AN APPLICATION OF NONDIFFERENTIABLE OPTIMIZATION IN OPTIMAL CONTROL

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FOREWORD

How to solve dynamic discrete-time optimization problems is important in applied systems analysis. For example, such problems occur in analyses of future energy supply, forest industry development, and agricultural production.

Thus, IIASA has investigated methods of solving such problems, particularly when they have a large scale and involve decomposition.

This paper presents a new approach to these problems of dynamic discrete-time optimization that is based on the techniques of nondifferentiable optimization. Specifically, this approach first represents the nonlinear discrete-time state equations by means of an exact penalty function, then decomposes the resulting large-scale problem, and finally uses nondifferentiable optimization algorithms to coordinate the solutions to the subproblems.

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Numerical Optimisation of Dynamic Systems
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E.A. Nurminski

The problem of optimal control for the nonlinear dynamic system with discrete time is considered. Using a nondifferentiable penalty function it is possible to transform the initial problem into an unconditional one. Special structure of this problem makes it possible to develop the specific method which is some composition of the gradient-like method of nondifferentiable optimization and the method of coordinate minimization.

1. INTRODUCTION

We consider here the problem of optimal control of the system which is governed by the equations

$$x(t+1) = g(x(t),u(t)), t = 0,1,..,T-1 \tag{1}$$

where $x(t)$ is the phase vector of the system at some discrete instant t, $u(t)$ is the corresponding control vector. Both $x(t)$ and $u(t)$ are the elements of the finite dimensional spaces E^n and E^m respectively. In x-space E^n we shall sometimes use the norm

$$|X| = \max_{i=1,..,n} |X_i| \tag{2}$$

preserving the usual notation for the euclidean norm

$$\| a \| = \left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} = (a,a)^{\frac{1}{2}}.$$

We shall consider the problem of optimal control in the sense that there is some objective function $f: E^n \to E^1$ and control variables are to be chosen in a way that a minimum

$$\min f(x(T)) \tag{3}$$

is achieved. We will not put any constraints on the control variables $u(t), t = 0,1,...,T-1$.

Problem (1),(3) has been studied by many authors and great efforts have been undertaken to develop computational procedures for this problem. Monographs [1,2,3,4] give examples of the development. The aim of this paper is to describe some ideas in optimal control which originate from nondifferentiable optimization - the special field in mathematical programming which deals with the problem of minimizing the functions which do not have existing derivatives everywhere. The close relationship between these two problems stems from the fact that under some reasonable conditions the problems (1),(3) and the problem of minimizing the function

$$\Phi_\lambda(x,u) = f(x(T)) + \lambda \max_{t=0,1,\ldots,T-1} |x(t+1) - g(x(t),u(t))| \tag{4}$$

over the variables

$$x = (x(1),\ldots,x(T)) \in E^{nT}$$

$$u = (u(0),\ldots,u(T-1)) \in E^{mT}$$

are equivalent for λ large enough.

A proof of this fact is given in Section 2 of the present paper.

This equivalence opens, at least in principle, the possibility of using the methods of nondifferentiable optimization [5,6] to solve (1),(3). A direct application of these methods is hampered, however, by the size of the resulting problem: the total number of variables in the problem

$$\min_{x,u} \Phi_\lambda(x,u) \tag{5}$$

is equal to $(n+m)T$ a possibly huge number even for low-dimensional systems.

Therefore, the second idea consists of using the specific structure of the function (4) and it gives the possibility of taking into account the equations (1)and excluding the state variables from the problem (5) as independent ones.

Finally, the iterative procedure for finding the optimal control u^* is a modification of subgradient search which is widely used in nondifferentiable optimization for the different kinds of nondifferentiable functions. The corresponding statement on the convergence of this method is given in Section 4.

The numerical aspects of this approach are considered in Section 5. Within the framework of the method an auxiliary extremum problem appears and the solution to this problem satisfies a specific system of linear equations. Remarkably, this solution can be found in a dynamic-like way.

