

**WHY A POPULATION CONVERGES TO STABILITY**

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## ABSTRACT

A central theorem in mathematical demography tells us that the age distribution of a closed population with unchanging fertility and mortality behavior must converge to a fixed and stable form. Proofs rely on ready-made theorems borrowed from linear algebra or from asymptotic transform theory, notably the Perron–Frobenius and the Tauberian theorems. But while these are efficient and expedient, they give little insight into the mechanism that forces the age distribution to converge.

This paper proposes a simple argument for convergence. An elementary device allows us to view the birth sequence as the product of an exponential sequence and a weighted smoothing process. Smoothing progressively damps out the peaks and hollows in the initial birth sequence; thus the birth sequence gradually becomes exponential, and this forces the age distribution to assume a fixed and final form.



## PREFACE

From the early days of its existence, IIASA has had a lively interest in the measurement and dynamics of population variables. This paper takes a new look at one of the central results in population dynamics, the strong ergodic theorem of demography.



A large part of mathematical demography is built upon one fundamental theorem, the “strong ergodic theorem” of demography. If the fertility and mortality age-schedules of a population remain unchanged over time, its age distribution, no matter what its initial shape, will converge in time to a fixed and stable form. In brief, when demographic behavior remains unchanged, the population, it is said, converges to stability.

There are two basic ways to prove that this is so, depending on whether demographic behavior is described in discrete time or in continuous time. For the discrete case, proof amounts to showing that an infinite product of the Leslie transition matrix achieves a limiting constant form. This is the principle behind the proofs of Leslie (1945), Lopez (1961), Parlett (1970), and many others. For the continuous case, proof amounts to solving the Lotka integral renewal equation and studying the asymptotic behavior of its solution terms. This is the principle behind the proofs of Lotka and Sharpe (1911), Lotka (1939), Coale (1972), as well as many others. Some of the modern papers [see, for example, Cohen (1979)] show that the theorem can be extended to a stochastic form, but the underlying principles remain largely the same.

While neither form of proof is mathematically difficult, neither offers much in the way of direct and ready insight. The problem is that both forms are built on borrowed theory, either on positive matrix theory or on asymptotic integral equation theory. Both forms of proof are not self-contained.\* The mechanism forcing the age structure to converge in each case therefore remains partially hidden within the borrowed theory and becomes difficult to see. Those who do not want to steep themselves in the theory of primitive matrices or in  $k$ th-order roots of integral equations are therefore left curious. Why should it

\*Lotka did publish a self-contained proof in 1922. He sandwiched the initial age distribution between two boundary curves that close in over time, eventually coinciding to trap the age distribution within a fixed shape. The proof is ingenious but the logic is loose, and the mechanism forcing convergence is difficult to see.

be that a population converges? What is it about the process of regeneration of population numbers that means the age structure will converge to a stable form, and population growth will converge to a constant rate? What mechanism underlies population convergence?

This short paper presents a new argument for the convergence of the age structure, one that is self-contained, and that brings the mechanism behind convergence into full view. The idea is simple. Looked at directly, the dynamics of the age distribution say little to our normal intuition. Looked at from a slightly different angle, though, population dynamics define a smoothing or averaging process over the generations – a process comfortable to our intuition. This smoothing and resmoothing turns out to be the mechanism that forces the age structure toward a fixed and final form.

*The Problem.* The problem can be stated simply enough. Assuming constant fertility and mortality behavior, with no in- or out-migration, a population evolves in discrete time according to the dynamics

$$B_t = \sum_x B_{t-x} p_x m_x \quad (1)$$

where  $B_t$  is the number of births in year  $t$ ,  $m_x$  is the probability of reproducing at age  $x$ , and  $p_x$  is the probability of surviving until age  $x$ . Present births, in other words, are the sum of births born to people at childbearing ages who still survive. Summation is taken over the age-groups 1 to  $M$  (where  $M$  is an upper limit to childbearing). And the numbers in the initial “generation,”  $B_M, \dots, B_1$ , are assumed given. Neither  $B$ , nor  $p$ , nor  $m$ , of course, is negative.

The age composition, or proportion at age  $a$  at time  $t$ , is given by

$$c_{a,t} = \frac{B_{t-a} p_a}{\sum_x B_{t-x} p_x} \quad (2)$$

the numbers at age  $a$ , divided by the total population. We seek to prove that the distribution  $c_{a,t}$  converges to a limiting constant function  $c_a^*$ .

