NUMERICAL SOLUTION OF PARABOLIC PROBLEMS WITH NON-SMOOTH SOLUTIONS

P. Markowich

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

This paper deals with the convergence of stable and consistent one-step approximations for linear parabolic initial-boundary-value problems with non-smooth solutions. The proofs given may be extended to semilinear parabolic problems using H.B. Keller's stability concept. Finally an extension to Lax's convergence theorem is given.

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In this paper we consider the problem:

$$\text{I)}\quad U_t = a(x,t)U_{xx} + b(x,t)U_x + c(x,t)U + f(x,t),$$

$$(x,t) \in (0,1) \times (0,T)$$

$$\text{II)}\quad U(x,0) = U_o(x) \qquad , \; x \in [0,1] \qquad , \; T > 0$$

$$\text{III)}\quad U(0,t) = \gamma_0(t),\; U(1,t) = \gamma_1(t),\; t \in (0,T] \quad .$$

(I) is called a linear inhomogenous parabolic differential equation in one space variable x, (II) the initial condition and (III) the boundary conditions.

For the following we make the assumptions:

(A) $a, b, c, f \in C^r([0,1] \times [0,T])$, r sufficiently large

(B) $a(x,t) \geq k > 0$, $(x,t) \in [0,1] \times [0,T]$. stability condition

(C) $U_o(0) = \gamma_o(0)$, $U_o(1) = \gamma_1(0)$ continuity of initial and boundary functions.

We know that the initial and boundary functions determine the differentiability (smoothness) of the solution U in the points (0,0) and (1,0), which is important for the smallness of the local error of a consistent numerical procedure.

If U_o, γ_o and γ_1 are continuous functions then a unique solution U exists, which is continuous on $[0,1] \times [0,T]$ and therefore bounded in the closed set $[0,1] \times [0,T]$, and if $U_o \in C^3([0,1])$; $\gamma_0, \gamma_1 \in C^2([0,T])$ and $\dot{\gamma}_0(0)(\dot{\gamma}_1(0))$, $U_o''(0)$, $U_o'(0)$, $U_o(0)$ $(U_o''(1), U_o'(1), U_o(1))$, set for U_t, U_{xx}, U_x, U into the differential equation I), fulfill I), then U, U_t, U_x, U_{xx} are continuous and bounded on $[0,1] \times [0,T]$. See [1] and [2].

We gain a numerical procedure by choosing numbers N and M, and by forming the step sizes $n = 1./N$ in x- direction and $k = 1./M$ in t- direction, and by substituting appropriate difference approximations for U_t, U_x, U_{xx} in the net-points (x_i, t_n) with $x_i = ih$ and $t_n = nk$. So we can write our procedure in the following form assuming that $h = h(k)$ with $\lim_{k \to 0} h(k) = 0$

$$(*) \begin{cases} \frac{1}{k}[B_0(k,t_n)U^n - B_1(k,t_{n-1})U^{n-1} - R(k,t_n)] - \hat{f}(t_n) = 0 \\ n = 1(1)M \text{ with} \\ U^o = U_o = (U_0(X_1), \ldots , U_0(X_{N-1}))^T \end{cases} .$$

If $B_0(k,t_n) \equiv I$, we call the scheme explicit, otherwise implicit. The U^n's are (N-1)- vectors with the approximate solutions on the n-th time level, $R(k,t_n)$ is the (N-1)- vector with worked-in boundary-conditions on the n-th time level, $\hat{f}(t_n)$ is the vector with the approximations for $f(x_i,t_n)$, $i = 1(1)N-1$, i.e., $\| \hat{f}(t_n) - (f(x_1,t_n), \ldots , f(x_{N-1},t_n))^T \| \to 0$ for $k \to 0$ with some appropriate norm, and $B_o(k,t_n)$, $B_1(k,t_{n-1})$ are (N-1)- square matrices derived from the difference approximations for the derivatives.