2. THE CONDITIONS FOR THE EQUIVALENCE OF THE OPTIMAL CONTROL PROBLEM AND THE NONDIFFERENTIAL EXTREMUM PROBLEM

A well-known method of solving conditional extremum problems is the penalty function approach which reduces the problem

$$\begin{aligned} &\min f(x) \\ &g_i(x) \leq 0 \ , \ i = 1,2,\ldots,I \end{aligned} \tag{6}$$

to the unconditional one

$$\min_x \{f(x) + \psi_\lambda(g_i(x),\ i=1,..,I)\} \tag{7}$$

The different aspects of this approach are discussed for instance in (7). Recently, due to the development of nondifferentiable optimization methods, the nondifferentiable penalty functions became the object of many studies. A great advantage of this function is the possibility of constructing for the conditional problem (6) the exactly equivalent unconditional problem (7). Equivalence here means that the solution of one problem is the solution to the other and vice versa.

A typical example of such penalty functions is

$$\psi_\lambda(g_i(x),\ i=1,2,..,I) = \lambda \max_{i=1,.,I} g_i^+(x)$$

where

$$g_i^+(x) = \max\{0, g_i(x)\}$$

positive part of $g_i(x)$

For this and similar penalty functions it was proved by many authors that under some mild conditions for $\lambda > 0$ large enough every solution of (7) is a solution of (6). There were also many attempts to estimate the lower bounds for such λ. Usually such estimates involve values of dual variables or Lagrange multipliers at the extremum point. Fortunately in our case it is possible to derive more useful and constructive bounds.

Let us denote for fixed control u

$$x_u = (x_u(1), x_u(2), \ldots, x_u(T))$$

the state variables which satisfy the equations (1) with given initial state x(0). We start by establishing the following result.

Theorem 1. If in the region $Z \times V \subset E^n \times E^m$ $g(z,v)$ satisfies a Lipschitz condition on z with the constant L, i.e.

$$|g(z',v) - g(z'',v)| \leq L|z' - z''|$$

for all $v \in V$, $z',z'' \in Z$ and f(z) satisfies in Z a Lipschitz condition with the constant M respectively then for

$$\lambda > M(L^T - 1)(L - 1)^{-1}$$

and

$$u \in V^T = \mathop{X}_{i=1}^{T} V$$

$$\min_x \Phi_\lambda(x,u) = \Phi_\lambda(x_u,u) = f(x_u(T))$$

Proof. First we estimate the distance between x(T) and $x_u(T)$ in a recursive way. For some instant t

$$|x_u(t+1) - x(t+1)| = |g(x_u(t),u(t)) - x(t+1)| \leq$$

$$\leq |g(x_u(t),u(t)) - g(x(t),u(t))| + |x(t+1) - g(x(t),u(t))| \leq$$

$$\leq L\,|x_u(t) - x(t)| + \max_{t=0,1,..,T-1} |x(t+1) - g(x(t),u(t))|$$

Let us denote

$$|x_u(t) - x(t)| = \Delta_t$$

$$\max_{t=0,1,..,T-1} |x(t+1) - g(x(t),u(t))| = \gamma$$

Then in this notation

$$\Delta_{t+1} \leq L\,\Delta_t + \gamma \qquad (8)$$

Inequality (8) together with the initial condition $\Delta_0 = 0$ establishes an upper bound for Δ_t. It is easy to verify that

$$\Delta_T \leq \gamma \sum_{t=0}^{T-1} L^t = \gamma(L^T - 1)(L - 1)^{-1}$$

Now the completion of the proof presents no difficulties:

$$\Phi_\lambda(x,u) - \Phi_\lambda(x_u,u) = f(x(T)) - f(x_u(T)) + \lambda\gamma \geq$$

$$\geq -M\,|x(T) - x_u(T)| + \lambda\gamma \geq \gamma(\lambda - M(L^T - 1)(L - 1)^{-1}) \geq 0$$

Q.E.D.

It immediately follows from the theorem that

$$\min_u f(x_u(T)) = \min_u \min_x \Phi_\lambda(x,u) = \min_{x,u} \Phi_\lambda(x,u)$$

and hence the problem (1),(3) and (5) are equivalent.

We shall suppose that the conditions of this theorem are valid throughout the paper and that λ is large enough for the conclusions of the theorem to hold.

