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Essays in multistate mathematical demography; special IIASA issue

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Editorial

I am pleased that this issue of the journal can be another 'special' concerned with mathematical demography. This is the fourth special issue largely concerned with this field. The series began in 1973 (Volume 5, number 1) with a collection suggested by Andrei Rogers who has also edited the current issue (and one of the others—associating him with three out of the four). The area of work was first attacked systematically by Dr Rogers in 1966, and the 1973 issue reflected a rapid rise of interest in the field. (The paper by Nathan Keyfitz is of particular interest in this context in that it includes a useful historical review.) In 1975 (Volume 7, number 7) we published selected papers from an IIASA conference on national settlement systems and strategies, which included a number of papers on migration, and in 1978 we published another special IIASA issue on migration and settlement. The current issue consists of papers presented at a session of the annual conference of the Population Association of America but contains a strong IIASA component and it is good to see the Institute's continuing interest in the field.

The papers in this issue reflect both the development of the level of technical skill, in particular the drawing together of diverse areas of work under the heading of *multistate* mathematical demography, and the increasing importance of population and various population-state projections for urban and social policy. At a time of serious economic recession and of associated cuts in public expenditure, it becomes all the more important to have accurate forecasts of the demand for public services (many of which vary significantly by age—such as schools, colleges and universities, health services and social services), educational skills for manpower planning and unemployment, and measures of local and regional differentiation. The population mix by local authority and the associated demand for services, for example, are particularly important in Britain at the present time in view of the government's proposals to reform the basis of the Rate Support Grant.

Although many government agencies are beginning to use more advanced techniques, applications are far from widespread. We hope, therefore, that this issue will contribute not only to research, but also to what should be a topical interest in development, particularly at local scales.

A G Wilson

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Foreword

The papers in this special issue of *Environment and Planning A* were first presented at the session on mathematical demography held at the 1979 Annual Meeting of the Population Association of America in Philadelphia, 26–28 April. They are representative examples of work currently under way in a relatively new branch of mathematical demography becoming known as *multistate demography*. The authors come from diverse backgrounds and represent different countries. Philip Rees is British and a lecturer in the School of Geography at the University of Leeds in England; Jacques Ledent is French and is a research scholar at the International Institute for Applied Systems Analysis in Laxenburg, Austria; Frans Willekens, a Belgian, is Research Director at Mens en Ruimte in Brussels; Kao-Lee Liaw is Canadian and teaches geography at McMaster University in Hamilton, Ontario; and Nathan Keyfitz, of the USA, is Andelot Professor of Sociology at Harvard University in Cambridge, Massachussetts.

I wish to express my thanks to Professor Charles Nam, the PAA conference chairman, for the invitation to organize the session on mathematical demography and to the authors listed above for agreeing to participate in it. My thanks go also to Alan Wilson, the editor of *Environment and Planning A*, for agreeing to publish this collection of papers as a special issue of his journal.

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Introduction to multistate mathematical demography

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Abstract. The study of the transitions that individuals experience over time, in the course of passing from one state of existence to another, is a fundamental dimension in much of mathematical demography. Recent work in multistate demographic analysis has led to a generalization of traditional demographic techniques for analyzing such problems. The papers in this issue are representative examples of work currently being carried out on this subject. A unifying thread is the use of matrix algebra to express multidimensional demographic processes in a compact and notationally elegant form which often leads to analytical insights that otherwise may be hidden in the more complicated nonmatrix formulations.

1 Introduction

Common to most topics in mathematical demography is an underlying concern with the transitions that people experience over time in the course of passing from one state of existence to another: for example, transitions from being single to being married, from being alive to being dead, from being employed to being unemployed, from being in school to having graduated. The study of transition patterns generally begins with the collection of data and the estimation of missing observations, continues with the calculation of the appropriate rates and corresponding probabilities, and often ends with the generation of simple projections of the future conditions that would arise were these probabilities to remain unchanged. In short, much of mathematical demography deals with problems of *measurement* and *dynamics* in *multistate* population systems.

Recent work in multistate demographic analysis has produced a generalization of classical demographic techniques that unifies most of the methods for dealing with transitions between multiple states of existence. For example, it is now clear that multiple decrement mortality tables, tables of working life, nuptiality tables, tables of educational life, and multiregional life tables are all members of a general class of increment-decrement life tables called *multistate life tables* (Hoem, 1970; Hoem and Fong, 1976; Krishnamoorthy, 1979; Ledent, 1978; Rogers, 1973a; 1973b; 1975; Rogers and Ledent, 1976; Schoen and Nelson, 1974; Schoen, 1975; Schoen and Land, 1977). It is also now clear that projections of populations classified by multiple states of existence can be carried out using a common methodology of *multistate projection*, in which the core model of population dynamics is a multistate generalization either of the continuous age-time model of Lotka (LeBras, 1971; Rogers, 1973a) or of the discrete age-time model of Leslie (Rogers, 1966; 1968; 1973a; Feeney, 1970).

