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On analysis of complex social-environmental systems: Integration of models

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1. Operating with models

1.1 Introduction

Global challenges – a critical state of the environment, limits in natural resources, a growing social inequality, world economic crises – create acute demand for systems analysis – an integrated scientific discipline that would focus on characterizing possible futures of complex large-scale social-environmental systems and actions that may lead to these futures.

There is a strong diversification in 'definitions' of systems analysis. We sharpen our view if we agree, at least temporarily, that systems analysis lies, by definition, in the field of methodology. We sharpen our view even stronger if we say that systems analysis develops and applies methods for exploration of complex social-environmental systems. We hold this viewpoint in the present discussion paper.

In this section (essentially following Kryazhimskiy, 2014) we make an attempt to characterize a basic set of rules in operation with the principal tools of systems analysis – models of large-scale social-environmental systems. All those rules have certainly been repeatedly used in research; our summary may hopefully serve as a zero approximation to a future 'road map for a systems analyst'.

In Section 2 we focus on a challenging and yet not so well explored issue of integration of models –synthesis of integrated knowledge from pieces of 'partial' knowledge provided by 'partial' models viewing a complex social-environmental system from different perspectives.

1.2 Historical data, imitation methods and models

A historical data set tells us about the real changes, which the complex social-environmental system under investigation survived in the past. The historical data set is highly important from a systems analysis perspective; it reflects all the phenomena (nonlinearities, cross-scale interactions, heterogeneities, etc.) that drive the system.

Theoretically speaking, all sorts of data possess that property. However, in the spectrum of all data that are principally available, some types of data are more informative than the others if we take into account the goal of research. Selection and collection of most informative data is a special important (sometimes very expensive

and time consuming[1]) task in systems analysis[2]. Here, for the sake of brevity, we do not discuss this task in any detail (leaving it for further discussions), and focus on techniques for utilizing the data.

Suppose we are given a historical data set (say, a time series) and want to convert it into a meaningful statement about the system's future. To do so, we are bound to use some method for imitation of the system's dynamics; here, we call such methods imitation methods.

Choosing an adequate imitation method is the key technical challenge. The complexity of a large-scale social-environmental system is usually so high and our knowledge about it is (despite the data at hand) so poor that we are principally unable to construct a method that would capture the system's dynamics precisely[3].

In other words, in systems analysis we are forced to deal with methods that fail to mimic the systems dynamics to a degree, at which we can (like in the case of mechanical systems) claim, in advance, that our method-based forecasts are accurate, or at least not misleading. In this situation, prior to the use of the chosen imitation method one needs to assess the method in order to understand its forecasting ability.

In a traditional view, an imitation method is a model – a set of mathematical formulas representing (to a certain degree of accuracy) the system design. In our discussion, we follow that tradition to a considerable extent; we associate basic types of imitation methods with models. In Section 2 go further and consider multi-model imitation methods involving several (many) models.

1.3 Stages in model-based research

Models representing the systems through mathematical relations are test beds for various analytic tools – from rigorous mathematical analyses to brute force simulations, which help us, indirectly, understand the operation of the systems.

In this section we briefly announce typical stages in model-based research (see Table 1). In sections 5 – 8 we comment on these stages in some detail.

A model-based research starts with choosing a modeling paradigm and a model's state space – two basic 'coordinates' of a model as a research instrument. Next two stages are construction of a model with the chosen basic 'coordinates', and assessment of the model, including a diagnostic analysis. Generation of a model-based forecast finalizes the research effort.

Typically, the first three research stages are not separated in time. Quite often, one updates the model, based on results of model assessment, and one changes the initially chosen modeling paradigm or the initially chosen state space, based on

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[1] For example, Wittemyer (2011) *assessed the relations between African elephant mortality and various economic data (livestock and maize prices, change in national and regional GDP, the normalized difference vegetation index, and others) in Kenya to* find *economic metrics serving as indicators of changes in human use of and resulting effects on natural resources.*

[2] Here, for brevity, we do not discuss how to cope with data errors, which can be significant and even misleading (we slightly touch upon this issue in Section 2; see footnote 12).

[3] In this context, the complexity of social-environmental systems is of a different type than that of mechanical systems; a mechanical system, no matter how complicated it is, is fully described by the mathematical model coupling the system's design with mechanical laws.

experience in constructing and assessing the model. The updating process may go through several iterations.

Table 1: Stages in model-based research

Stage 1	Choosing a modeling paradigm and a state space
Stage 2	Constructing a model
Stage 3	Assessing the model
Stage 4	Generating a model-based forecast

1.4 Stage 1. Modeling paradigms and state spaces

A modeling paradigm is a disciplinary niche for a model, understood in a mathematical sense. The modeling paradigm pre-defines the style of the model-based research. A model can follow approaches in modeling of dynamical systems (see, e.g., Ljung and Glad, 1994) and pretend to imitate the system's dynamics by imitating relations between the system's components; it can follow theory of optimal central planner control (Pontryagin, 1962; see also Aseev and Kryazhimskiy, 2007, presenting a control-theoretic technique applicable to problems of economic growth) and represent global optimization principles; it can follow theory of differential games (Isaacs, 1965; Krasovskii and Subbotin, 1988; Basar and Olsder, 1982) and implement global equilibrium principles; it can follow theory of evolutionary games (see, e.g., Weibull, 1995) and demonstrate learning/adaptation schemes. The list can certainly be extended. Complex models can lie under several modeling paradigms (occupy several disciplinary niches) simultaneously.

A model's state space is a set, within which the model's states are allowed to vary. The model's state space characterizes the model's ability to capture details, its resolution scale. The more complex is the model's state space, the more details are captured by the model and the finer is the model's resolution scale.

Finite-dimensional vector spaces (whose points are finite-dimensional vectors of given dimensions) are the state spaces for widely used deterministic models described by finite-dimensional ordinary differential equations and their discrete-time analogues. Deterministic models described by partial differential equations and infinite-dimensional ordinary differential equations operate in infinite-dimensional functional spaces whose points are functions defined on solid sets[4]. By choosing a deterministic model, one 'declares', implicitly, that one has a good understanding of the mechanism driving the system.

Stochastic models deal with probability distributions of points. Their state spaces are structured as probability spaces, in which points act as elementary events. Usually, one chooses a stochastic model if the mechanism that drives the system is uncertain though one has an understanding of a statistics related to its operation.

The points serving as states for the deterministic models (as elementary events for the stochastic models) are as usual regarded as direct prototypes of real, 'physical' values

[4] Here, by a solid set we mean a set whose cardinality (a mathematical generalization for the number of elements) is not less than that of the set of all real numbers (the cardinality of the continuum).

characterizing (not necessarily entirely) the real systems; the models and systems 'speak the same language'.