3. SOME DISCRETE MIN-MAX THEORY

Problem (5) is a problem of nondifferentiable optimizations because of specific features of the max operation. It is worth remarking that function (4) in general does not satisfy convexity conditions in the nonlinear case. There are many classes of nondifferentiablity. The collection[6] gives several examples of these types of nondifferentiabilty. Another example of the different classes on nondifferentiable functions are [8 - 11] and this list may be extended.

A sufficiently general class of nondifferentiable functions that contains under rather mild conditions the function (4) is a class of weakly convex functions as it was defined in [9]. A similar class of nondifferentiable functions was considered also in [8].

The definition of the weakly convex function is given below.

Definition. A continuous function $f(x)$ is said to be the weakly convex function if for any fixed x a nonempty set $G(x)$ of vectors g exists such that for every y

$$f(y) - f(x) \geq (g, y - x) + r(x,y) \tag{9}$$

where the residual term $r(x,y)$ has uniform smallness with respect to the difference $\|x - y\|$, that is in every compact set K

$$r(x,y)\ \|x - y\|^{-1} \to 0 \tag{10}$$

when $\|x - y\| \to 0$, $x, y \in K$ 1/. The vector g in the equality (9) we shall call subgradient of the weakly convex function $f(x)$ by analogy with convex functions.

The class of weakly convex functions has a remarkable property that it is closed under the operation of finite maxima, i.e. if a finite family $f_i(x)$, $i \in I$ of the weakly convex functions is given then the function

$$f(x) = \max_{i \in I} f_i(x) \tag{11}$$

1/ Note: $\|x - y\| \to 0$ does not mean that $y \to x$ for some fixed x .

is also a weakly convex function. So far as the problem (1),(3) is concerned the differentiability of the right parts of the equations (1) and the weak convexity of the objective function f will guarantee the weak convexity of the function (4). It allows the consideration of the sufficiently general class of dynamic system (1), (3) and guarantees the wide applicability of the theoretical developments. Another advantage is that for the weakly convex functions the convergence of the gradient-like methods has already been proved in [9].

An important question which immediately arises is the calculation of subgradients g which satisfy the inequality (9). In the general case this problem is rather difficult but for the special classes of functions it is possible to develop some kind of differential calculus and get some constructive rules for computing the subgradients. Particularly important is the class of maximum functions (11). In this case it is easy to show that if $G_i(x)$ is a subgradient set for the function $f_i(x)$ at the point x then subgradient set G(x) for the function f at the same point is given by the expression

$$G(x) = co\{G_i(x),\ i \in I(x)\},\ I(x) = \{i: f_i(x) = f(x)\}$$

Taking into account that for smooth functions a subgradient coincides with the usual gradient it defines subgradients for a fairly large set of functions.

What we are going to do is to use the fact that the function $\Phi_\lambda(x,u)$ is a finite maximum of differentiable functions. In the sequel we shall assume that the objective f(x(T)) is continuously differentiable. Then the function $\Phi_\lambda(x,u)$ may be represented in the following way ($\phi_i(\cdot)$ is defined below):

$$\Phi_\lambda(x,u) = \max_{i=1,.,N} \phi_i(x,u)$$

and one can obtain some useful results on the structure and properties of the subgradient set G(x,u) .

For the notational simplicity we replace the pair $(x,u) \in E^{nT} x$ E^{mT} by $z \in E^{(n+m)T}$ which yields

$$\Phi_\lambda(z) = \max_{i=1,.,N} \phi_i(z) \tag{12}$$

and consider the set

$$Z^* = \{z^*: \phi_i(z^*) = \Phi_\lambda(z^*),\ i = 1,..,N\}$$

For the exact coincidence of (12) and (4) we may choose $N = 2nT$ and

$$\phi_k(z) = f(x(T) + \lambda(x_i(t+1) - g_i(x(t),\ u(t)))$$

for

$$k = nt + i\ ,\ i = 1,...,n \quad t = 0,1,..,T-1\ ;$$

$$\phi_k(z) = f(x(T)) - \lambda(x_i(t+1) - g_i(x(t),u(t)))$$

for

$$k = T - 1 = nt + i \qquad i = 1,..,n \qquad t = 0,1,..,T-1\ .$$

It is useful to notice that if $z \in Z^*$ then the state and control variables are linked by the dynamic equations:

$$x(t+1) = g(x(t),u(t)),\quad t = 0,1,..,T-1$$

Moreover for the $z \in Z^*$ the subgradient set $G(z)$ is given by the expression:

$$G(z) = \mathrm{co}\{\phi_i'(z),\ i = 1,..,N\}$$

It follows from the continuity of $\phi_i'(z)$ that $G(z)$ is continuous on Z^*. The continuity of the $G(z)$, $z \in Z^*$ in turn induces the continuity of the value of the problem

$$\max_{g \in G(z)} \min_{g' \in G(z')} (g,g') = v(z,z') \tag{13}$$

on the product set $Z^* \times Z^*$.

It is well known that the problem (13) for $z' = z$ has the same external solution g_* such that

$$\min_{g' \in G(z)} (g_*,g') = \max_{g \in G(z)} \min_{g' \in G(z^{\frac{1}{2}})} (g,g')$$

as the problem

$$\min_{g \in G(z)} \|g\|^2 \tag{14}$$

Due to the strict convexity of the problem (14) the solution g_* is a continuous function of z and if we denote by g_* the solution of (14) for $z = z'$ then

$$v(z,z') \geq \min_{g' \in G(z')} (g_*, g') = \min_{g' \in G(z')} (g_*' + g_* - g_*', g') \geq$$

$$\geq \min_{g' \in G(z')} (g_*', g') - \|g_* - g_*'\| \max_{g' \in G(z')} \|g'\| = V(z',z') -$$

$$- \|g_* - g_*'\| \max_{g' \in G(z')} \|g'\|$$

Let us now assume that $0 \bar{\in} G(z)$ and consequently $0 \bar{\in} G(z')$ for z' close enough to z. In that case we may suppose that

$$v(z,z) \geq 4\delta \ , \ v(z',z') \geq 2\delta$$

and

$$\min_{g' \in G(z')} (g_*, g') \geq 2\delta - \|g_* - g_*'\| \max_{g' \in G(z')} \|g'\| \geq \delta > 0$$

for z' close enough to z and $z' \in Z^*$. The latter means that there exists an $\varepsilon > 0$ such that

$$\min_{g' \in G(z')} (g_*, g') \geq \delta > 0 \qquad (15)$$

for

$$\|z - z'\| \leq \varepsilon \ , \ z' \in Z^*$$

We are also interested in developing an analogous inequality for all neighbors z' not only for $z' \in Z^*$. As G(z) for $z \in Z^*$ is in a certain sense a maximal set it is easy to prove that (15) holds for all z close to z.

We start our proof with the statement that for every $\gamma > 0$ exists such $\varepsilon > 0$ that for all $z' \in Z^*$ and for all z'' such that

$$\|z' - z\| \leq \varepsilon, \ \|z'' - z\| \leq \varepsilon$$

holds

$$G(z') + \gamma S \supset G(z'')$$

where S - unit ball.

In fact if we suppose the contrary then there exist such sequences

$$z'_n, z''_n \to z$$

where

$$z'_n \in Z^*$$

and there exists $g_n \in G(z''_n)$ such that

$$g_n \overline{\in} G(z''_n) + \gamma S$$

for some $\gamma > 0$.

Without any loss of generality we may assume that $g_n \to g''$ and for g'' two inclusions hold:

(i) $g'' \overline{\in} \text{int}(G(z) + \gamma S)$

It follows from (16) and continuity of G(z) on Z* and

(ii) $g'' \in G(z)$

which follows from uppersemicontinuity of G(z).

Obviously (i) and (ii) contradict each other which proves (15).

4. SUBGRADIENT-COORDINATE METHOD

On the basis of the results obtained above the following method for solving the problem (1),(3) may be developed:

STEP 0. Set some initial control variables

$$u^0 = (u^0(0),\dots,u^0(T-1))$$

and compute through (1) the corresponding trajectory of the dynamic system. The initial point $x^0(0)$ is given. The counter k of iterations set equal to zero: $k = 0$.