We define the local error of the procedure (*) for the parabolic problem I), II), III) in the solution U as the sequence of vectors.

$$L^n(U,k) = \frac{1}{k}[B_0(k,t_n)U(t_n) - B_1(k,t_{n-1})U(t_{n-1}) - R(k,t_n)] - \hat{f}(t_n),\ n = 1(1)M$$

where $U(t_i)$ are the vectors containing the solution U evaluated in the net-points of the i-th-time-level. Further we say that (*) is consistent with I), II), III) in U of order l if $\|L^n(u,k)\| \leq C(U)k^l$, where C(U) is bounded and independent of n. We can show by Taylor's expansion that C(U) is a finitelinear combination of bounds of partial derivatives of U on the rectangle $[0,1] \times [0,T]$, if $\|\cdot\|$ is the maximum norm. The second important concept concerned with difference approximations is stability. We call the difference scheme (*) stable, if $B_0(k,t_n)$ is invertible for $k \leq k_o$ and for all $n \leq N$ and if $\|B_0^{-1}(k,t_n)\| \leq P$ for $k \leq k_o$ and $n \leq N$ where P is independent of k and n and if

$$\left\|\prod_{i=n}^{m} B_0^{-1}(k,t_i)B_1(k,t_{i-1})\right\| \leq L \text{ for } k \leq k_o,\ t_n = nk \in (0,T]$$

with $1 \leq m \leq n$, where L is independent of n, m and k. Further we say that (*) is convergent to U, if for $t = t_n = nk$ fixed, $\lim_{\substack{k\to 0\\ n\to\infty}} \|U^n(k) - U(t_n)\| = 0$ uniformly in $t\,(U^n(k) = U^n)$.

The sequence of vectors $E^n(k) = U^n(k) - U(t_n)$ is called global error. We easily conclude convergence from stability and consistency. By solving the recursive relation (*) for $U^n = U^n(k)$ we find: $\|U^n(k)\| \leq L\|U^o\| + P(TL+1) \max_{1\leq i\leq n} \|\hat{f}(t_i)\|$ presuming

$\gamma_0 = \gamma_1 = 0$. That means that $U^n(k)$ depends continuously on the initial condition U^o and on the disturbance $\hat{f}$ (in the norm $\| \ \|$). For the following we set $\|X\| = \max_{i=1(1)N-1} |X_i|$ for

$X = (X_1, \ldots, X_{N-1})^T \in \mathbb{R}^{N-1}$. Now we can prove:

<u>Theorem 1</u>: consider the parabolic problem I), II) and III) with the assumptions (A), (B) and (C). Let (*) be a finit difference approximation to I), II) and III), which is stable and consistent of the order l with problems of the form I*, II*, III with

solutions in $C^m([0,1] \times [0,T])$ (problem-I), II), III) with inhomogenity in $C^{m-2}([0,1] \times [0,T])$ and changed initial function) and let U_o, γ_0, γ_1 of the given problem fulfill:

a) $\gamma_0(0) = a(0,0)U_o''(0) + b(0,0)U_o'(0) + c(0,0)U_o(0) + f(0,0)$

b) $\gamma_1(0) = a(1,0)U_o''(1) + b(1,0)U_o'(1) + c(1,0)U_o(1) + f(1,0)$

with $\gamma_0, \gamma_1 \in C^m([0,T])$, $U_o \in C^3([0,1)]$, then the numerical procedure (*) is convergent for the given problem I), II) and III) in the maximum norm.

Proof: as mentioned before there exists a unique solution U of the given problem, so that U, U_t, U_x, U_{xx} are continuous and bounded in $[0,1] \times [0,T]$. (Proof in [1]).

Now let $\varepsilon > o$ be fixed. We construct the sequence of Bernstein polynomials to U on $[0,1] \times [0,\Gamma]$

$$B_n(U,x,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{i}\binom{n}{j} U\left(\frac{i}{n},\frac{Tj}{n}\right)(1-x)^{n-i}x^i\left(1-\frac{t}{T}\right)^{n-j}\left(\frac{t}{T}\right)^j$$

and know that:

$$B_n(U,.,.) \to U$$

$$\frac{\partial}{\partial t} B_n(U,.,.) \to U_t$$

$$\frac{\partial}{\partial x} B_n(U,.,.) \to U_x$$

$$\frac{\partial^2}{\partial x^2} B_n(U,.,.) \to U_{xx}$$

uniformly on $[0,1] \times [0,T]$ for $n \to \infty$.

