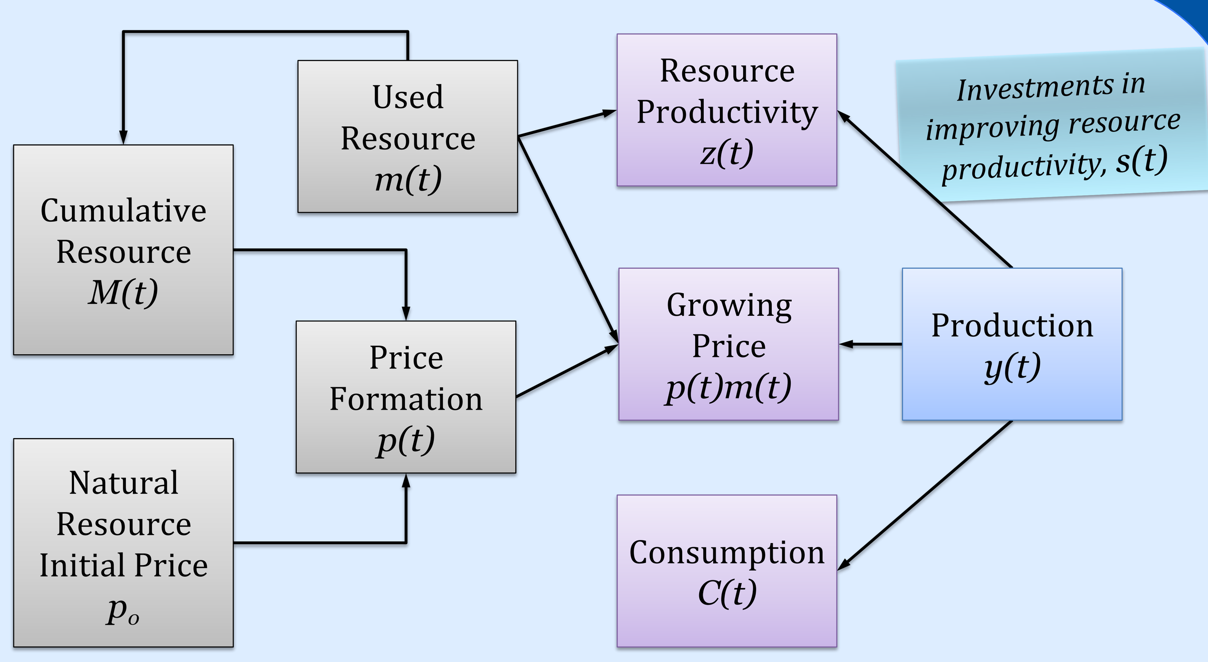
Optimization of Resource Productivity as a Driver of Economic Growth

by Alexander Tarasyev and Bing Zhu

A dynamic optimization model of investment in improvement of the resource productivity index is analyzed for obtaining balanced economic growth trends including both the consumption index and natural resources use. The research is closely connected with the problem of shortages of natural resources stocks, the security of supply of energy and materials, and the environmental effectiveness of their consumption. The main idea of the model is to introduce an integrated environment for elaboration of a control policy for management of the investment process in development of basic production factors such as capital, energy and material consumption. An essential feature of the model is the possibility to invest in economy's dematerialization. Another important construction is connected with the price formation mechanism which presumes the rapid growth of prices on exhausting materials. The balance is formed in the consumption index which negatively depends on growing prices on materials. The optimal control problem for the investment process is posed and solved within the Pontryagin maximum principle. Specifically, the growth and decline trends of the Hamiltonian trajectories are examined for the optimal solution.

It is proved that for specific range of the model parameters there exists the unique steady state of the Hamiltonian system. The steady state can be interpreted as the optimal steady trajectory along which investments in improving resource productivity provide raising resource efficiency and balancing this trend with growth of the consumption index. The fact of existence of the steady state demonstrates the possibility of the growth path in an economy with exhausting resources. Sensitivity analysis of steady state solutions is implemented to demonstrate adequate trends of the model trajectories. As a result of analysis, one can elaborate investment strategies in economy's dematerialization, resource and environmental management for improving the resource productivity and for shifting the economic system from non-optimal paths to the trajectory of sustainable development.