In a less straightforward model design a model's states are images of the 'physical' values; the transformation converting the 'physical' values into their images is not necessarily one-to-one. The transformation 'compresses' the 'physical' space by not distinguishing between the 'physical' points having the same image. The use of such image states can be efficient in forecasting radical structural changes (catastrophes)[5]. Indeed, on the one hand, the image variables representing clusters of 'physical' points are by construction less sensitive to weak perturbations in the dynamics than the 'physical' points themselves. On the other hand, transitions between the clusters reveal remarkable irregularities in the model's behavior and can signal on upcoming critical changes (catastrophes).

Table 2 summarizes our preliminary classification of modeling paradigms and state spaces as 'basic coordinates' for models.

Table 2: Modeling paradigms and state spaces

Modeling paradigms					
Disciplinary niches	**Models' functions**				
Modeling of dynamical systems	Imitation of the systems' dynamics				
Optimal control	Implementation of global optimization principles				
Differential games	Implementation of global equilibrium principles				
Evolutionary games	Implementation of learning/adaptation schemes				
		Vector spaces	Probability spaces	Image spaces	**State spaces**

1.5 Stage 2. Construction of models

As mentioned in Subsection 1.3, construction of a model is not separated from the choice of a modeling paradigm and a state space, and from the choice of a model assessment method (see Table 1, stages 2 and 3). Accordingly, in this subsection we partially overlap with the previous one (discussing modeling paradigms and state spaces) and the next one (discussing model assessment methods).

We dare to say here that choosing a modeling paradigm is, conceptually, not a too difficult task. Principally, one can have in mind the following simplified pattern.

If the mechanism that drives the system does not include non-specified time-varying inputs or actions (the system is 'closed'), or if we, for some reason, decide to ignore

[5] For example, Kryazhimskiy and Beck (2002) use binary, 'minus' and 'plus', images of short-term transitions to assess tendencies towards catastrophes; the transitions regarded as dangerous have 'minus' images and those regarded as safe have 'plus' images. Keilis-Borok *et al.,* (2003) use binary images (codes) for crime trends to forecast homicide surges.

such actions, approaches in modeling dynamical systems will be an appropriate disciplinary niche and our model's mission will be to imitate the system's dynamics.

On the contrary, if we view the system as an 'open' system driven by time-varying actions not given in advance, we construct an 'open' model and equip it with an action selection principle, based on which we study the model's behavior by generating selected (extreme, most likely, etc.) paths within the infinite pool of the model's potentially allowable action-driven trajectories.

Action selection principles base on several methodological approaches.

The scenario-based approach is widely used. Scenarios represent exceptional 'typical' actions (as given functions of time) and 'encircle', in some sense, the set of all the system's allowable paths into the future. Usually, the design of a set of scenarios results from an informal analysis, and the number of scenarios, especially for complex models, is small. Each scenario generates a 'closed' model with 'frozen' actions. Each 'closed' model represents a variant of the system's behavior and fits with the modeling dynamical systems approach as a disciplinary niche.

The agent-based modeling approach (see, e.g., Bonabeau, 2002) aims at exploring repeated interactions of multiple agents that from their actions based on 'typical' individual feedbacks. Combinations of the agents' 'typical' feedbacks result in a global action selection principle and transform the original 'open' model representing the agents' society with 'free' actions into a 'closed' one.

A central planner action selection principle aimed at finding the globally optimal scenario for the system's performance brings us to optimal control theory as a disciplinary niche. A well-known example of the use of the global optimization principle is Nordhaus's Dynamic Integrated Climate Economy (DICE) model (Nordhaus, 1994), which optimizes the global social welfare utility index under a feedback from the global climate system. Of special interest are cases where the optimal scenario agrees with historical data (checking this is part of the model assessment task – see Table 1 and Subsection 1.6 below). In such cases the optimal scenario turns into a business-as-usual one, which gives us a strong reason for conjecturing that optimization (with respect to the given long-term performance index) is an underlying law in the system's performance (see Figure 2 in Subsection 1.6 for an example). As usual, the process of construction of a model realizing the central planner's optimal action scenario includes a serious control-theoretic analysis, through which the original 'open' model is converted into the final 'closed' one.

If the system is driven by several agents pursuing individual interests, theory of differential games or theory of evolutionary games will be an appropriate disciplinary niche.

Theory of differential games (as well as the related theory of multi-stage games) suggests methods for selecting the equilibrium (mutually acceptable and therefore most likely from a theoretical perspective) action strategies for 'forward looking' agents that measure their benefits based on the system's overall performance. A well-developed branch of theory of differential games, the theory of zero-sum differential games, covers in particular a typical situation, in which the single agent steers the system affected by non-predictable and non-observable dynamical disturbances (treated as actions of the agent's opponent whose interest is opposite to that of the agent). The theory suggests techniques for constructing the agent's optimal feedback action strategy that maximizes the agent's global benefit under the worst action strategy of the opponent.

Theory of evolutionary games departs from an assumption that interacting and interdependent agents act 'myopically', 'boundedly rational', trying to adapt and, if possible, win in the changing environment. The theory is close in spirit to the agent-based modeling paradigm and refers primarily to models of biological evolution and models of social behavior (Levin *et al.*, 2012).

Action selection principles help us explore futures of 'open' systems by modeling exceptional ('boundary', 'most likely', 'most typical') behaviors. A technical motivation for the use of this 'selective' approach is evident: it is hardly possible to model all possible futures. However, in some situations, advanced control-theoretic and computational techniques allow one to construct, for any given future 'target' point in time, the attainability domain of an 'open' model – the set of all the model's states reachable at the 'target' point under all possible action scenarios (see Figure 1).

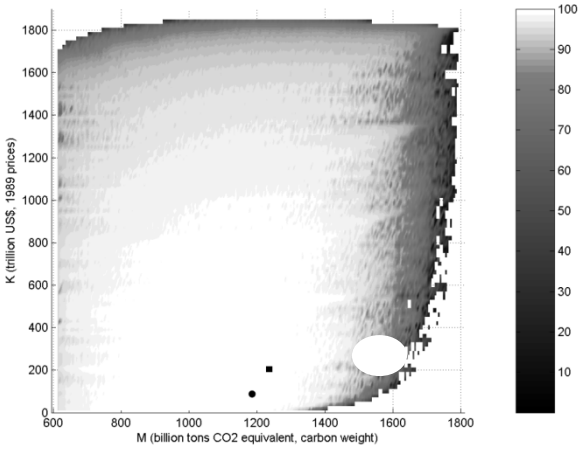

Figure 1: The attainability domain for Nordhaus's simplified DICE model (Nordhaus, 1994) for year 2100, with 1965 as the starting year, and the landscape of the values of the global social welfare utility index; the small black circle is Nordhaus's optimal point (Smirnov, 2005).