2 Multistate mathematical demography

The life table is a central concept in classical single-state demography. Its use to express the facts of mortality in terms of survival probabilities and their combined impact on the lives of a cohort of people born at the same moment has been so successful that, in the words of Keyfitz (1968, page 3), "we are incapable of thinking of population change and mortality from any other starting point". The natural starting point for thinking about multistate population change, therefore, is the multistate life table, its theoretical derivation, and its empirical calculation.

2.1 *Multistate life tables*

Multistate increment-decrement tables come in two forms: those with a single radix (*uniradix tables*) or those with a multiple radix (*multiradix tables*). Figure 1 illustrates these two kinds of multistate life tables.

In uniradix life tables everyone is born into the same state of existence, that of being *outside* the particular state of interest, be it labor force, school, or marriage. Entry, by members of the initial radix, into the state of interest occurs at some given age, and from then on the individual may transfer out of and back into that state a number of times during his lifetime. Death may occur at any age and in any state. Moreover, the 'in' and the 'out' states may both be disaggregated into several substates.

Multiradix life tables allow several cohorts (radices) to interact during the process of multistate demographic evolution. The most common application is a disaggregation by different regions of birth, as in multiregional mathematical demography (Rogers, 1975). For example, consider some of the possible interactions between a cohort of rural-born babies and a cohort of urban-born babies. A rural youth might migrate to an urban area to go to school or to join the urban labor force; he might return several years later as an adult having married an urban-born wife; if unsuccessful in entering the rural labor market, he might decide to migrate once again, raising his children in yet another urban or rural region. As in uniradix multistate life tables, death may occur at any age and in any state, and each *regional* state can be disaggregated into substates, some of which may be particular 'in' and 'out' states of interest, for example, being *out* of school *in* region 2.

If the life table is the natural starting point for thinking about population change and mortality, the natural starting point for thinking about the life table itself is the differential equation that defines $l(x)$, the probability of surviving to age x, if the chance of dying between age a and $a+\mathrm{d}a$ for those aged a is $\mu(a)\mathrm{d}a$:

$$\frac{\mathrm{d}}{\mathrm{d}x} l(x) = -\mu(x) l(x) . \tag{1}$$

The solution to equation (1) is

$$l(x) = \exp\left[-\int_0^x \mu(a)\,\mathrm{d}a\right] , \tag{2}$$

and the probability that an individual at exact age x will survive to exact age $x+h$ is

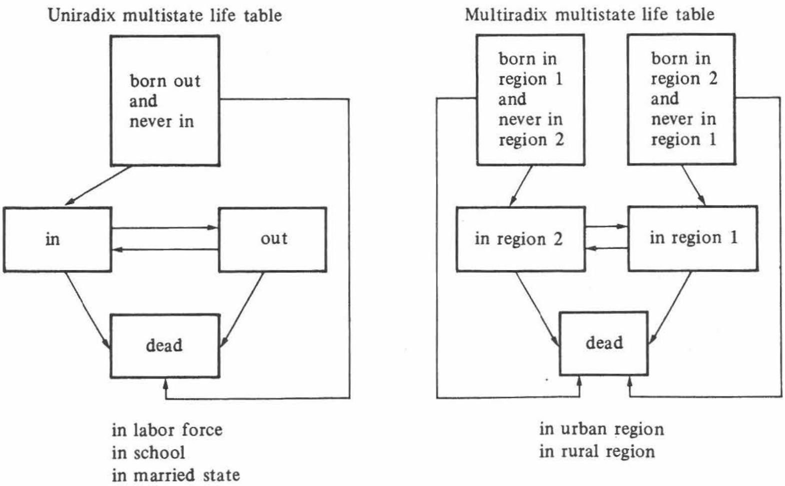

Figure 1. The two kinds of multistate life table.

therefore

$$p(x) = \frac{l(x+h)}{l(x)} = \exp\left[-\int_x^{x+h} \mu(a)\mathrm{d}a\right] . \tag{3}$$

The multistate life table is founded on an equation that is similar to equation (1) but with matrices replacing scalars:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{l}(x) = -\boldsymbol{\mu}(x)\mathbf{l}(x) . \tag{4}$$

The definitions and arrangements of the elements of these matrices are described in the third and fourth papers of this issue and need not concern us here. The general solution for $\mathbf{l}(x)$ is somewhat more complicated than in the single-state model, but, in the special case when $\boldsymbol{\mu}(a)$ is a constant matrix within the age interval x to $x+h$, we can write

$$\mathbf{l}(x+h) = \exp[-h\mathbf{M}(x)]\mathbf{l}(x) , \tag{5}$$

where $\mathbf{M}(x)$ is the finite approximation to $\boldsymbol{\mu}(x)$. Observing that the age-specific matrix of transition probabilities between states, $\mathbf{P}(x)$, is $\mathbf{l}(x+h)\,\mathbf{l}(x)^{-1}$, and expanding the exponential in equation (5) to its first two terms, gives

$$\mathbf{P}(x) = \mathbf{I} - h\mathbf{M}(x) , \tag{6}$$

where $\mathbf{I}$ is the identity matrix. We can improve the approximation by premultiplying both sides of equation (5) by $\exp[h\mathbf{M}(x)/2]$ and then expanding to find

$$\left[\mathbf{I}+\frac{h}{2}\mathbf{M}(x)\right]\mathbf{l}(x+h) = \left[\mathbf{I}-\frac{h}{2}\mathbf{M}(x)\right]\mathbf{l}(x) , \tag{7}$$

whence

$$\mathbf{P}(x) = \left[\mathbf{I}+\frac{h}{2}\mathbf{M}(x)\right]^{-1}\left[\mathbf{I}-\frac{h}{2}\mathbf{M}(x)\right] . \tag{8}$$

