ON THE SENSITIVITY OF LINEAR DYNAMIC MODELS

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September 1979
WP-79-80

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INTRODUCTION

This paper represents the results of a three month study, in which several Junior Scientists from many countries took part during the summer of 1979 at IIASA. While many of these results are not fully completed, and some represent only preliminary directions of research, we feel that the documentation of the efforts of the Junior Scientists is justified.

ABSTRACT

The exact expression of the trajectory deviation due to parameter changes is obtained for the linear dynamic models.

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The sensitivity of a dynamic system to variations of its parameters is one of the basic aspects in the treatment of dynamic systems, since there is always a certain discrepancy between the actual system and its mathematical model. Let the mathematical model of the dynamic system be given by the general discrete-time vector equation

$$\vec{x}_{t+1,0} = \vec{f}_t(\vec{x}_{t,0}; \vec{u}_t) \quad , \tag{1}$$

where $\vec{x}_{t,0}$ represents the state vector at the time moment t, and $\vec{u}_t$ represents the control vector at time t which brings the system from initial state $\vec{x}_0$ to the final state $\vec{x}_{T,0}$ and, in doing so, delivers a minimum value of a performance index of the form

$$I = \sum_{t=0}^{T} F_t(\vec{x}_{t,0}; \vec{u}_t) \quad . \tag{2}$$

Usually a real system cannot be identified exactly and the function f_t in (1) represents the nominal law of the system behavior but it can be changed due to the parameter variations and becomes

$$\vec{x}_{t+1} = \vec{\rho}_t(\vec{x}_t; \vec{u}_t) \quad . \tag{3}$$

Consequently (3) will be called the actual law of the system behavior.

Suppose that the control vector $\vec{u}_t$ has been found to be optimal with particular regard to nominal system behavior law (1). The actual law (3) can change the state vector from $\vec{x}_{t,0}$ value to $\vec{x}_t$, and it is important to find the performance index deviation ΔI in order to judge whether the solution of an optimization problem is of practical use in view of the given system behavior tolerance.

The great number of optimization problems of practical importance are taken into account by linear (Propoi 1979).

$$I = \sum_{t=0}^{T-1} (\vec{c}_t \cdot \vec{x}_t + \vec{d}_t \cdot \vec{u}_t) + \vec{c}_T \cdot \vec{x}_T \quad , \tag{4}$$

or square (see, for example, Brison 1979)

$$I = \sum_{t=0}^{T-1} (\vec{x}_t^* \cdot C_t \cdot \vec{x}_t + \vec{u}_t^* \cdot D_t \cdot \vec{u}_t) + \vec{x}_T^* \cdot C_T \cdot \vec{x}_T \quad , \tag{5}$$

performance indexes. For these cases the deviation ΔI can be found easily if the actual state vector $\vec{x}_t$ can be written in the form

$$\vec{x}_t = \vec{x}_{t,0} + \Delta\vec{x}_t \quad . \tag{6}$$

In this case after substituting (6) for (4) or (5) the performance index can be written as follows

$$I = \sum_{t=0}^{T} F_t(\vec{x}_{t,0}; \vec{u}_t) + \Delta I \quad . \tag{7}$$

It is a common practice in sensitivity theory (Frank 1978) to represent the changing parameters of the system by a vector

$$\vec{\alpha}_t = \vec{\alpha}_{t,0} + \Delta\vec{\alpha}_t , \tag{8}$$

(where $\vec{\alpha}_t$ and $\vec{\alpha}_{t,0}$ are actual and nominal parameter vectors) and to define a so-called sensitivity function S which relates the elements of the set of the parameter deviations $\Delta\vec{\alpha}_t$ to the elements of the set of the parameter-induced errors of the state vector $\Delta\vec{x}_t$ by the linear equation

$$\Delta\vec{x}_t \approx S(\vec{\alpha}_{t,0}) \cdot \Delta\vec{\alpha}_t . \tag{9}$$

This relation is the first order approximation with respect to $\Delta\alpha_t$ and is valid only for small parameter variations, i.e., $\| \Delta\vec{\alpha} \| << \| \vec{\alpha}_{t,0} \|$. Moreover, for the investigation of the sensitivity with respect to controller parameters sensitivity functions of higher orders should be applied (Wierzbicki 1977).

For the important class of linear dynamic models

$$\vec{x}_{t+1,0} = A_t \cdot \vec{x}_{t,0} + B_t \cdot \vec{u}_t , \tag{10}$$

which describes a great variety of practical problems, the state vector deviation $\Delta\vec{x}_t$ due to changes of the system behavior law (induced by changes of parameter matrices $A_t \to A_t + a_t, B_t \to B_t + b_t$)

$$\vec{x}_{t+1} = (A_t + a_t) \cdot \vec{x}_t + (B_t + b_t) \cdot \vec{u}_t , \tag{11}$$

can be derived exactly as a function of system parameters A, B, parameter changes a, b, control vector $\vec{u}$ and intiial state vector $\vec{x}_0$.

