COMPARATIVE DYNAMICS OF THREE DEMOGRAPHIC MODELS OF URBANIZATION

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FOREWORD

Roughly 1.6 billion people, 40 percent of the world's population, live in urban areas today. At the beginning of the last century, the urban population of the world totaled only 25 million. According to recent United Nations estimates, about 3.1 billion people, twice today's urban population, will be living in urban areas by the year 2000.

Scholars and policy makers often disagree when it comes to evaluating the desirability of current rapid rates of urban growth in many parts of the globe. Some see this trend as fostering national processes of socioeconomic development, particularly in the poorer and rapidly urbanizing countries of the Third World; whereas others believe the consequences to be largely undesirable and argue that such urban growth should be slowed down.

As part of a search for convincing evidence for or against rapid rates of urban growth in developing countries, the Human Settlements and Services Area initiated in 1977 a research project to study the process of structural transformation in nations evolving from primarily rural—agrarian to urban—industrial societies. Data from several countries selected as case studies are being collected, and the research is focusing on spatial population growth and economic development, and on their resources and service demands.

This paper examines the comparative dynamics of three related demographic models of urbanization. It sets out, for each model, a differential equation that traces the impacts of different patterns of natural increase and net migration on the evolution of the urban to rural population ratio.

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CONTENTS

INTRODUCTION

Since the beginning of the last century, the world's population has grown rapidly, increasing from approximately one billion in 1800 to four billion in 1975. At the same time, urban population growth has been even more explosive: the urban population totals 1.6 billion today versus 25 million in 1800. Thus, the proportion of the world's population living in urban areas has increased from 2.5 percent in 1800 to 40 percent today. According to the latest UN projections (United Nations 1979), this past trend of population growth and urbanization is likely to continue: by the end of this century, slightly more than half the world's population will be living in urban areas.

Clearly, urbanization results from the differential growth of rural and urban areas, i.e., it depends on the rural–urban differentials in natural increase as well as the net transfer of population from rural to urban areas. In the past, there has been little analytical work done to clarify this dependence. Most of the research has concentrated on descriptive generalizations such as the demographic transition resulting from the joint and simultaneous occurrence of the vital and mobility revolutions.*

By contrast, our purpose is to examine the process of urbanization from an analytical point of view. Such an objective is performed by examining and comparing the dynamics of recently devised models of rural and urban population change. For each of the three models considered, the analysis is established on the basis of a simple differential equation – describing the evolution of the urban to rural population ratio – which is arrived at by combining the original differential equations describing the rural and urban populations.

Note that our intention here is not to test the validity of these alternative models but rather to use these models to facilitate the comprehension of the relationship between urbanization and its component factors at various stages of socioeconomic development.

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*The vital revolution is the process by which societies advance from high birth and death rates to low birth and death rates. The mobility revolution is a similar process by which they move from low to high mobility rates.

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This paper consists of three sections. Section I makes use of the Keyfitz model (Keyfitz 1978) in which the migration exchange between rural and urban areas is seen as a rural net outmigration flow representing a constant fraction of the rural population. Section II is based on a continuous version of a two-region components-of-change model (Rogers 1968) whose relevance in such a context was first suggested by Ledent (1978a, b). In contrast to the Keyfitz model, this model presents a symmetric treatment of the migration flows between the rural and urban areas: each sector exhibits a constant gross outmigration rate. Finally, Section III utilizes an extended version of the Rogers model that exhibits a varying regime of rural—urban migration (United Nations 1979): the gross migration flows out of each sector are introduced through a gravity model. Note that all of the aforementioned models assume constant natural increase differentials between urban and rural regions; however, the case of varying regimes of natural increase differentials is briefly examined, at the end of Section III, in relation to the third and last model.

I THE KEYFITZ MODEL

Basically, Keyfitz (1978) considers a rural–urban population system, initially entirely rural, in which the rural and urban sectors are submitted to constant rates of natural increase, denoted by r and u, respectively. In addition, he views the migration exchange between the two sectors as a net outmigration flow from the rural sector, equal to a constant fraction m of the rural population (m is assumed to be positive).

DERIVATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

The evolution of such a rural–urban population system can be described by the following system of differential equations:

$$\frac{dP_r(t)}{dt} = (r - m)P_r(t) \tag{1}$$

and

$$\frac{dP_u(t)}{dt} = mP_r(t) + uP_u(t) \tag{2}$$

where $P_r(t)$ and $P_u(t)$ are the rural and urban populations at time t.

Letting $S(t)$ denote the ratio $P_u(t)/P_r(t)$ of the urban to rural population, we have

$$\frac{dS(t)}{S(t)dt} = \frac{dP_u(t)}{P_u(t)dt} - \frac{dP_r(t)}{P_r(t)dt} \tag{3}$$

Note that if one retains $S(t)$ as the index of urbanization, this last equation can be interpreted as follows: the growth rate of urbanization is equal to the difference between the urban and rural population growth rates (United Nations 1979). Then, since the rural growth rate is constant and the urban growth rate is a

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simple function of $S(t)$, substituting (1) and (2) into (3) and rearranging terms leads to the following differential equation in $S(t)$

$$\frac{dS(t)}{dt} - (u + m - r)S(t) = m \qquad (4)$$

EVOLUTION OF THE URBANIZATION LEVEL AND GROWTH RATE

Recalling that, by assumption, the system is initially entirely rural, we thus obtain the solution of (4) as

$$S(t) = \frac{m}{u + m - r} \{\exp[(u + m - r)t] - 1\} \qquad (5)$$

Expression (5) shows that the urban to rural population ratio (or urbanization index) only depends on the rural—urban differential in natural increase $r - u$ and the rural net outmigration rate m.

Differentiating (5) with respect to time leads to

$$\frac{dS(t)}{dt} = m \exp[(u + m - r)t] \qquad (6)$$

which is positive for all values of t. Consequently, the urban to rural population ratio monotonically increases as t increases.

What is the long-term behavior of $S(t)$? We must consider two cases here (Figure 1):

(a) if $u + m - r > 0$, $S(t)$ increases indefinitely at the exponential rate $(u + m - r)$;

(b) if $u + m - r < 0$, $S(t)$ tends toward a limit equal to $m/[r - (u + m)]$.

In fact, virtually all actual population systems are characterized by parameters corresponding to the first case (Ledent 1978b). Thus, we impose the following restriction

$$u + m - r > 0 \qquad (7)$$

so that $S(t)$ is an *exponential function* of t. Thus, in the long run, the system becomes predominantly urban.

Then, how does the growth rate of urbanization $dS(t)/S(t)dt$ evolve? From (4), we have

$$\frac{dS(t)}{S(t)dt} = u + m - r + \frac{m}{S(t)} \qquad (8)$$

Recalling the variations of $S(t)$, we thus obtain the result that the growth rate of urbanization monotonically decreases from $+\infty$ (for $t = 0$) to $u + m - r$ (as $t \longrightarrow +\infty$), a quantity which remains positive as a consequence of (7). It is
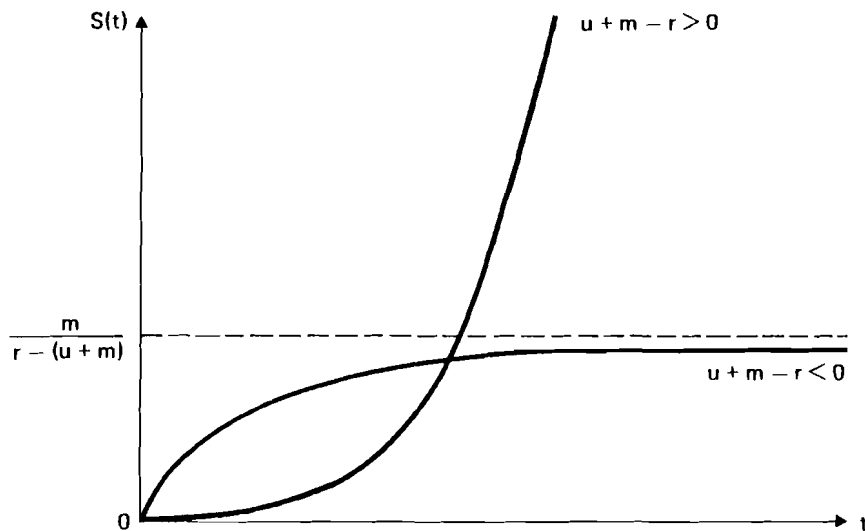
FIGURE 1 The Keyfitz model: the variations of $S(t)$.

easily established that the second derivative of $dS(t)/S(t)dt$ is always positive: the growth rate of urbanization is described by a convex curve (Figure 2).

EVOLUTION OF THE PROPORTION OF THE POPULATION THAT IS URBAN

By definition, the proportion $\alpha(t)$ of the population that is urban is such that

$$\alpha(t) = \frac{P_u(t)}{P_r(t) + P_u(t)} = \frac{S(t)}{1 + S(t)} \tag{9}$$

Differentiating $\alpha(t)$ with respect to time leads to

$$\frac{d\alpha(t)}{\alpha(t)dt} = \frac{dS(t)}{S(t)dt} - \frac{dS(t)}{[1 + S(t)]dt} = \frac{dS(t)}{S(t)[1 + S(t)]dt} \tag{10}$$

Thus, $\alpha(t)$ monotonically increases over time: from zero (for $t = 0$) to 1 (as $t \rightarrow +\infty$). But, what is the shape of the curve describing $\alpha(t)$?

