THE WEBSTER METHOD OF APPORTIONMENT

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ABSTRACT

Several results concerning the problem of U.S. Congressional apportionment are given which together indicate that a method first proposed by Daniel Webster (also known as "Major Fractions") is fairest judged on the basis of common sense, Constitutional requirement, and precedent.

Key words: Congress/representation/fair division/U.S. Constitution.

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§1. Introduction

The Constitution of the United States requires that the House of Representatives be apportioned among the several states according to their census populations. Various methods for so doing have been advanced over the years, beginning in 1792 after the first census. Four different methods have been used. In studying the differences between the methods there emerge several criteria which we believe to be most important by reason of common sense, Constitutional requirement, and precedent.

The aim of this note is to set down, for the record, several results describing the interplay between these criteria which together indicate that one method best answers the needs. Detailed proofs will appear elsewhere.

Definitions and Elementary Properties

An apportionment problem is specified by an s-vector ($s \geqq 2$) of rational numbers $\underset{\sim}{p} = (p_1, \ldots, p_s)$, all $p_i > 0$, and an integer house size $h \geqq 0$. An apportionment of h among s is an integer s-vector $\underset{\sim}{a} = (a_1, \ldots, a_s) \geqq \underset{\sim}{0}$ with $\sum_i a_i = h$. An apportionment method is a multi-valued function $\underset{\sim}{M}(\underset{\sim}{p};h)$ so that, for each $\underset{\sim}{p} > 0$ and $h \geqq 0$, $\underset{\sim}{M}$ is a set of apportionments $\underset{\sim}{a}$ of h among s (sometimes unique, sometimes not). $\underset{\sim}{M}^{(s)}$ denotes a method for fixed s; $\underset{\sim}{M}^{(s,h)}$ for fixed s and h. A particular $\underset{\sim}{M}$-solution is a single-valued function $\underset{\sim}{f}$, with $\underset{\sim}{f}(\underset{\sim}{p};h) = \underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p},h)$.

The quota of state i for h,s is $q_i = p_i h/\sum_j p_j$. The lower quota is $\lfloor q_i \rfloor$; the upper quota $\lceil q_i \rceil$.*

The following elementary properties define more explicitly what is meant by a method that apportions 'according to numbers'. Method $\underset{\sim}{M}$ is homogeneous when $\underset{\sim}{a} \in \underset{\sim}{M}(\lambda \underset{\sim}{p};h)$ if and only if $\underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p};h)$ for all rational $\lambda > 0$. It is proportional if $\underset{\sim}{a} = \underset{\sim}{q}$ is unique in $\underset{\sim}{M}(\underset{\sim}{p};h)$ whenever the quotas q_i are all integer. These properties are essential to the very idea of proportionality. A method is symmetric if for any permutation π of $1,\ldots,s$, $(a_{\pi(1)},\ldots,a_{\pi(s)}) \in \underset{\sim}{M}((p_{\pi(1)},\ldots,p_{\pi(s)});h)$ if and only if $\underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p};h)$. Thus only the numbers count, not the names of states.

Finally, a method is non-degenerate if $\underset{\sim}{p}^n \to \underset{\sim}{p}$ and $\underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p}^n;h)$ for all n implies $\underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p};h)$. So, if the $\underset{\sim}{p}^n$ are a sequence of increasingly accurate estimates of the true population $\underset{\sim}{p}$, all of which admit the apportionment $\underset{\sim}{a}$ by $\underset{\sim}{M}$, then so does $\underset{\sim}{p}$. This is a technical property that allows for a just handling of ties.

These four properties are met by all methods which have, to our knowledge, ever been proposed, and we assume them in the sequel unless otherwise noted.