A *a*

of model construction. Traditionally, researchers aim to achieve the highest possible degree of precision in imitation of the systems' dynamics. A common tendency is to use complex models operating in high-dimensional state spaces and incorporating numerous relations between their compartments[6]. Such models are usually mathematically intractable in the sense that they leave no room for rigorous mathematical analyses, appealing, primarily, for brute force simulations. Moreover, for such models (which are inevitably inaccurate), there is always a danger that small failures distributed across the models' compartments and links will multiply and result in fatal modeling and forecasting failures. Model assessment exercises with the use of historical data (see Subsection 1.6) can help estimate a probability of fatal failures; however, the cost of such assessment exercises – in terms of time, effort and reliability of the result – grows dramatically with the increase of the model's complexity.

[6] This tendency is represented by, for example, complex agent-based models.

Simplifying the situation, we can say that reliability of a model-based forecast is inverse proportional to the model's complexity and to the dimension of the model's state space.

In this context, aggregated, low-dimensional, mathematically tractable (and well assessed against data – see section 7 below) models produce the most reliable forecasts; a drawback of such forecasts is that they are highly aggregated and miss many important details. Models operating in simple image spaces (whose elements represent clusters of 'physical' points – see Subsection 1.4) and having extremely rough resolution scales have also a potential for being highly reliable from a prognostic point of view; on the other hand, such models are targeted to forecasting strong events (catastrophes) only and unable to capture any smooth trends.

Finding the optimal compromise in the tradeoff between a model's complexity and its ability to produce reliable forecasts is the key challenge at the stage of model construction.

1.6 Stage 3. Model assessment

Three 'universal' model assessment techniques are calibration, retrospective forecasting and sensitivity analysis.

Calibration is a procedure, through which one adjusts a model to data. Calibration is usually understood as identification of the model's parameter values, which give the best fit with the historical data. For deterministic models the best-fit parameter values minimize the distances between the models' trajectories and the historical ones. For stochastic models producing bundles of trajectories with different probabilities, the best-fit parameter values are usually defined to be the ones that provide the maximum likelihoods for the historical trajectories.

Retrospective forecasting is a diagnostic stage of analysis. A goal is to qualify the model's ability to mimic the system's dynamics in the past. If we find that the model is satisfactory in this respect, we arrive at an important diagnostic conclusion that the system operated (in the past) in agreement with the chosen modeling paradigm and the chosen model design. This gives us a basis for stating that the model will most likely produce a satisfactory forecast ('most likely' implies here that we assume that the system's dynamics will not survive a serious change in a subsequent period).

Retrospective forecasting is organized as a test for checking the ability of the calibrated model to produce forecasts in retrospect. To perform the test we split the historical time series in two periods – a virtual past and a virtual future, the latter following the former. Then we 'travel to the past' – virtually, we bring ourselves to the end of the virtual past. We use our model to process the data in the virtual past and to produce a forecast for the virtual future. Finally, we 'travel back to the present' and compare our model-based forecast for the virtual future with the data series for the virtual future. Assessing the fit between the model-based forecast and data in the virtual future, we make our decision on the model's ability to forecast. To get better knowledge on the model's ability to forecast, one carries out retrospective forecasting tests several (many) times, each time choosing new virtual past and virtual future periods. Based on the resulting set of forecast-data fits, one assesses the model's ability to forecast statistically (see Figure 2 for an illustration)[7].

[7] To make the statistics fitter to standard definitions of probability theory, one can carry out 'statistically identical' retrospective forecasting tests with fix lengths of the virtual past and virtual future periods.

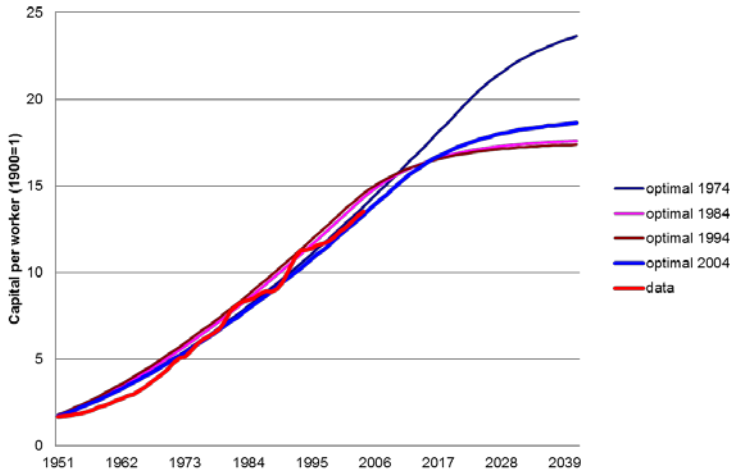

Figure 2: A retrospective forecasting test (Krasovskii and Tarasyev, 2010). The red curve is the capital-per-worker time series for the UK (in ratios to the level of 1950) in period 1950-2006 (Groningen Total Economy Data Base, http://www.ggdc.net/). The four other curves are the simulated optimal capital-per-worker growth trajectories for the UK for the same period (each simulated trajectory is optimal in the sense that it maximizes the integrated consumption index – a standard growth criterion used in theory of endogenous economic growth). The optimal trajectories are simulated using an aggregated optimal economic growth model. The model was calibrated four times, with four virtual past data series – 1950-1974 (dark blue); 1950-1984 (pink); 1950-1994 (lavender); and 1950-2004 (blue). In the three former cases, the retrospective forecasts for virtual futures cover, respectively, 1975-2006, 1984-2006 and 1995-2006, and show good fits with data, telling us that the model produces reliable forecasts. Based on success in retrospective forecasting, the researchers extended the simulated trajectories beyond 2006 to provide forecasts for the real (not virtual) future. These forecasts show a trend change in a not far-distant future – linear growth (seen in the time series) switches to saturation.

Sensitivity analysis is an instrument for testing robustness of model- and data-based forecasts to errors in data and in the model's design. The underlying phenomenon is bifurcation. Bifurcation occurs if a small change in a model's parameter values makes the model switch to a radically different behavior. Such behavior switches happen if the changed parameter values cross the so-called bifurcation curve

For a simple model, one can find the bifurcation curve analytically and see if the vector of the reference parameter values identified through calibration lies far enough from the bifurcation curve, or close to it. In the former case, the reference model configured by the reference parameter values is robust in the sense that if one assumes that the calibration result is (slightly) inaccurate, one can still believe that the model- and data-based forecast given by the reference model is essentially correct; let us call this situation regular. If the reference parameter values lie close to the bifurcation curve (the irregular situation), the reference model is sensitive – the model configurations corresponding to different behaviors provide alternative forecasts. In both situations, the model-based forecasts (supported by a successful retrospective forecasting exercise) can be considered as reliable (here we come back to our earlier statement on reliability of simplified models – see Subsection 1.5).