Substituting (5) into (9) leads to an explicit expression of $\alpha(t)$:

$$\alpha(t) = \frac{\exp[(u + m - r)t] - 1}{\exp[(u + m - r)t] + (u - r)/m} \tag{11}$$

which suggests the consideration of two cases.
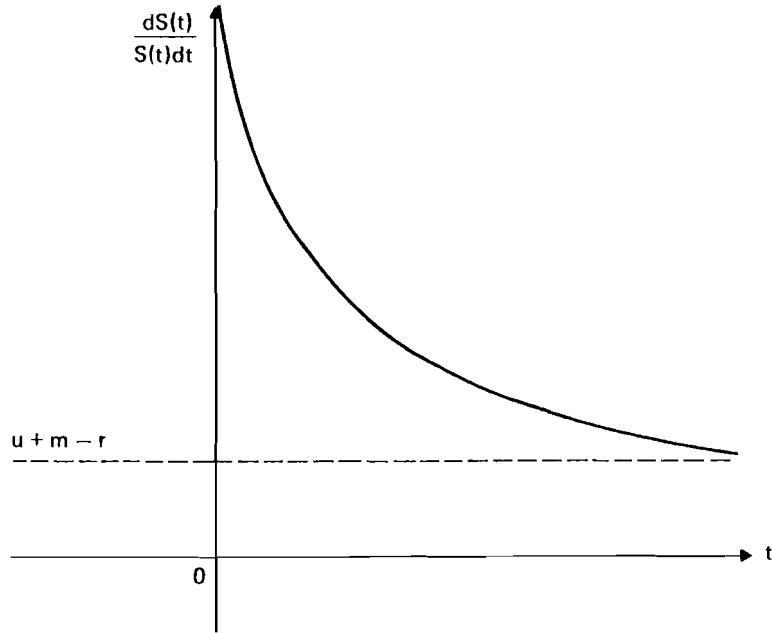
FIGURE 2   The Keyfitz model: the variations of the growth rate of urbanization.

(a) If $r \leqslant u$, the right-hand side of (11) represents a logistic function of time. Because only positive values of t are relevant to the variations of $\alpha(t)$, it is important to determine whether the point of inflection of this logistic function occurs for a negative or a positive value of t.

Differentiating the right-hand side of (11) twice with respect to time indicates that the second derivative of $\alpha(t)$ has the sign of

$$x(t) = - \frac{\exp[(u + m - r)t] - (u - r)/m}{\exp[(u + m - r)t] + (u - r)/m} \tag{12}$$

It is then readily established that the point of inflection occurs for

$$t_\alpha = \frac{1}{u + m - r} \ln\left(\frac{u - r}{m}\right) \tag{13}$$

an expression which shows that the sign of t_α depends on the respective values of r and $u - m$.

As shown in Figure 3, it follows that:

(i) if $r < u - m$, t_α is positive and the curve describing the variations of $\alpha(t)$ (the solid curve of Figure 3(a)) is S-shaped;
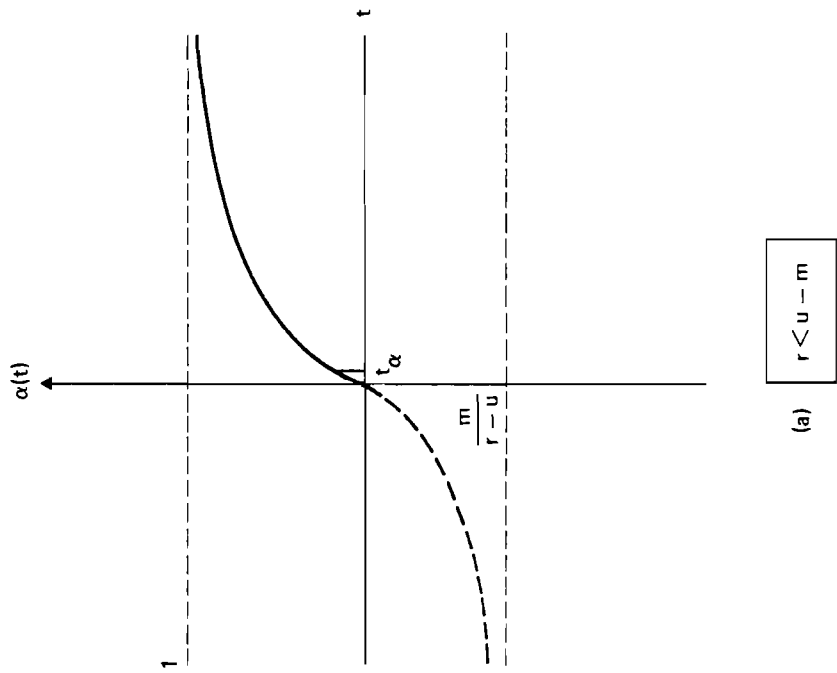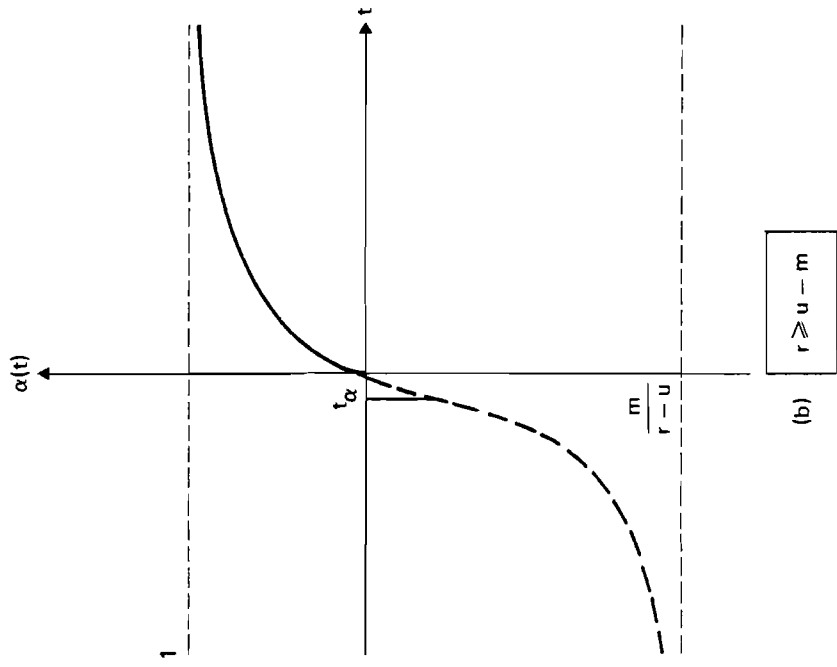
FIGURE 3   The Keyfitz model: the variations of $\alpha(t)$ for $r \leqslant u$.

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(ii) if $r \geqslant u - m$, t_α is negative and the curve describing the variations of $\alpha(t)$ (the solid curve of Figure 3(b)) is shaped downward.

(b) If $r > u$, the right-hand side of (11) is no longer a logistic function of time. Its variations are slightly more complicated and are represented in Figure 4. But since $x(t)$ is negative for all values of t, the curve describing the variations of $\alpha(t)$ (the solid curve of Figure 4) is simply shaped downward.

In practice, since the rural rate of natural increase is higher or only slightly less than the urban rate of natural increase, situation (b) of Figure 3 or that of Figure 4 is typical. In other words, $\alpha(t)$ – which, in all cases, monotonically increases from zero to one – is described by a curve shaped downward (concave).

EVOLUTION OF THE RURAL AND URBAN POPULATIONS

To analyze such an evolution, the explicit derivation of expressions of $P_r(t)$ and $P_u(t)$ as functions of time (Keyfitz 1978) is not necessary. In fact, it is sufficient to look at the sign of the rural and urban population growth rates.

FIGURE 4   The Keyfitz model: the variations of $\alpha(t)$ for $r > u$.

Indeed, we immediately have from (1) that $P_r(t)$ varies exponentially, increasing indefinitely if $r > m$ or decreasing toward zero if $r < m$.

To obtain the variations of $P_u(t)$, we rewrite (2) as

$$\frac{dP_u(t)}{P_u(t)dt} = \frac{m}{S(t)} + u \tag{14}$$

It follows that the urban growth rate monotonically decreases from $+\infty$ (for $t = 0$) to u (as $t \to +\infty$). Consequently,
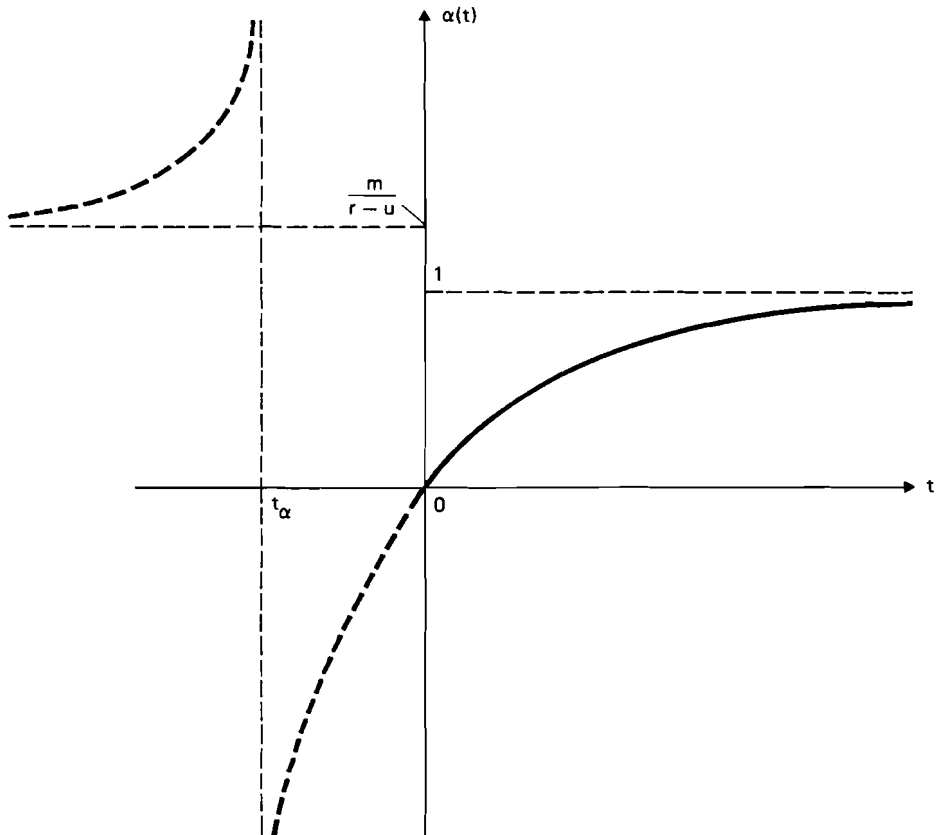
(a) if $u > 0$, the urban population monotonically increases as $t \to +\infty$;

(b) if $u < 0$, the urban population increases and then decreases toward zero as $t \to +\infty$.