Model Variables

- Production: $y = y(t)$
- Used materials: $m = m(t), \quad m(0) = m^*$
- Cumulative resource consumption:

$$M(t) = \int_0^t m(s)ds, \qquad M(0) = M^* = 0$$

- Resource productivity: $z(t) = \frac{y(t)}{m(t)}$

Price Formation Mechanism

Due to the limitation or exhaustion of natural materials its prices raise. It is assumed that prices are growing according to the inversely proportional rule of resources exhaustion:

$$p(t) = \frac{p_0}{\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}},$$

M_0 – limitation of natural resources,
p_0 – initial prices on natural resources,
γ – non-negative elasticity coefficient of the price formation mechanism

Balance Equation

Production $y(t)$ in period t is shared between consumption $c(t)$ and the growing cost of natural resources $p(t) \cdot m(t)$ plus investment $s(t)$ in improving the resource productivity:

$$y(t) = c(t) + p(t) \cdot m(t) + s(t)$$

Investment level is bounded with positive constant parameter s^0: $0 \le s(t) \le s^0 < y(t)$

Balance equation in relative variable:

$$1 = \frac{c(t)}{y(t)} + p(t)Z(t) + u(t),$$
$$0 \le u(t) \le u^0 < 1$$

Production Function and Consumption

An exponential production function of the Cobb-Douglas type:

$$y(t) = ae^{bt}(m(t))^{\alpha}$$

Positive parameter a is a scale factor;

Non-negative growth rate b indicates the growth process of production $y(t)$ due to development of basic production factors such as capital, labor, technology, etc.;

The symbol α denotes non-negative elasticity coefficient of natural resources: $0 \le \alpha < 1$.

A *production factor* is the diminishing return to scale of natural resources.

Consumption intensity expressed through the resources consumption $m(t), M(t)$ by substituting relations of the price formation mechanism $p(t)$ and the production function $y(t)$ to the relative balance relation:

$$\frac{c(t)}{y(t)} = 1 - \frac{p_0}{ae^{bt}\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}}(m(t))^{1-\alpha} - u(t)$$

Model Dynamics

The relative raise in the resource productivity $z(t)$ is proportional to the portion of the assigned investment $u(t)$

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \beta u(t)$$

Non-negative parameter β describes the effectiveness of investments investment $u(t)$ in raising the resource productivity.

Since $z(t) = y(t)/m(t)$ the rate of the resource productivity can be decomposed into two components: the production rate $\dot{y}(t)$ and the rate of the resource consumption $\dot{m}(t)$:

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \frac{1}{y(t)}\frac{dy(t)}{dt} - \frac{1}{m(t)}\frac{dm(t)}{dt}$$
$$\Rightarrow \frac{1}{m(t)}\frac{dm(t)}{dt} = \frac{1}{1-\alpha}\left(b - \beta u(t)\right)$$

The rate of the resource consumption is influenced by the production growth rate b and can be reduced only by investment $u(t)$ in raising the resource productivity.

If investment is equal to zero, $u(t) = 0$, then the rate of the resource consumption should be proportional to the production growth rate b.

Phase Variables

New phase variables:

$$x_1(t) = e^{-\frac{b\gamma t}{1-\alpha-\gamma}}\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}$$
$$x_2(t) = e^{-\frac{bt}{1-\alpha-\gamma}}m(t)$$

Dynamics of new variables:

$$\dot{x}_1(t) = -\frac{b\gamma}{1-\alpha-\gamma}x_1(t) - \frac{\gamma}{M_0}(x_1(t))^{1-\frac{1}{\gamma}}x_2(t)$$
$$\dot{x}_2(t) = \frac{1}{1-\alpha}\left(-\frac{b\gamma}{1-\alpha-\gamma} - \beta u(t)\right)x_2(t)$$
$$x_1(0) = 1, \qquad x_2(0) = m^*$$