STEP 1. Compute for given u^k and x^k the subgradient of the function $\Phi_\lambda(x,u)$ defined by (4) at the point x^k,u^k :

$$g^k = (g_x^k, g_u^k)$$

STEP 2. Change the control variables u^k in accordance with the formula

$$u^{k+1} = u^k - \rho_k\, g_u^k$$

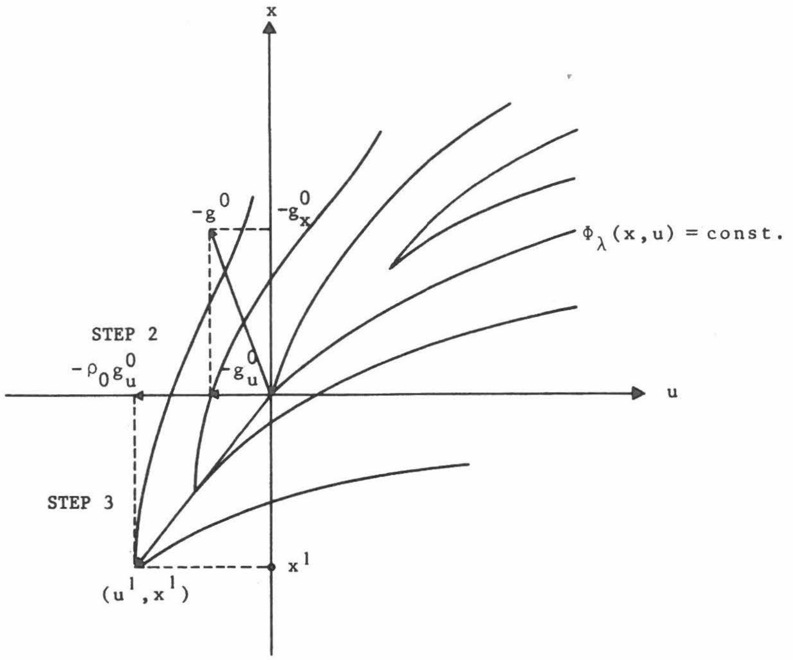
where ρ_k -step multipliers specified below.

STEP 3. Compute the new trajectory x^{k+1} from the equations (1) for $u = u^{k+1}$.

STEP 4. Set $k := k+1$ and go to STEP 1.

From the general point of view the proposed method is a composition of two well-known methods for which a convergence was proved before. In fact STEP 2 is a gradient-like method which has been investigated under the different assumptions. Due to Theorem 1, STEP 3 may be considered as a coordinate optimization method which also has well-known convergence properties. At this stage we compute the exact minimum of the function $\Phi_\lambda(x,u)$ with respect to x. Due to Theorem 1, this computation can be performed through dynamic equations (1) rather than by minimizing this function. This stage of the method is similar to the iteration of the cyclic coordinate optimization method, the classical version of which belongs to Gauss - Zaidel, where the sequential calculations of the exact minimum of the objective with respect to different variables are performed. In this case, special measures should be undertaken to guarantee the convergence of the whole process. Generally this process will not converge as it is shown on Figure 1 where the level sets of the function Φ_λ are drawn in the case of single dimensional x and u.

FIGURE 1



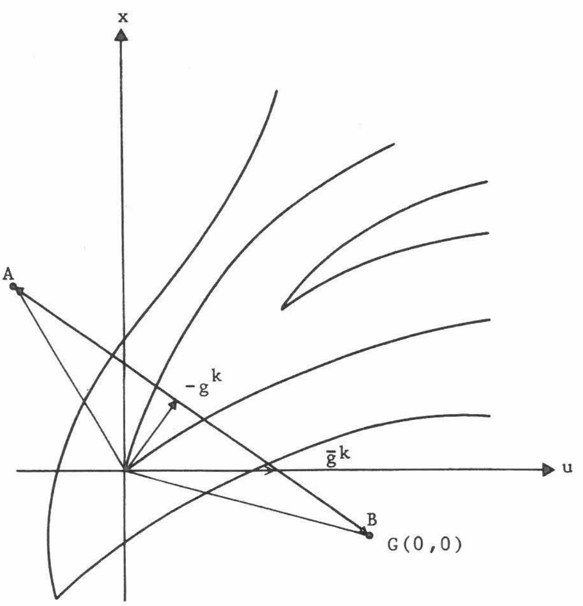
It is easy to see that for the given subgradient g^0, calculated at the origin any shift in the control variable u in the direction $-g_u^0$ will result in the transition to some point (x^1, u^1) where $u^1 = u^0 - \rho_0 g_u^0$ and x^1 satisfies the corresponding equations (1) for $u = u^1$ and the value of objective function

$$\Phi_\lambda(x^1, u^1) = f(x^1(T))$$

is larger than at the initial point. At the point (x^1, u^1) this situation may repeat and as a result we will have a sequence $\{u^k\}$ which does not converge to the solution of (1),(3).