Hence, we impose a further restriction that the urban rate of natural increase is positive, i.e.,

$$u > 0 \tag{15}$$

From the above variations of $P_r(t)$ and $P_u(t)$, we may conclude that the fact that the system becomes predominantly urban as $t \to +\infty$ reflects that either the rural population vanishes (if $r < m$) or the urban population becomes infinitely large with regard to the rural population (if $r > m$).

The dynamics of the Keyfitz model — a model characterized by a constant rural net outmigration rate

$$m_r(t) = m \tag{16}$$

where m is positive and subject to restrictions (7) and (15) — are summarized in Table 1.

APPLICATION TO ACTUAL RURAL–URBAN POPULATION SYSTEMS

Since $S(t)$ may take any positive value as t increases, it follows that any actual two-sector system — characterized by a ratio $\bar{S}$ of urban to rural population — appears to be identical to the subsequent state of an initially entirely rural population system subject to the same parameters r, u, and m. The time t_D at which this hypothetical population reaches the ratio $\bar{S}$ is given by the solution of $S(t) = \bar{S}$, i.e., (Keyfitz 1978),

$$t_D = \frac{1}{u + m - r} \ln\left(1 + \frac{u - m - r}{m}\bar{S}\right) \tag{17}$$

Thus, if one observes an actual population system in year y, the ratio of the urban to rural population in year $y + T$ is given by

$$S(y + T) = \frac{m}{u + m - r}\left[\left(1 + \frac{u + m - r}{m}\bar{S}\right)\exp[(u + m - r)T] - 1\right] \tag{18}$$

TABLE 1   The Keyfitz model: the variations of the main functions.

Function		t = 0	t = +∞
$P_r(t)$	(a) $r < m$	$P_r(0)$	0
	(b) $r = m$	$P_r(0)$	$P_r(0)$
	(c) $r > m$	$P_r(0)$	+∞
$P_u(t)$		0	+∞
$S(t)$		0	+∞
$\dfrac{dS(t)}{S(t)dt}$		+∞	$u + m - r$
$\alpha(t)$		0	1
$m_r(t)$		m	m

As an illustration, Table 2 indicates the pace of urbanization that would occur in two actual rural—urban systems on the basis of the Keyfitz model: those of India and the U.S.S.R.

Rogers and Willekens (1976) report that the urban population of India was growing at an annual rate of 37.2 per thousand during the late sixties. This rate was the sum of a rate of natural increase of 19.5 and a net migration rate of 17.7 per thousand. At the same time, the rural population was growing at an annual rate of 17.15 per thousand which was the sum of a rate of natural increase of 21.50 per thousand and a net migration rate of —4.35 per thousand. Then, in this system

$$r = 21.5 \times 10^{-3}, \quad u = 19.5 \times 10^{-3}, \quad m = 4.35 \times 10^{-3}, \quad \bar{S} = 0.245$$

The left-hand side of Table 2 indicates that, if the above rates remain constant, the urbanization process of India will be slow. For example, the percentage of the population that is urban will increase, in 25 years, from 19.7 percent to only 27.1 percent. About 130 years will be necessary for the urban population to exceed the rural population.

As for the U.S.S.R. — observed in the early seventies — appropriate data can be found in Rogers (1976):

$$r = 10.0 \times 10^{-3}, \quad u = 9.0 \times 10^{-3}, \quad m = 20.8 \times 10^{-3}, \quad \bar{S} = 1.291$$

TABLE 2   The Keyfitz model: application to India and the U.S.S.R.

India			U.S.S.R.	
$r = 21.5 \times 10^{-3}, u = 19.5 \times 10^{-3},$ $m = 4.35 \times 10^{-3}$			$r = 10 \times 10^{-3}, u = 9 \times 10^{-3},$ $m = 20.8 \times 10^{-3}$	
S	α (percentage)	T	S	α (percentage)
0.245	19.70	0	1.291	56.35
0.270	21.27	5	1.527	60.43
0.295	22.79	10	1.787	64.12
0.372	27.12	25	2.729	73.19
0.507	33.62	50	5.031	83.42
0.800	44.45	100	14.604	93.59
1.502	60.03	200	101.850	99.03
4.929	83.13	500	—	100.00
20.070	95.25	1,000	—	100.00

The right-hand side of Table 2 indicates that, on the basis of these rates, the urbanization process will remain strong in the future: the percentage of the population that is urban will increase from 56.4 percent to 73.2 percent in 25 years and to 83.4 percent in 50 years.

Note that there exists an important contrast between the India and U.S.S.R. cases. Whereas the rural population increases indefinitely in the former case, it decreases toward zero in the latter (since r is less than m): the rural population of the U.S.S.R., unlike that of India, vanishes in the long run.

SENSITIVITY ANALYSIS

Because eq. (5), which expresses the ratio of urban to rural population, is simple, it is easy to differentiate it with respect to the basic parameters m and $r - u$.

In particular, differentiating $S(t)$ with respect to m leads to:

$$\frac{dS(t)}{S(t)dm} = \frac{[u - r + tm(u + m - r)] \exp[(u + m - r)t] - (u - r)}{m(u + m - r)\{\exp[(u + m - r)t] - 1\}} \quad (19)$$

It is readily established that the numerator of the right-hand side of (19) is an increasing function of time taking the value zero for $t = 0$. It thus follows that $dS(t)/dm$ is positive so that, as expected, a higher rural net outmigration rate tends to hasten the pace of the urbanization phenomenon.

In order to assess more accurately the impact of the value of m on the urbanization level, we have simulated the growth of the Indian system for different values of the rural net outmigration rate (while keeping r and u identical to their observed values). Table 3 indicates that a 0.001 increase of the rural net outmigration rate produces a small acceleration in the pace of urbanization:

TABLE 3   The Keyfitz model: impact of the rural net outmigration rate on the percentage of the Indian population that is urban 50 years hence ($r - u = 2.0 \times 10^{-3}$).

m'	0.001	$m/2$	m	$m + 0.001$	$2m$	$3m$	$4m$
$\alpha(+50)$	22.01	26.29	33.62	36.75	46.21	56.45	64.77

TABLE 4   The Keyfitz model: the impact of the rural–urban natural increase differential on the percentage of the Indian population that is urban 50 years hence ($m = 4.35 \times 10^{-3}$).

$r - u$	−0.002	−0.001	0	0.001	0.002	0.003	0.004
$\alpha(+50)$	37.22	36.29	35.39	34.50	33.62	32.76	31.92

the urban proportion reaches 36.8 percent (versus 33.6 percent) after 50 years. Indeed, a doubling or a tripling of the rural net outmigration rate creates a dramatic speeding up of the urbanization process: after 50 years, the urban proportion reaches 46.2 and 56.5 percent, respectively.

Similarly, differentiating $S(t)$ with respect to the rural–urban natural increase differential leads to:

$$\frac{dS(t)}{S(t)d(r - u)} = - \frac{1 + [(u + m - r)t - 1] \exp[(u + m - r)t]}{\exp[(u + m - r)t] - 1} \quad (20)$$

It can be seen that the numerator of the right-hand side of (20) is an increasing function of time taking the value zero for $t = 0$. It follows that $dS(t)/d(r - u)$ is negative so that, as expected, a smaller rural–urban natural increase differential tends to speed up the urbanization phenomenon.

The impact of the value of $r - u$ on the urbanization level is assessed by simulating the growth of the Indian system for different values of $r - u$ (while keeping the rural net outmigration equal to its observed value). Table 4 indicates that a relatively small change in the natural increase differential only produces a small acceleration of the urbanization process: for example, a 0.001 decrease in the rural–urban natural increase differential causes the percentage of the population that is urban after 50 years to increase from 33.6 percent to 34.5 percent. This impact is much less than the one caused by a similar increase in the rural net outmigration rate: let us recall that a 0.001 increase in the latter causes the urban percentage to increase to 36.8 percent.

In addition, note that, because the rural and urban rates of natural increase generally take on similar values, the impact on $\alpha(+50)$ of plausible variations in the value of $r - u$ is rather small. As indicated by the figures

displayed in Tables 3 and 4, the impact caused by plausible variations of m is much more important.

In the less developed countries, the rural natural increase rate r is generally higher than the urban natural increase rate u, and the difference tends to decline with economic development. In these countries, economic development promotes urbanization as a consequence of both declining rural—urban natural increase differentials and increasing net outmigration rates. However, as shown above, the influence through migration exchange is likely to be preponderant.

II THE ROGERS MODEL

As an alternative to the Keyfitz model, Ledent (1978a, b) suggests using a continuous version of a two-region components-of-change model (Rogers 1968). This model, still characterized by constant rates of natural increase in both sectors, presents a more symmetric consideration of the migration exchange between the two sectors. In each sector, a constant fraction of the population is assumed to move to the other sector.

DERIVATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

Let o_r and o_u denote the gross migration rates out of the rural and urban sectors, respectively (o_r and o_u are positive). Then the evolution of the rural–urban population system is described by the following:

$$\frac{dP_r(t)}{dt} = (r - o_r)P_r(t) + o_u P_u(t) \tag{21}$$

$$\frac{dP_u(t)}{dt} = o_r P_r(t) + (u - o_u)P_u(t) \tag{22}$$

Since both rural and urban growth rates are simple functions of $S(t)$, substituting (21) and (22) into (3) and rearranging terms leads to the following differential equation in $S(t)$:

$$\frac{dS(t)}{dt} = o_r + [u - o_u - (r - o_r)]S(t) - o_u[S(t)]^2 \tag{23}$$

In the mathematic literature, (23) is referred to as a Riccati equation.

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The right-hand side of (23) is a polynomial in $S(t)$ of the second order which admits two real roots since its discriminant $\Delta = [u - o_u - (r - o_r)]^2 + 4o_ro_u$ is positive. Moreover, since their product $-o_r/o_u$ is negative, these two roots have opposite signs.