As Butzer has shown in [3] for functions U in $C^1([0,1]^2)$, we can prove it for our case.

Now we set $U_\varepsilon = B_n(U,.,.)$ with $n > N(\varepsilon)$ fixed so that

$$\|U-U_\varepsilon\|_\infty + \|U_t-U_{\varepsilon t}\|_\infty + \|U_x-U_{\varepsilon x}\|_\infty + \|U_{xx}-U_{\varepsilon xx}\|_\infty \leq \varepsilon$$

and define: $v_\varepsilon = U_\varepsilon - [(1-x)(U_\varepsilon(0,t)-\gamma_0(t)) + x\,(U_\varepsilon(1,t)-\gamma_1(t))]$.

We have $\begin{cases} v_\varepsilon(0,t) = \gamma_0(t) \\ v_\varepsilon(1,t) = \gamma_1(t) \end{cases}$ and v_ε is a function

in $C^m([0,1] \times [0,T])$, because γ_0, γ_1 are in $C^m([0,T])$
$B_n(U,..) = U_\varepsilon$ is in $C^\infty([0,1] \times [0,T])$ and moreover:

$$\|U-v_\varepsilon\|_\infty + \|U_t-v_{\varepsilon t}\|_\infty + \|U_x-v_{\varepsilon x}\|_\infty + \|U_{xx}-v_{\varepsilon xx}\|_\infty \leqslant 2\varepsilon+2\varepsilon+2\varepsilon+\varepsilon = 7\varepsilon \quad .$$

That means, that we have constructed a function v_ε in $C^m([0,1] \times \times [0,T])$ which has the boundary values as U and which approximates U, U_t, U_x and U_{xx} uniformly on the closed rectangle $[0,1] \times [0,T]$.

We consider the neighboring problem:

$$\text{I*)}\quad v_t = a(x,t)v_{xx} + b(x,t)v_x + c(x,t)v + f(x,t) + $$
$$+ (v_{\varepsilon t}-a(x,t)v_{\varepsilon xx}-b(x,t)v_{\varepsilon x}-c(x,t)v_\varepsilon-f(x,t))$$
$$(x,t) \in (0,1] \times (0,T]$$

$$\text{II*)}\quad v(x,0) = v_\varepsilon(x,0),\ x \in [0,1]$$

$$\text{III)}\quad v(0,t) = \gamma_0(t),\ v(1,t) = \gamma_1(t),\ t \in [0,T] \quad [\text{III* = III}]$$

which has the unique solution $v = v_\varepsilon$.

We set: $Z_\varepsilon = v_{\varepsilon t}-a(x,t)v_{\varepsilon xx}-b(x,t)v_{\varepsilon x}-c(x,t)v_\varepsilon-f(x,t)$,

$$Z_\varepsilon \in C^{m-2}([0,1] \times [0,1]),$$

and conclude

$$\|Z_\varepsilon\|_\infty \leqslant \|U_t-a(x,t)U_{xx}-b(x,t)U_x-c(x,t)U-f(x,t)\| +$$
$$+\|U_t-v_{\varepsilon t}-a(x,t)(U_{xx}-v_{\varepsilon xx})-b(x,t)(U_x-v_{\varepsilon x})-c(x,t)(U-v_\varepsilon)\| \leqslant$$
$$\leqslant 0+(1+\|a\|_\infty+\|b\|_\infty+\|c\|_\infty)\varepsilon = C_1\varepsilon,\ C_1 \in \mathbb{R} \quad .$$

The numerical procedure for I*), II*), III) has the form

$$(\bar{v})\begin{cases}\frac{1}{k}[B_0(k,t_n)V_\varepsilon^n - B_1(k,t_{n-1})V_\varepsilon^{n-1} - R(k,t_n)] = \hat{f}(t_n) + \hat{z}_\varepsilon(t_n), \\ \qquad\qquad\qquad\qquad\qquad\qquad , n = 1(1)M \\ V_\varepsilon^o = (v_\varepsilon(x_1,0), - , v_\varepsilon(x_{N-1},0))^T\end{cases}$$

and converges to v_ε of order 1, that means:
$\| V_\varepsilon^n(k) - V_\varepsilon(t_n)\| \leqslant C(\varepsilon)k^1$, because the order of convergence is the same as the order of consistency in the case of smooth solutions.

