MULTISTAGE BENDERS' DECOMPOSITION APPLIED TO MULTIPERIOD, MULTICOMMODITY PRODUCTION, DISTRIBUTION AND INVENTORY SYSTEM

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ABSTRACT

It has become more and more important for some industries to have an efficient program for their long range activities. Such a program usually means a production, distribution, and inventory plan of multicommodity over a multiperiod range. The network flow model is a standard way to represent the problem. Recent advances in the computational aspect of the generalized network (Glover et al. 1978:24, 1209-1220) gives us an indication of broader areas of application. However, the real world imposes complicated constraints upon us which can not be represented in the network models, not even in generalized network models.

In a previous paper (Tone 1977a:20, 77-93), the author tried a decomposition of network type constraints and non-network type constraints (called pattern constraints) by using Benders' partitioning procedure (Benders 1962:4, 238-252). The computational experiments show that the decomposition technique works well.

In this paper, the author develops a method to handle the multiperiod problem, where the problems in each period are coupled with the succeeding one by the existence of the inventory activities. Our system is doubly decomposable; by the existence of the pattern constraints and by inventory activities. The algorithm consists of two parts, one for solving the network flow problem in each period and the other for solving the pattern and coupling constraints which may be called a master problem. Finite convergence is guaranteed.

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NETWORK REPRESENTATION OF THE MODEL

The general diagram os the model is shown in Figure 1.1, which has n period horizons, connected by the inventory activities.

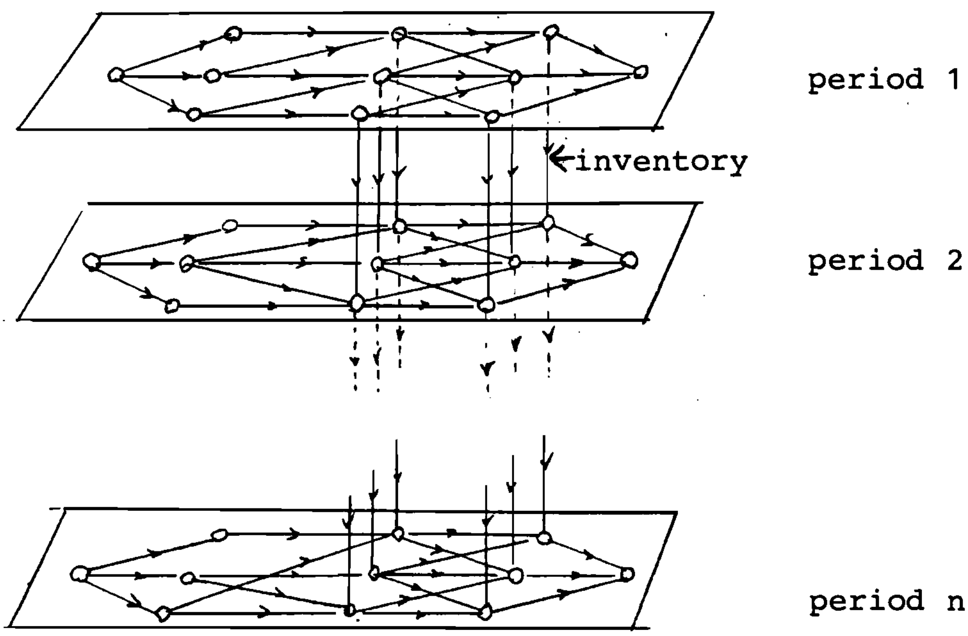

Figure 1.1 . General model.

In each period, the flow of materials is represented by the network model which includes supply of raw materials, production, and distribution of products. There are seasonal variations in the supply of raw materials and the demands for final products. We should take into account the variety of products and the mutual dependency between different products due to the ingredients ratio of raw materials and the characteristics of the processing units. Also, the safety stock level and the maximum and minimum level of production are important matters to be considered.

The objective of our model is the minimization of the total cost which includes the costs of raw materials, operation of processing units, distribution of products, and inventory.

Separation of the Network Flow Constraints

In addition to the usual network flow constraints, there exist several "pattern flow" constraints which destroy the network flow structure of the system and make it difficult to solve the whole system by the usual network flow algorithms. Some examples are as follows.

