SELECTION OF TIME INDEPENDENT TECHNOLOGIES UNDER AN UNCERTAIN PLANNING HORIZON

J.-P. Ponssard and T.N. Srinivasan

September 1974 WP-74-44

SELECTION OF TIME INDEPENDENT TECHNOLOGIES UNDER AN UNCERTAIN PLANNING HORIZON

by

J. P. Ponssard

and

T. N. Srinivasan

IIASA, Laxenburg, Austria

(On leave from the Centre d'Enseignement Superieur du Management Public, 94112, Arcueil, and from Groupe de Gestion des Organisations Ecole Polytechnique, 75005, Paris, France)

July 1974

1. Description of the Problem and Summary

The problem concerns the selection of a technology among a set of available technologies $I = \{i\}$ each of which is characterized by a capital investment cost k_i and an operating cost per unit time c_i. These costs are assumed independent of time.

The planning horizon is not known. (A typical case might be the arrival of some more efficient technology at some unknown date in the future [Manne]). Hence discounted costs cannot be compared. The procedure adopted here is to determine the subset of efficient technologies; that is, to delete those which could not be chosen whatever the time horizon is. Suppose for instance that $k_1 > k_2$ and $c_1 > c_2$ then clearly technology 1 can be deleted. It may be seen that, while sufficient, this condition is not necessary. However a simple enough condition for the determination of efficient technologies is derived.

If the planning horizon may be described by a random variable then it is shown that it may be replaced by a point estimate. This point estimate, interpreted as a certainty equivalent, may then be used to select the optimal technology among all efficient ones.

Finally, the expected value of perfect information on the planning horizon is derived for the case of a constant rate of substitution between capital and operating costs. Some numerical results illustrate the model.

2. The Subset of Efficient Technologies

It will be convenient to represent a technology $i \in I$ as a point in a two-dimension diagram, capital investment and operating costs respectively (see Figure 1). The set I will be assumed closed and bounded. Let t be the planning horizon, then the discounted cost associated with technology i may be written

$$V_i(t) = k_i + c_i \int_o^t e^{-\rho\nu} d\nu = k_i + c_i (1-e^{-\rho t})/\rho \quad .$$

A technology $i \in I$ is said to be inefficient with respect to I if and only if the following holds:

(2-1) $\quad \forall t \in [o,\infty], \; \exists j_t \in I-\{i\}: \; V_{j_t}(t) \le V_i(t) \quad .$

A technology which is not efficient is called efficient. If all technologies in I are efficient with respect to I then the set I itself will be called efficient.