This example also gives an answer to the question in what way the process Step 0 - Step 3 has to be modified. It is clear that we need to make a special choice of the subgradient g^k in the Step 1. For the above example the set of the subgradients G(0,0) calculated at the origin is plotted in Figure 2 as an interval A B .

FIGURE 2



There are many possible choices of the vector g^k which will guarantee the convergence of the process and some of them are already known. Remarkably, if we choose a vector g^k which has a zero x-component, namely:

$$\bar{g}^k = (0, g_u^k)$$

then the u-component g_u^k will coincide with the derivative of the objective $f(x_u(T))$ as a complex function of u and may be obtained in a traditional way with the help of the conjugate (or adjoint) system

$$\lambda(t-1) = g_x'(x_u(t), u(t))\,\lambda(t), \quad t = T, T-1, .., 1$$

with the terminal condition $\lambda(T) = -f_x'(x_u(T))$.

It is useful to notice that when x^k realizes

$$\min_x \Phi_\lambda(x,u^k)$$

then such a vector g^k always exists. This approach is widely known and thus we will not discuss here its advantages and disadvantages. Our aim is to propose some alternative for this choice and to express the hope that the future research will make comparison possible.

The idea is to choose vector $g^k \in G(x^k,u^k)$ in the Step 1 as a vector of the minimal length in this set that is

$$\|g^k\|^2 = \min_{g \in G(x^k,u^k)} \|g\|^2 \tag{17}$$

The corresponding choice of g from the set G(0) is shown on the Figure 2.

The convergence of the method is stated in the following theorem:

<u>Theorem 2</u>. In STEP 1 the subgradient g^k be chosen as (17), let the sequence of step multipliers satisfy the conditions

$$\rho_k \to +0, \ \sum \rho_k = \infty$$

and assume the sequence $\{(x^k,u^k)\}$ generated by this method to be bounded. Then every limit point of the sequence $\{u^k\}$ realizes the minimum of the function

$$f(x_u(T)) = \min_x \Phi_\lambda(x,u)$$

Proof. Consider the set (which might be empty)

$$U^* = \{u^*: \min_x \Phi_\lambda(x,u^*) = \min_{x,u} \Phi_\lambda(x,u)\}$$

We shall prove that all limit points of the sequence lie in this set. Because the sequence $\{u^k\}$ has a nonempty set of limit points the following is a constructive proof of nonemptiness of the set U^* as well.

First assume that the limit point u' of the sequence exists such that $u' \bar{\in} U^*$

$$0 \bar{\in} G(x_{u'}, u')$$

and consequently

$$0 \bar{\in} G(x^k, u^k) \tag{18}$$

for (x^k, u^k) which are close enough to (x'_u, u'). In fact it is enough to require the closeness of u' and u^k. If u' is a single limit point of the sequence $\{u^k\}$ then (18) holds for all k large enough and all (x^k, u^k) lie in an arbitary small neighborhood of the point (x'_u, u') for k large enough. In that case the results of the Section 2 are applicable which gives the possibility to estimate a decrease of the objective function:

$$\Phi_\lambda(x^{k+1}, u^{k+1}) \leq \Phi_\lambda(x^k - \rho_k g_x^k, u^k - \rho_k g_u^k) \leq$$

$$\leq \Phi_\lambda(x^k, u^k) - \rho_k(g^k, g) + r(x^k - \rho_k g_x^k, u^{k+1}; x^k, u^k)$$

where $g \in G(x^k - \rho_k g_x^k, u^{k+1})$ and the weak convexity of the function $\Phi_\lambda(x,u)$ was used. Under the suppositions of the proof an inequality similar to (15) holds

$$(g^k, g) \geq \delta > 0$$

for any g from the set $G(x^k - \rho_k g_x^k, u^{k+1})$ and for k large enough

$$r(x^k - \rho_k g_x^k, u^{k+1}; x^k, u^k) \leq \frac{\delta}{2} \rho_k$$

so

$$\Phi_\lambda(x^{k+1}, u^{k+1}) \leq \Phi_\lambda(x^k, u^k) - \frac{\delta}{2} \rho_k, k \geq K \tag{19}$$

