Solution of Evolutionary Games via Hamilton-Jacobi-Bellman Equations

Nikolay Krasovskii*, Arkady Kryazhimskiy**, Alexander Tarasyev**,***

* Ural State Agrarian University, Ekaterinburg, Russia

** International Institute for Applied System Analysis, Laxenburg, Austria

*** N.N. Krasovskii Institute of Mathematics and Mechanics, UrB RAS, Ekaterinburg, Russia

Introduction

The paper is focused on construction of solution for bimatrix evolutionary games basing on methods of the theory of optimal control and generalized solutions of Hamilton-Jacobi-Bellman equations. It is assumed that the evolutionary dynamics describes interactions of agents in large population groups in biological and social models or interactions of investors on financial markets. Interactions of agents are subject to the dynamic process which provides the possibility to control flows between different types of behavior or investments. Parameters of the dynamics are not fixed a priori and can be treated as controls constructed either as time programs or feedbacks.

Payoff functionals in the evolutionary game of two coalitions are determined by the limit of average matrix gains on infinite horizon. The notion of a dynamical Nash equilibrium is introduced in the class of control feedbacks within Krasovskii's theory of differential games.

Elements of a dynamical Nash equilibrium are based on guaranteed feedbacks constructed within the framework of the theory of generalized solutions of Hamilton-Jacobi-Bellman equations. The value functions for the series of differential games are constructed analytically and their stability properties are verified using the technique of conjugate derivatives.

The equilibrium trajectories are generated on the basis of positive feedbacks originated by value functions. It is shown that the proposed approach provides new qualitative results for the equilibrium trajectories in evolutionary games and ensures better results for payoff functionals than replicator dynamics in evolutionary games or Nash values in static bimatrix games. The efficiency of the proposed approach is demonstrated by applications to construction of equilibrium dynamics for agents' interactions on financial markets.

Evolutionary Game

Let us consider the system of differential equations which describes behavioral dynamics for two coalitions:

$$\dot{x} = -x + u, \qquad \dot{y} = -y + v. \tag{1}$$

Parameter x, $0 \le x \le 1$ is the probability of the fact that a randomly taken individual of the first coalition holds the first strategy. Parameter y, $0 \le y \le 1$ is the probability of choosing the first strategy by an individual of the second coalition. Control parameters u and v satisfy the restrictions $0 \le u \le 1$, $0 \le v \le 1$ and can be interpreted as signals for individuals to change their strategies. The system dynamics (1) is interpreted as a version of controlled Kolmogorov's equations [5] and generalizes evolutionary games dynamics [1, 2, 3, 9].

The terminal payoff functionals of coalitions are defined as mathematical expectations corresponding to payoff matrixes $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $i, j = 1, 2$ and can be interpreted as "local" interests of coalitions:

$$g_A(x(T), y(T)) = C_A x(T) y(T) - \alpha_1 x(T) - \alpha_2 y(T) + a_{22}, \tag{2}$$

at a given instant T. Here parameters C_A, α_1, α_2 are determined according to the classical theory of bimatrix games [12]:

$$C_A = a_{11} - a_{12} - a_{21} + a_{22}, \quad \alpha_1 = a_{22} - a_{12}, \quad \alpha_2 = a_{22} - a_{21}. \tag{3}$$

The payoff function g_B for the second coalition is determined according to coefficients of matrix B.

"Global" interests J_A^∞ of the first coalition are defined as:

$$J_A^\infty = [J_A^-, J_A^+], \quad J_A^- = \liminf_{t \to \infty} g_A(x(t), y(t)),$$
$$J_A^+ = \limsup_{t \to \infty} g_A(x(t), y(t)). \tag{4}$$

Interests J_B^∞ of the second coalition are defined analogously.

We consider the solution of the evolutionary game basing on the optimal control theory [10] and differential games [8]. Following [4, 7, 8, 9] we introduce the notion of a dynamical Nash equilibrium in the class of closed-loop strategies (feedbacks) $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$.