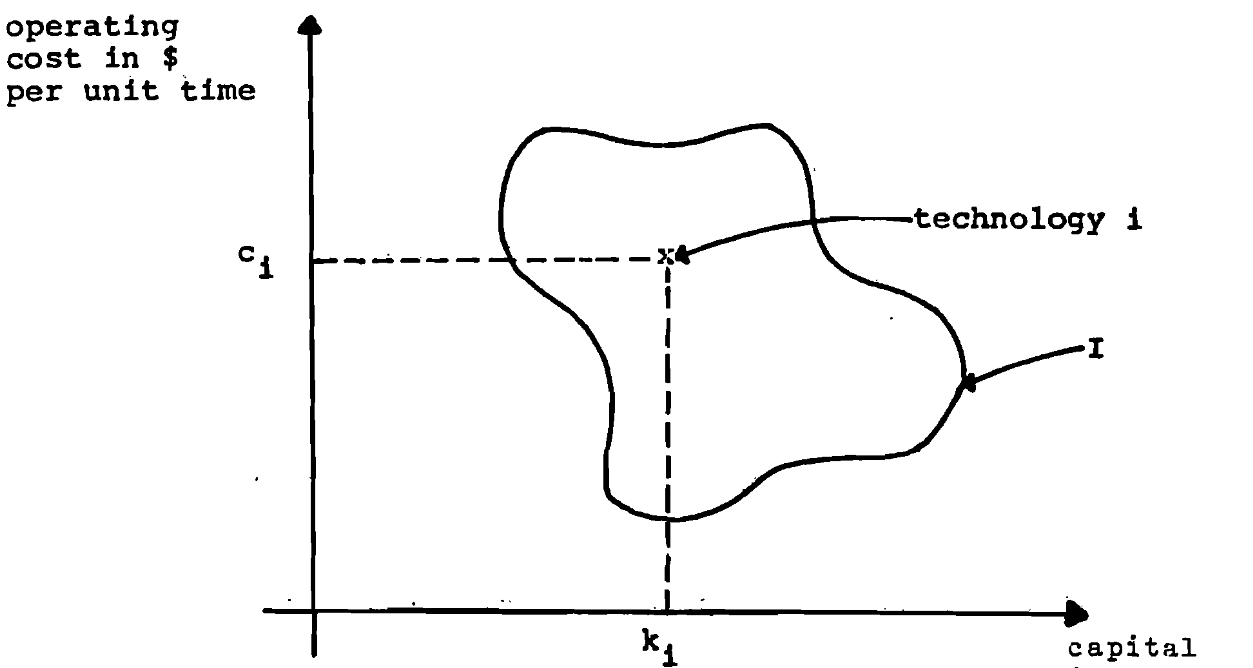


Figure 1

Graphical representation of the set of technologies I

Our objective is to determine the maximal efficient subset, if it exists, which is included in a given set I.

Let $\theta = 1 - e^{-\rho t}$, then condition (2-1) may be rewritten as

(2-2) $\forall\ \theta \in [0,1],\ \exists\ j_\theta \in I - \{i\}:\ V_{j_\theta}(\theta) \leq V_i(\theta)$,

in which $V_i(\theta)$ is a linear function as depicted in Figure 2.

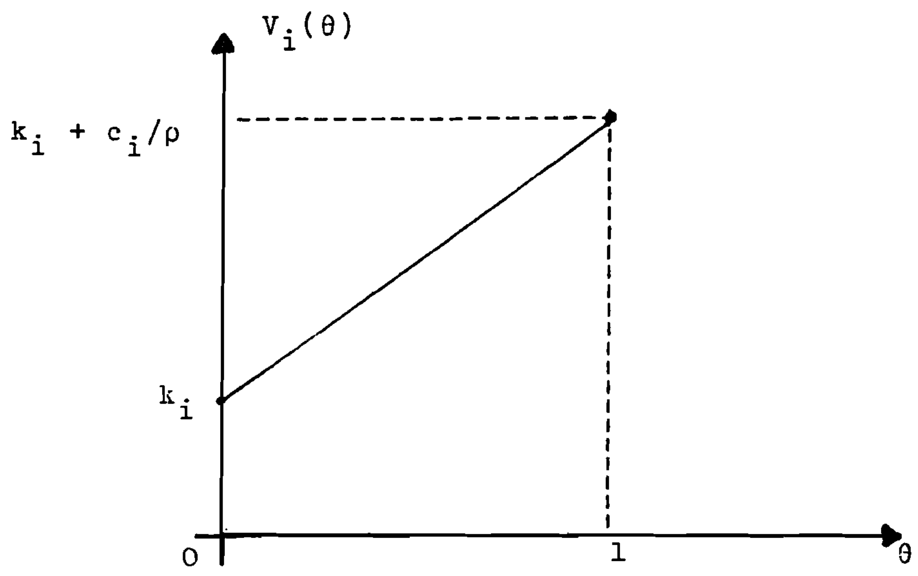


Figure 2

Three simple lemma follow directly.

Lemma 1

Let I = {1,2} then technology 1 is inefficient with respect to I if and only if

(2-3) $k_1 \geq k_2$ and $k_1 + c_1/\rho \geq k_2 + c_2/\rho$.

Lemma 2

Let $I = \{1,2,3\}$. Assume that the subset $\{1,3\}$ is efficient and that $k_1 \leq k_2 \leq k_3$.

Then technology 2 is inefficient with respect to I if and only if:

(2-4) $$c_2 \geq \frac{k_3-k_2}{k_3-k_1} c_1 + \frac{k_2-k_1}{k_3-k_1} c_3 \quad .$$

Proof: Let

$$\mu = (k_3 - k_2)/(k_3 - k_1)$$

$$c_\mu = \mu c_1 + (1- \mu) c_3$$

and $V_\mu(\theta) = k_2 + c_\mu \theta/\rho$.

Then

(i) $\forall \, \theta \in [0,1] \; : \; V_\mu(\theta) \geq \text{Min} \, [V_1(\theta), V_3(\theta)]$,

(ii) $\exists \, \theta_{13} \in [0,1] \; : \; V_\mu(\theta_{13}) = V_1(\theta_{13}) = V_3(\theta_{13})$.