Summing (19) from k> K to N-1 we get

$$\Phi_\lambda(x^N, u^N) < \Phi_\lambda(x^K, u^K) - \frac{\delta}{2} \sum_{s=K}^{N-1} \rho_s \to -\infty$$

when $N \to \infty$. This obviously contradicts the boundness of the Φ_λ and proves that the point u' is not a single limit point of the sequence $\{u^k\}$.

Let us denote the subsequence converging to u' as $\{u^{n_k}\}$. From above it follows that for some arbitrary small $\varepsilon > 0$ indices m_k exist such that

$$m_k = \min_{m > n_k} m \ : \ \|x^m - x^{n_k}\| > \varepsilon$$

Then for $n_k \leq m < n_k$ the (19) is valid and thus

$$\Phi_\lambda(x^{m+1}, u^{m+1}) \leq \Phi_\lambda(x^m, u^m) - \frac{\delta}{2}\rho_m \tag{20}$$

Again summing (20) for $n_k \leq m < m_k$ we get

$$\Phi_\lambda(x^{m_k}, u^{m_k}) \leq \Phi_\lambda(x^{n_k}, u^{n_k}) - \frac{\delta}{2} \sum_{n=n_k}^{m_k - 1} \delta_m$$

On the other hand

$$\varepsilon < \| \sum_{m=n_k}^{m_k - 1} \rho_m g^m \| \leq C \sum_{m=n_k}^{m_k - 1} \rho_m$$

which yields

$$\sum_{m=n_k}^{m_k - 1} \rho_m \geq \frac{\varepsilon}{C}$$

and finally

$$\Phi_\lambda(x^{m_k}, u^{m_k}) \leq \Phi_\lambda(x^{n_k}, u^{n_k}) - \frac{\delta\varepsilon}{2C}$$

Passing to the limit as $k \to \infty$ we get

$$\overline{\lim_{k\to\infty}} \Phi_\lambda(x^{m_k}, u^{m_k}) < \lim_{k\to\infty} \Phi_\lambda(x^{n_k}, u^{n_k}) \tag{21}$$

Inequality (21) guarantees the convergence of the process due to the general results of the convergence of iterative processes of non-linear programming obtained in [10].

5. COMPUTATIONAL ASPECTS

The application of the proposed method involves through Step 1 the solution of the problem

$$\min_{g \in G(x^k, u^k)} \|g\|^2 \tag{22}$$

This problem seems to be a large-scale quadratic problem but some considerations allow us to determine an explicit solution.

If we denote the full derivative of the

$$\psi_i^k(t) = x_i^k(t+1) - g_i(x^k(t), u^k(t))$$

in respect to the variables x^k, u^k as $\psi'_{ik}(t)$ and as $\psi'_k(t)$ - the (n+m) Txn matrix with the column $\psi'_{ik}(t)$, i=1,.,n the subgradient set $G(x^k, u^k)$ may be presented in a form

$$G(x^k, u^k) = f'(x^k(T)) + \lambda P_k$$

where the set P_k has a form

$$P_k = \{p : p = \sum_{t=0}^{T-1} \psi'_k(t)\theta^k(t)\}$$

where the n-vectors

$$\theta^k(t) = (\theta_1^k(t), \ldots, \theta_n^k(t))$$

may be represented as

$$\theta_i^k(t) = \alpha_i^k(t) - \beta_i^k(t)$$

and

$$\alpha_i^k(t), \beta_i^k(t) \geq 0$$

$$\sum_{t=0}^{T-1} \sum_{i=1}^{n} (\alpha_i^k(t) + \beta_i^k(t)) = 1$$

Note that the set P_k has a central symmetry that is if $p \in P_k$ then $-p \in P_k$ as well. It follows from the symmetry of P_k that zero belongs to the relative interior of the set P_k :