The procedure for I), II), III) is:

$$(\bar{v}\bar{v})\begin{cases}\frac{1}{k}[B_0(k,t_n)U^n - B_1(k,t_{n-1})U^{n-1} - R(k,t_n)] = \hat{f}(t_n) \\ U^o = U_o = (U_o(x_1), \ldots, U_o(x_{N-1}))^T \quad .\end{cases}$$

We subtract $(\bar{v})$ from $(\bar{v}\bar{v})$ and get:

$$\frac{1}{k}[B_0(k,t_n)(U^n-V_\varepsilon^n) - B_1(k,t_n)(U^{n-1}-V_\varepsilon^{n-1})] = -\hat{z}_\varepsilon(t_n)$$

$$U^o-V_\varepsilon^o = (U_o(x_1) - v_\varepsilon(x_1,0), \ldots, U_o(x_{N-1}) - v_\varepsilon(x_{N-1},0))^T$$

We use that the solution of a difference equation of this form depends continuously on the initial condition and on the disturbance, if the boundary conditions are homogenous:

$$\| U^n-V_\varepsilon^n\| \leqslant L\| U^o-V_\varepsilon^o\| + P(LT+1)\max_{1\leqslant i\leqslant h}\| \hat{z}_\varepsilon(t_n)\| \leqslant (7L+P(LT+1)C_1)\varepsilon = C_2\varepsilon \quad .$$

We get for $t = nk$ fixed in $(0,T]$:

$$\| U(t)-U^n(k)\| \leqslant \| U(t)-V_\varepsilon(t)\| + \| V_\varepsilon(t)-V_\varepsilon^n(k)\| + \| V_\varepsilon^n(k)-U^n(k)\| \leqslant$$

$$\leqslant 7\varepsilon + C(\varepsilon)k^1 + C_2\varepsilon = (7+C_2)\varepsilon + C(\varepsilon)k^1 \quad .$$

For $k<(\frac{\varepsilon}{C(\varepsilon)})^{\frac{1}{l}}$ we get $\|U(t)-U^n(k)\|\leq(8+C_2)\varepsilon$, where C_2 is independent of n, ε and k. If we start the proof with $\frac{\varepsilon}{8+C_2}$ convergence follows.

Our second step is to neglect the conditions a) and b) in Theorem 1. So we prove:

Theorem 2: consider the numerical procedure (*) for I), II) and III) under the same assumptions as in Theorem 1. Let (A), (B) and (C) be valid. If $U_o \varepsilon C([0,1])$ and $\gamma_0,\gamma_1 \varepsilon C^m([0,T])$, then the numerical procedure (*) is convergent to the unique solution of I), II) and III).

Proof: Let $\varepsilon>0$ be fixed. Then we choose a function $\bar{U}_o^\varepsilon$ in $C^\infty[(0,1])$, so that $\|U_o-\bar{U}_o^\varepsilon\|_\infty<\varepsilon$. The existence of $\bar{U}_o^\varepsilon$ is a consequence of the approximation theorem of Weierstrass. We define:

$$U_o^\varepsilon = \bar{U}_o^\varepsilon-[x(\gamma_1(0)-\bar{U}_o^\varepsilon(1)) + (1-x)(\gamma_0(0)-\bar{U}_o^\varepsilon(0))]$$

We get: $U_o^\varepsilon(0) = \gamma_0(0)$ and $U_o^\varepsilon(1) = \gamma_1(0)$ and

$$\|U_o-U_o^\varepsilon\|\leq\|U_o-\bar{U}_o^\varepsilon\|+|x|\varepsilon+|1-x|\varepsilon\leq 2\varepsilon \quad .$$

Now we choose a function $y^\varepsilon(x)\ \varepsilon\ C^3([0,1])$ fulfilling $y^\varepsilon(0) = = y^\varepsilon(1) = 0$ and $\|y^\varepsilon\|_\infty\leq\varepsilon$ and form $V_o^\varepsilon = U_o^\varepsilon + y^\varepsilon$. The function V_o^ε shall satisfy:

1) $\gamma_0(0) = a(0,0)V_o^{\varepsilon''}(0) + b(0,0)V_o^{\varepsilon'}(0) + c(0,0)V_o^\varepsilon(0) + f(0,0)$

2) $\gamma_1(0) = a(1,0)V_o^{\varepsilon''}(1) + b(1,0)V_o^{\varepsilon'}(1) + c(1,0)V_o^\varepsilon(1) + f(1,0)$

That means:

1a) $\gamma_0(0) - [f(0,0) + a(0,0)U_o^{\varepsilon''}(0) + b(0,0)U_o^{\varepsilon'}(0) +$

$+ c(0,0)U_o^\varepsilon(0)] = a(0,0)y^{\varepsilon''}(0) + b(0,0)y^{\varepsilon'}(0)$

1b) $\gamma_1(0) - [f(1,0) + a(1,0)U_o^{\varepsilon''}(1) + b(1,0)U_o^{\varepsilon'}(1) +$

$+ c(1,0)U_o^\varepsilon(1)] = a(1,0)y^{\varepsilon''}(1) + b(1,0)y^{\varepsilon'}(1) \quad .$

We choose $y^{\varepsilon\prime}(0) = y^{\varepsilon\prime}(1) = 0$ and compute $y^{\varepsilon\prime\prime}(0) = y_1$ and $y^{\varepsilon\prime\prime}(1) = y_2$ from the equations 1a) and 2a) and construct:

$$y^{\varepsilon}(x) = \left\{\begin{array}{ll} \frac{y_1}{2t_1^2} x^2 (x-t_1)'' & 0 \leqslant x \leqslant t_1 \\ 0 & t_1 \leqslant x \leqslant t_2 \\ \frac{y_2}{2t_2^2}(x-1)^2(x-t_2)^4 & t_2 \leqslant x \leqslant 1 \end{array}\right\} \varepsilon C^3([0,1])$$

with $0 < t_1 < \min\left(\frac{1}{2}, \sqrt[4]{\frac{729\varepsilon}{8|y_1|}}\right)$ for $y_1 \neq 0$ and

$0 < t_2 < \min\left(\frac{1}{2}, \sqrt[4]{\frac{729\varepsilon}{8|y_2|}}\right)$ for $y_1 \neq 0$.

Otherwise there is no restriction on t_1 resp t_2 (only $0<t_1<t_2<1$).

Now we consider:

$$(\Delta)\begin{cases} V_t = a(x,t)V_{xx}+b(x,t)V_x+c(x,t)V+f(x,t) , & (x,t)\varepsilon(0,1]\times(0,T] \\ V(x,0) = V_o^{\varepsilon}(x) & x\varepsilon[0,1] \\ V(0,t) = \gamma_0(t) & t\varepsilon(0,T] \\ V(1,t) = \gamma_1(t) & \end{cases}$$

We have: $V_o^{\varepsilon}\varepsilon C^3([0,1])$, γ_0, $\gamma_0 \varepsilon C^m([0,T])$, $V_o^{\varepsilon}(0) = \gamma_0(0)$, $V_o^{\varepsilon}(1) = \gamma_1(0)$ and V_o^{ε}, γ_0, γ_1 fulfill the condition a) and b) in theorem 1. So we can conclude, that this problem has a unique solution V_{ε}, so that V_{ε}, $V_{\varepsilon t}$, $V_{\varepsilon x}$, $V_{\varepsilon xx}$ are continuous in $[0,1]\times[0,T]$. Also we can conclude that $Z = U-V_{\varepsilon}$ is the unique solution of

$$(\Delta\Delta)\begin{cases} Z_t = a(x,t)Z_{xx} + b(x,t)Z_x + c(x,t)Z \\ Z(x,0) = U_o(x) - V_o^{\varepsilon}(x) \\ Z(0,t) = Z(1,t) \equiv 0 \end{cases}$$