Condition (2-4) is equivalent to

$$\forall \, \theta \in [0,1] \; : \; V_2(\theta) \geq V_\mu(\theta).$$

Thus (i) and (ii) correspond to the sufficient and necessary parts of the lemma respectively. (see Figure 3 for a graphical representation). ||

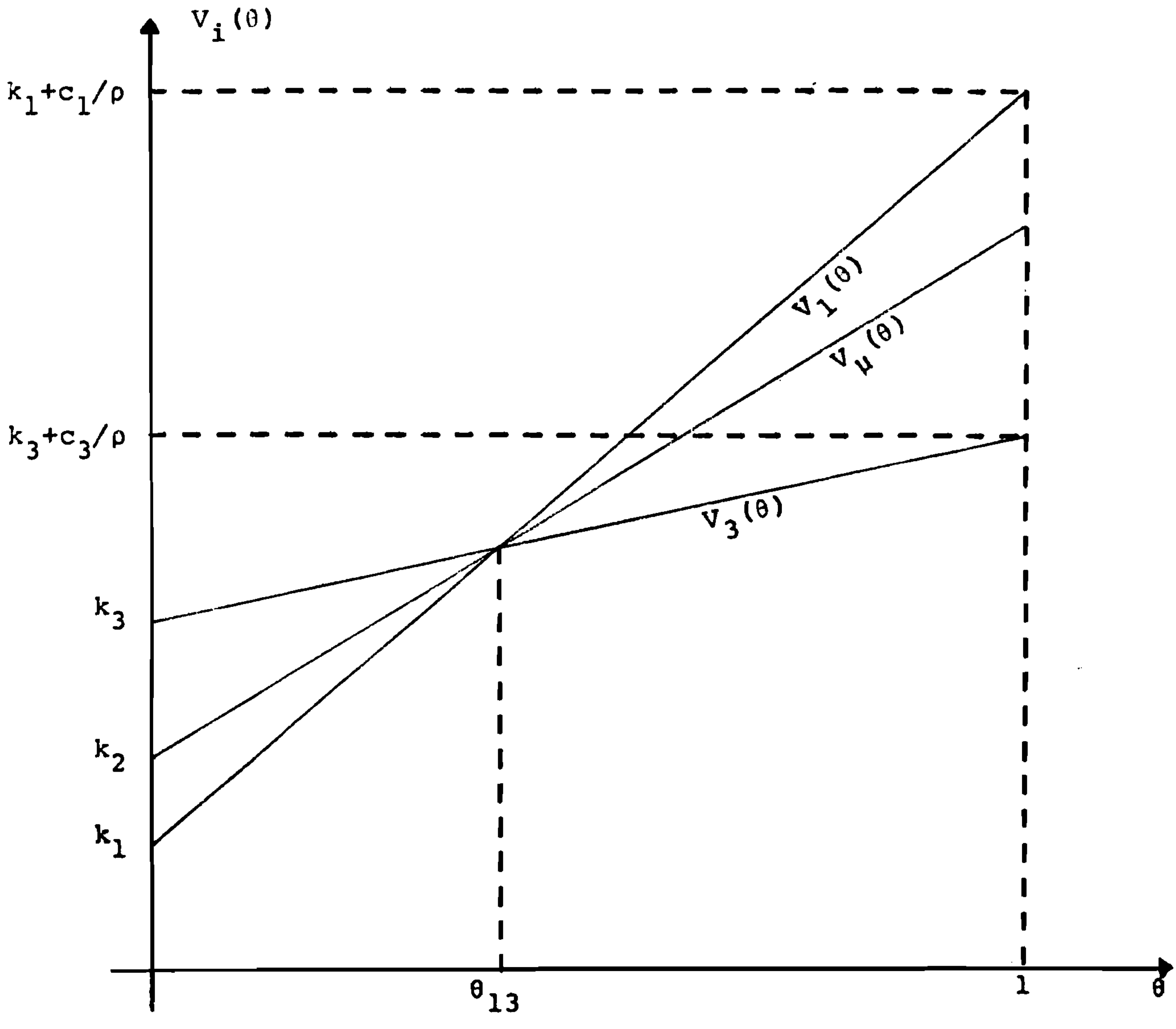

$V_i(\theta)$
k_1+c_1/ρ
k_3+c_3/ρ
k_3
k_2
k_1
$V_1(\theta)$
$V_\mu(\theta)$
$V_3(\theta)$
θ_{13}
1
θ

Figure 3

Lemma 3

Technology $i \in I$ is inefficient with respect to I if and only if it is inefficient with respect to J in which $J \subset I$ and consists of three points at most (including i).