Definition 1. *Let $\varepsilon > 0$ and $(x_0, y_0) \in [0, 1] \times [0, 1]$. A pair of feedbacks $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ is called a Nash equilibrium for an initial position (x_0, y_0) if for any other feedbacks $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ the following condition holds: the inequalities:*

$$J_A^-(x^0(\cdot), y^0(\cdot)) \ge J_A^+(x_1(\cdot), y_1(\cdot)) - \varepsilon,$$
$$J_B^-(x^0(\cdot), y^0(\cdot)) \ge J_B^+(x_2(\cdot), y_2(\cdot)) - \varepsilon, \tag{5}$$

are valid for all trajectories:

$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0), \ (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0),$$
$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V).$$

Here the symbol X stands for the set of trajectories, which start from the initial point (x_0, y_0) and are generated by the corresponded strategies (U^0, V^0), (U, V^0), (U^0, V).

Dynamic Nash equilibrium can be constructed by pasting positive feedbacks u_A^0, v_B^0 and punishing feedbacks u_B^0, v_A^0 according to relations [4]:

$$U^0 = u^0(t, x, y, \varepsilon) \begin{cases} u_A^\varepsilon(t), & \|(x, y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon, \\ u_B^0(x, y), & \text{otherwise}, \end{cases} \tag{6}$$

$$V^0 = v^0(t, x, y, \varepsilon) \begin{cases} v_B^\varepsilon(t), & \|(x, y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon, \\ v_A^0(x, y), & \text{otherwise}. \end{cases} \tag{7}$$

Value Function for Positive Feedback

The main role in construction of dynamic Nash equilibrium belongs to positive feedbacks u_A^0, v_B^0, which maximize with guarantee the mean values g_A, g_B on the infinite horizon $T \to \infty$. For this purpose we construct value functions w_A, w_B in zero sum games with the infinite horizon. Basing on the method of generalized characteristics for Hamilton-Jacobi-Bellman equations we obtain the analytical structure for value functions. For example, in the case when $C_A < 0$ the value function w_A is determined by the system of four functions:

$$w_A(x, y) = \psi_A^i(x, y), \quad \text{if} \quad (x, y) \in E_A^i, \quad i = 1, \ldots, 4, \tag{8}$$
$$\psi_A^1(x, y) = a_{21} + \frac{((C_A - \alpha_1)x + \alpha_2(1 - y))^2}{4 C_A x (1 - y)},$$
$$\psi_A^2(x, y) = a_{12} + \frac{(\alpha_1(1 - x) + (C_A - \alpha_2)y)^2}{4 C_A (1 - x) y},$$
$$\psi_A^3(x, y) = C_A x y - \alpha_1 x - \alpha_2 y + a_{22},$$
$$\psi_A^4(x, y) = v_A = \frac{a_{22} C_A - \alpha_1 \alpha_2}{C_A}.$$

Here v_A is the value of the static game with matrix A. The value function w_A is presented in the Figure 1.

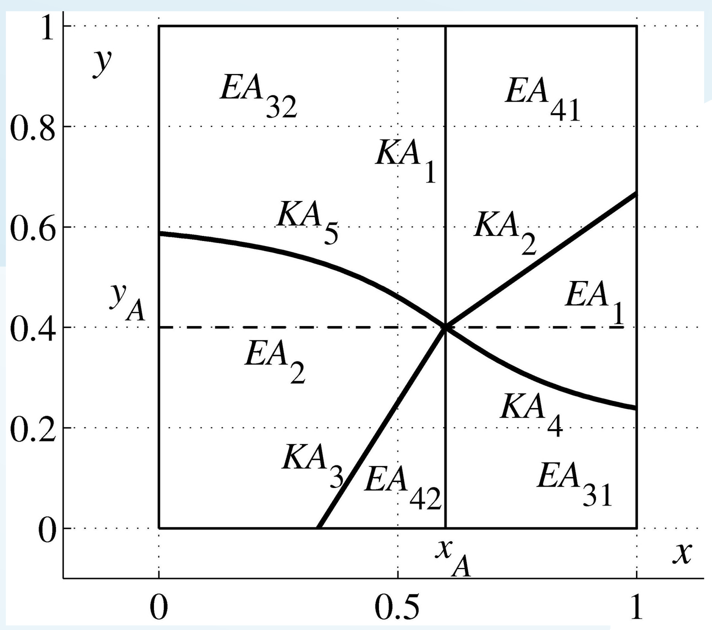

Fig. 1. Structure of the value function w_A.