For a complex model, we have, typically, no way to find the bifurcation curve (manifold) analytically and, consequently, no way to understand in advance if the situation is regular or irregular. We approach some understanding if we carry out a numerical sensitivity analysis – run the model several (many) times, with different parameter values concentrated around the reference ones. In the regular situation the simulated trajectories lie close to each other, in the irregular one we see divergent trajectories. The cost for sensitivity analysis – in terms of time, effort and reliability of the result –

grows with a model's dimension and complexity. For super-complex models having thousands of parameters (including those that are not measurable in principle), various combinations of which can, potentially, configure bifurcation manifolds, it is hardly possible to carry out meaningful sensitivity analyses. As mentioned in section 6, forecasts based on very complex models cannot be regarded as reliable ones.

1.7 Stage 4. Model-based forecasts

A model-based forecast results from extension of the model's trajectory into the future. If the model is satisfactorily assessed against historical data, we are quite confident that the forecast is reliable[8]. However (in contrast to the case of mechanical systems), we can never guarantee that our model-based forecast is correct. A model assessment exercise (see Subsection 1.6), no matter how accurate and successful it is, is coupled with the system's history (past data). The latter may not capture some 'hidden' phenomena in the system's dynamics. A model agreeing with the system's history may fail to adequately represent the system's dynamics in a subsequent period if a 'hidden' phenomenon becomes active.

If we do expect a new 'hidden' phenomenon to become active in the future and if we understand (to some extent) the way it acts, we can, accordingly, modify our model design prior to forecasting. Having no data on the operation of the new phenomenon, we are bound to modify the model based only on our theoretical understanding of the expected change. To compensate for the fact that the model's parameters responsible for the new phenomenon are not calibrated against data, we can vary these parameters in some reasonable range and produce a corresponding set of the model's future trajectories. The latter set serves then as a 'fuzzy' forecast showing us a range of the system's possible paths into the future under the action of the new phenomenon that is not understood in all detail and has never operated in the past.

The model assessment techniques discussed in Subsection 1.6 – calibration, retrospective forecasting and sensitivity analysis – aim at raising reliability of model-based 'quantitative' forecasts expressed in terms of values of 'real' variables characterizing behavior of particular real systems under investigation. There are forecasts of a different type – 'qualitative' ones. A qualitative forecast characterizes trends and relations expected to occur in the system's future, without specifying these in terms of numbers[9]. A qualitative forecast results straightforwardly from a theoretical or numerical assessment of a model in no strong connection to real data.

1.8 Monitoring and updating

We expect systems analysis to be a process of constant generation of new knowledge about complex social-environmental systems. Generation of new knowledge includes refining knowledge through learning from experience. Forecasts produced earlier provide solid bases for learning. In the course of time, a forecast produced earlier overlaps with a recent history and turns into a retrospective one (see Subsection 1.6);

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[8] Based on the results of model assessment (see Subsection 1.5), we complement the forecast with estimates for the forecasting errors.

[9] The statement 'species X will extinct', with no indication of any time horizon, is an example of a qualitative forecast; see Figure 6 for another example.

one can see if it fits with recent historical data[10]. As time moves on, a basis for retrospective model assessment is automatically extended, which appeals for carrying out retrospective model assessment repeatedly, on-line. Periodically, the changes in the current assessment results will make the researchers update their instrumental research components – the modeling paradigms, state spaces, models and estimates of the models' forecasting abilities (as mentioned in Subsection 1.3, the research stages are not separated from each other). On-line monitoring/updating will be a promising approach to constantly enhancing our knowledge about complex social-environmental systems and their futures.

2. Integration of models

2.1 A multi-model approach

Here we come back to the statement that the laws that drive a complex social-environmental system are never perfectly understood (see Subsection 1.2). Any model captures the system's operation only partially. The use of a single model can be misleading[11]. This understanding leads us to a conjecture that analysis of complex social-environmental systems should tend to employ a multi-model approach. Following the multi-model approach, we use a family of models showing the system from different angles and complementing each other (see Figure 3 for a metaphoric illustration).

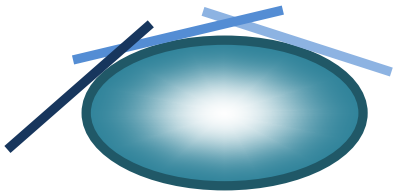

Figure 3: Earth maps as a metaphor for the multi-model approach. A detailed map is an image of a part of the Earth surface. Due to the curvature of the Earth surface, the image, which is quite precise at the center, loses its precision at the periphery. If we want to get a better precision in some small area in the periphery, we look at another map centered in that area. Thus, two, or more maps serve as complementary models of a part of the Earth surface. (This approach is used in differential geometry in studies of complex manifolds using families of local Euclidean maps.)

Once we decide to use a family of models, we abandon the area of (single-)model-based research (discussed in Subsections 1.3 – 1.7) and enter a broader area of research based on multi-model imitation methods (we use the terminology introduced in Subsection 1.2).

A multi-model imitation method employs alternative models that may differ structurally. The core of multi-model imitation is integration of models – methods for generation of

[10] Simulated time series for 1995-2027 (Lutz *et al.*, 2002), shown in Figure 5 were generated as forecasts. Now they appeal for comparison with data for the last decade.

[11] For example, any model from a given family of models can fail to be satisfactory in retrospective forecasting (see Subsection 1.6).

integrated knowledge about the systems, based on pieces of knowledge provided by the 'partial' models employed (see Figure 4 for an illustration).

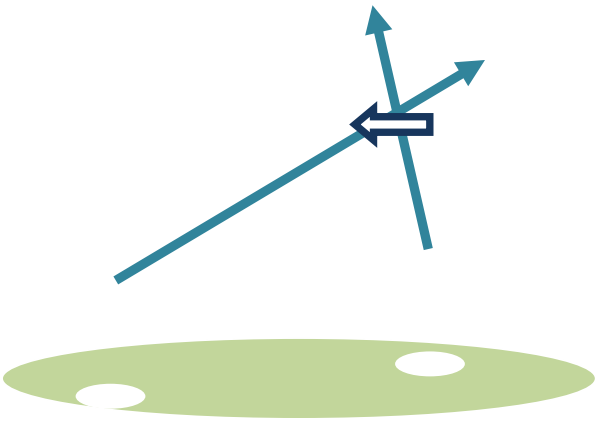

Figure 4: An illustration to integration of models. A plane is observed from two observation stations. Each station provides limited information on the location of the plane – the straight line connecting the station and the plane. The two lines are two alternative models for the plane's location. The plane is located at the point, at which the lines intersect. Intersecting the lines is a method to integrate the models.

Model integration has already been used in studies of complex social-environmental systems (see Figure 5 and Figure 6 for examples); in particular, the idea of model integration lies behind participatory methods (see, e.g., Pahl-Wostl, 2002), which synthesize knowledge from experts' opinions serving as individual models.