Proof: The "if" part is obvious. Let us prove the "only if" part.

Let

$$V(\theta) = \underset{j \in I}{\text{Min}} \{V_j(\theta)\} \quad .$$

For technology i to be inefficient with respect to I it is necessary that

$$\forall \theta \in [0,1] : V_1(\theta) \geq V(\theta) \quad .$$

Since the $\{V_j(\theta)\}_{j \in I}$ are linear functions this implies that there exists at most two technologies in I, j_1 and j_2, and a convex combination (μ_1, μ_2) such that

$$\forall \theta \in [0,1] : V_i(\theta) \geq \mu_1 V_{j_1}(\theta) = \mu_2 V_{j_2}(\theta) \quad .$$

Using lemma 2 this shows that the subset $j = \{i_1, j_i, j_2\}$ is the required subset. ||

These three lemmas characterise the subset of efficient technologies. Lemma 1 and 2 give necessary and sufficient conditions whenever the set I contains two or three technologies respectively.

Lemma 3 ensures that only comparisons between two or three technologies need be considered. Graphical representations of lemma 1 and 2 are depicted in figures 4 and 5.

All together the graphical characterization of the maximal efficient subset for a given set I is shown in figure 6.

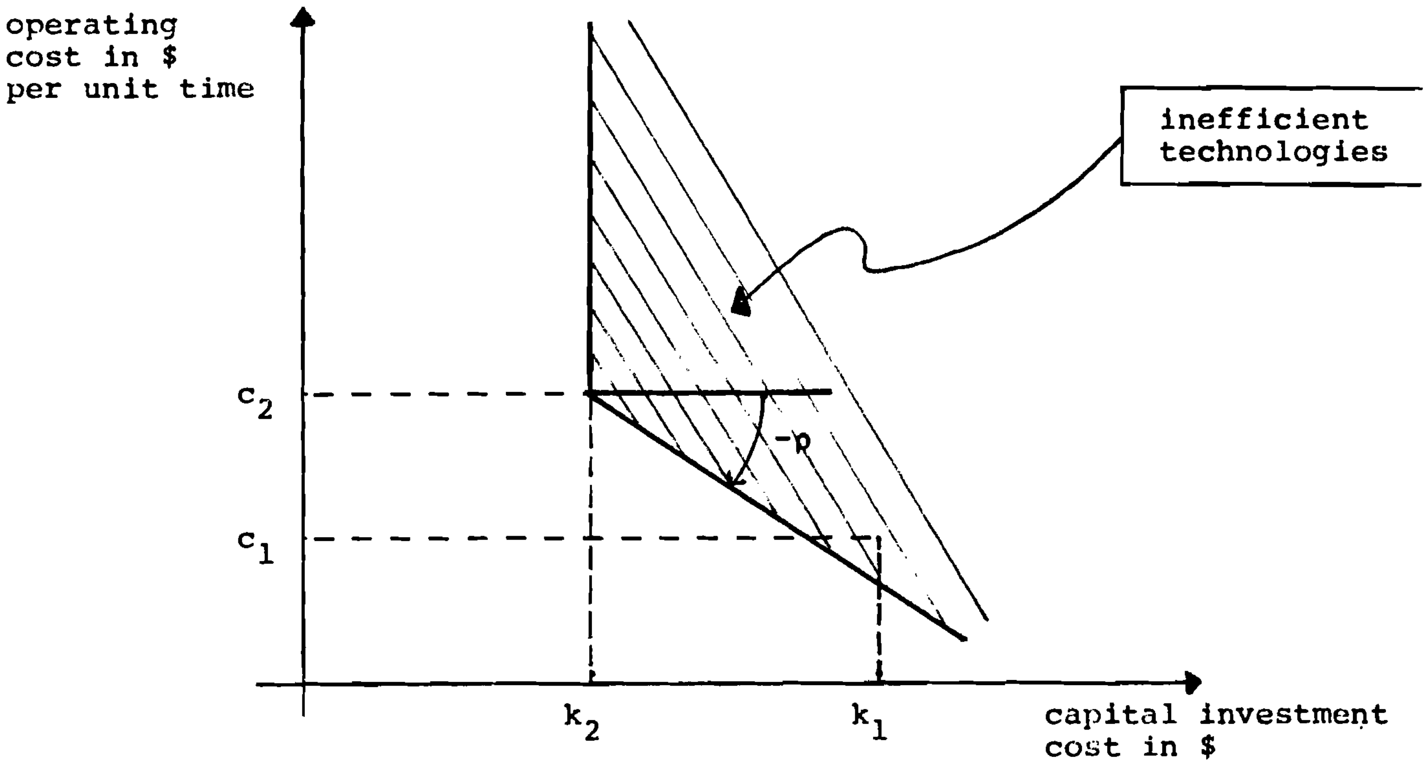


Figure 4
Graphical Representation of Lemma 1

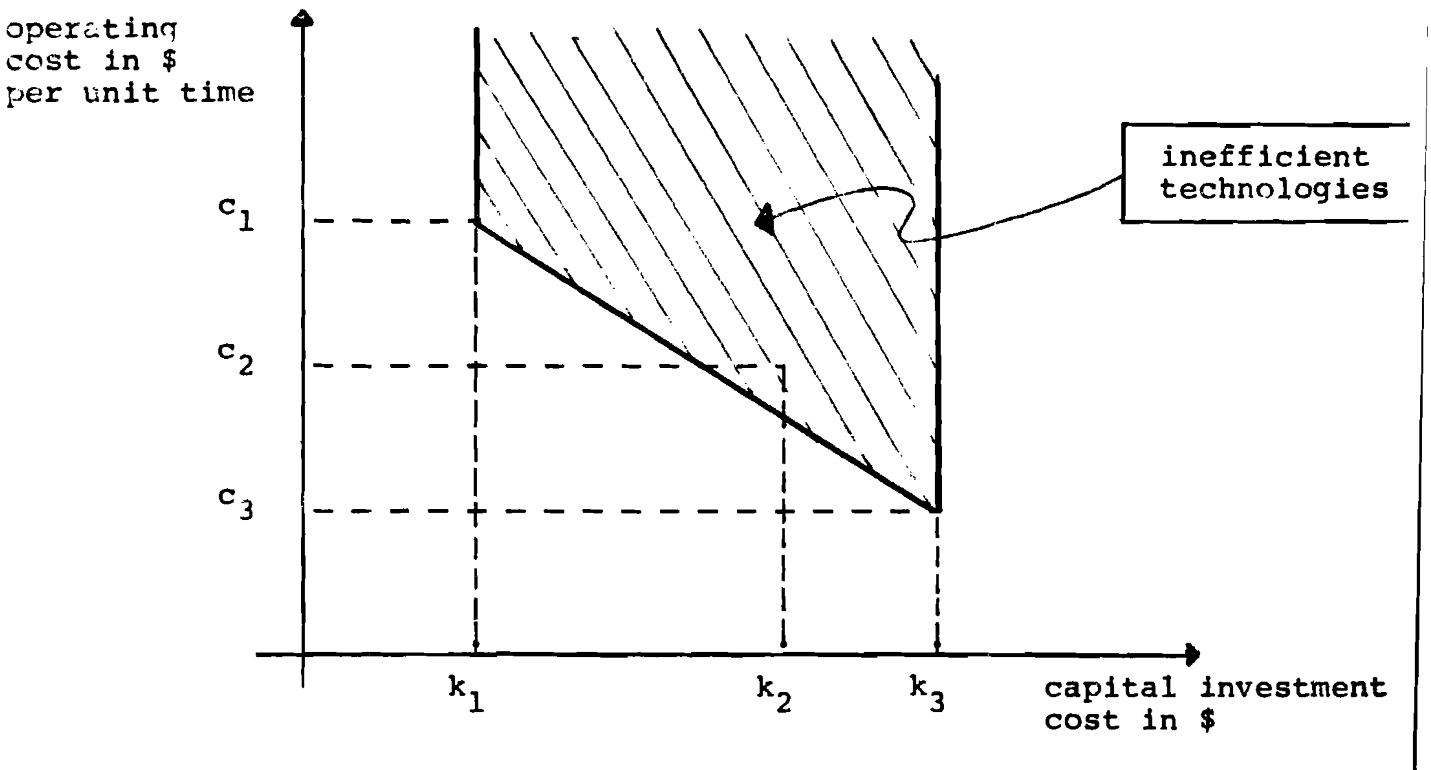


Figure 5