It is shown that the value function w_A has properties of u-stability and v-stability [6, 8] which can be expressed in terms of conjugate derivatives [11]:

$$D_* w_A(x, y)|(s) \le H(x, y, s), \quad (x, y) \in (0, 1) \times (0, 1),$$
$$s = (s_1, s_2) \in R^2, \tag{9}$$

$$D^* w_A(x, y)|(s) \ge H(x, y, s), \quad (x, y) \in (0, 1) \times (0, 1),$$
$$w_A(x, y) < g_A(x, y), \quad s = (s_1, s_2) \in R^2. \tag{10}$$

Here the conjugate derivatives $D^* w_A$, $D_* w_A$ and the Hamiltonian H are determined by:

$$D^* w_A(x, y)|(s) = \sup_{h \in R^2} (\langle s, h \rangle - \partial_- w_A(x, y)|(h)), \tag{11}$$

$$D_* w_A(x, y)|(s) = \inf_{h \in R^2} (\langle s, h \rangle - \partial_+ w_A(x, y)|(h)), \tag{12}$$

$$H(x, y, s) = -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\}. \tag{13}$$

Model Applications

Application 1. Let us consider payoff matrices for two players on financial markets of bond and assets. Matrices A, B reflect the behavior of "bulls" and "bears", respectively:

$$A = \begin{pmatrix} 10 & 0 \\ 1.75 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 3 \\ 10 & 0.5 \end{pmatrix}.$$

In the Figure 2 we depict the static Nash equilibrium NE, switching lines K_A, K_B for feedback strategies, the new equilibrium at the point ME of their intersection, and equilibrium trajectories T_1, T_2, T_3. The new equilibrium point ME differs essentially from the static Nash equilibrium NE and provides better results for payoff functions of both players.

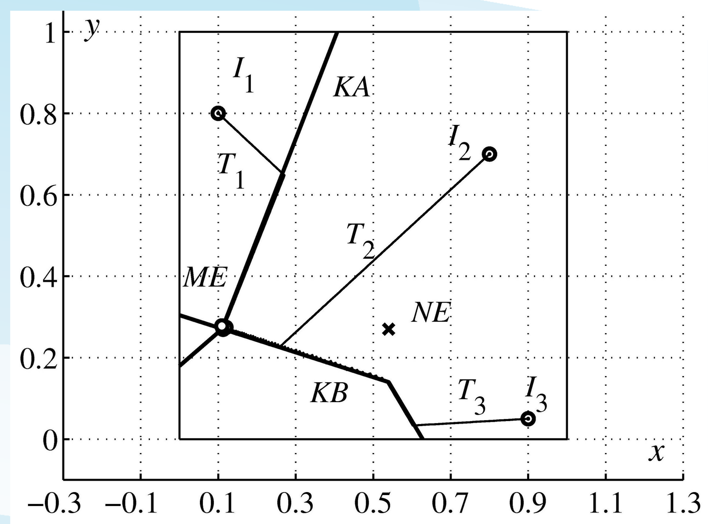

Fig. 2. Equilibrium trajectories for the financial markets game.