Let S_A denote the positive root

$$S_A = \frac{u - o_u - (r - o_r) + [\{u - o_u - (r - o_r)\}^2 + 4o_ro_u]^{1/2}}{2o_u} \qquad (24)$$

and S_B the negative one: it is identical to S_A except that the sign preceding the square root term is a minus instead of a plus. Then, one can rewrite (23) as:

$$\frac{dS(t)}{dt} = o_u[S_A - S(t)][S(t) - S_B] \qquad (25)$$

Since the urban–rural population system is initially entirely rural (i.e., $S(0) = 0$), it is clear that the variations of $S(t)$ are represented by part of a *logistic* function: $S(t)$ monotonically increases from 0 to S_A over the time continuum $[0, +\infty]$, i.e.,

$$0 < S(t) < S_A \qquad \forall t > 0 \qquad (26)$$

Thus, in contrast to the Keyfitz model, the Rogers model leads to a long-run stable equilibrium.

Further, rearranging terms in (25) leads to

$$\frac{\frac{dS(t)}{dt}}{S(t) - S_B} - \frac{\frac{-dS(t)}{dt}}{S_A - S(t)} = o_u(S_A - S_B) \qquad (27)$$

Upon observing that (26) holds, the integration of (27) yields

$$\frac{S(t) - S_B}{S_A - S(t)} = -\frac{S_B}{S_A} \exp[o_u(S_A - S_B)t] \qquad (28)$$

or, alternatively,

$$S(t) = \frac{S_A S_B\{1 - \exp[o_u(S_A - S_B)t]\}}{S_A - S_B \exp[o_u(S_A - S_B)t]} \qquad (29)$$

Note that, as suggested by eq. (24), S_A – as well as S_B – are functions of the rural and urban rates of natural increase through their difference. Thus, the

urban to rural population ratio only depends on the rural–urban differential in the natural increase $r - u$ and the gross migration rates out of both sectors.

As mentioned above, the variations of $S(t)$ are described by a truncated logistic curve. The question then is one of knowing whether this curve presents a point of inflection or not.

Differentiating eq. (29) with respect to time indicates that $d^2S(t)/dt^2$ has the sign of

$$y(t) = u - o_u - (r - o_r) - 2o_uS(t) \tag{30}$$

We thus obtain the following.

(a) If $u - o_u > r - o_r$, $d^2S(t)/dt^2$ is positive (negative) for all t such that

$$S(t) \underset{(<)}{>} \frac{u - o_u - (r - o_r)}{2o_u} \tag{31}$$

i.e.,

$$t \underset{(<)}{>} t_s = \frac{1}{o_u(S_A - S_B)} \ln\left(-\frac{S_A}{S_B}\right) \tag{32}$$

Then, $S(t)$ appears to be an S-shaped curve (Figure 5(a)).

(b) If $u - o_u < r - o_r$, it is clear from (30) that $d^2S(t)/dt^2$ is negative so that $S(t)$ is shaped downward (Figure 5(b)).

Since the actual values of u and r are roughly similar, the existence of a point of inflection depends, for a large part, on the comparative values of o_r and o_u. Thus in practice if the rural outmigration rate is much higher than the urban outmigration rate, the curve describing the variations of $S(t)$ exhibits a point of inflection.

Let us now turn to the examination of the evolution of the growth rate of urbanization $dS(t)/S(t)dt$. From (25), we have

$$\frac{dS(t)}{S(t)dt} = -o_u\left[S(t) - (S_A - S_B) + \frac{S_AS_B}{S(t)}\right] \tag{33}$$

The first derivative of this expression with respect to time has the sign of $-o_u\{1 - S_AS_B/[S(t)]^2\}$, which is negative for all values of t (the product S_AS_B is negative). Thus, the growth rate of urbanization monotonically decreases from $+\infty$ (for $t = 0$) to zero (as $t \to +\infty$).

Recalling the interpretation of $dS(t)/S(t)dt$ as the urban–rural growth rate difference, we conclude to the constant reduction of this difference which eventually vanishes (as a consequence of the stability result).

It is easily established that the second derivative of $dS(t)/S(t)dt$ is positive so that the variations of the growth rate of urbanization are described by a convex curve (Figure 6).
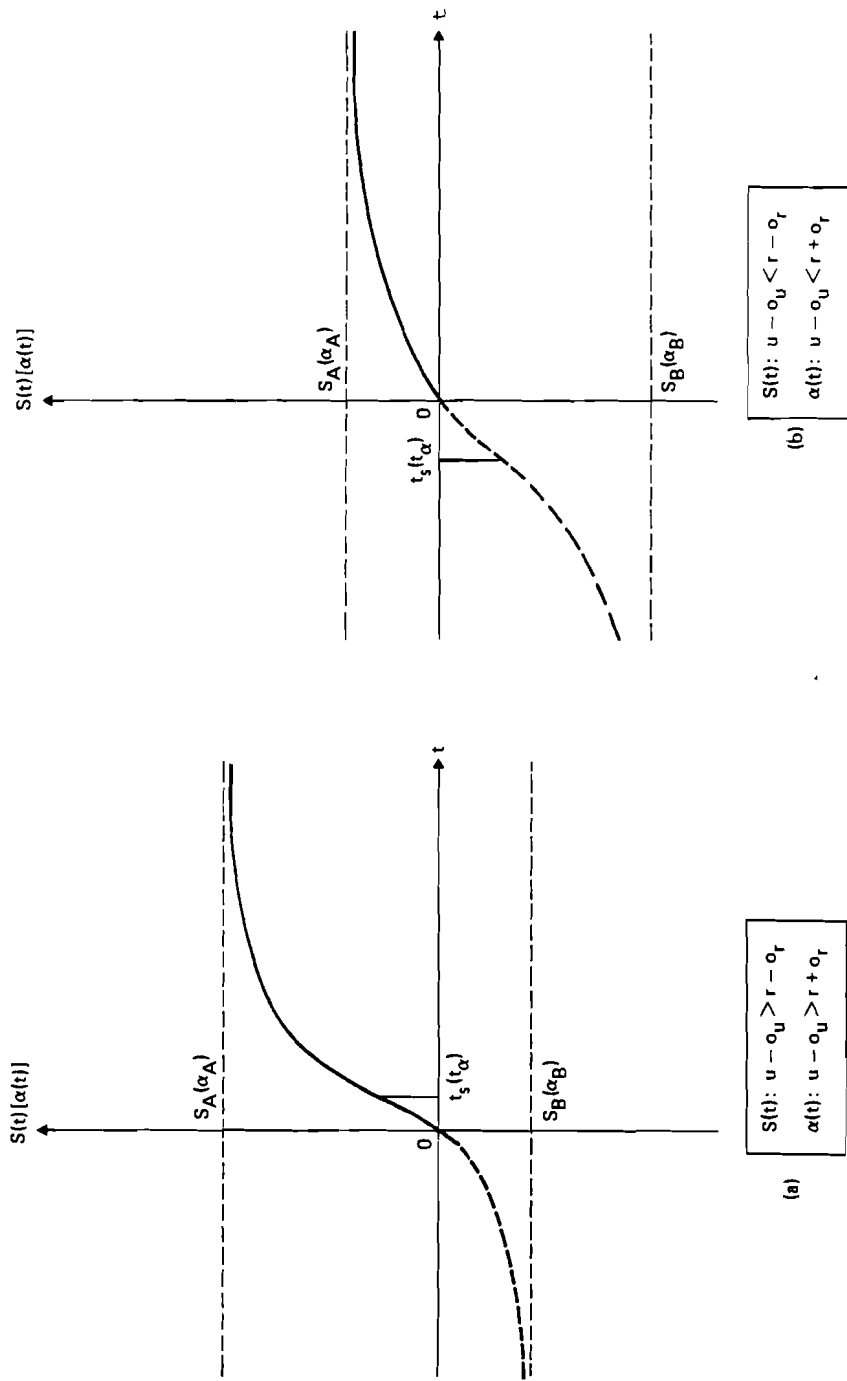
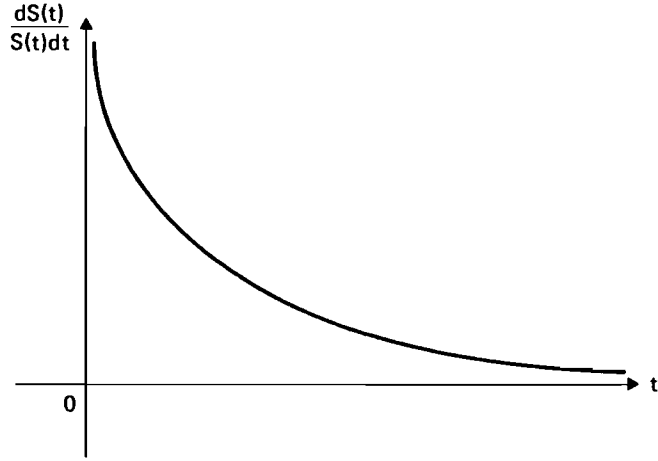FIGURE 5   The Rogers model: the variations of $S(t)$ and $\alpha(t)$.

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FIGURE 6   The Rogers model: the variations of the growth rate of urbanization.

EVOLUTION OF THE PROPORTION OF THE
POPULATION THAT IS URBAN

Substituting (29) into (9) yields an expression of the proportion $\alpha(t)$ of the population that is urban:

$$\alpha(t) = \frac{S_A S_B \{1 - \exp[o_u(S_A - S_B)t]\}}{S_A(1 + S_B) - S_B(1 + S_A)\exp[o_u(S_A - S_B)t]} \tag{34}$$

This last expression shows that the variations of $\alpha(t)$ are also described by a truncated *logistic* curve.

Clearly, $\alpha(t)$ monotonically increases from zero (for $t = 0$) to $\alpha_A = S_A/(1 + S_A)$ (as $t \to \infty$).