Two observations will help us to furnish a proof. First, note that it is enough to prove that  $B_t$  converges to an exponential form,  $B_t \rightarrow B^* e^{rt}$ , where  $B^*$  and  $r$  are constants. For if this is true, it follows by substituting for  $B_{t-a}$  and  $B_{t-x}$  in equation (2) that the age distribution becomes fixed and unchanging with time:

$$c_{a,t} \rightarrow \frac{e^{-ra} p_a}{\sum_x e^{-rx} p_x} = c_a^* \quad (3)$$

Armed with this fact we can confine our attention to why the birth sequence,  $B_t$ , should become exponential. Second, note that convergence to an exponential form is hard to prove – the target is moving, as it were – but convergence to a



fixed value is easy. Therefore we will normalize or redefine the problem to one of convergence to a fixed value.

*Smoothing Process.* Begin with the dynamics

$$B_t = \sum_x B_{t-x} p_x m_x \quad (4)$$

and divide both sides by  $e^{rt}$ ,

$$\frac{B_t}{e^{rt}} = \sum_x \frac{B_{t-x} e^{-rx}}{e^{rt} e^{-rx}} p_x m_x \quad (5)$$

Renaming  $B_t e^{-rt}$  to be the variable  $\hat{B}_t$  – the “growth-corrected” birth sequence – the new, but equivalent, dynamics become

$$\hat{B}_t = \sum_x \hat{B}_{t-x} e^{-rx} p_x m_x \quad (6)$$

We will speak somewhat loosely of  $\hat{B}$  in what follows as “births,” remembering though that these “births” differ from real births by an exponential factor.

We now need only show that for some value of  $r$ ,  $\hat{B}$  eventually becomes constant over time. Allowing ourselves some foresight, we choose  $r$  to satisfy

$$1 = \sum_x e^{-rx} p_x m_x \quad (7)$$

Finally, renaming  $e^{-rx} p_x m_x$  as  $\psi_x$ , we may write the new but equivalent dynamics as

$$\hat{B}_t = \sum_x \hat{B}_{t-x} \psi_x \quad (8)$$

where, by virtue of equation (7),

$$1 = \sum_x \psi_x$$

The original dynamics have been changed but little;  $B_t$  has merely been normalized to the new variable  $\hat{B}_t$ . Notice though, in the new system for  $\hat{B}_t$ , the coefficients  $\psi(x)$  sum to one –  $\psi$  is a weighting function. The new dynamics therefore describe a continuous smoothing process:  $\hat{B}_t$  is the weighted average of the  $M$  immediate past values of  $\hat{B}$ ;  $\hat{B}_{t+1}$  is the weighted average of  $\hat{B}_t$  and the  $M - 1$  immediate past values of  $\hat{B}$ ;  $\hat{B}_{t+2}$  is the weighted average of  $\hat{B}_{t+1}$ ,  $\hat{B}_t$ , and the  $M - 2$  immediate past values of  $\hat{B}$ . And so on. This constant averaging, then averaging of the averages, we would suspect, will converge  $\hat{B}$  to a fixed value  $B^*$  (as in Figure 1), and equivalently will converge  $B$  to the exponential form  $B^* e^{rt}$ . Why?

The reason is easy to sketch when all  $\psi(x)$  are strictly positive (greater

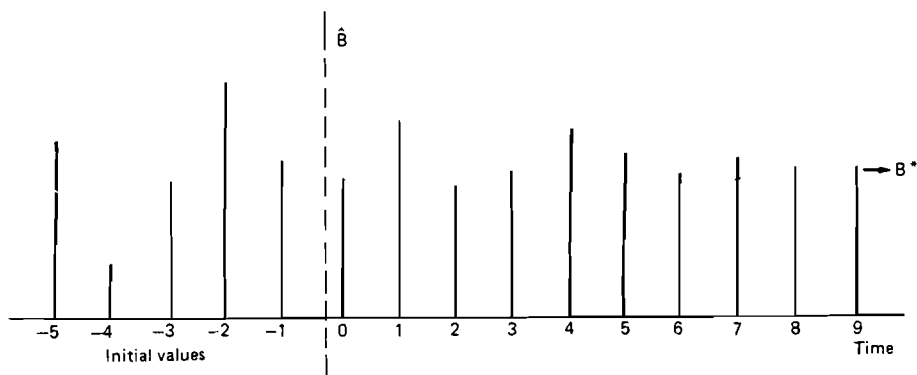


FIGURE 1

than  $\epsilon$ , say). Mark the largest of the initial birth values as  $\hat{B}_{-p}$ , the smallest as  $\hat{B}_{-q}$ , the difference between them being  $d$ . The initial given values therefore lie within a spread of  $d$  units. Now, the first value  $\hat{B}_0$  generated by the process will fall short of the greatest value  $\hat{B}_{-p}$  by at least  $\epsilon d$  units:

$$\begin{aligned}\hat{B}_0 &= \sum_{x \neq q} \hat{B}_{-x} \psi_x + \hat{B}_{-q} \psi_q \\ &= \sum_{x \neq q} \hat{B}_{-x} \psi_x + \hat{B}_{-p} \psi_q - \hat{B}_{-p} \psi_q + \hat{B}_{-q} \psi_q\end{aligned}$$

so that

$$\hat{B}_0 \leq \hat{B}_{-p} - \psi_q (\hat{B}_{-p} - \hat{B}_{-q}) < \hat{B}_{-p} - \epsilon d \quad (9)$$

Similarly,  $\hat{B}_0$  must exceed the smallest value  $\hat{B}_{-q}$  by at least  $\epsilon d$  units. Therefore  $\hat{B}_0$  will lie strictly inside the initial spread of birth values – inside by a fixed factor  $1 - 2\epsilon$ . The same argument applies all the more so to  $\hat{B}_1$ , and again to  $\hat{B}_2$ , and so on until  $\hat{B}_{M-1}$ . The spread of the entire new generation of  $\hat{B}$  values therefore lies strictly within that of the old one, and by a specified uniform factor. Repeating the argument over the generations, the generational spread in  $\hat{B}$  diminishes geometrically to zero.  $\hat{B}_t$  therefore converges to a fixed value  $B^*$  and  $B_t$  therefore converges to exponential growth,  $B^*e^{rt}$ .

So far so good. But what if some of the  $\psi_x$  values are zero as in real populations where no reproduction takes place at certain ages? Will the process always converge? The answer is no. Consider the four-age-group population in Figure 2, with  $\psi_1 = \psi_3 = 0$ , and  $\psi_2 = \psi_4 = 1/2$ . Childbearing occurs only in the second and fourth age-groups. This population will oscillate indefinitely. Here the smoothing process does not smooth: something is wrong. To see what, we need to look at smoothing more closely.

*Smoothing – A Closer Look.* In general, assume that some, or several, of the  $\psi$  values are zero. The value  $\hat{B}_0$  then depends directly on only certain of the original  $\hat{B}$  values. Similarly,  $\hat{B}_1$  depends directly only on the neighbors of

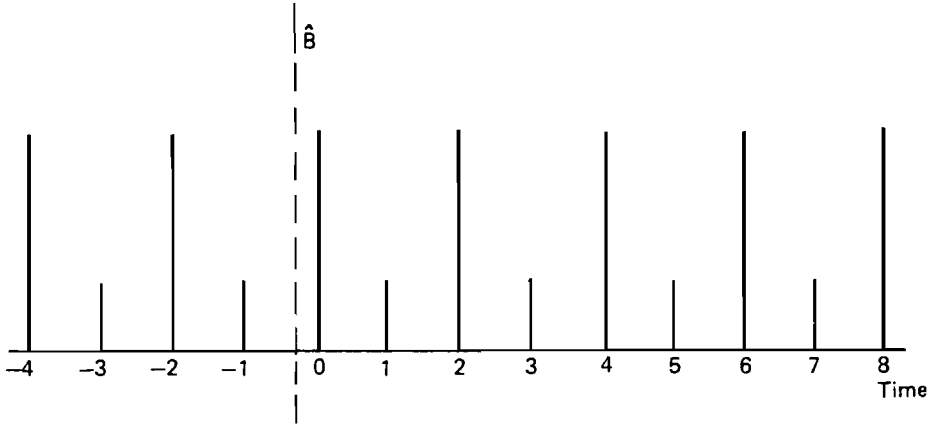


FIGURE 2

these values. For the system

$$\hat{B}_t = \hat{B}_{t-3}\psi_3 + \hat{B}_{t-4}\psi_4 ; \quad \psi_1, \psi_2 = 0$$

we can graph this dependence as in Figure 3A, picturing each birth cohort as a point, with a directed arrow drawn between them if dependent.