Logarithmic Consumption Index

Logarithmic consumption index in time t:

$$\ln c(t) = \ln a + \alpha \ln x_2(t) + \frac{(1-\gamma)b}{1-\alpha-\gamma}t + \ln\left(1 - \frac{p_0}{a}\frac{(x_2(t))^{1-\alpha}}{x_1(t)} - u(t)\right)$$

The integrated logarithmic index discounted with the discount rate ρ, $\rho > 0$,

$$J(x_1(\cdot), x_2(\cdot), u(\cdot)) = \int_0^T e^{-\rho t}\ln c(t)\,dt,$$
$$0 < T \le +\infty$$

is the *utility function* in the considered control problem.

Optimal Control Problem. Special Case

The problem is investigated for the special case when the elasticity coefficient γ in the price formation mechanism has the unit value, $\gamma = 1$. Hence, phase variables x_1, x_2 are rewritten in the form:

$$x_1(t) = e^{\frac{b}{\alpha}t}\left(1 - \frac{M(t)}{M_0}\right), \qquad x_2(t) = e^{\frac{b}{\alpha}t}m(t)$$

and satisfy the dynamical system:

$$\dot{x}_1(t) = \frac{b}{\alpha}x_1(t) - \frac{1}{M_0}x_2(t), \qquad x_1(0) = 1$$
$$\dot{x}_2(t) = \frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u(t)\right)x_2(t), x_2(0) = m^*$$

Utility functional looks as follows:

$$J(x_1(\cdot), x_2(\cdot), u(\cdot)) = \int_0^T e^{-\rho t}\ln c(t)\,dt, T \le +\infty$$

where consumption level is found by formula:

$$c(t) = x_2^{\alpha}(t)\left(1 - \frac{p_0}{a}\frac{(x_2(t))^{1-\alpha}}{x_1(t)} - u(t)\right)$$

The main goal of the posed optimal control problem is to raise the resource productivity.

Hamiltonian Function. Control Regimes

$$H(x_1, x_2, u, t, \psi_1, \psi_2) = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - u\right) + \psi_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u\right)x_2\psi_2$$

Values of control variable u maximizing the Hamiltonian function are the following:

$$\hat{u} = \begin{cases} 0, & \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - \frac{1-\alpha}{\beta\psi_2 x_2} > 1 \\ u^*, & 1 - u^0 \le \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - \frac{1-\alpha}{\beta\psi_2 x_2} \le 1 \\ u^0, & \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - \frac{1-\alpha}{\beta\psi_2 x_2} < 1 - u^0 \end{cases}$$

Intermediate maximum control value:

$$u^* = 1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} + \frac{1-\alpha}{\beta\psi_2 x_2}.$$

For the intermediate control regime the adjoint variable ψ_2 is strictly negative.

Qualitative Analysis. Steady States

There is an unique steady state at the domain of the intermediate control regime $u = u^*$, if and only if model parameters are located in area Ω depicted at Figure

$$\Omega = \left\{ \begin{array}{l} (\rho, b): \quad \rho > 0, \qquad 0 < b < \alpha\rho, \\ \dfrac{\left(\rho - \frac{\beta}{2}\right)^2}{\left(\frac{\beta^2}{4(1-\alpha)}\right)} + \dfrac{\left(b - \frac{\alpha\beta}{2}\right)^2}{\left(\frac{\alpha\beta^2}{4}\right)} < 1 \end{array} \right\}$$

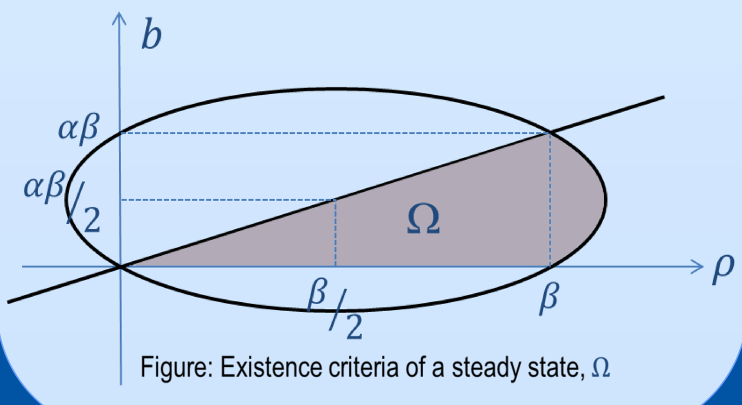


Figure: Existence criteria of a steady state, Ω

Steady State Coordinates

If model parameters are located at the set Ω then steady state coordinates are found analytically:

$$x_1^* = \frac{\alpha}{bM_0}\left(\frac{\beta\rho b p_0 M_0}{a(\alpha\rho - b)\big((\rho - b) + \alpha(\beta - \rho)\big)}\right)^{\frac{1}{\alpha}},$$
$$x_2^* = \left(\frac{\beta\rho b p_0 M_0}{a(\alpha\rho - b)\big((\rho - b) + \alpha(\beta - \rho)\big)}\right)^{\frac{1}{\alpha}},$$
$$z_1^* = \frac{\alpha(\rho(1-\alpha) + \alpha\beta - b)}{\alpha\rho(1-\alpha)(\beta - \rho) + b(\alpha\beta - b)},$$
$$z_2^* = -\frac{\alpha\rho(1-\alpha)}{\alpha\rho(1-\alpha)(\beta - \rho) + b(\alpha\beta - b)}$$

Steady state coordinates have the property of *well-posedness* when model parameters meet restrictions in Ω:

$$x_1^* > 0, \qquad x_2^* > 0,$$
$$z_1^* = x_1^*\psi_1^* > 0, \qquad z_2^* = x_2^*\psi_2^* < 0$$

Optimal Control at the Steady State

Estimation of the optimal control value u^* at the steady state $(x_1^*, x_2^*, z_1^*, z_2^*)$:

$$u^* = \frac{b}{\alpha\beta}$$

Due to the *Ω-restrictions* the value of the optimal control u^* is located in the proper range

$$0 < u^* < 1$$

This fact means that it is reasonable to make an assumption:

The upper bound u^0 for the control parameter u should satisfy to the following condition:

$$\frac{b}{\alpha\beta} \le u^0 < 1.$$

Consumption Level at the Steady State

Consumption index c^* have strictly positive value at the steady state

$$c^* = a(x_2^*)^{\alpha}\left(1 - \frac{b}{\alpha\beta}\right) - \frac{b}{\alpha}p_0 M_0 = \frac{b^2 p_0 M_0(\alpha\beta - b + (1-\alpha)(\beta - \rho))}{\alpha(\alpha\rho - b)(\rho - b + \alpha(\beta - \rho))} > 0$$

Consumption intensity v^* at the steady state is determined by relation

$$v^* = \frac{c^*}{y^*} = \frac{b(\alpha\beta - b + (1-\alpha)(\beta - \rho))}{\alpha\beta\rho} > 0$$

Consumption and Investment

Proportion between consumption c^* and investment s^* is given by the ratio

$$w^* = \frac{c^*}{s^*} = \frac{\alpha\beta - b + (1-\alpha)(\beta - \rho)}{\rho}$$

Proportion w^* is linear in parameters α, β, b and can be rewritten as follows

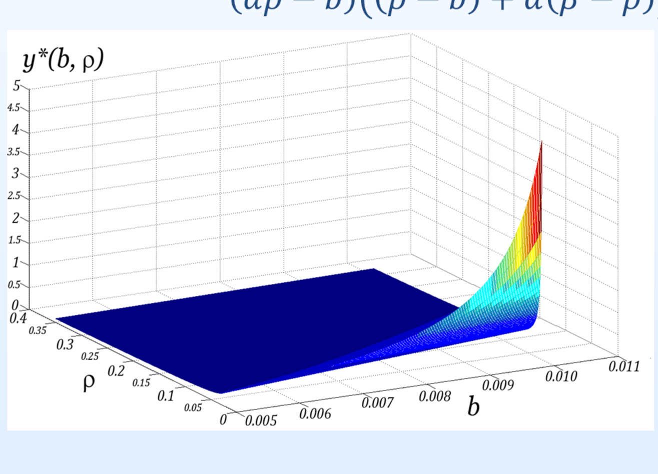
$$w^* = \alpha + \frac{\beta}{\rho} - \frac{b}{\rho} - 1.$$

Proportion w^* **increases** in parameters α and β and **decreases** in parameters b and ρ.