Does the curve describing the variations of $\alpha(t)$ present a point of inflection or not? Differentiating $\alpha(t)$ twice with respect to time shows that $d^2\alpha(t)/dt^2$ has the sign of

$$z(t) = -\left\{\frac{1 + S_B}{S_B} + \frac{1 + S_A}{S_A}\exp[o_u(S_A - S_B)t]\right\} \tag{35}$$

an expression which is positive for all values of t less than

$$t_\alpha = \frac{1}{o_u(S_A - S_B)}\ln\left(\frac{S_A(1 + S_B)}{-S_B(1 + S_A)}\right) \tag{36}$$

There exist such values only if t_α is positive, i.e., if $-S_A(1 + S_B)/S_B(1 + S_A) > 1$

or $S_A + S_B + 2S_AS_B > 0$. Recalling the values of the sum and product of the two roots of (23), we thus obtain that:

(a) if $u - o_u > r + o_r$, $d^2\alpha(t)/dt^2$ is first positive for $t < t_\alpha$ and then negative for $t > t_\alpha$; $\alpha(t)$ then appears to be an S-shaped curve (Figure 5(b));

(b) if $u - o_u < r + o_r$, $d^2\alpha(t)/dt^2$ is negative and the curve describing the variations of $\alpha(t)$ is directed downward (Figure 5(b)).

In practice, since the rural and urban rates of natural increase are of the same magnitude, situation (b) is typical.

To summarize, the Rogers model — like the Keyfitz model — leads to a proportion $\alpha(t)$ of the population that is urban which is an increasing and concave function of time. However, there exists a major difference between the two models in the long run: the Rogers model leads to stability ($\alpha_A < 1$) unlike the Keyfitz model ($\alpha_A = 1$).

EVOLUTION OF THE RURAL AND URBAN POPULATIONS

How does the rural and urban population vary over time? For this purpose, the availability of the expressions of $P_r(t)$ and $P_u(t)$ as functions of time — which have been derived elsewhere (Ledent 1978a) — is not necessary. As with the Keyfitz model, an answer to such a question can be obtained with relatively little effort by determining the sign of the rural and urban population growth rates.

Equation (21) suggests that the rural growth rate $dP_r(t)/P_r(t)dt$ is positive (negative) if $S(t)$ is greater (less) than $(o_r - r)/o_u$. Therefore:

(a) If $S_A \geqslant (o_r - r)/o_u$, $dP_r(t)/dt$ is positive as $t \rightarrow +\infty$, i.e., $P_r(t)$ increases indefinitely. Two subcases must be considered here:

(i) if $r \geqslant o_r$, $dP_r(t)/dt$ is positive for all positive values of t so that $P_r(t)$ monotonically increases toward $+\infty$;

(ii) if $r < o_r$, $dP_r(t)/dt$ is first negative for all t less than

$$t_r = \frac{1}{o_u(S_A - S_B)} \ln\left(\frac{o_r + S_A(o_r - r)}{o_r + S_B(o_r - r)}\right) \tag{37}$$

and positive afterwards, i.e., $P_r(t)$ monotonically decreases as t increases from 0 to t_r and then monotonically increases toward $+\infty$.

(b) If $S_A < (o_r - r)/o_u$, $dP_r(t)/dt$ is negative and $P_r(t)$ monotonically decreases toward zero.

As for the variations of the urban population, eq. (22) rewritten as

$$\frac{dP_u(t)}{P_u(t)dt} = u - o_u + \frac{o_r}{S(t)} \tag{38}$$

suggests that the urban growth rate $dP_u(t)/P_u(t)dt$ monotonically decreases

from $+\infty$ to its long-term value which is also the long-term rural growth rate. Thus:

(a) if $S_A \geqslant (o_r - r)/o_u$, $dP_u(t)/dt$ is positive for all t and $P_u(t)$ monotonically increases toward $+\infty$;

(b) if $S_A < (o_r - r)/o_u$, $dP_u(t)/dt$ is first positive for all t less than a certain value t_u

$$t_u = \frac{1}{o_u(S_A - S_B)} \ln\left(\frac{o_u(1 + S_A) - u}{o_u(1 + S_B) - u}\right) \tag{39}$$

and negative for $t > t_u$. Thus, $P_u(t)$ monotonically increases as t increases from 0 to t_u and then monotonically decreases toward zero.

Clearly, the case of vanishing rural and urban populations is of no interest to us, and we thus impose the restriction

$$S_A \geqslant (o_r - r)/o_u \tag{40}*$$

Recalling (24), which expresses S_A in terms of the basic parameters, it is readily established that (40) holds if

(i) $u + r > o_u + o_r$

or

(ii) $u + r < o_u + o_r$ and $(u - o_u)(r - o_r) - o_r o_u < 0$

EVOLUTION OF THE RURAL NET OUTMIGRATION RATE

A question of importance here is the evolution of the rural net outmigration rate implied by the Rogers model. Clearly,

$$m_r(t) = o_r - o_u \frac{P_u(t)}{P_r(t)} = o_r - o_u S(t) \tag{41}$$

an equation which shows that $m_r(t)$ is also described by a branch of a logistic curve (Figure 7): it monotonically declines from o_r (for $t = 0$) to $o_r - o_u S_A$ ($t \rightarrow \infty$) and exhibits (does not exhibit) a point of inflection when $S(t)$ does not (does).

This property of a declining rural net outmigration rate thus seems to reduce the applicability of the Rogers model to already somewhat developed countries.

*It is easy to establish that this condition is equivalent to

$$r + uS_A \geqslant 0 \tag{40'}$$

an inequality which ensures that the population of the whole system does not vanish. Note that this condition is less restrictive than the corresponding condition (15) in the Keyfitz model (u is positive); if u is negative, the Rogers model still applies as long as r is positive and such that (40) holds.
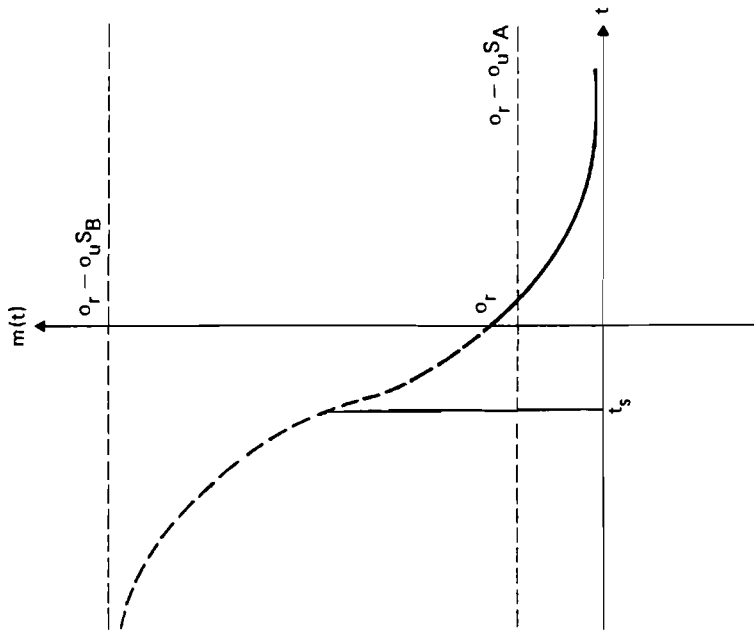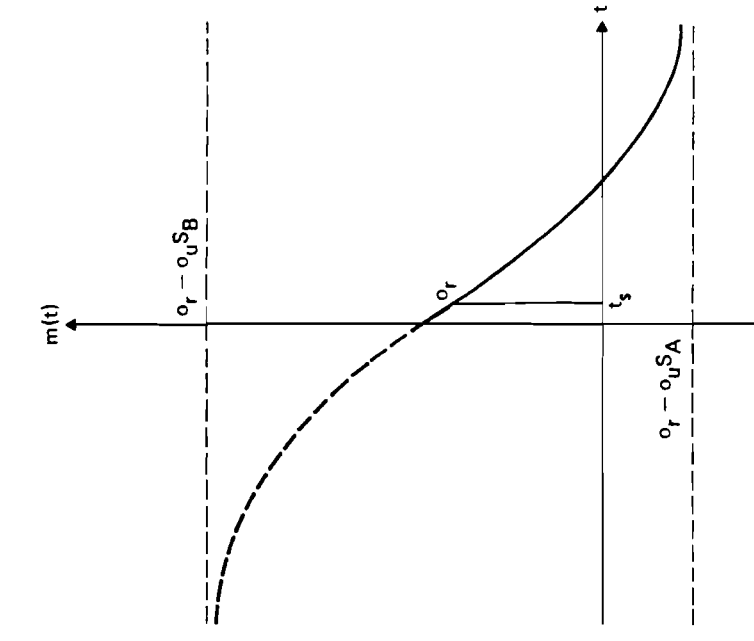
FIGURE 7 The Rogers model: the variations of $m(t)$.

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How large is the drop in the rural net outmigration rate? Recalling (24) which expresses S_A in terms of the basic parameters, we have

$$m_r(\infty) = \frac{1}{2}\left(r + o_r + o_u - u - \{[u - o_u - (r - o_r)]^2 + 4o_r o_u\}^{1/2}\right) \quad (42)$$

Let

$$G = \frac{1}{2}(r + o_r + o_u - u) \quad (43a)$$

$$H = \frac{1}{2}\{[u - o_u - (r - o_r)]^2 + 4o_r o_u\}^{1/2} \quad (43b)$$

and let us calculate $G^2 - H^2$. After several manipulations, we obtain that

$$G^2 - H^2 = o_r(r - u) \quad (44)$$

Consequently,

$$m_r(\infty) = G - H = \frac{o_r(r - u)}{G + H} \quad (45)$$

and, since $G + H = o_r - o_u S_B$, we finally have

$$m_r(\infty) = \frac{o_r(r - u)}{o_r - o_u S_B} \quad (46)$$

This last equation suggests two interesting conclusions:

(a) If the urban rate of natural increase is higher than the corresponding rural rate, the direction of the rural—urban migration exchange is reversed at some point in time.

(b) Since S_B is negative, $o_r/(o_r - o_u S_B)$ is less than one and therefore the absolute value of the long-run rural net migration is less than the rural—urban differential in natural increase, i.e., a value generally close to zero.

To summarize, a built-in property of the Rogers model is a sharp drop in the rural net outmigration rate toward a small value (either positive or negative) less in absolute value than the rural—urban differential in natural increase.