"And" constraint. In Figure 2.1, the ratio of amounts of product 1 and product 2 is determined by the characteristics of the processing units and raw materials. We cannot have one product independent of the other.

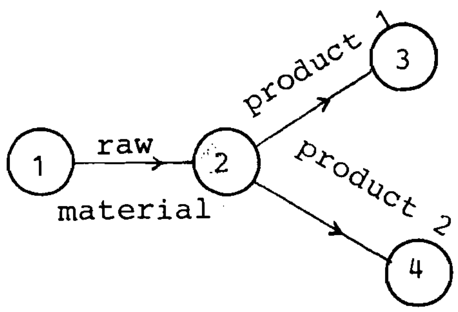

Figure 2.1 Pattern flow.

In the network model including such processing units, the flow on the arc (1, 2) must be divided into the flows on its succeeding arcs (2, 3) and (2, 4) in proportion to the given ratio, for example, 0.7:0.3. Then we have the pattern constraint

$$0.3x_{2,3} = 0.7x_{2,4} \quad .$$

"Either Or" constraint. Either product 1 or product 2 must be processed in the unit. In the corresponding network model, the flow $x_{1,2}$ cannot be divided and must go through the arc (2, 3) or (2, 4) as a whole. This is an example of combinatorial pattern flows.

Multicommodity case. There are situations where a multicommodity flow model is different from a single-commodity one only in the existence of several arcs whose capacities are shared by the multicommodity flows. In such cases, it may be possible to transform the former into the latter with pattern constraints.

For example, in Figure 2.2, if the sum of inventories of product 1 and product 2 is limited by the capacity of the warehouse, we introduce a constraint such as

$$x_1 + x_2 \leq c \quad ,$$

where x_1 and x_2 are inventory of product 1 and 2, and c is the warehouse capacity.

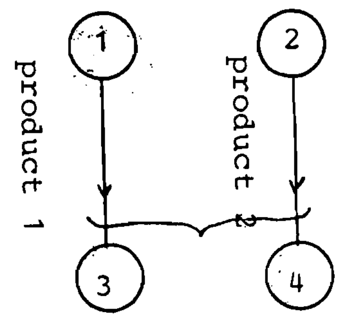

Figure 2.2 Inventory.

Most of the multicommodity constraints can be transformed into a single-commodity flow model which obeys pattern constraints, by a technique similar to the above and by the skillful introduction of artificial arcs. The above mentioned are rather simple examples of pattern constraints. It should be noted that if all the pattern constraints and cost structures are linear, our method is a dual version of the Dantzig-Wolfe decomposition method.

FORMULATION

Notations used:

- t: period $t = 1, 2, \ldots, n$,
- x_t: the vector of flows on the ordinary arcs in the period t, i.e., concerning only flow conservation and capacity constraints,
- y_t: the vector of flows concerning the pattern constraints in period t,
- z_t: the vector of the inventory flows from period t to period t + 1. z_0 and z_{n+1} are given,
- N_t, F_t, G_t: the node-arc incidence matrices with respect to x_t, y_t, and z_t in that order
- c_t: the cost vector of x_t,
- $f_t(y_t)$: the cost of y_t,
- $g_t(z_t)$: the cost of z_t, in many cases, linear with respect to the elements of z_t,
- b_t: the supply and demand vector corresponding to the nodes, in period t. The elements of b_t are positive, negative, or zero in accordance with the supply node, demand node or intermediate node,
- S_t: the set of y_t, satisfying given pattern constraints,
- T_t: the set of z_t, satisfying given pattern constraints, if any.

We can now formulate the problem as follows:

[P1]

Minimize

$$\sum_{t=1}^{n} \{c_t x_t + f_t(y_t) + g_t(z_t)\} \quad , \tag{3.1}$$

subject to:

node equations

$$N_t x_t + F_t y_t + G_{t-1} z_{t-1} + G_t z_t = b_t \quad , \tag{3.2}$$

$$(t = 1, 2, \ldots, n) \quad ,$$

capacity constraints

$$0 \leq x_t \leq a_t \quad , \tag{3.3}$$

pattern constraints

$$y_t \in S_t \quad , \quad z_t \in T_t \quad . \tag{3.4}$$

Figure 3.1 shows the structure of the node equations (3.2) which have the feature of decomposition.