Production at the Steady State

Production y^* has strictly positive values at the steady state

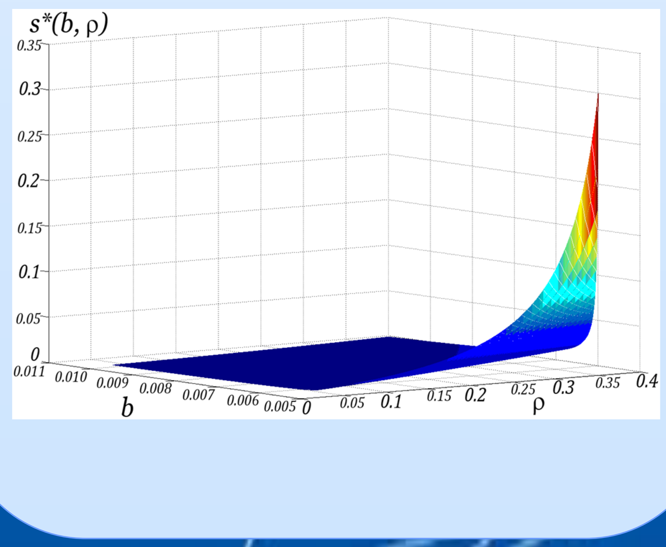
$$y^* = a(x_2^*)^{\alpha} = \frac{\beta\rho b p_0 M_0}{(\alpha\rho - b)\big((\rho - b) + \alpha(\beta - \rho)\big)}$$



Investment at the Steady State

The optimal absolute value of investment s^* is given by formula

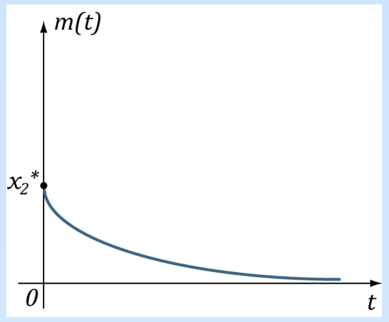
$$s^* = u^* y^* = \frac{b^2 \rho p_0 M_0}{\alpha(\alpha\rho - b)\big((\rho - b) + \alpha(\beta - \rho)\big)} > 0$$



Model Solution at the Steady State

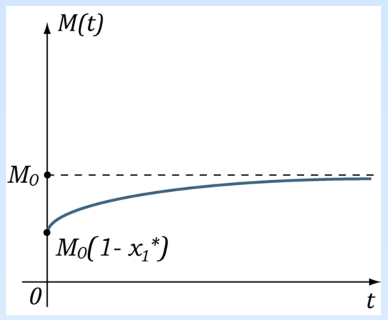
At the steady state the current resource use $m(t)$ decreases to zero according to the exponential law:

$$m(t) = x_2^* e^{-\frac{b}{\alpha}t}$$



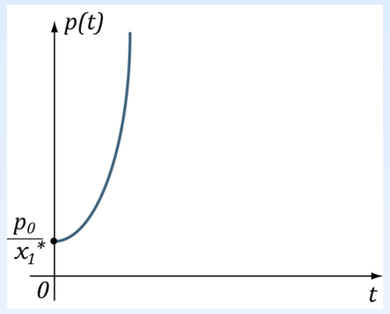
The cumulative resource consumption $M(t)$ increases with saturation at the limit level M_0 of natural resources according to the logistic growth law:

$$M(t) = M_0\left(1 - x_1^* e^{-\frac{b}{\alpha}t}\right)$$



Price $p(t)$ generated by price formation mechanism increases exponentially at the steady state:

$$p(t) = \frac{p_0}{x_1^*}e^{\frac{b}{\alpha}t}$$

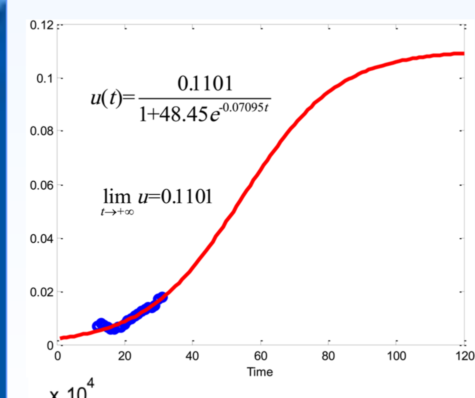


Transition Trajectories for China Economy

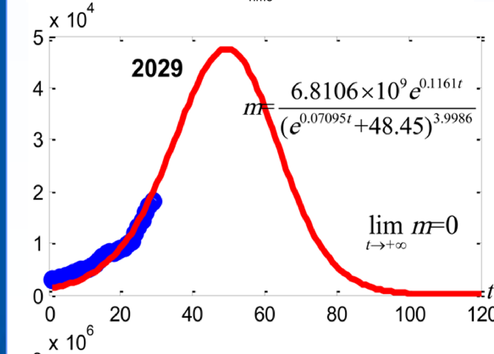
For year 1980, $t = 1$

Saturation level: $u^* = 0.1101$

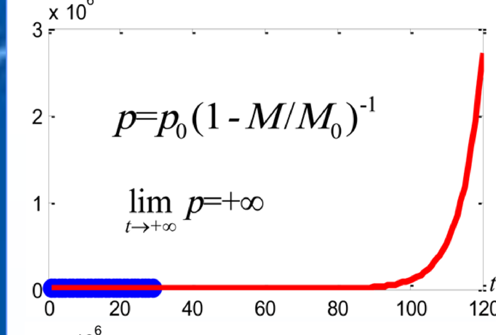
Growth rate of $u = u(t)$: $r = 0.07095$



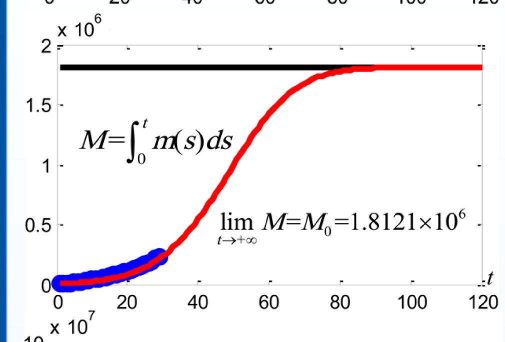
R&D Expenditure Intensity



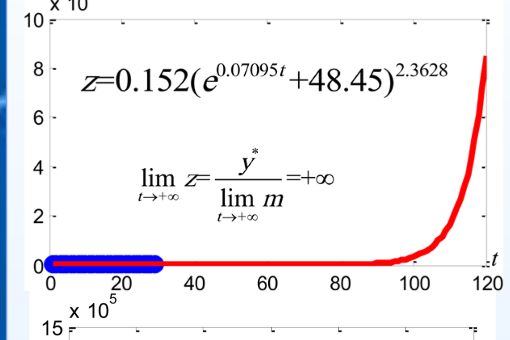
Used Materials



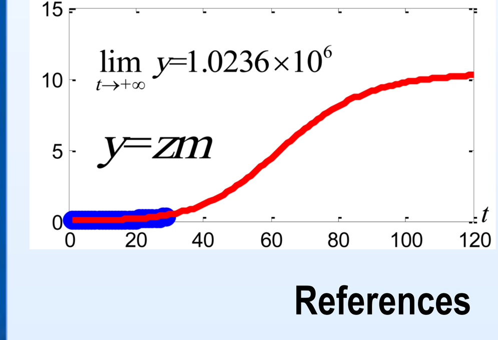
Resource Price



Cumulative Resource Consumption



Resource Productivity



Gross Domestic Product (Production)

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