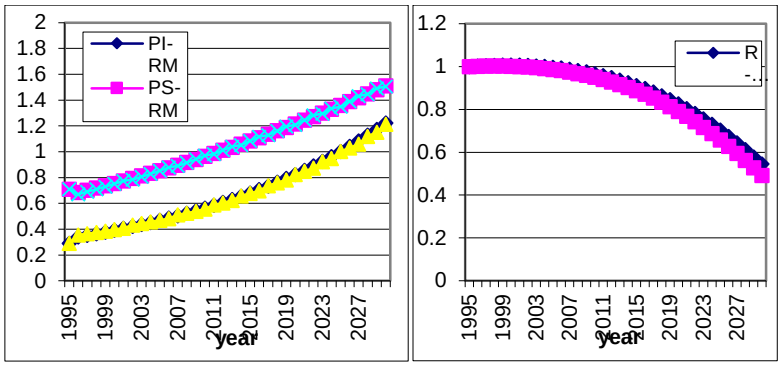

Figure 5: Simulated time series of the food-secure (PS) and food-insecure (PI) rural populations (left) and the resource stock (right) for the detailed (EM) and simplified (RM) PEDA (population, environment, development, agriculture) models for Mali in 1995-2027 (Lutz *et al.*, 2002). The simplified model disregards the urban population, aggregates over education, age and sex, and reduces the number of population states from 1600 to two (food-secure and food-insecure rural population). A good fit in the simulation results (which is also seen in other simulations corresponding to different parameter values) shows that the composition of the population states and the resource degradation process are insensitive to the education, age and sex structures. The latter observation is an integrated knowledge obtained from the use of two alternative models.

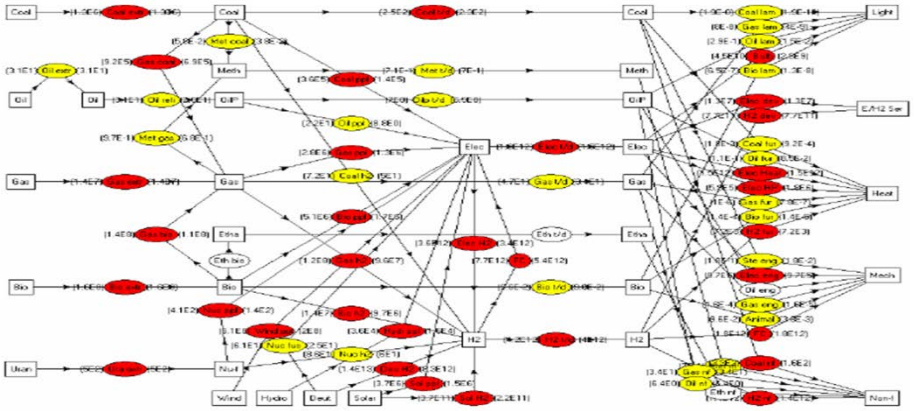

Figure 6: A snapshot from a typical model run of a model conceptualizing the emergence of new technologies and their combinations as a random process (reflecting the unpredictability of technological innovation) subject to resource constraints and economic incentives (Ma *et al.*, 2008). Ellipses represent primary energy technologies, which combine into alternative technology chains. The technology color codes indicate the level of development of various technologies ranging from large (red), very small (either emerging embryonic technologies or technologies being phased out, yellow) to technologies not used at all (white). Arrows indicate the directions of linkages, whereas the extent of linkages is given as numerical values of the corresponding energy flows. The individual trajectories obtained in 200 runs can be viewed as alternative models for historical development of the global energy system. The researchers analyzed the alternative histories and revealed their common features. For example, they found that the simulated evolution of the global energy system is characterized by increase in complexity (in terms of the number of energy chains), a 'complexity peak', and decline in complexity; another finding is that in all simulations there is a powerful tendency towards "decarbonization" – decrease in the carbon intensity of energy systems. These and other general conclusions (qualitative forecasts – see section 8) resulting from analysis of multiple simulations integrate pieces of knowledge provided by the individual; models.

Although model integration does not seem to be commonly recognized as a promising research avenue today, it may have a strong potential to develop into a powerful instrumental framework for systems analysis. One can think of a 'model calculus' – a structured family of partially formalized model integration techniques. Below we suggest a sketch on possible model integration methods.

In the model integration toolbox, cross-verification of models will be an important instrument. Models viewing a complex uncertain system from different perspectives cross-verify each other by registering identical phenomena in the system's behavior; in this situation we get a strong evidence for regarding the registered phenomenon to be a true feature of the system[12].

Quite often, due to modeling and observation errors, the outcomes from different models disagree with each other (even observation data resulting from different observation methods can differ essentially[13]); the models seem to be inconsistent.

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[12] A good fit in the simulation results for the full and simplified PEDA models in the Mali case study (see Figure 5), and the common features of the simulated alternative trajectories of the global energy system (see a comment to Figure 6) are illustrations for the model cross-verification phenomenon.

[13] See, e.g., Nilsson, *et al.,* (2007) for analysis of uncertainties in estimates for regional terrestrial biota full carbon accounts.

Situations of a similar type are addressed by theory of ill-posed problems (Tikhonov and Arsenin, 1974), which deals with solutions in poorly defined cases including those where the constraints for the sought solution are incompatible. The theory suggests regularization techniques allowing one to 'reconcile' the constraints and find appropriate 'surrogate' solutions. We expect that this theoretical background can be used in the context of systems analysis.

In analysis of complex social-environmental systems, in situations where models' outcomes disagree with each other, one is usually inclined to look for the 'most accurate' model, implying that the models are ordered in accuracy (though the order is unknown). In this context, a challenging question would be if there is a test for identifying the 'most accurate' model.

One can expect that, typically, the models whose outcomes disagree with each other are not ordered in accuracy ('model A' can in some aspects be more accurate and in some aspects less accurate than 'model B'). In this situation, the models can be assumed to be 'equally inaccurate'. We come to a need to develop a methodology for synthesizing an 'integrated outcome' (an analogue of a 'surrogate' solution in theory of ill-posed problems) that would incorporate the features of the models' alternative outcomes. A thorough analysis of extreme (possibly, very rare) cases where the models' outcomes 'come to agreement' could be the key.

Often, in analysis of a complex social-environmental system, one departs from 'disciplinary' models representing different compartments of the system and capturing the impacts of other compartments through exogenous inputs[14]. Traditionally, one applies the integrated assessment modeling paradigm (see, e.g., Argent, 2004) and establishes physical on-line links between the 'disciplinary' models – by letting part of the output variables of, say, 'model A' enter 'model B' as exogenous inputs, and vice versa. The interlinked 'disciplinary' models form a higher dimensional 'interdisciplinary' model. We call this model integration technique hard integration – as opposed to soft integration discussed below.

Hard integration has a clear motivation. Its potential drawbacks are not so obvious but can be critical. The complexity of the resulting integrated model (in terms of the dimension of the state space and the number of links between the model's compartments) is higher than the 'sum' of the complexities of the original 'disciplinary' models. Adding more 'disciplinary' modules, we find that the complexity of the integrated model grows much faster than the number of the 'disciplinary' modules. A highly complex integrated model can turn out to be not reliable enough (in Subsection 1.5 we argued that 'reliability of a model-based forecast is inverse proportional to the model's complexity and to the dimension of the model's state space'). Moreover, due to nonlinear integration of the system's parts in the entire complex system, simple mechanical interlinking the 'disciplinary' models can result in an inadequate representation of the entire system even if all 'disciplinary' models represent the corresponding parts of the system satisfactorily.