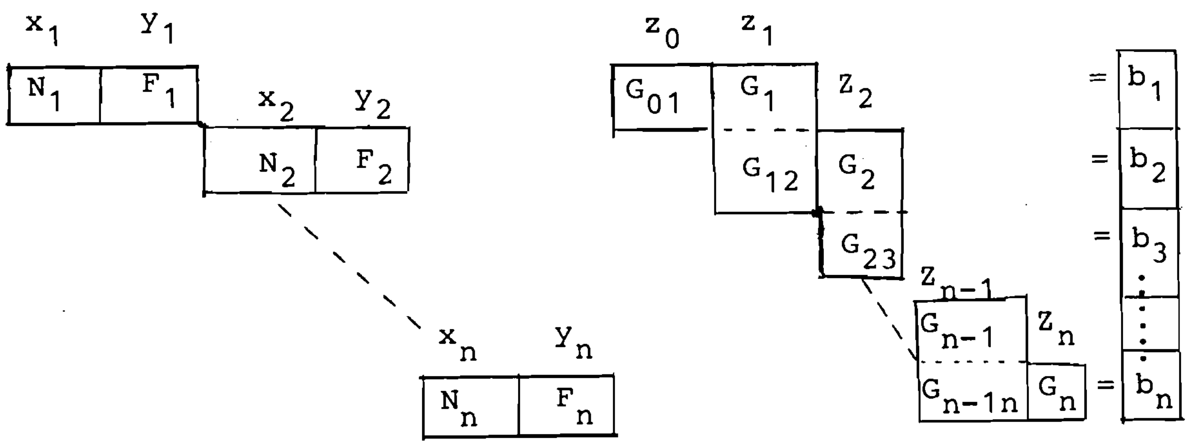

Figure 3.1 Structure of the model.

Algorithm Based on Benders' Decomposition

Step 0. Initialization. For a given set of $(\bar{y}_t, \bar{z}_t)$ $(t = 1,..., n)$ satisfying pattern constraints i.e. $\bar{y}_t \in S_t$ and $\bar{z}_t \in T_t$ $(t = 1,...,n)$, we consider the following network flow problem.

[P2]

Minimize

$$\sum_{t=1}^{n} \{c_t x_t + f_t(\bar{y}_t) + g_t(\bar{z}_t)\} \quad , \tag{4.1}$$

subject to

$$Nx_t = b_t - F_t\bar{y}_t - G_{t-1,t}\bar{z}_{t-1} - G_t\bar{z}_t \quad , \tag{4.2}$$

$$(t = 1,..., n) \quad ,$$

$$0 \leq x_t \leq a_t \quad , \quad (t = 1,\ldots,n) \quad . \tag{4.3}$$

The dual problem of [P2] is as follows:

[P3]

Maximize

$$\sum_{t=1}^{n} \{u_t^T(b_t - F_t\bar{y}_t - G_{t-1,t}\bar{z}_{t-1} - G_t\bar{z}_t) + v_t^T a_t\} \quad , \tag{4.4}$$

subject to

$$u_t^T N_t + v_t \geq c_t \quad , \quad (t = 1,\ldots, n) \quad , \tag{4.5}$$

$$v_t \geq 0 \quad , \tag{4.6}$$

where u_t and v_t are the dual variables corresponding to (4.2) and (4.3), respectively. Note that [P3] is decomposed with respect to t (t = 1,..., n). We denote the decomposed problem by $[P3]_t$.

Step 1. Solving the decomposed problems. Solve [P3]. The dual of [P3] is the mimimum cost network flow problem [P2]. Therefore, we can solve it quickly.

(1) If [P3] is infeasible, then the original problem [P1] is unbounded below or infeasible. Stop.

Otherwise,

(2) if [P3] has an optimal solution $(\bar{u}_t, \bar{v}_t)$ (t = 1,..., n), let the sets V and W be

$$V = \{(\bar{u}_t, \bar{v}_t)\} (t = 1,\ldots, n) \quad ,$$

$$W = \emptyset \quad .$$

Go to Step 2.

(3) If [P3] is unbounded upper, then some of the $[P3]_t$ must also be unbounded upper. Let them be $[P3]_{k1},\ldots, [P3]_{k\ell}$ and let the direction vector of the unboundedness be $(\hat{u}_{ki}, \hat{v}_{ki})$ (i = 1,..., m) where $\ell + m = n$.