For this purpose we rewrite the system equations (11) in the form

$$\vec{x}_{t+1} = \vec{x}_{t+1,0} + (A_t + a_t) \cdot \Delta\vec{x}_t + a_t \cdot \vec{x}_{t,0} + b_t \cdot \vec{u}_t . \tag{12}$$

It is not difficult to see that (12) is the recurrant relation

$$\Delta\vec{x}_{t+1} = (A_t+a_t) \cdot \Delta\vec{x}_t + a_t \cdot \vec{x}_{t,0} + b_t \cdot \vec{u}_t , \qquad (13)$$

and obtain by means of iterative procedure the following expression for $\Delta\vec{x}_{t+1}$

$$\Delta\vec{x}_{t+1} = \prod_{i=1}^{t}(A_i+a_i)\Delta\vec{x}_1 + \sum_{k=2}^{t} \prod_{i=k}^{t} (A_i+a_i)(a_{k-1}\vec{x}_{k-1,0}+b_{k-1}\cdot\vec{u}_{k-1}) + $$
$$+ a_t\cdot\vec{x}_{t,0}+b_t\cdot\vec{u}_t , \qquad (14)$$

where $\Delta\vec{x}_1 = a_0 \cdot \vec{x}_0 + b_0 \cdot \vec{u}_0$.

After exclusion of $\vec{x}_{k-1,0}$ from (14) with the help of equation (10), expression (14) takes the form

$$\Delta x_{t+1} = \Delta x^{(1)}_{t+1}(a,\vec{x}_0) + x^{(2)}_{t+1}(a,\vec{u}) + \Delta x^{(3)}_{t+1}(a,b,\vec{u}) , \qquad (15)$$

where

$$\Delta x^{(1)}_{t+1} = [\prod_{i=1}^{t} (A_i+a_i)\cdot a_0 + \sum_{k=2}^{t} \prod_{i=k}^{t} (A_i+a_i)\cdot a_{k-1}\prod_{j=0}^{k-2} A_j + a_t \prod_{i=0}^{t-1} A_i]\vec{x}_0 ,$$

$$\Delta x^{(2)}_{t+1} = \sum_{k=2}^{t} \prod_{i=k}^{t} (A_i+a_i)a_{k-1}(\sum_{p=1}^{k-2} \prod_{j=p}^{k-2} A_j B_{p-1}\vec{u}_{p-1}+B_{k-2}\vec{u}_{k-2}) +$$
$$+ a_t(\sum_{k=1}^{t-1} \prod_{i=k}^{t-1} A_i B_{k-1}\vec{u}_{k-1}+B_{t-1}\vec{u}_{t-1}) , \qquad (16)$$

$$\vec{x}^{(3)}_{t+1} = \prod_{i=1}^{t}(A_i+a_i)b_0\vec{u}_0+b_t\vec{u}_t + \sum_{k=2}^{t} \prod_{i=k}^{t} (A_i+a_i)b_{k-1}\vec{u}_{k-1} .$$

Expressions (15) and (16) are rather complex but they do not contain the dependence on nominal system trajectroy.

It may be interesting to compare exact expressions (14)-(16) with those derived in the first order approximation with respect to parameter changes. It can be seen from relation (13) that in order to derive the first order approximation one should put a_i equal to zero in all factors A_i+a_i in (14)-(16).

As a conclusion it should be noted that if the possible changes of control vector $\vec{u}_t$ from the nominal vector $\vec{u}_{t,0}$

$$\Delta\vec{u}_t = \vec{u}_t - \vec{u}_{t,0} \quad , \tag{17}$$

are known they can also be taken into account in the same way as b_t. In this case the expression for $\Delta\vec{x}_{t+1}$ takes the following form

$$\Delta\vec{x}_{t+1} = \Delta\vec{x}^{(1)}_{t+1}(a,\vec{x}_0)+\Delta\vec{x}^{(2)}_{t+1}(a,\vec{u})+\Delta\vec{x}^{(3)}_{t+1}(a,b,\vec{u})+\Delta\vec{x}^{(4)}_{t+1}(a,b,\Delta\vec{u}) \quad , \tag{18}$$

where u is the nominal control vector, $\Delta\vec{x}^{(1)}_{t+1}$, $\Delta\vec{x}^{(2)}_{t+1}$, $\Delta\vec{x}^{(3)}_{t+1}$ are defined by (16) and

$$\Delta\vec{x}^{(4)}_{t+1}(a,b,\Delta\vec{u}) = \sum_{k=2}^{t}\prod_{i=k}^{t}(A_i+a_i)(B_{k-1}+b_{k-1})\Delta\vec{u}_{k-1}+(B_t+b_t)\Delta\vec{u}_t \quad . \tag{19}$$

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