Graphical Representation of Lemma 2

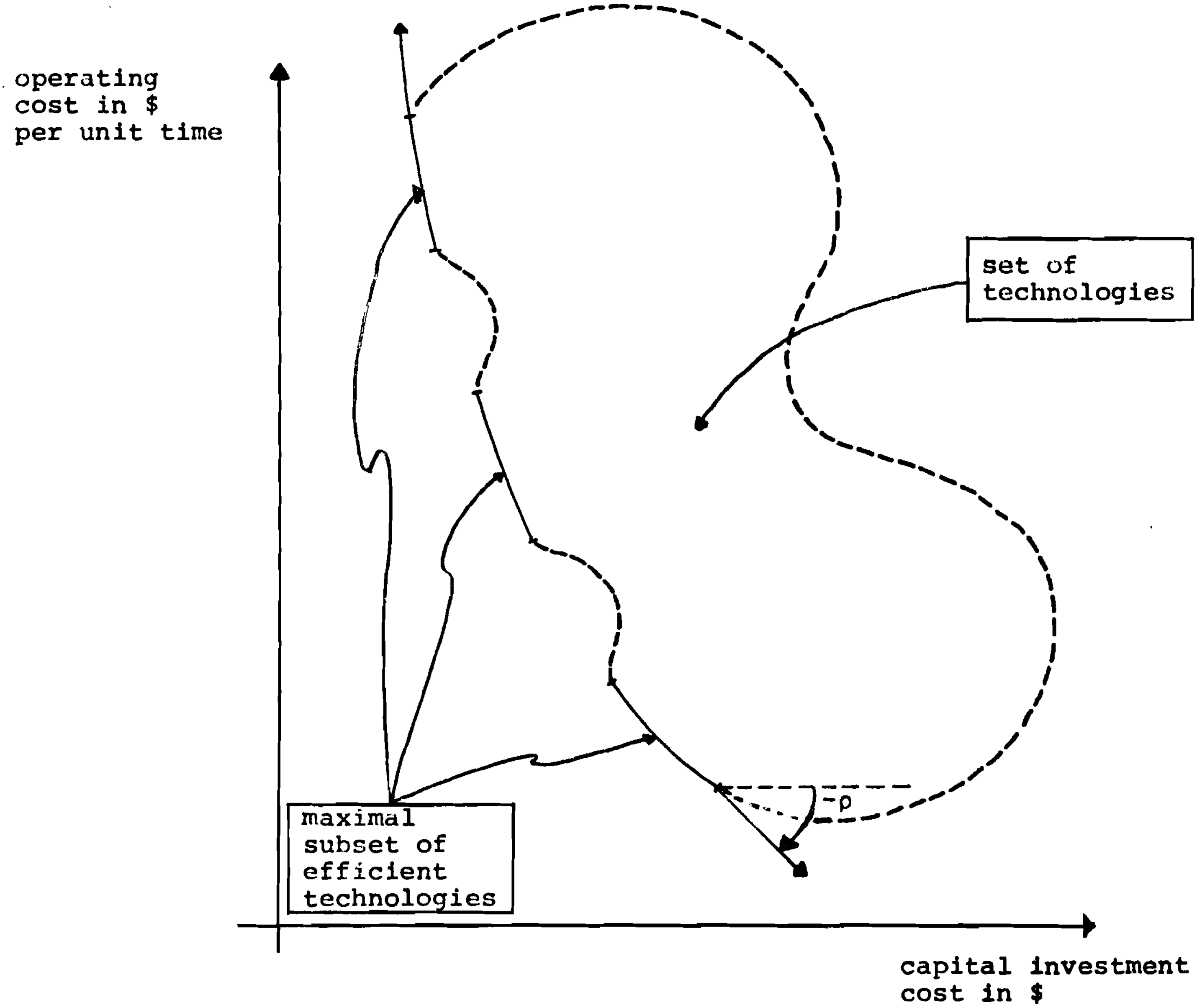


Figure 6

Graphical Characterization of the
Maximal Subset of Efficient Technologies

3. Selection of an Optimal Technology under a Probabilistic Planning Horizon

In this section we shall assume that the set of available technologies is efficient. It will also be assumed to be finite. Technologies will be labelled 1,2,...N corresponding to $k_1 < k_2, \ldots < k_n$ so that the higher the label the more capital intensive the technology and presumably the more appropriate would such a technology be the longer the planning horizon. Indeed this will be easily shown.