(U is the unique solution of the given problem).
We know that the solution Z depends continuously on the initial data Z(x,0), so we have:

$$\|Z\|_\infty = \|U-V_\varepsilon\| \leq C.\|U_o - V_o^\varepsilon\| \leq C\varepsilon \quad .$$

The numerical procedure to the given problem has the form:

$$\frac{1}{k}[B_0(k,t_n)U^n_{(k)} - B_1(k,t_{n-1})U^{n-1}_{(k)} - R(k,t_n)] = \hat{f}(t_n)$$

and to (Δ)

$$\frac{1}{k}[B_0(k,t_n)V_\varepsilon^n(k) - B_1(k,t_{n-1})V_\varepsilon^{n-1}(k) - R(k,t_n)] = \hat{f}(t_n)$$

$$V_\varepsilon^o = V_o^\varepsilon$$

We conclude by subtracting:

$$\frac{1}{k}[B_0(k,t_n)(U^n(k)-V_\varepsilon^n(k)) - B_1(k,t_{n-1})(U^{n-1}(k)-V_\varepsilon^{n-1}(k))] = 0$$

$$U^o - V_\varepsilon^o = U_o - V_o^\varepsilon$$

We get by stability: $\|U^n(k)-V_\varepsilon^n(k)\| \leq L\|U_o-V_o^\varepsilon\| \leq 3L\varepsilon$.
Applying theorem 1 we conclude, that there is a $k_o(\varepsilon)>0$ so that for all $k<k_o(\varepsilon)$, $\|V_\varepsilon(t) - V_\varepsilon^n(k)\| \leq \varepsilon$ for $t = nk$ fixed in [0,T].
So,

$$\|U(t)-U^n(k)\| \leq \|U(t)-V_\varepsilon(t)\| + \|V_\varepsilon(t)-V_\varepsilon^n(t)\| + \|V_\varepsilon^n(t)-U^n(k)\| \leq$$

$$\leq C\varepsilon + \varepsilon + 3L\varepsilon = (C+1+3L)\varepsilon \quad .$$

And that means convergence.

Putting the used proof-methods on a more formal level we can derive an extension to Lax's convergence theorem for stable approximations to linear operator equations which are consistent for data in a dense set. Consider the linear and invertible operator $F : (A,\| \ \|_A) \rightarrow (B,\| \ \|_B$ where A, B are appropriate linear spaces and let $\|F^{-1}\|_B$ be bounded by k_1. That means that the solution U of the equation FU = g depends continuously on the data g. For the numerical computation of U we use approximations $F_h U_h = g_h$ with the following properties:

1) $F_h: (A_h, \| \|_{A_h}) \to (B_h, \| \|_{B_h})$ for $0<h\leq h_o$ (step-size, grid parameter), where A_h, B_h are appropriate linear spaces.

2) F_h is linear and invertible and $\|F_h^{-1}\|_{B_h} \leq k_2$ for all $h \leq h_o$.

The last property of F_h is called stability:

3) There exist linear and uniformly bounded operators,

$\Delta_h^A; (A, \| \|_A) \to (A_h, \| \|_{A_h})$

$\Delta_h^B; (B, \| \|_B) \to (B_h, \| \|_{B_h})$

4) $\|\Delta_h^B(g) - g_h\|_{B_h} = o(1)$ for $h \to o$.

5) The scheme $F_h U_h = g_h$ is consistent with FU = g for all $g \in X \subset B$, where X is dense in B, i.e.,

$\|F_h(\Delta_h^A U) - g_h\|_{B_h} = o(1)$ for $h \to 0$

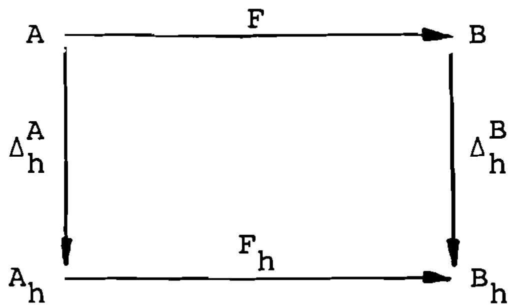
where U is the solution of FU = g.