The dynamics of the Rogers model — as defined by eqs. (21) and (22) and subject to restriction (40) — are summarized in Table 5.

APPLICATION TO ACTUAL RURAL–URBAN POPULATION SYSTEMS

Since $S(t)$ can take any value between zero and S_A as t increases, any actual two-sector system — characterized by the basic parameters r, u, o_r and o_u such that (40) holds and a ratio $\bar{S}$ of urban to rural population such that (47) holds — is identical to the subsequent state of an initially entirely rural population system subject to the same basic parameters.

$$\bar{S} < S_A \quad (47)$$

TABLE 5   The Rogers model: the variations of the main functions.

Function	t		
	0	t_r	$+\infty$
$P_r(t)$ (a) $r < o_r$	$P_r(0)$	$P_r(t_r)$	$+\infty$
(b) $r \geqslant o_r$	$P_r(0)$		$+\infty$
$P_u(t)$	0		$+\infty$
$S(t)$	0		S_A
$\dfrac{dS(t)}{S(t)dt}$	$+\infty$		0
$\alpha(t)$	0		$\dfrac{S_A}{1+S_A}$
$m(t)$	o_r		$o_r - o_u S_A$

The time t_D at which this hypothetical population reaches the ratio $\bar{S}$ is given by the solution of $S(t) = \bar{S}$, i.e.,

$$t_D = \frac{1}{o_u(S_A - S_B)} \ln\left(\frac{S_A(\bar{S} - S_B)}{-S_B(S_A - \bar{S})}\right) \qquad (48)$$

On the basis of this, if one observes an actual population system characterized as above in year T, the ratio of the urban to rural population in year $y + T$ is given by:

$$S(y + T) = \frac{S_B(S_A - \bar{S}) + S_A(\bar{S} - S_B)\exp[o_u(S_A - S_B)T]}{S_A - \bar{S} + (\bar{S} - S_B)\exp[o_u(S_A - S_B)T]} \qquad (49)$$

Table 6 indicates the urbanization that would occur on the basis of (49) in the two actual rural—urban systems considered previously. It turns out that the long-term equilibrium is reached in about 400 years in the case of India and in less than 200 years in the case of the U.S.S.R. Note the relatively low value of the long-term percentage of the population that is urban in the case of India: 37.7 percent versus 19.7 percent initially. By contrast, the corresponding figures for the U.S.S.R. are 75.3 and 56.4 percent, respectively.

In addition, the comparison of the figures of Table 6 with those of Table 5

TABLE 6 The Rogers model: application to India and the U.S.S.R.

India				U.S.S.R.		
$r = 21.5 \times 10^{-3}$ $u = 19.5 \times 10^{-3}$				$r = 10.0 \times 10^{-3}$ $u = 9.0 \times 10^{-3}$		
$o_r = 6.8 \times 10^{-3}$ $o_u = 10.0 \times 10^{-3}$				$o_r = 35.0 \times 10^{-3}$ $o_u = 11.0 \times 10^{-3}$		
S	α (percentage)	m (per thousand)	T	S	α (percentage)	m (per thousand)
0.245	19.70	4.35	0	1.291	56.35	20.80
0.269	21.21	4.11	5	1.512	60.19	18.37
0.292	22.61	3.88	10	1.721	63.24	16.07
0.353	26.10	3.27	25	2.088	67.62	12.03
0.433	30.20	2.47	50	2.745	73.30	4.81
0.528	34.54	1.52	100	3.011	75.02	1.97
0.590	37.13	0.90	200	3.045	75.28	1.51
0.605	37.68	0.75	500	3.045	75.28	1.51
0.605	37.68	0.75	1,000	3.045	75.28	1.51

indicates that, in spite of their totally differing long-term behavior, the Keyfitz and Rogers models do not show well-marked differences in the pace of urbanization over the first 25 years. For example, after 25 years, the percentage of the population that is urban, with the Rogers model, is 26.1 percent in the case of India and 67.6 percent in the case of the U.S.S.R., whereas the comparable figures obtained with the Keyfitz model are 27.1 and 73.2 percent, respectively. As expected, since the Rogers model implies a continuous decrease of the rural net outmigration rate, it leads to a slightly slower urbanization process than the Keyfitz model.

What is the shape of the curve describing the variations of the ratio $S(t)$ of the urban to rural populations? First, it is clear from the values of the basic parameters that the curve $S(t)$ associated with the actual systems considered above does not admit a point of inflection in the case of India but admits one in the case of the U.S.S.R. In the latter case, the question is then one of knowing if the point of inflection occurs before or after the time at which the hypothetical population system presents the same characteristics as the observed one. Clearly, the answer to this follows from the relative values of t_D and t_S. From a comparison of (32) and (48), it follows that t_D is smaller (greater) than t_S if $\bar{S}$ is smaller (greater) than the half sum of S_A and S_B, i.e.,

$$\bar{S} \underset{(<)}{>} \frac{u - o_u - (r - o_r)}{2 o_u} \tag{50}$$

In the case of the hypothetical population system of the U.S.S.R., t_D is greater than t_S. Consequently, the urbanization process of both India and the U.S.S.R. on the basis of the Rogers model implies a continuous slowing down

of the growth rate of the urbanization index $S(t)$ after the observed period.

How do the urban and rural populations evolve over time? The urban population monotonically increases toward $+\infty$ in both cases (Table 6). The rural population monotonically increases in the case of India, whereas it first decreases, passes through a minimum, and then increases indefinitely in the case of the U.S.S.R.

Finally, we note the continuous decline of the rural net outmigration rates which, as expected, take on small long-term values. In the Indian case, m decreases from 4.35 per thousand to about one-sixth of this value (0.75 per thousand), while in the case of the U.S.S.R., it decreases from 20.9 per thousand to one-thirtieth of this value (1.5 per thousand).

SENSITIVITY ANALYSIS

What is the impact of small changes in the basic parameters on the level of the long-term equilibrium? Differentiating S_A with respect to the urban outmigration rate leads to

$$\frac{dS_A}{do_u} = \frac{o_r - o_u S_A}{o_u[(u - o_u - (r - o_r)^2 + 4o_r o_u]^{1/2}} - \frac{S_A}{o_u} \tag{51}$$

an expression which, it can be shown, is always negative. As expected, a higher urban outmigration rate tends to reduce the equilibrium urbanization level. As shown in Table 7, an immediate increase of the urban outmigration rate by one point leads to a decline of the long-term percentage α_A of the Indian population that is urban from 36.7 percent to 35.6 percent. Table 7 displays values of α_A corresponding to a set of various values of o_u. A value of the urban gross migration rate as small as $o_u = 0.001$ implies a rather large value of α_A (83.7 percent) while a value of o_u, two and a half times the initial value, leads to a quasi-stationary system: α_A reaches 20.4 percent versus the initial 19.7 percent. Indeed, if there is no migration from the urban to rural areas, the model becomes the Keyfitz model as the percentage of the population that is urban tends toward a hundred percent.

A change in o_u has a sensible impact not only on the long-term urban proportion but also on the urban proportion of the years following the initial period (see Table 7 which displays the values of the urban proportion 50 years hence for various values of o_u).

Differentiating S_A with respect to the rural outmigration rate leads to:

$$\frac{dS_A}{do_r} = \frac{S_A + 1}{\{[u - o_u - (r - o_r)]^2 + 4o_r o_u\}^{1/2}} \tag{52}$$

Clearly, this derivative is always positive, which shows that a higher rural outmigration rate tends to increase the urbanization level at equilibrium. As shown in Table 8, an immediate increase of the rural outmigration rate by

TABLE 7 The Rogers model: impact of the urban outmigration rate on the percentage of the Indian population that is urban ($r - u = 2 \times 10^{-3}$; $o_r = 6.8 \times 10^{-3}$).

o_u'	0.001	$o_u/2$	o_u	$o_u + 0.001$	$3o_u/2$	$2o_u$	$5o_u/2$
$\alpha(+50)$	39.75	35.06	30.20	29.34	26.22	22.94	20.24
α_A	83.68	53.41	37.68	35.63	29.29	24.01	20.36

TABLE 8 The Rogers model: impact of the rural outmigration rate on the percentage of the Indian population that is urban ($r - u = 2 \times 10^{-3}$; $o_u = 10.0 \times 10^{-3}$).

o_r'	0.001	$o_r/2$	o_r	$o_r + 0.001$	$2o_r$	$3o_r$	$4o_r$
$\alpha(+50)$	14.16	21.28	30.20	32.59	44.63	55.50	63.72
α_A	7.79	22.75	37.68	41.11	55.63	65.62	72.04

0.001 leads to a rise in the long-term urban proportion in India from 37.7 percent to 41.1 percent. Table 8 also displays values of α_A corresponding to a set of various values of o_r; observe that the doubling of o_r leads to a 55.6 percent equilibrium while its quadrupling yields a 72.0 percent equilibrium. In the case of there being no migration from the rural to urban areas, the model would become a model polar to that of Keyfitz in that the population would become predominantly rural.

Finally, differentiating S_A with respect to the rural–urban natural increase differential yields

$$\frac{dS_A}{d(r-u)} = -\frac{S_A}{\{[u - o_u - (r - o_r)]^2 + 4o_r o_u\}^{1/2}} \tag{53}$$

so that an immediate decrease in $r - u$ brings about a higher urbanization level.

Table 9 shows the values of α_A corresponding to some likely values of $r - u$. The impact of plausible changes in $r - u$ is to remain relatively modest since a 4 per thousand decline leads to an increase of α_A from 37.7 percent to only 43.4 percent.

Thus, with regard to the relative impacts of changes in the natural increase and migration regimes, the Rogers model leads to conclusions similar to those obtained with the Keyfitz model: variations in the migration regimes have a larger influence on the pace of urbanization than variations in the fertility–mortality regimes.

TABLE 9 The Rogers model: impact of the rural–urban natural increase differential on the percentage of the Indian population that is urban ($o_r = 6.8 \times 10^{-3}$; $o_u = 10 \times 10^{-3}$).