Lemma 4

Assume $i < j$ then

$$\text{if } t \leq t_{ij} \qquad V_i(t) \leq V_j(t) \quad ,$$

$$\text{if } t \geq t_{ij} \qquad V_i(t) \geq V_j(t) \quad ,$$

in which

$$t_{ij} = -\frac{1}{\rho} \operatorname{Log} \left[1 - (k_j - k_i)\ \rho/(c_i - c_j)\right] \quad .$$

Proof: Recall that in terms of the θ variable ($\theta = 1-e^{-\rho t}$), $V_i(\theta)$ is linear. Let $\theta_{ij} = \rho(k_j - k_i)/(c_i - c_j)$. Since the technologies are efficient, using lemma 1, $\theta_{ij} \in [0, 1]$ and

$$\text{if } \theta \leq \theta_{ij} \qquad V_i(\theta) \leq V_j(\theta) \quad ,$$

$$\text{if } \theta \geq \theta_{ij} \qquad V_i(\theta) \geq V_j(\theta) \quad . \qquad ||$$

Lemma 5

Assume $i < j < e$ then $t_{ij} < t_{je}$.

Proof: Since the transformation t into θ is monotonically increasing $t_{ij} < t_{je}$ is equivalent to $\theta_{ij} < \theta_{je}$ which in turn corresponds to

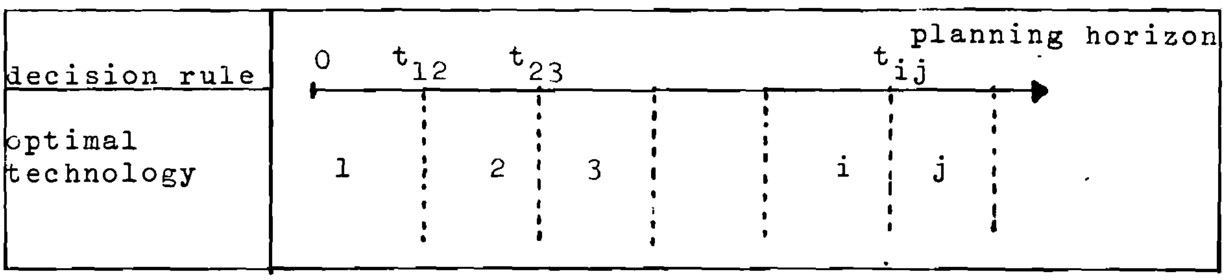
$$\frac{k_j - k_i}{c_i - c_j} < \frac{k_e - k_j}{c_j - c_e}$$

or

$$c_j < \frac{k_e - k_j}{k_e - k_i} c_i + \frac{k_j - k_i}{k_e - k_i} c_e$$

The three technologies i, j, e are efficient then using lemma 2 it is clear that this inequality is satisfied. ||

The combination of lemma 4 and lemma 5 would allow for a very simple decision rule under a known planning horizon. This is summarized by the following diagram.



Under an unknown planning horizon this decision rule is the best that can formally be done and then it would be up to the decision maker to integrate his subjective feelings and select a technology.

However, if these subjective feelings may be expressed as a probability distribution then the probabilistic planning horizon may be replaced by a certainty equivalent according to the following lemma.

Lemma 6

Assume that the planning horizon is a random variable with probability distribution $F(t)$. Then it may be replaced by a certainty equivalent $\bar{t}^d$, such that

$$\bar{t}^d = -(\text{Log } g(\rho))\ /\rho$$

in which the function $g(\rho)$ is the Laplace transform of $F(t)$.