To compensate for possible errors in the outcomes from the complex integrated model, one can apply the cross-verification instrument (see above) involving the original 'disciplinary' models and/or partially integrated models.

A promising line in model integration research will be developing 'dialogues' between

[14] In analyses of spatially distributed systems one can use 'regional' models instead of 'disciplinary' ones, or both types of models.

complex agent-based models and conceptual aggregated models. The agent-based models operate at microscopic scales. They flexibly implement detailed assumptions about the agents' behaviors but do not serve as instruments for generating general conclusions. Conceptual aggregated models operate at macroscopic scales. They capture general trends but are unable to interpret them at the micro-level. A challenging task will be to interlink agent-based and aggregated models so that the aggregated models will convert the microscopic trajectories generated by the agent-based models into macroscopic trends, and, conversely, the agent-based models will interpret the macroscopic trends suggested by the aggregated models in terms of microscopic behaviors.

We presume that research employing the multi-model approach will go, typically, through the same stages as research employing a single model (see Table 1 in Subsection 1.3). One starts with choosing modeling paradigms and state space for the models to be employed. Next, one constructs and assesses models and model integration techniques. Generation of a forecast is the final stage in research.

2.2 Soft integration

In the rest of this section we consider two approaches to soft integration of models. Soft integration, in contrast to hard integration (see Subsection 2.1), suggests that integrated knowledge is synthesized through analysis of the models' outputs, without interference with the models' operation.

Obvious positive features of soft integration are the following.

- Soft integration is applicable in cases where hard integration is impossible – for example, in cases where the models are represented by data sets – see Subsection 2.3.
- Soft integration does not raise the complexity of the models and, as a consequence, does not reduce the reliability of the overall analysis (see Subsection 2.1).
- As opposed to hard integration requiring serious coordinated efforts of model designers and programmers, soft integration requires coordination in delivering the models' outputs to the integration group only.

In what follows we essentially base on Kryazhimskiy *et al.,* (2014) and Kryazhimskiy (2014).

2.3 Model integration in climate research

Climate change is the research field, in which recognition of the value of the use of multiple models has pioneered. For example, the Fourth IPCC Report relies on the results of 23 global climate models (Randall *et al.,* 2007) and considers their mean; for the Fifth IPCC Report, the IPCC Expert group developed recommendations on good practice in assessing multi-model climate projections and combining those with advanced statistical approaches (Knutti *et al.,* 2010a).

In a multi-model ensemble each model is usually attributed with its intrinsic uncertainties, often grouped into uncertainties in initial conditions, uncertainties in boundary conditions, parameter uncertainties and structural uncertainties (Tebaldi and Knutti 2007). For that reason, each model's outcome is commonly represented as a

random variable (or a random process). A systems analyst deals then with a family of probability distributions providing alternative descriptions to the same object.

In particular applications, researchers facing such phenomena employ specific features of the systems under investigation to reconcile alternative pieces of information and generate integrated knowledge (see, e.g., Nilsson *et al*., 2007). However, the subjectivity of the experts' makes those research efforts vulnerable for criticism. Development of a well-justified tool for integration of different viewpoints into a single picture becomes a challenge in systems analysis.

There have been impressive attempts undertaken to create a formal methodology for integration of alternative models-based results. To our knowledge, such attempts concentrate, primarily, on the question of weighting (in an appropriate way) the models that form a multi-model ensemble, based on assessment of the models' performance for the past and the present (Rajagopalan *et al.,* 2002, Robertson *et al.,* 2004, Tibaldi and Knutti 2007); the weights may also incorporate information on the degree of the models' interdependence (see, e.g., Knutti *et al.,* 2010b for an overview of currently available approaches). Knutti *et al.,* 2010a, pointing out serious difficulties in weighting the models, claims that "a robust approach to assigning weights to individual model projections of climate change has yet to be identified. … Studies should employ formal statistical frameworks rather than using ad hoc techniques."

2.4 Posterior integration of probability distributions. Background

Weighting, or prioritizing models is necessarily based on additional information on the models' performance (e.g., as mentioned above, assessment of the models' success in the past). Often such information is not available; all the models are then 'equal partners'. Here, we present a method for soft integration of models in this 'equity' situation. We focus on a case where each model in a model family is a probability distribution on a given set of objects (which can be numbers – though not necessarily); we call it the *object set*.

We assume that one of the objects in the object set is *true* – though unknown to us – and all the other ones are *false*. We can then associate each model with a method to observe the true object. The method is used many times. Each time a random error (whose distribution is not known to us) shifts the true object within the object set. Objects appear as observation results with some frequencies. The model – a probability distribution on the admissible object set – is the histogram of those frequencies. The model tells us that each object whose probability is greater than zero can be the true one, and suggests a likelihood of its being the true one.

For simplicity we deal here with a finite family of models, which we number $1, \dots, n$, and with a finite object set, which we denote Z. Model $i = 1, \dots, n$ is a probability distribution p_i on Z, which assigns, for each object $z \in Z$, a probability for z to be the true object, $p_i(z)$.

If the original object set is an infinite subset of a finite-dimensional Euclidean space[15] and the original probability distributions (the original models) have densities, one can use an approximate finite object set and approximate models in the integration analysis discussed below. The approximate object set is defined by putting a grid on the original object set and viewing the centers of the grid cells as approximate objects. For every original probability distribution on the original object set (an original model), one defines

[15] For the sake of mathematical rigor, one also needs to require the object set to be Borel measurable (e.g., closed) and bounded.

its approximation (an approximate model) as the probability distribution on the centers of the grid cells, which equips the center of each grid cell with the probability given by the original probability distribution to that grid cell. Varying the grid, one can construct a family of approximate object sets and approximate models and carry out the integration analysis for every approximation. If the results of the analyses are close to each other, one can conjecture that the resulting information on the location of the true object is reliable.

We assume $p_i(z) > 0$ for every $z \in Z$ and every $i = 1, \dots, n$[16]. Furthermore, we assume that the probability distributions $p_1, \dots, p_n$ are independent, i.e., the probability of the fact that models $1, \dots, n$ view, respectively, objects $z_1, \dots, z_n$ as the true one, equals $p_1(z_1) \dots p_n(z_n)$.