$r - u$	−0.002	−0.001	0	0.001	0.002	0.003	0.004
$\alpha(+50)$	32.86	32.18	31.51	30.85	30.20	29.56	28.94
α_A	43.40	41.93	40.48	39.06	37.68	36.35	35.06

III   THE UNITED NATIONS MODEL

Very recently, the Population Division of the United Nations (1979) proposed a model of urbanization extending the Rogers model in the direction of realism: gross outmigration rates and natural increase rates are allowed to vary. This extension is presented here in two stages: first, we introduce gravity-type migration flows and, second, we add declining urban—rural natural increase differentials.

INTRODUCING GRAVITY-TYPE MIGRATION FLOWS

As an alternative to constant outmigration rates from rural and urban sectors, the United Nations assumes that the probability of moving from one sector to the other is a linear function of the proportion of the total population that is located in that other sector, i.e.,

$$o_r(t) = i + j \frac{P_u(t)}{P_r(t) + P_u(t)} \tag{54}$$

and

$$o_u(t) = k + l \frac{P_r(t)}{P_r(t) + P_u(t)} \tag{55}$$

in which all coefficients are constants.

It follows that $o_r(t)$ and $o_u(t)$ are simple functions of the ratio of urban to rural population:

$$o_r(t) = i + j \frac{S(t)}{1 + S(t)} \tag{56}$$

and

$$o_u(t) = k + l \frac{1}{1 + S(t)} \tag{57}$$

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Are there any *a priori* restrictions regarding the coefficients i, j, k, and l? First, i and k are positive so that $o_r(t)$ and $o_u(t)$ are always positive. Second, j is assumed to be positive because it is likely that the gross outmigration rate from the rural sector increases as the urban proportion increases. By contrast, there is no *a priori* sign for the parameter l in the urban gross outmigration rate equation: l is positive (negative) if $o_u(t)$ declines (increases) over time.

Note that

$$o_r(t) = i + j - j \frac{1}{1 + S(t)} \tag{58}$$

Hence, the comparison of (57) and (58) suggests that

$$j > |l| \tag{59}$$

because urban outmigration rates are generally regarded as being less sensitive to changes in socio-economic conditions than rural outmigration rates.

Recalling eqs. (21) and (22) in which o_r and o_u are now time-dependent, we obtain the result that the growth rates of the two populations are still simple functions of $S(t)$:

$$\frac{dP_r(t)}{P_r(t)dt} = r - i + kS(t) + (l - j)\frac{S(t)}{1 + S(t)} \tag{60}$$

and

$$\frac{dP_u(t)}{P_u(t)dt} = u - k + \frac{i}{S(t)} + (j - l)\frac{1}{1 + S(t)} \tag{61}$$

Substituting (57) and (58) into (3) and rearranging terms then yields the following differential equation in $S(t)$

$$\frac{dS(t)}{dt} = i + [(u - l - k) - (r - j - i)]S(t) - k[S(t)]^2 \tag{62}$$

This last equation has exactly the same functional form that was derived in the case of constant gross outmigration rates (Riccati equation). The only differences are that:

- the constant terms in $o_r(t)$ and $o_u(t)$, i and k, respectively, are substituted for o_r and o_u;
- the constant rates of natural increase r and u are replaced by $r - j$ and $u - l$, respectively.

The main consequence of this observation is that the above model leads to a pattern of urbanization similar to that of the Rogers model. The ratio of

urban to rural populations $S(t)$ and the percentage $\alpha(t)$ of the population that is urban are given by formulas similar to (29) and (34) respectively. S_A and S_B are now replaced by S_A' and S_B' which also have opposite signs:

$$S_A' = \frac{(u-l-k)-(r-j-i)+\{[(u-l-k)-(r-j-i)]^2+4ik\}^{1/2}}{2k} \quad (63)$$

whereas S_B' is identical to S_A' except that the sign preceding the square root term is a minus instead of a plus. (Note that the existence of these two roots of opposite signs follows from the assumption that both i and k are positive.)

By contrast to the evolution of $S(t)$ and $\alpha(t)$, the evolution of the rural and urban populations is not easily obtained. Only in the case of the urban population can we derive interesting results. Differentiating (61) with respect to time leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}P_u(t)}{P_u(t)\mathrm{d}t}\right) = -\left[\frac{i}{[S(t)]^2}+\frac{j-l}{[1+S(t)]^2}\right] \quad (64)$$

Because of inequality (59), the right-hand side of (64) is negative and the urban growth rate thus monotonically decreases from $+\infty$ to its long-term value. It follows that, as in the Rogers model, the urban population either increases monotonically toward $+\infty$ or increases and then decreases toward zero. Indeed, only the first case is of interest to us: it corresponds to the situation in which substituting S_A' for $S(t)$ in (60) or (61) yields a positive value, i.e.,

$$kS_A'^2+(r+k+l-i-j)S_A'+r-i>0 \quad (65)$$

The adoption of this restriction (65) — replacing the restriction (40) of the Rogers model — thus allows the urban population to increase monotonically toward $+\infty$. Because the model admits a long-term equilibrium, the rural population also becomes infinite as t increases but its variations are not necessarily simple over the time continuum.

Summarizing the above results, we could conclude that the United Nations model does not significantly differ from the Rogers model. However, this statement is proved wrong by the evolution of the rural net outmigration rate.

From (56) and (57), we have

$$m_r(t) = o_r(t)-o_u(t)S(t) = i-kS(t)+(j-l)\frac{S(t)}{1+S(t)} \quad (66)$$

Differentiating this expression with respect to time leads to

$$\frac{\mathrm{d}m_r(t)}{\mathrm{d}t} = \left[-k+\frac{j-l}{[1+S(t)]^2}\right]\frac{\mathrm{d}S(t)}{\mathrm{d}t} \quad (67)$$

Consequently, the rural net outmigration rate does not necessarily decrease monotonically as in the Rogers model. Its evolution is as follows, according to the parameter values:

(a) if $j - l > k(1 + S_A')^2$, $m_r(t)$ monotonically increases;

(b) if $k < j - l < k(1 + S_A')^2$, $m_r(t)$ increases, passes through a maximum and then decreases;

(c) if $j - l < k$, $m_r(t)$ monotonically decreases.

Thus, for some adequate parameter values (case (b)), the United Nations model may allow for an evolutive pattern of rural—urban migration which resembles the historical trend observed in today's developed nations.

The above model is also applicable to actual rural—urban systems as long as the observed urban to rural populations $\bar{S}$ is less than the quantity S_A', calculated from the model parameters using (63). We have simulated the evolution of the two population systems of India and the U.S.S.R. assuming that the constant terms appearing in the gross migration rate equations are equal to half the value of the corresponding observed rates:

$$i = o_r/2 \quad k = o_u/2$$

As indicated in Table 10, the urban proportions tend toward larger equilibrium values than in the case of constant gross migration rates: α_A reaches 65.0 percent instead of 37.7 percent (for India) and 85.1 percent versus 75.3 percent (for the U.S.S.R.). Indeed, this larger urbanization level is due to increasing rural outmigration rates and decreasing urban outmigration rates; in the Indian case o_r rises from 6.8 to 14.6 per thousand while o_u declines from 10.0 to 7.2 per thousand.

Nevertheless, in both cases, the rural net outmigration rate monotonically decreases over the simulation period. Note that, in the Indian case, the parameters are such that, in the corresponding hypothetical population system, $m_r(t)$ increases and then decreases. However, the maximum reached after 42 years is slightly higher than the observed value: 4.58 versus 4.35 per thousand; this explains why $m_r(t)$ appears to be quasistationary over the first hundred years of the simulation period (Table 10).

The values of j and l, implied by the above assumption concerning i and k, are, in the Indian case, equal to 0.0154 and 0.0065, respectively. But how sensitive is the model to changes in these migration multipliers? For that purpose, we have simulated the Indian system by keeping j (or l) constant — and equal to the value just derived — and by letting l (or j) vary. On the one hand, Table 11 indicates that if the urban migration multiplier is kept constant, the long-term urban proportion increases from 40.8 percent (when $j = 0$) to 74.9 percent (if j is increased by 50 percent). On the other hand, Table 12 shows the dependence of the long-term urban proportion on the urban migration multiplier if the rural migration multiplier is kept constant: it decreases from 75.9 percent (when $l = 9.75 \times 10^{-3}$) to 44.5 percent (when $l = -13.0 \times 10^{-3}$).

TABLE 10   The United Nations model (stage 1): application to India and the U.S.S.R. ($i = o_r/2$; $k = o_u/2$).

India						U.S.S.R.				
S	α (percent)	o_r (per thousand)	o_u (per thousand)	m (per thousand)	T	S	α (percent)	o_r (per thousand)	o_u (per thousand)	m (per thousand)
0.245	19.70	6.80	10.00	4.35	0	1.291	56.35	35.00	11.00	20.80
0.270	21.28	7.07	9.90	4.40	5	1.524	60.38	36.25	10.49	20.26
0.296	22.83	7.34	9.81	4.44	10	1.771	63.91	37.35	10.05	19.55
0.376	27.31	8.11	9.53	4.53	25	2.567	71.96	39.85	9.03	16.66
0.518	34.14	9.29	9.10	4.57	50	3.849	79.38	42.15	8.10	10.98
0.823	45.15	11.19	8.42	4.27	100	5.286	84.09	43.62	7.50	3.95
1.361	57.64	13.35	7.64	2.96	200	5.714	85.11	43.93	7.38	1.78
1.831	64.68	14.59	7.19	1.33	500	5.729	85.14	43.94	7.37	1.70
1.853	64.95	14.61	7.18	1.30	1,000	5.729	85.14	43.94	7.37	1.70

TABLE 11 The United Nations model (stage 1): impact of variations in the rural outmigration multiplier on the long-term equilibrium of the Indian population ($l = 6.5 \times 10^{-3}$).

j ($\times 10^{-3}$)	0	3.85	7.70	11.55	15.40	19.25	23.10	26.95	30.80
α'_A		40.79	45.48	51.49	58.34	64.95	70.52	74.91	78.32	80.98
$o_r(\infty)$	6.80	7.91	9.54	11.80	14.61	17.76	21.10	24.50	27.95
$o_u(\infty)$	8.69	8.39	8.02	7.59	7.18	6.84	6.56	6.35	6.18

TABLE 12 The United Nations model (stage 1): impact of variations in the urban outmigration multiplier on the long-term equilibrium of the Indian population ($j = 15.4 \times 10^{-3}$).

l ($\times 10^{-3}$)	−13.00	−9.75	−6.50	−3.25	0	3.25	6.50	9.75
α'_A	44.52	46.47	48.86	50.27	53.69	58.29	64.95	75.94
$o_r(\infty)$	11.08	11.42	11.83	12.08	12.67	13.46	14.61	16.51
$o_u(\infty)$	12.70	12.08	11.36	10.95	10.00	8.80	7.18	4.75

The conclusion here is that the level of urbanization at equilibrium is heavily dependent on the values of the rural and urban migration multipliers, j and l, respectively. However, the urbanization path is similar in all cases and is, as shown earlier, germane to that offered by the Rogers model.