Let

$$V = (\bar{u}_{h_i}, \bar{v}_{h_i}) \ (i = 1,\ldots, m) \quad ,$$

and

$$W = \{(\hat{u}_{k_i}, \hat{v}_{k_i})\}(i=1,\ldots, \ell).$$

Go to Step 2.

Step 2. Solving the master problem. Solve the following minimizing problem with respect to $\{y_t\}$, $\{z_t\}$ and to:

[P4]

Minimize t_0

subject to

$$\begin{aligned} &\bar{u}_{t_i}^T (b_{t_i} - F_t y_t - G_{t-1t} z_{t-1} - G_t z_t) \\ &+ \bar{v}_{t_i}^T a_t + f_t(y_t) + g_t(z_t) \le t_0 \quad , \quad (\forall (\bar{u}_{t_i}, \bar{v}_{t_i}) \in V) \quad , \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\hat{u}_{t_i}^T (b_t - F_t y_t - G_{t-1t} z_{t-1} - G_t z_t) + \hat{v}_{t_i}^T a_t \le 0 \\ &(\forall (\hat{u}_{t_i}, \hat{v}_{t_i}) \in W) \quad , \end{aligned} \tag{4.8}$$

$$t_0 \ge -M \text{ (M is a sufficiently large positive number)} \tag{4.9}$$

$$y_t \in S_t \quad , \quad z_t \in T_t \quad . \tag{4.10}$$

(1) If [P4] is infeasible, then the original problem [P1] is infeasible. Stop.

(2) Otherwise, let an optimal solution be $(\hat{t}_0, \hat{y}_t, \hat{z}_t)$, $(t = 1,\ldots, n)$. The element of V whose corresponding constraints do not contribute to determine $\hat{t}_0$, is to be removed from set V. Go to Step 3.

[P5]

Maximize

$$w_0 = \sum_{t=1}^{n} \{(u_t^T(b_t - F_t\hat{y}_t - G_{t-1t}\hat{z}_{t-1}$$

$$- G_t\hat{z}_t - v_c^Ta_t + f_t(\hat{y}_t) + g_t(\hat{z}_t)\} \ , \tag{4.11}$$

subject to

$$u_t^TN_t + v_t \geq - c_t \quad , \quad (t = 1,\ldots, n) \quad , \tag{4.12}$$

$$v_t \geq 0 \quad , \quad (t = 1,\ldots, n) . \tag{4.13}$$

[P5] is again composed of the separable problem $[P5]_t$, $(t = 1,\ldots, n)$.

(1) If [P5] has an optimal solution $(\hat{u}_t, \hat{v}_t)$, $(t = 1,\ldots,n)$ with the objective function value $\hat{w}_0$, then check the equality:

$$\hat{t}_0 = \hat{w}_0 \quad . \tag{4.14}$$

(1a) If equality (4.14) holds, then $\{\hat{y}_t, \hat{z}_t\}$ is an optimal solution of the origianl problem [P1], and we can have the optimal $\{\hat{x}_t\}$ as the dual solution of $[P5]_t$. Stop.

(1b) Otherwise, if

$$\hat{t}_0 > \hat{w}_0 \quad , \tag{4.15}$$

then add the vertices $(\hat{u}_t, \hat{v}_t)$ $(t = 1, \ldots, n)$ to set V. Go back to Step 2.

(2) If [P5] is unbounded upper, some of $[P5]_t$ must also be unbounded upper. Let them be $[P5]_{k_1}, \ldots, [P5]_{k_\ell}$ and let the direction vector of the unboundedness be $(\hat{u}_{h_i}, \hat{v}_{h_i})$ $(i = 1,\ldots, \ell)$. The remaining $[P5]_t$, if any, have their optimal solution $(\hat{u}_{h_i}, \hat{v}_{h_i})$ $(i = 1,\ldots, m)$ where $\ell + m = n$.

Let

$$V = V \cup \{(\bar{u}_{h_i}, \bar{v}_{h_i})\} \quad , \quad (i = 1,\ldots, m) \quad ,$$

and

$$W = W \cup \{(\hat{u}_{k_i}, \hat{v}_{k_i})\} \quad , \quad (i = 1,\ldots, \quad) \quad .$$

Go back to Step 2.

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