Divisor Methods

A rank-index $r(p,a)$, $a \geq 0$ integer and $p > 0$ rational is any real valued function satisfying $r(p,a) > r(p,a+1)$. The Huntington method based on $r(p,a)$ [6] is .

$$\underset{\sim}{M}(\underset{\sim}{p};h) = \{\underset{\sim}{a} \geq \underset{\sim}{0} : a_i \text{ integer}, \textstyle\sum a_i = h, \max_i r(p_i,a_i) \leq \min_{a_j>0} r(p_j,a_j-1)\}.$$

A rank-index determines a method by assigning priorities in the allocation of seats by the following recursive rule on the size of the house $(h' \leq h)$: at $h' = 0$ set all $a_i = 0$; if $\underset{\sim}{a}$ apportions $h' < h$, then an apportionment of $h'+1$ seats is found by giving one more seat to some state maximizing $r(p_i,a_i)$.

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*$\lfloor x \rfloor$ denotes the greatest integer $\leq x$, $\lceil x \rceil$ the smallest integer $\geq x$.

A divisor criterion $d(a)$, $a \geq 0$ integer, is any real valued monotone increasing function. The divisor method based on $d(a)$ is the Huntington method based on $r(p,a) = p/d(a)$. We adopt the convention that $p > q$ implies $p/0 > q/0$.

A divisor method is regular if either $a < d(a) \leq a + 1$ for all a, or $a \leq d(a) < a + 1$ for all a.

Lemma 1. A divisor method is proportional if and only if it is regular.

It is of interest to know that virtually all of the methods proposed -- with the notable exception of Hamilton's -- have been regular divisor methods. These have received different names and descriptions in various countries and times. To the best of our knowledge they should be credited in terms of earliest discovery as follows. John Quincy Adams' method [1] has $d(a) = a$; James Dean's method [17] (he was Professor of Astronomy and Mathematics at Dartmouth and the University of Vermont) has $d(a) = 2a(a+1)/(2a+1)$. E.V. Huntington's method of equal proportions [12,13] (he was Professor of Mathematics at Harvard) has $d(a) = \sqrt{a(a+1)}$. Daniel Webster's method [17] has $d(a) = a + 1/2$. Thomas Jefferson's method [15] has $d(a) = a + 1$. These are all regular, hence proportional. Huntington unified these "historic five methods" through his test of inequality approach [12,13] and showed how they could be computed recursively using divisor functions. In the eighteenth and nineteenth centuries the methods were described in different (though equivalent) terms using the idea of an ideal district size or common divisor, λ. First a λ is specified, then the numbers p_i/λ are used to determine the apportionments a_i, whose sum determines h. For example, Adams' method rounds up all fractions, that is, sets $a_i = \lceil p_i/\lambda \rceil$; Jefferson's drops all fractions, that is, sets $a_i = \lfloor p_i/\lambda \rfloor$; and Webster's method rounds to the nearest integer, that is, sets $a_i = \lfloor p_i/\lambda + 1/2 \rfloor$.

Jefferson's method was used for the apportionments based on the censuses of 1790 through 1840. Webster's method was used for 1910 and 1930. Huntington's method of equal proportions was used for 1930 -- it happened to agree with Webster's -- and since 1940 it has been the law of the land.

House Monotonicity

Another early method is Alexander Hamilton's [11], re-invented and used for the censuses of 1850 through 1900 under the name "Vinton's Method of 1850". It first gives to each state i its lower quota $\lfloor q_i \rfloor$; then assigns one additional seat to each of the $\sum(q_i - \lfloor q_i \rfloor)$ states having the largest remainder $q_i - \lfloor q_i \rfloor$. But it admits the infamous Alabama paradox in which an increase in the house can result in some states losing seats.

A method $\underset{\sim}{M}$ is house monotone if there exists for any $\underset{\sim}{p}$ some $\underset{\sim}{M}$-solution $\underset{\sim}{f}$ for which $\underset{\sim}{f}(\underset{\sim}{p};h+1) \geqq \underset{\sim}{f}(\underset{\sim}{p};h)$ for all h. Congressional debate makes clear that only house monotone methods can be countenanced. All Huntington methods are house monotone; indeed the quest for house monotone methods is what motivated Huntington's work (see also Willcox [18]).