$$0 \in \text{ri } P_k \tag{23}$$

Let us consider the least linear manifold L_k which contains P_k and the orthogonal complimentary manifold L_k'. Due to (23) any vector q belonging to L_k may be absorbed by the set λP_k for $\lambda > 0$ large enough. Then if we split the derivative f' on two vectors f_1' and f_2'

$$f' = f_1' + f_2'$$

where

$$f_1' \in L_k, \; f_2' \in L_k'$$

then the problem (22) may be rewritten as

$$\min_{p \in P_k} \|f_2'\|^2 + \|f_1' + \lambda p\|^2 = \|f_2'\|^2 + \min_{p \in \lambda P_k} \|f_1' + p\|^2$$

Due to the absorbing properties of the set P_k for $\lambda > 0$ large enough

$$-f_1' \in \lambda P_k$$

and so the second term is equal zero. Finally we get

$$\min_{g \in G(x^k, u^k)} \|g\|^2 = \|f_1'\|^2$$

So the problem (22) is equivalent to the problem of finding the distance between f' and the linear manifold L_k.

The linear manifold L_k may be presented in a form

$$L_k = \{q : q = \sum_{t=0}^{T-1} \psi_k'(t)\lambda(t)\}$$

where $\lambda(t) \in E^n$ and the explicit solution of the (22) should satisfy the linear system

$$\psi_k'(t)^* f_2' = 0, \quad t = 0,1,\dots,T-1$$

where the * means transposition.

Eventually $\lambda(t)$ should be chosen in such a way that

$$\psi_k'(t)^* (\psi_k'(t)\lambda(t) + f'(x^k(T))) = 0 \qquad (24)$$

$$t=0,1,\dots,T-1$$

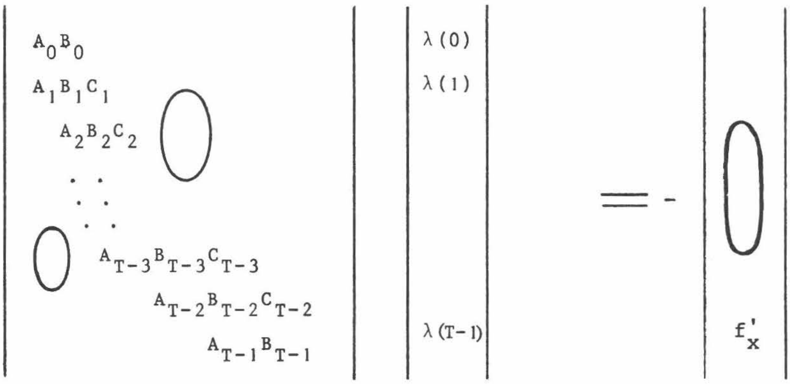
As far as the matrices $\psi_k'(t)$, $t=0,\dots,T-1$ have a vast number of zeroes the system (24) has a block-diagonal structure shown on the Figure 3.

If we suppose that the vectors $\lambda(t)$, $t = 0,1,\dots,T-1$ are linked by the relation

$$\lambda(t+1) = U_t\lambda(t) + \nu(t) \qquad (25)$$

where U_t - n x n matrices and $\nu(t)$ - n - vectors then for U_t and $\nu(t)$

FIGURE 3



We have an equation:

$$B_t U_{t-1} + C_t U_t U_{t-1} + A_t = 0 \qquad (26)$$

with the terminal condition

$$U_{T-2} = - B_{T-1}^{-1} A_{T-2}$$

and

$$C_t \nu(t) + (B_t + C_t U_t)\nu(t-1) = 0 \qquad (27)$$

with the terminal condition

$$\nu(T-2) = - f_x'(x(T))$$

The equations (26),(27) solved in reverse time permit through (25) the recurrent definition of the $\lambda(t)$, $t=0,1,\ldots,T-2$ with the initial state $\lambda(0)$ defined as follows

$$(B_0 + C\, U_0)\, \lambda(0) + C_0\, \nu(0) = 0$$

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