The graph extends indefinitely downward. Notice, though, that while  $\hat{B}_0$  depends directly on only two of the initial values,  $\hat{B}_4$  depends on three of them, and  $\hat{B}_8$  on all four of them. If we so chose, we could therefore write the dynamics with present  $\hat{B}$  values specifying  $\hat{B}$  eight steps ahead:

$$\hat{B}_{t+8} = \hat{B}_{t-1}\psi'_1 + \hat{B}_{t-2}\psi'_2 + \hat{B}_{t-3}\psi'_3 + \hat{B}_{t-4}\psi'_4$$

This process, with new weights  $\psi^1$ , describes the evolution of  $\hat{B}$  perfectly well; moreover, it remains a smoothing process as we can see by following the weights backward from  $\hat{B}_8$ : they divide up but continue to sum to one. Most important, it is a function of *all* the initial values and is strictly positive in all its coefficients. We could therefore apply the above convergence argument, showing that the spread in any four consecutive values must be reduced eight steps ahead by a fixed factor. Taking  $\hat{B}$  values now twelve at a time (the original initial four plus the intervening eight), generational spread once again reduces geometrically;  $\hat{B}$  converges, even though we started with some  $\psi$  weights as zero.

What then went wrong with the case where  $\psi_1$  and  $\psi_3$  were zero? Forming its graph (Figure 3B), we see there is no future  $\hat{B}$  value that is a function of all the original given values. Even-indexed  $\hat{B}$ 's depend on even-indexed  $\hat{B}$ 's; odd ones depend on odd ones. Here two separate but identical processes are going on: the even process never "sees" the initial values of the odd process and vice versa. Both processes iterate their initial values to a limit: but there is an even

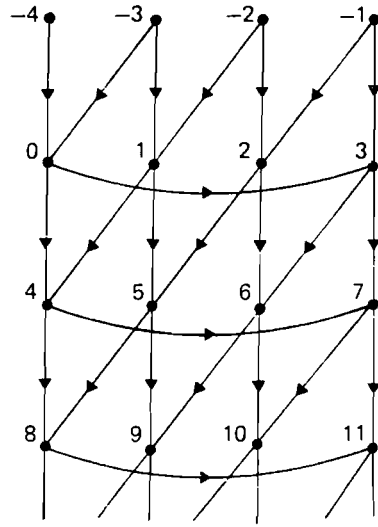


FIGURE 3A

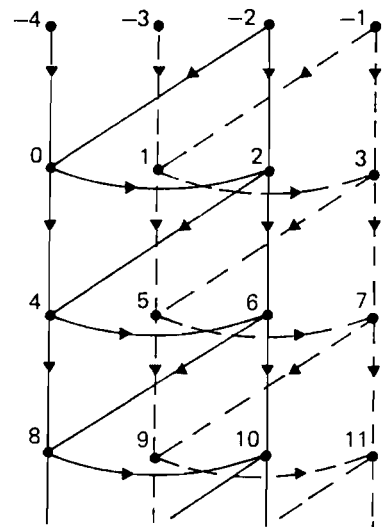


FIGURE 3B

limit and an odd limit. The process oscillates indefinitely between them.

We can deduce some general principles. If  $\hat{B}_0$  depends on initial points  $-k_1, \dots, -k_j$  that share a common divisor, the process will not converge. To see this, take the greatest common divisor,  $n$ , and label all the points modulo this number. We then see  $n$  separate, but identical, processes going on.  $\hat{B}$  will have cluster points but no single limit; the process will be periodic. The converse, that  $\hat{B}$  converges if  $B_0$  depends on at least two initial points that are relatively prime, is not difficult to show but requires slightly more apparatus than we permit ourselves here. Sufficient for our purposes is the observation that the process converges if there are at least two consecutive positive reproductive ages. In this case any  $\hat{B}$  value sufficiently in the future will have a dependence graph that progressively fans out backward to include all the initial values. Only one smoothing process then happens; hence  $\hat{B}$  converges to a limiting value  $B^*$ ; hence  $B_t$  converges to steady exponential growth  $B^*e^{t'}$ ; hence the age composition converges to a fixed and stable form.

*The Limiting Coefficient.* One question remains. How can we determine  $B^*$ , the limiting coefficient of the exponential birth sequence? One possibility is to look for a quantity that is invariant, that is carried along unchanged over the generations. Such a quantity would enable us to relate  $B^*$  at the end of the process to the  $\hat{B}$  values at the beginning. Now, each generation at any time can donate to the future a certain number of direct descendants. All future population must be built on these direct descendants — they are the system's "reproductive potential" or "reproductive value" as it were. We might suspect this reproductive potential, *in the growth-corrected dynamics* we have defined, to

be invariant. A little algebra shows that this turns out to be the case.