Uniformity

An inherent principle of fair division is: every subdivision of a fair division must be fair. In the context of apportionment this principle can be formulated as follows: $\underset{\sim}{M}$ is uniform [8] if $(\underset{\sim}{a},\underset{\sim}{b}) \in \underset{\sim}{M}(\underset{\sim}{p},\underset{\sim}{q};h)$ implies (i) $\underset{\sim}{a} \in \underset{\sim}{M}(\underset{\sim}{p};\sum_i a_i)$, and (ii) if also $\underset{\sim}{a}' \in \underset{\sim}{M}(\underset{\sim}{p};\sum a_i)$ then $(\underset{\sim}{a}',\underset{\sim}{b}) \in \underset{\sim}{M}(\underset{\sim}{p},\underset{\sim}{q};h)$. That is, an apportionment acceptable for all states is acceptable if restricted to any subset of states considered alone; moreover, if the restriction admits a different apportionment of the same number of seats then using it instead results in an alternate acceptable apportionment for the whole.

Theorem 1. If a method is uniform and proportional, then it is house monotone.

In fact the proof requires, in addition to uniformity, only that two states having identical populations cannot have apportionments differing by more than one seat. (This result was later independently noted by Hylland [14].) Since the Hamilton method is not house monotone it is not uniform.

Theorem 2. A method is uniform and proportional if and only if it is a Huntington method.

This follows directly from an earlier characterization of Huntington methods [6] and Theorem 1.

Population Monotonicity

Uniformity inherently bears the idea that a method should be applicable to all problems with all possible house sizes and numbers of states. A critic might counter that in many situations s and h are fixed: in the United States h = 435 and s = 50. So, let us fix s and h.

A census provides populations $\underset{\sim}{p} = (p_1,\ldots,p_s)$. But these change over time, and errors in census numbers may yield various $\underset{\sim}{p}$'s. A method must behave reasonably when applied to different $\underset{\sim}{p}$'s. Many definitions for such behavior are conceivable. The obvious mathematical choice is to compare two $\underset{\sim}{p}$'s identical in all state populations save one, and ask that a method never assign to the one state having greater population fewer seats. Actual population changes over the years do not produce such situations.

A method $\underset{\sim}{M}^{(s,h)} = \underset{\sim}{M}^*(\underset{\sim}{p})$ (having fixed s and h) is population monotone if $\underset{\sim}{a} \in \underset{\sim}{M}^*(\underset{\sim}{p})$, $\underset{\sim}{a}' \in \underset{\sim}{M}^*(\underset{\sim}{p}')$ and $p_i'/p_j' \geqq p_i/p_j$ imply that $a_i' < a_i$ and $a_j' > a_j$ occurs only if $p_i'/p_j' = p_i/p_j$ and $(a_1,\ldots,a_i',\ldots,a_j',\ldots,a_s) \in \underset{\sim}{M}^*(\underset{\sim}{p})$. This avers that if populations change, apportionments should not change by giving more seats to a state with relatively smaller population and less seats to a state with relatively greater population (unless there is a "tie").

Theorem 3. Fix $s \neq 3$ and h. $\underset{\sim}{M}^{(s,h)}$ is population monotone if and only if it is a divisor method.

The result is not true for s = 3: a counter-example exists for h = 7. And, of course, the divisor is a function of s and h.

Corollary. $\underset{\sim}{M}$ is uniform and population monotone if and only if it is a regular divisor method.

Invoking uniformity together with population monotonicity results in a divisor independent of s and h, which is what one would naturally expect. In fact, we have shown that uniformity and proportionality, together with the very weak demand that

$p_i > p_j$ must imply $a_i \geqq a_j$, suffices to characterize divisor methods (actually a somewhat more general result obtains if proportionality is dropped) [2]. Hylland [14] has recently found a similar result.

Satisfying Quota

The most primitive request for a method of apportionment is that it should guarantee to each state at least its lower quota and at most its upper quota, $\lfloor q_i \rfloor \leqq a_i \leqq \lceil q_i \rceil$, for all i. Methods with this property are said to satisfy quota. The Hamilton method is predicated on it, as is the Quota method [3,9]. It is an unfortunate fact that it is simply impossible to have a method which satisfies quota together with other fundamental criteria.

Theorem 4. There is no uniform method that satisfies quota.