To integrate the models, we argue as follows. Let us consider combinations of independent individual observations carried out with the use of methods (models) $1, \dots, n$ as independent random tests. Combinations of individual observation results, $z_1, \dots, z_n$, provided by methods (models) $1, \dots, n$, respectively, are elementary events generated in those random tests. Clearly, an elementary event $(z_1, \dots, z_n)$ is true (realized actually) if each of the objects $z_1, \dots, z_n$ is the true one; the latter property implies $z_1 = \dots = z_n$. Therefore, the event that comprises all $(z_1, \dots, z_n)$ satisfying $z_1 = \dots = z_n$ is realized with a guarantee. We call it the posterior event. Note that any smaller event cannot be said to be realized with a guarantee, since it misses $(z, \dots, z)$ for some $z \in Z$ which may turn out be the true object.

Since the posterior event is realized with a guarantee, a conditional probability distribution of the elementary events, conditioned to the posterior event gives us a posterior probability distribution of the elementary events, P. The posterior probability distribution is concentrated on the elementary events of form $(z, \dots, z)$ and is given by the Bayes formula

$$P(z) = P(z, \dots, z) = \frac{p_1(z) \dots p_n(z)}{\sum_{w \in Z} p_1(w) \dots p_n(w)}.$$

We view P as the result of posterior integration of models $p_1, \dots, p_n$.

We use an appropriate measure of concentration to compare P with $p_1, \dots, p_n$ in informativeness. We understand a measure of concentration of probability distributions on Z to be a function evaluating every probability distribution on Z with a real number and reaching its minimum value at the point-concentrated probability distributions only. A simple measure of concentration is the one-minus-max-measure evaluating every probability distribution, p, on Z with $1 - max_{z \in Z} p(z)$. If all $z \in Z$ are numbers, every probability distribution, p, on Z determines a random variable on Z, which we call the one associated with p; in that case an obvious measure of concentration is the

[16] If some model, i, assigns a zero probability to a certain $z \in Z$, it becomes questionable if we should anyway allow z to be the true object. If we allow z to be the true object, a next question would be whether we should believe in model i at all – without believing in its assessment of z. This brings us back to the question of prioritizing the models, which we want to avoid in this subsection. If we decide that model i assesses z correctly, i.e., z cannot be the true object indeed, we simply reduce Z by removing z from it. Doing so for every model, we come to $p_i(z) > 0$ for every $z \in Z$ and every $i = 1, \dots, n$ (unless the models are totally inconsistent, i.e., every $z \in Z$ is viewed as not probable by some model).

variance – the function evaluating every probability distribution, p, on Z with the variance of the random variable associated with p.

Intuitively, it is clear that the smaller is the value of a measure of concentration of a probability distribution p, the more informative is p. In the limit case where p provides the minimum value to some measure of concentration, this holds for any measure of concentration as well, since p is then point-concentrated, being maximally informative. Conversely, if p does not provide the minimum value to some measure of concentration, this holds for any measure of concentration either, and p is not point-concentrated. In such intermediate (typical) case the result of comparison of probability distributions in informativeness through the use of a measure of concentration depends on the choice of the latter. In this context, we call a probability distribution p' more (less) informative than a probability distribution p'' with respect to a chosen measure of concentration.

Suppose a measure of concentration, μ, is chosen. If P, the result of posterior integration of models $p_1, \dots, p_n$, is more informative than each of those with respect to μ, posterior integration increases our knowledge about the true object; $p_1, \dots, p_n$ are not in contradiction and complement each other – from the viewpoint of the chosen measure of concentration, μ.

If P is less informative than any of $p_1, \dots, p_n$, posterior integration reduces our prior knowledge about the true object; $p_1, \dots, p_n$ contradict each other – from the viewpoint of μ. In this situation at least one of models $p_1, \dots, p_n$is misleading; identifying a misleading model and understanding in what sense it is misleading is a special task, which we do not discuss here. Although the situation itself is of a negative character, a new knowledge about contradiction between the models can form a basis for planning further research.

Finally, if P is more informative than part of $p_1, \dots, p_n$ and less informative than the other part of $p_1, \dots, p_n$, we can view the latter part of the models as an instrument for improving knowledge provided by the former part of the models.

2.5 Posterior integration of probability distributions. A manual

Here, we provide a 'manual' that may help researchers use the posterior integration technique (see Kryazhimskiy *et al.*, 2014, for an example).

Initial conditions

You are given an object set, Z, containing a deterministic true object; Z is either a finite set or an infinite, closed and bounded subset of a finite-dimensional Euclidean space (see footnote 15). The location of the true object within the object set is not known to you.

You are given n probability distributions on the object set, $p_1, \dots, p_n$, which act as models for the location of the true object within the object set. If the object set is an infinite subset of an Euclidean space, every probability distribution has a density.

If the object set is finite, every model assigns a non-zero probability to every object in the object set, i.e., $p_i(z) > 0$ for every $z \in Z$ and every $i = 1, \dots, n$.

The probability distributions are mutually independent in the sense that the probability of a combination of any n objects to appear in a combination of n random tests

corresponding to the given probability distributions equals the product of the probabilities of those objects to appear in the individual random tests[17].

Step 0: Approximation

This step applies only if the object set is an infinite subset of a finite-dimensional Euclidean space (and the probability distributions have densities). A goal is to construct an approximate finite object set and approximate probability distributions.

Put a grid on the object set and take the set of the centers of the grid cells for the approximate object set (the latter is necessarily finite).

For every originally given probability distribution, define its approximation to be the probability distribution on the centers of the grid cells, which equips the center of each grid cell with the probability given by the original probability distribution to that grid cell.

Make sure that $p_i(z) > 0$ for every $z \in Z$ and every $i = 1, \dots, n$.

From now on, deal with the approximate object set (denoted further, again, as Z) and approximate probability distributions on it (denoted further, again, as $p_1, \dots, p_n$).

Step 1: Posterior integration

Find the result of posterior integration of models $p_1, \dots, p_n$ -- the posterior probability distribution, P:

$$P(z) = \frac{p_1(z) \dots p_n(z)}{\sum_{w \in Z} p_1(w) \dots p_n(w)}.$$

Step 2: Informativeness assessment

Choose a measure of concentration, μ, of probability distributions on Z. If all $z \in Z$ are numbers, it is recommended to take the variance for a measure of concentration. The variance is the function evaluating every probability distribution, p, on Z with the variance of the random variable associated with p.

Compute the values of the measure of concentration μ for all given probability distributions, $\mu(p_1), \dots, \mu(p_n)$, and for the posterior probability, $\mu(P)$.

If $\mu(P)$ is greater than each of $\mu(p_1), \dots, \mu(p_n)$, state that P is more informative (with respect to μ) than each of the originally given probability distributions, $p_1, \dots, p_n$ and those are not in contradiction.

If $\mu(P)$ is smaller than each of $\mu(p_1), \dots, \mu(p_n)$, state that P is less informative (with respect to μ) than each of the originally given probability distributions, $p_1, \dots, p_n$, implying that at least one of those is misleading.