Proof: By definition of the Laplace transform,

$$g(\rho) = \int_0^\infty e^{-\rho t} dF(t).$$

The expected discounted cost associated with technology i may be written as

$$\bar{V}_i = \int_0^\infty V_i(t)\ dF(t)$$

$$= k_i + c_i\ \left[1 - \int_0^\infty e^{-\rho t} dF(t)\right]/\rho$$

$$= k_i + c_i\ \left[1 - g(\rho)\right]/\rho$$

Substituting $g(\rho)$ in terms of $\bar{t}^d$ gives the lemma. ||

It is easily seen that for $\rho = 0$, $\bar{t}^d = \int_0^\infty t\ dF(t)$ and that as a first order appromimation in ρ

$$\bar{t}^d \sim \text{mean} - \frac{\text{var.}\rho}{2}\ .$$

Hence variance and discount rate operates in the same direction, both tend to shorten an uncertain planning horizon as compared with its mean value. $\bar{t}^d$ may then be interpreted as a "discounted mean value" thus the notation.

4. Expected Value of Information on the Planning Horizon

Loosely speaking the expected value of information is the difference between the minimal expected costs with and without information. As such it provides an interesting insight to determine whether further inquiry may result in a substantial reduction of cost [Raiffa].

The analysis will be formally pursued under the following two assumptions:

(i) substitution between capital investment cost and operating cost per unit time may be expressed as a Cobb-Douglas production function; hence after proper rescaling we have

$$k^{\alpha} c^{1-\alpha} = 1 \quad ,$$

(ii) uncertainties about the planning horizon may be expressed in terms of an exponential probability distribution such that

$$F(t) = 1 - e^{-\lambda t}$$

Then, using elementary calculus, we obtain the minimum expected cost under a probabilistic planning horizon,

$$V(\theta) = \alpha^{-\alpha}(1 - \alpha)^{\alpha-1} \left[\frac{\theta}{\rho}\right]^{1-\alpha}$$

and the certainty equivalent $\bar{t}^d$,

$$\bar{t}^d = - \frac{1}{\rho} \text{Log } (\lambda/\lambda+\rho)$$

so that

$$\bar{\theta}^d = 1 - e^{-\rho \bar{t}^d} = \rho/(\rho + \lambda) \quad ,$$

$$V(\bar{\theta}^d) = \alpha^{-\alpha} (1 - \alpha)^{\alpha-1} (\lambda + \rho)^{\alpha-1} \quad .$$

Now if we were to know the planning horizon t, which is distributed according to $\frac{dF(t)}{dt} = e^{-\lambda t}$ (or equivalently θ which is distributed according to $\frac{dG(\theta)}{d\theta} = \frac{\lambda}{\rho} (1-\theta)^{\frac{\lambda}{\rho}-1}$), we would select the best technology given t. A priori, we may expect a minimum cost

$$\bar{V} = \int_0^1 V(\theta)\, dG(\theta) \quad ,$$

$$= \alpha^{-\alpha}(1 - \alpha)^{\alpha-1} \lambda\rho^{\alpha-2}\, \Gamma(2 - \alpha)\, \Gamma(\lambda/\rho)/\Gamma(1 - \alpha + \lambda/\rho)$$

in which $\Gamma(n)$ is the gamma function $(\Gamma(n+1) = n\Gamma(n))$. Some numerical values are given in the following tables, assuming an elasticity coefficient for α=.25.

λ \ ρ	.05	.10	.15
.05	13.8	11.0	9.2
.10	8.1	6.9	6.1
.15	5.7	5.1	4.6

Table 1

The certainty equivalent planning horizon $\bar{t}^d$

λ \ ρ	.05	.10	.15
.05	9.86	7.28	5.86
.10	7.28	5.86	4.96
.15	5.86	4.96	4.33

Table 2

The minimum expected cost $V(\bar{\theta}^d)$.

λ \ ρ	.05	.10	.15
.05	.04	.03	.02
.10	.05	.04	.03
.15	.06	.05	.04

Table 3

The (relative) expected value of information $(V(\bar{\theta}^d)-\bar{V})/V(\bar{\theta}^d)$.

quit It may be seen that the expected value of information is quite low. However, this does not mean that uncertainty plays no role in the selection of the optimal technology. Indeed if uncertainty were to be ignored and the mean value of the planning horizon used as a point estimate then the cost would increase from 10 to 20% depending on the discount rate. This emphasises the significance of the certainty equivalent as defined in section 3.

REFERENCES

[1]. Manne, A.S., "Waiting for the Breeder" Review of Economic Studies, forthcoming, 1974.

[2]. Ponssard, J.P., "Notes on the Value of Information about the Arrival Date of a new Technology" IIASA, January 1974.

[3]. Raiffa, H. "Decision Analysis, Introductory Lectures on Choices under Uncertainty" Addison-Wesley 1968