boldsymbol{R}^p$ is the ordinary inverse; if $\boldsymbol{R}$ is singular and of rank N, i.e., the number of instruments exceeds the number of targets, then $\boldsymbol{R}^p$ defines a minimum norm solution to (2.3), and if $\boldsymbol{R}$ is singular and of rank K, i.e., the targets exceed the instruments in number, then $\boldsymbol{R}^p$ defines a solution to (2.3) that minimizes the squared deviations between the desired and the realized values of the target variables. No explicit objective function has been specified, but it is implicit in the minimizing properties of the generalized inverses. The interesting feature of generalized inverses is that they provide an analytical solution to policy models.

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