If $\mu(P)$ is smaller than part of $\mu(p_1), \dots, \mu(p_n)$ and bigger that the other part of $\mu(p_1), \dots, \mu(p_n)$, state that, $p_1, \dots, p_n$ are in contradiction – though part of those improve – through posterior integration – information provided by the other part of those.

[17] This formulation is applicable if the object set is finite. If the object set is an infinite subset of an Euclidean space E (see the previous paragraph), the mutual independency condition needs to be formulated in terms of the probabilities of objects to fall into arbitrary boxes in E (we omit further details).

2.6 Soft integration of ODE models. Background

Often, models for the dynamics of social and environmental systems have the form of finite-dimensional ordinary differential equations (ODEs),

$$\dot{x}(t) = f(x(t), y(t)), \quad x(0) = x^0;$$

here $x(t)$ is a finite-dimensional vector characterizing the system's state at time t; x^0 is the initial state at a the initial time 0; $\dot{x}(t)$ is the time derivative of $x(t)$ at time t, characterizing the instantaneous rate of change in the system' state at time t; and $y(t)$ is a time-varying parameter representing an exogenous input scenario. For simplicity, assume that $x(t)$ represents the dynamics of the system's economic components developing under an environmental input scenario $y(t)$; in this sense the above ODE acts as an 'economic' model.

The environmental dynamics, $y(t)$, can, in turn, be modeled by an 'environmental model' – an ODE, in which the economic dynamics, $x(t)$, acts as an input scenario,

$$\dot{y}(t) = g(x(t), y(t)), \quad y(0) = y^0.$$

The coupled economic-environmental system is then described by the system of two ODEs given above – the 'economic' and 'environmental' ones.

A straightforward way to represent the coupled system is to apply the hard integration technique (see Subsection 2.1) – to implement the system of the two ODEs as an integrated computer-based model. However, if both ODEs are complex and multi-dimensional (which is often the case), such hard coupling can be extremely difficult in a technical aspect. A complex technical problem can be implementation of on-line transportation of the current outcomes (states) of each model to the other one. This task can be comparable in complexity with writing and testing a new code for the coupled system. Moreover, as argued earlier (see Subsection 2.1), hard coupling implies that the errors built in each model are multiplied, which reduces the reliability of the modeling results.

Soft integration of the 'economic' and 'environmental' ODEs can be interpreted as their repeated 'dialogue'. One chooses initial input scenarios for the 'economic' and 'environmental' models, $y_0(t)$ and $x_0(t)$. One generates a response of the 'economic' model, $x_1(t)$, to scenario $y_0(t)$ as the solution to

$$\dot{x}(t) = f(x(t), y_0(t)), \quad x(0) = x^0,$$

and a response of the 'environmental' model, $y_1(t)$, to scenario $x_0(t)$ as the solution to

$$\dot{y}(t) = g(x_0(t), y(t)), \quad y(0) = y^0.$$

Next, one generates a response of the 'economic' model, $x_2(t)$, to scenario $y_1(t)$ as the solution to

$$\dot{x}(t) = f(x(t), y_1(t)), \quad x(0) = x^0,$$

and a response of the 'environmental' model, $y_2(t)$, to scenario $x_1(t)$ as the solution to

$$\dot{y}(t) = g(x_1(t), y(t)), \quad y(0) = y^0.$$

Generally, given kth responses of the 'economic' and 'environmental' models, $x_k(t)$ and $y_k(t)$, one generates a $(k+1)$th response of the 'economic' model, $x_{k+1}(t)$, as the solution to

$$\dot{x}(t) = f(x(t), y_k(t)), \quad x(0) = x^0,$$

and a $(k+1)$th response of the 'environmental' model, $y_{k+1}(t)$, as the solution to

$$\dot{y}(t) = g(x_k(t), y(t)), \quad y(0) = y^0.$$

It can be stated mathematically that under assumptions standard for the theory of ODEs[18] the sequence of the models' responses, $(x_k(t), y_k(t))$, converges to the solution of the system of the 'economic' and 'environmental' ODEs uniformly on any bounded time interval $[0, T]$. Moreover, one can derive an analytic upper estimate for the convergence rate. The estimate tells us that the shorter is the time interval $[0, T]$, the faster is the convergence rate. For small T the convergence rate is of type q^k with $q < 1$, implying that an appropriate approximation accuracy can achieved in a few iterations in the above 'dialogue'. A practical stopping criterion involves the distance between the subsequent responses, $(x_k(t), y_k(t))$ and $(x_{k-1}(t), y_{k-1}(t))$. Ones that distance is small enough, $(x_k(t), y_k(t))$ lies close to the solution of the system of the 'economic' and 'environmental' ODEs.

The statements are generalizable for the case of several ODE models, each of which is fed back by the outcomes of the other ones.

2.7 Soft integration of ODE models. A manual

Here we summarize the above material in the form of a 'manual' (we restrict ourselves for the case of two ODE models).

Initial conditions

You are given a finite-dimensional ODE

$$\dot{x}(t) = f(x(t), y(t)), \quad x(0) = x^0, \quad t \in [0, T]$$

call it ODE 1 – and a finite-dimensional ODE

$$\dot{y}(t) = g(x(t), y(t)), \quad y(0) = y^0, \quad t \in [0, T]$$

call it ODE 2. In ODE 1 $y(t)$ is an input variable having the dimension of the state variable of ODE 2, and in ODE 2 $x(t)$ is an input variable having the dimension of the state variable of ODE 1.

Step 0: Choosing initial input scenarios

Choose an initial input scenario for ODE 1, $y_0(t)$, and an initial input scenario for ODE 2, $x_0(t)$.

Step 1: Generation of first responses

Generate a first response of ODE 1, $x_1(t)$, as the solution to

$$\dot{x}(t) = f(x(t), y_0(t)), \quad x(0) = x^0, \quad t \in [0, T]$$

and a first response of of ODE 2, $y_1(t)$, as the solution to

$$\dot{y}(t) = g(x_0(t), y(t)), \quad y(0) = y^0, \quad t \in [0, T].$$

…

Step k: Generation of *k*th responses

Given a kth response of ODE 1, $x_{k-1}(t)$, and a kth response of ODE 2, $y_{k-1}(t)$, generate a kth response of ODE1, $x_k(t)$, as the solution to

[18] It is sufficient to assume f and g to be Lipschitz continuous.

$$\dot{x}(t) = f(x(t), y_{k-1}(t)), \quad x(0) = x^0, \quad t \in [0, T]$$

and a kth response of ODE 2, $y_k(t)$, as the solution to

$$\dot{y}(t) = g(x_{k-1}(t), y(t)), \quad y(0) = y^0, \quad t \in [0, T].$$

Compute the uniform distance between $(x_k(t), y_k(t))$ and $(x_{k-1}(t), y_{k-1}(t))$. If the distance is smaller than an accuracy parameter ε chosen in advance, take $(x_k(t), y_k(t))$ for the approximate solution of the system of ODE 1 and ODE 2. Otherwise go to step $k + 1$.

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