Application 2. Let us consider an example of coordination games. These games envisage coordinated solutions. Such situation describes the investment process in parallel projects:

$$A = \begin{pmatrix} 10 & 0 \\ 6 & 20 \end{pmatrix}, \quad B = \begin{pmatrix} 20 & 0 \\ 4 & 10 \end{pmatrix}.$$

Figure 3 presents the case with three static Nash equilibria N_1, N_2, N_3. The intersection point of switching lines K_A, K_B does not attract equilibrium trajectories T_1, T_2, T_3, T_4. Trajectories converge to intersection points of lines K_A, K_B with the edges of the unit square and provide better payoff results than the Nash equilibrium N_2.

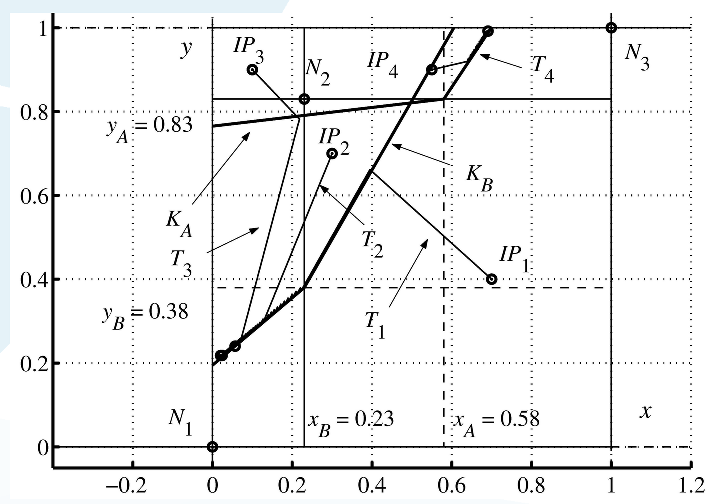

Fig. 3. Equilibrium trajectories in the coordination game of investments.

References

[1] *Basar T., Olsder G.J.*, Dynamic Noncooperative Game Theory. London: Academic Press, 1982. 519 p.

[2] *Friedman D.*, Evolutionary Games in Economics, Econometrica. 1991. Vol. 59, No. 3. P. 637–666.

[3] *Hofbauer J., Sigmund K.*, The Theory of Evolution and Dynamic Systems. Cambridge, Cambridge Univ. Press. 1988. 341 p.

[4] *Kleimenov A.F.*, Nonantagonistic Positional Differential Games, Ekaterinburg, Nauka, 1993. 184 p.

[5] *Kolmogorov A.N.*, On Analytical Methods in Probability Theory // Progress of Mathematical Sciences. 1938. Vol. 5. P. 5–41.

[6] *Krasovskii A.N., Krasovskii N.N.*, Control Under Lack of Information. Boston etc.: Birkhauser, 1994. 320 p.

[7] *Krasovskii N.A., Kryazhimskiy A.V., Tarasyev A.M.*, Hamilton-Jacobi Equations in Evolutionary Games // Proceedings of IMM UrB RAS. 2014. Vol. 20. 3. P. 114-131.

[8] *Krasovskii N.N., Subbotin A.I.*, Game-Theoretical Control Problems, NY, Berlin, Springer, 1988. 517 p.

[9] *Kryazhimskii A.V., Osipov Y.S.*, On Differential-Evolutionary Games // Proceedings of Math. Institute of RAS. 1995. Vol. 211. P. 257-287.

[10] *Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., and Mischenko E.F.*, The Mathematical Theory of Optimal Processes. New York: Interscience, 1962. 360 p.

[11] *Subbotin A.I., Tarasyev A.M.*, Conjugate Derivatives of the Value Function of a Differential Game, Soviet Math. Dokl., 1985, Vol. 32, No. 2, P. 162-166

[12] *Vorobyev N.N.*, Game Theory for Economists and System Scientists, Moscow, Nauka, 1985. 271 p.

Acknowledgments

The research is supported by the Russian Science Foundation grant No. 15-11-10018.

Contact Information

E-mails: nkrasovskiy@gmail.com (Nikolay Krasovskii)
tarasiev@iiasa.ac.at, tam@imm.uran.ru (Alexander Tarasyev).