At time  $t$ , age-groups  $\hat{B}_{t-M}, \dots, \hat{B}_{t-1}$ , taken together, contribute  $V_t$  direct descendants to the future – to the period from  $t$  onward:

$$\begin{aligned} V_t = & \hat{B}_{t-M} \psi_M + \hat{B}_{t-M+1} (\psi_{M-1} + \psi_M) + \dots \\ & + \hat{B}_{t-1} (\psi_1 + \psi_2 + \dots + \psi_M) \end{aligned} \quad (10)$$

Similarly age-groups  $\hat{B}_{t+1-M}, \dots, \hat{B}_t$  contribute  $V_{t+1}$  to the period from  $t+1$  onward:

$$\begin{aligned} V_{t+1} = & \hat{B}_{t-M+1} \psi_M + \hat{B}_{t-M+2} (\psi_{M-1} + \psi_M) + \dots \\ & + \hat{B}_t (\psi_1 + \psi_2 + \dots + \psi_M) \end{aligned}$$

Noting that the coefficient of  $\hat{B}_t$  is one, and using equation (8) to replace  $\hat{B}_t$ , we find

$$\begin{aligned} V_{t+1} = & \hat{B}_{t-M+1} \psi_M + \hat{B}_{t-M+2} (\psi_{M-1} + \psi_M) + \dots \\ & + \hat{B}_{t-1} \psi_1 + \dots + \hat{B}_{t-M+1} \psi_{M-1} + \hat{B}_{t-M} \psi_M \end{aligned} \quad (11)$$

Comparing equation (10) with equation (11) term for term, we see that  $V_t = V_{t+1}$ ;  $V_t$  is indeed an invariant quantity  $V$ .

At the start

$$\begin{aligned} V_0 = V = & \hat{B}_{-M} \psi_M + \hat{B}_{-M+1} (\psi_{M-1} + \psi_M) + \dots \\ & + \hat{B}_{-1} (\psi_1 + \psi_2 + \dots + \psi_M) \end{aligned} \quad (12)$$

And in the limit

$$\begin{aligned} V = & B^* \psi_M + B^* (\psi_{M-1} + \psi_M) + \dots + B^* (\psi_1 + \dots + \psi_M) \\ V = & B^* (\psi_1 + 2\psi_2 + \dots + M\psi_M) \end{aligned} \quad (13)$$

Since  $\psi$  is the distribution of childbearing in the population, the coefficient of  $B^*$  is the mean age of childbearing, denoted  $A_r$ . Putting equations (12) and (13) together yields the result we seek:

$$\begin{aligned} B^* = & A_r^{-1} \{ \hat{B}_{-M} \psi_M + \hat{B}_{-M+1} (\psi_{M-1} + \psi_M) + \dots + \hat{B}_{-1} (\psi_1 + \dots + \psi_M) \} \\ B^* = & A_r^{-1} \{ B_{-M} e^{rM} \psi_M + B_{-M+1} e^{r(M-1)} (\psi_{M-1} + \psi_M) + \dots \\ & + B_{-1} e^r (\psi_1 + \dots + \psi_M) \} \end{aligned}$$

$$B^* = A_r^{-1} \sum_{j=0}^{M-1} e^{-rj} \sum_{x=1}^{M-j} B_{-x} p_{x+j} m_{x+j} \quad (14)$$

The value  $B^*$  is directly determined by the initial birth sequence and the fertility and mortality age patterns.

*Conclusion.* To go back to the original question, why, in plain words, does a population converge? The argument presented here is both simple and new. Once the population's tendency to grow is eliminated – by dividing growth out of the dynamics – the process of population replacement, barring bizarre reproductive patterns, literally smoothes the generations out. Childbearing, and thus the function  $\psi$ , is not concentrated at one age but is spread over several years. Hence past humps and hollows in the birth sequence are thrown in together in the replacement process. They are averaged together – they smooth out.

Adding growth back means that a smooth exponential increase is reached in the long run – an exponential that is fully fixed, given information from the initial birth sequence and the net fertility pattern. And once the birth sequence reaches exponential increase, the age distribution must assume its stable shape, no matter how it started.

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