Theorem 5. Fix $s \geqq 4$ and h large ($h \geqq s+3$ suffices). There is no population monotone method $\underset{\sim}{M}^{(s,h)}$ that satisfies quota.

So even for fixed s there is no method $\underset{\sim}{M}^{(s)}$ which reconciles the primitive wish to satisfy quota with the necessity of population monotonicity. For $s = 3$ a special result obtains.

Theorem 6. The method of Webster is the unique divisor method which satisfies quota for $s = 3$.

Satisfying quota -- as desirable as it may be -- is incompatible with uniformity and with population monotonicity for fixed s and h. We conclude that it must be abandoned. And this, we will see, can be done at essentially no cost. In particular, we discard the Quota method as well as all quotatone methods [7].

We are left with the class of regular divisor methods.

Bias

Why has Huntington's method of equal proportions been retained for U.S. Congressional apportionment from among the five historic divisor methods? If one inspects examples, it is immediately evident that as application of Adams' method it succeeded by application of Dean's, then Huntington's, Webster's and Jefferson's, solutions tend more and more to favor large states over small. This behavior can be proved ([9], Theorem 1). Two reasons were used to adopt Huntington's method: (1) it is in the middle of the five from the point of view of favoring small as versus large*, (2) it is based on a measure of pairwise inequality of representation between states which (while arbitrary) seems preferable to those measures of inequality which characterize the other four methods ([10,16]).

In these reports no absolute standard for determining whether a method favors small as against large states was set down. The desire to choose a method which is "unbiased" in its award of seats to small and large states is well founded, and is rooted in the "historic compromise" in which the Senate was given representation independent of population, and each state was assured of at least one seat in the House.

Suppose that a pair of states with populations (p,q), $p > q$ receive (a,b) seats. If $a/p > b/q$ then the <u>larger state is favored</u> whereas if $a/p < b/q$ the <u>smaller state is favored</u>. Inherent to uniformity is the true-to-life fact that a state judges how well or how badly it has been treated by making comparisons with its sister states' allocations. Indeed, this observation was at the origin of Huntington's approach, although he then developed methods based on admittedly arbitrary measures of inequality between states' representation. By definition a uniform method apportions seats among every two states in the same manner as it would were the two considered alone. Therefore, consider the set $S(a,b)$ of all two state problems (p,q) (normalized, by homogeneity, to $p + q = 1$) which yield the apportionment (a,b), $a > b \geqq 1$ (implying, by population monotonicity, $p \geqq q$). A divisor method $d(\cdot)$ is <u>unbiased</u> if the measure of the subset of $S(a,b)$ of those populations for which the small state is favored is the same as the measure of the subset for which the large state is favored, for all pairs (a,b), $a > b \geqq 1$. So, independently of the magnitudes of a and b, an unbiased method $d(\cdot)$ neither favors small nor large over the set of all problems.

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*It was fortunate, for this logic, that the number of methods considered was odd.

<u>Theorem 7</u>. The unique uniform, population monotone, and unbiased method is that of Webster.

Dually, one might approach the concept of "bias" by fixing (p,q), $p+q = 1$, $p \neq q$, and considering the apportionments of $h = 1,2,3,\ldots,h^*$ seats, where h^* is the smallest integer for which ph^* and qh^* are integer. A method is "unbiased" if the number of times the small state is favored is the same as the number of times the large state is favored, for all pairs (p,q), $p+q = 1$, $p \neq q$. By this definition the method of Webster is again the unique uniform, population monotone, and "unbiased" method.

Specific apportionments for a given problem can be analyzed for bias. Inspect each pair of allocations to states (a,b) where $a > b \geq 1$ and define the <u>bias ratio</u> to be the number of times the smaller state is favored divided by the total number of comparisons. One cannot expect any regular divisor method to yield a perfect bias ratio of .5: for some problems the ratios tend to be high, for others low. Bias is a concept concerning many problems and so must be applied over many problems. We have taken the 19 census populations of the United States (1790-1970 inclusive) and found apportionments by each of the historic five methods together with their respective bias ratios for every case (see Table 2).