We can conclude:

<u>Theorem 3</u>: under the given assumptions on F and F_h the procedure $F_h U_h = g_h$ is convergent to the solution U of the equation FU = g, for all $g \in B$, i.e.,

$$\|\Delta_h^A(U) - U_h\|_{A_h} = o(1) \text{ for } h \to 0.$$

<u>Proof</u>: We have the following situation:



Let ε fixed be greater o.

For solving $FU = g$ we consider the scheme $F_h U_h = g_h$. Because X is dense in B we can choose $g_\varepsilon \varepsilon X$ so that $\| g-g_\varepsilon \|_B \leqslant \varepsilon$. Instead of $FU = g$ we now solve $FU_\varepsilon = g_\varepsilon$. We conclude $\| U-U_\varepsilon \|_A \leqslant \| F^{-1} \| \, \| g-g_\varepsilon \|_B$ that means:

A) $\| U-U_\varepsilon \|_A \leqslant k_1 \varepsilon$.

Now we consider $F_h U_{\varepsilon h} = g_{\varepsilon h}$ and we easily prove the convergence of $U_{\varepsilon h}$ to U_ε for $h \to o$ and fixed $\varepsilon > o$ by the usual consistency - stability method:

$$F_h U_{\varepsilon h} = g_{\varepsilon h}$$

$$\underline{F_h \Delta_h^A U_\varepsilon = g_{\varepsilon h} + C_\varepsilon(h)} \qquad \| C_\varepsilon(h) \|_{B_h} = o(1)$$ for $h \to o$ and fixed $\varepsilon > o$ because $g_\varepsilon \varepsilon X$.

$$F_h (U_{\varepsilon h} - \Delta_h^A U_\varepsilon) = -C_\varepsilon(h) \Rightarrow$$

B) $\| U_{\varepsilon h} - \Delta_h^A U_\varepsilon \|_{A_h} \leqslant \| F_h^{-1} \| \; \| C_\varepsilon(h) \|_{B_h} \leqslant k_2 c_\varepsilon(h)$

$c_\varepsilon(h) \to 0$ for $h \to o$
ε <u>fixed</u> greater than o.

Now we want to find a bound for $U_{\varepsilon h} - U_h$:

$$U_{\varepsilon h} - U_h = F_h^{-1}(g_{\varepsilon h} - g_h) = F_h^{-1}(g_{\varepsilon h} - \Delta_h^B g_\varepsilon + \Delta_h^B g_\varepsilon - \Delta_h^B g + \Delta_h^B g - g_h)$$

C) $\| U_{\varepsilon h} - U_h \| A_h \leqslant k_2 (\| g_{\varepsilon h} - \Delta_h^B g_\varepsilon \|_{B_h} + \| \Delta_h^B g_\varepsilon - \Delta_h^B g \| + \| \Delta_h^B - g_h \|)$

$$= k_2 d_\varepsilon(h) \to 0 \text{ for } h \to 0 \text{ and } \varepsilon > 0 \text{ fixed}$$

because of the assumptions 2), 3) and 4). So we can conclude from (A), (B) and (C):

$$\|\Delta_h^A U - U_h\|_{A_h} \leq \|\Delta_h^A U - \Delta_h^A U_\varepsilon\|_{A_h} + \|\Delta_h^A U_\varepsilon - U_{\varepsilon h}\|_{A_h} + \|U_{\varepsilon h} - U_h\|_{A_h} \leq \|\Delta_h^A\| \cdot$$

$$\cdot \; k_1 \varepsilon + k_2 c_\varepsilon(h) + k_2 d_\varepsilon(h) \quad .$$

We can find for every $\varepsilon > o$ a $h < h(\varepsilon)$ so that $\|\Delta_h^A U - U_h\|_{A_h} \leq C\varepsilon$ where C is independent of ε, h and that means convergence.

It is easy to extend Theorem 3 to cases where the difference scheme F_h is uniformly continuous in h (stable) in some components of the data-vector g, but not in all. The methods for doing this are the same as used in Theorem 2, because stability of one step difference - approximation means that the solutions $U^n(k)$ depend uniformly continuous (in the grid-parameter k) on the initial data and on the disturbance but not on the boundary values.

Remark

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