ADDING DECREASING URBAN–RURAL NATURAL INCREASE DIFFERENTIALS

In a second stage, the United Nations model allows for decreasing urban and rural rates of natural increase; however, it assumes that the urban–rural differential remains constant, in which case the urbanization process is identical to that obtained in the case of constant natural increase rates in both areas. Here, we suppose that both rural and urban natural increase rates are linearly decreasing with the ratio $S(t)$ of the urban to rural populations, but at a different rate:

$$r(t) = a - bS(t) \tag{68}$$

$$u(t) = c - dS(t) \tag{69}$$

where b and d are positive coefficients. Subtracting (69) from (68) leads to

$$r(t) - u(t) = a - c - (b - d)S(t) \tag{70}$$

TABLE 13 The United Nations model (stage 2): application to India ($i = o_r/2; j = o_u/2; f = 0.01$).

T	S	α (percent)	o_r (per thousand)	o_u (per thousand)	m (per thousand)	$r - u$ (per thousand)
0	0.245	19.70	6.80	10.00	4.35	2.00
5	0.270	21.28	7.07	9.90	4.40	1.97
10	0.296	22.83	7.34	9.81	4.44	1.95
25	0.376	27.34	8.12	9.52	4.53	1.87
50	0.522	34.28	9.32	9.09	4.57	1.72
100	0.842	45.71	11.29	8.38	4.23	1.39
200	1.463	59.44	13.66	7.53	2.63	0.76
500	2.148	68.23	15.18	6.98	0.19	0.06
1,000	2.193	68.68	15.25	6.95	0.01	0.02

TABLE 14 The United Nations model (stage 2): impact of variations in the natural increase multiplier on the long-term equilibrium of the Indian population.

f (per thousand)	0	2.0	4.0	6.0	8.0	10.0	12.0	14.0
α_A	64.95	65.64	66.35	67.10	67.87	68.68	69.53	70.42
$o_r (\infty)$	14.61	14.73	14.85	14.98	15.11	15.25	15.40	15.55
$o_u (\infty)$	7.18	7.14	7.10	7.05	7.00	6.95	6.90	6.84
$r - u (\infty)$	2.00	1.66	1.30	0.90	0.48	0.02	−0.49	−1.05

a relationship which shows that we necessarily have

$$f = b - d > 0 \tag{71}$$

if we suppose that the rural—urban differential in natural increase rates declines as the urban proportions rise.

Substituting (68) and (69) for r and u, respectively, into (62) yields

$$\frac{dS(t)}{dt} = i + [(c - l - k) - (b - j - i)] S(t) - (k - f)[S(t)]^2 \tag{72}$$

a new differential equation in $S(t)$ which still has the same functional form as the differential equation obtained with the Rogers model.

Typically, $f = b - d$ is expected to be small so that, in most current applications, the discriminant of the right-hand side of (72) is positive.

Thus, the introduction of a declining rural—urban differential in natural increase does not radically affect the pattern of urbanization which still remains similar to that of the Rogers model. Table 13 displays the evolution of the urban

proportion in the Indian system; in case (a) the natural increase multiplier f is chosen equal to 0.01, and in (b) the migration flows are described by a gravity model with $i = o_r/2$ and $k = o_u/2$. The long-term urban proportion appears to be equal to 68.7 percent versus 65.0 percent for the case $f = 0$ (i.e., $r - u$ remains constantly equal to its observed value).

Selected values of α_A corresponding to various values of f between 0 and 14 per thousand appear in Table 14. Thus, as the preceding results obtained by changing $r - u$ instantaneously could suggest, declining natural increase differentials between the urban and rural sectors have a rather small impact on urbanization indices such as S_A or α_A.

CONCLUSIONS

This paper has sought to examine analytically the relation between the urbanization phenomenon and the demographic parameters which affect it. In the process, many interesting conclusions have been drawn which concern the three alternative models used in the course of our investigations.

First, we have shown that the Keyfitz model (Keyfitz 1978) implies an urban to rural population ratio which increases exponentially over time and a proportion urban which increases monotonically (with a curve of variations shaped downward); it is a logistic function of time only if the rural rate of natural increase is larger than the urban one. However, the Keyfitz model appears of limited application because of:

(a) its assumption of fixed rural net outmigration rate;

(b) its asymmetric treatment of the migration flows between the rural and urban sectors which, in the long run, leads to some undesirable features such as the preponderance of the urban region and the possible emptying out of the rural region.

Second, we have shown that the continuous version of the two-region Rogers model (Rogers 1968) implies an urban to rural population ratio as well as a proportion urban which are described by a truncated logistic curve (with possibly the presence of a point of inflection in the case of the first index). Also, the Rogers model seems to be a more useful tool than the Keyfitz model to examine the urbanization phenomenon. Its more symmetric treatment of the rural and urban outmigration flows leads, in the long run, to more reasonable features: it admits a long-term equilibrium in which the rural and urban populations grow at the same rate. However, because it implies a continuous decline of the rural net outmigration rate (with a possible reversal in the direction of the rural—urban migration transfer), the Rogers model appears to be applicable only to nations which have already reached a certain level of economic development.
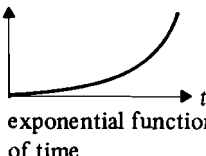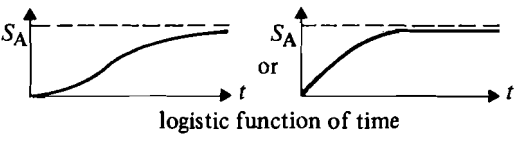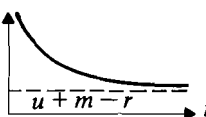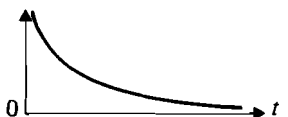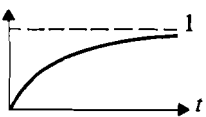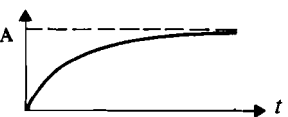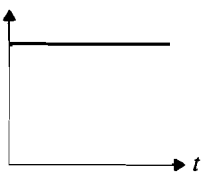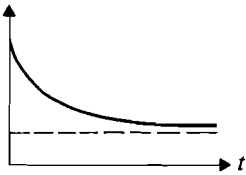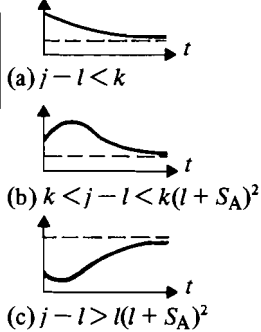
Third, we have shown that, although it relies on well-defined hypotheses

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(constant natural increase and gross outmigration rates in both rural and urban sectors), the Rogers model is quite general in form. As suggested by the United Nations (1979), various assumptions concerning the migration and natural increase regimes — e.g., gravity-type migration flows and natural increase rates declining linearly with the urban to rural population ratio — do not alter the pattern of urbanization stemming from the Rogers model. The only difference is that, for an adequate choice of the model parameters, the variations of the rural net outmigration rate may replicate the historical variations observed in today's developed nations: increase up to a maximum and then decrease.

The above findings concerning the comparative dynamics of the three alternative models are summarized in Table 15. Besides these findings, this paper has also permitted the derivation of interesting results about the relation between economic development and urbanization. We have shown that the former influences the latter through the rural—urban natural increase differential and the migration exchange between the two sectors, in such a way that both these factors have a direct (positive) impact on urbanization; however, the impact due to the natural increase factor is much less important. An important consequence of this is that, from a modeling point of view, a refining of the natural increase functions is not so rewarding as a realistic treatment of the migration function(s). Thus, a general strategy when building an urbanization model might be to suppose identical rural and urban natural increase rates — which considerably simplify the analytics (Keyfitz 1978) or ensure mathematical tractability (Ledent 1978c) — and to concentrate on the specification of the rural—urban migration exchange.

From a practical point of view, this paper has presented several numerical illustrations which have provided us with several interesting conclusions regarding the future urbanization process of India and the U.S.S.R. Perhaps the most significant one is that India is bound to remain a predominantly rural country for quite a while. For example, assuming an unchanged urban outmigration rate, the occurrence of a 50-percent urban proportion 50 years hence requires a sustained rural outmigration rate equal to 2.5 times its current value (see Table 8).

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TABLE 15 Comparative dynamics of the three alternative models: a tabular summary.

	The Keyfitz Model	The Rogers Model	The United Nations Model
Restrictions	$u + m - r > 0$ $u > 0; m > 0$	$S_A \geqslant \dfrac{o_r - r}{o_u} (uS_A + r \geqslant 0)$	eq. (65)
Long-term behavior	Urban population preponderant with rural population possibly vanishing	Rural—urban equilibrium	
$S(t)$	exponential function of time	logistic function of time	
$\dfrac{dS(t)}{S(t)dt}$	$\overline{u + m - r}$	0	
$\alpha(t)$	logistic function of time if $r > u$	logistic function of time	
$m(t)$			(a) $j - l < k$ (b) $k < j - l < k(l + S_A)^2$ (c) $j - l > l(l + S_A)^2$

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