To compare the overall tendencies of the five methods count for each method the number of times the smaller state is favored over all 19 problems and divide by the number of comparisons to obtain the bias ratio over the course of U.S. Congressional history (see Table 1). Huntington's method of equal proportions, now in use, has bias ratio .562 and decidedly favors the small states.

J.Q. Adams	J. Dean	E.V. Huntington	D. Webster	T. Jefferson
.780	.583	.562	.518	.199

<u>Table 1</u>. Bias ratio over 1790-1970 U.S. Censuses

	J.Q. Adams	J. Dean	E.V. Huntington	D. Webster	T. Jefferson
1790	.592	.592	.592	.426	.235
1800	.678	.456	.456	.456	.280
1810	.720	.472	.472	.472	.331
1820	.850	.643	.623	.481	.198
1830	.866	.508	.431	.431	.198
1840	.678	.401	.401	.401	.250
1850	.658	.460	.452	.452	.214
1860	.756	.653	.581	.481	.151
1870	.785	.637	.528	.491	.113
1880	.688	.416	.416	.395	.200
1890	.741	.542	.542	.484	.270
1900	.795	.637	.619	.570	.219
1910	.829	.684	.684	.657	.246
1920	.781	.618	.618	.522	.159
1930	.824	.621	.535	.514	.198
1940	.795	.588	.588	.568	.154
1950	.822	.656	.656	.614	.214
1960	.786	.542	.542	.506	.162
1970	.783	.548	.526	.472	.201

Table 2. Bias ratio for each U.S. Census population

The more detailed yearly figures of Table 2 show that for some specific problems (1880 is the one example) Huntington's method is less biased than Webster's, while for others (e.g., 1820, 1920) the reverse holds. This is unavoidable. Overall the statistics sustain the analysis: the Webster method is indicated if bias is to be avoided.

Minimum Requirements

The U.S. Constitution requires that each state receive a minimum of 1 seat, France assures each of its departements at least 2 seats, the European Parliament has fixed minimum numbers of seats attached to each of the countries and ranging between 6 and 36. None of the above developments has explicitly accounted for a minimum requirement other than zero. However, with appropriate modifications of definitions, the theorems can be extended and the fundamental conclusions are the same.

Conclusion

Methods of apportionment must be analyzed by identifying the criteria they satisfy (or do not satisfy) and by observing their behavior when used for actual problems.

The argument of this paper may be summarized as follows. Population monotonicity for fixed s (=50) and h (=435) means that a divisor method must be used. Adjoining uniformity narrows the choice to a regular divisor method defined independently of s and h, and guarantees house monotonicity. The requirement in addition that a method not be biased towards small or large states leaves but one method: that of Webster.

The major casualty appears to be the lack of a guarantee that apportionments satisfy quota. Insisting upon that guarantee would rule out all population monotone methods and all uniform methods. That is too great a price. In fact the method of Webster does "best" among the regular divisor methods in satisfying quota, and for three reasons.

First, as we have seen, it satisfies quota for s = 3, and is the only divisor method which does. Second, we say a method

$\underset{\sim}{M}$ is relatively well rounded -- "almost" satisfies quota -- if for $\underset{\sim}{a} \in \underset{\sim}{M}$ there is no pair of states with $a_i < q_i - 1/2$ and $a_j > q_j + 1/2$. The method of Webster is characterized as the unique divisor method which is relatively well rounded [4]. Third, empirical observation makes clear that the event of a Webster apportionment not satisfying quota is extremely unlikely. A Monte Carlo experiment confirms this: for s = 50, h = 435, 20,000 populations were chosen uniformly over the simplex $\{\underset{\sim}{p} ; \sum p_i = 1 , 435p_i \geqq .5\}$. The method of Webster violates quota 37 times. This extrapolates to less than one violation of quota in 5000 years.

We conclude with Daniel Webster, "...let the rule be, that the population shall be divided by a common divisor, and, in addition to the number of members resulting from such division, a member shall be allowed to each state whose fraction exceeds a moiety of the divisor" ([17], p.120).

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