Those familiar with single-state life-table construction methods will recognize in equation (8) the conventional formula for deriving life-table probabilities from observed rates. The only difference in the multistate version is that matrices appear in place of scalars.

The transition probabilities in $\mathbf{P}(x)$ refer to individuals who are at exact age x. For population projections, however, it is useful to derive the corresponding survivorship proportions, $\mathbf{S}(x)$, that refer to individuals in age group x to $x+h$ at the start of the projection. Here again it is a simple matter to show that the multistate analog of the conventional expression is

$$\mathbf{S}(x) = [\mathbf{I}+\mathbf{P}(x+h)]\mathbf{P}(x)[\mathbf{I}+\mathbf{P}(x)]^{-1} , \tag{9}$$

which yields the recursive expression

$$\mathbf{P}(x) = [\mathbf{I}+\mathbf{P}(x+h)-\mathbf{S}(x)]^{-1}\mathbf{S}(x) . \tag{10}$$

All life-table functions originate from a set of transition probabilities, defined for all ages. The first question in constructing such tables, therefore, is how to transform observed age-specific death and migration rates, $\mathbf{M}(x)$, or survivorship proportions, $\mathbf{S}(x)$, into the age-specific transition matrix, $\mathbf{P}(x)$. Equations (8) and (10) suggest two alternative procedures. The first focuses on observed rates, the second on observed proportions surviving. In Rogers (1975) these two estimation methods were called the 'option 1' and 'option 2' methods respectively.

"We begin with an examination of a probability estimation method that views migration data in the same way as mortality data, that is, as reported *events* ... we call this estimation method the Option 1 method. Then we develop a probability estimation method in which migration data are reported as changes of regions of residence from those at a fixed prior date. This method is ... the Option 2 method" (Rogers, 1975, page 81).

Operationally their definitions are the following.
Option 1: Given $\hat{\mathbf{M}}(x)$, find $\mathbf{P}(x)$ such that $\mathbf{M}(x) = \hat{\mathbf{M}}(x)$.
Option 2: Given $\hat{\mathbf{S}}(x)$, find $\mathbf{P}(x)$ such that $\mathbf{S}(x) = \hat{\mathbf{S}}(x)$.
The matrices with circumflexes above them are the empirical (observed) counterparts of the corresponding life-table measures, which are expressed without circumflexes. The estimation procedure in each instance seeks to find values of $\mathbf{P}(x)$ that will equate the life-table measures with their observed counterparts[(1)].

The two distinct perspectives implied by these options are carefully examined in the third paper of this issue.

2.2 *Multistate projection models*
An important and fundamental application of the survivorship probabilities and proportions found in a multistate life table is to population projection. Multistate projection models are of two kinds: continuous age-time Lotka models and discrete age-time Leslie models.

A *continuous age-time model* of a single-sex population may be defined for a multistate system by means of a straightforward generalization of the classical single-state Lotka model. Beginning with the number of female births in each state at time t, $B_i(t)$ say, we note that women aged a to $a+\mathrm{d}a$ in state i at time t are survivors of those born a years ago and now living in state i at age a, that is, $\sum_{j=1}^{r} B_j(t-a)\,{}_jl_i(a)\,\mathrm{d}a$, where $a \leqslant t$. At time t these women give birth to $\left[\sum_{j=1}^{r} B_j(t-a)\,{}_jl_i(a)\right] m_i(a)\mathrm{d}a$ children per year while in state i. Here ${}_jl_i(a)$ denotes the probability that a baby girl born in state j will survive to age a in state i, and $m_i(a)\mathrm{d}a$ is the annual rate of female childbearing among women aged a to $a+\mathrm{d}a$ in state i. Integrating this last expression over all ages a and focusing on the population at times beyond the last age of childbearing, β, gives the homogeneous equation system

$$B(t) = \int_0^\beta \mathbf{m}(a)\mathbf{l}(a)B(t-a)\mathrm{d}a \,. \tag{11}$$

The *discrete age-time model* of multistate demographic growth expresses by means of a matrix operation the population projection process; a multistate population set out as a vector, is multiplied by a growth matrix that projects that population forward through time. The projection calculates the state-specific and age-specific survivors of a multistate population of a given sex and adds to this total the new births that survive to the end of the unit time interval. This process may be described by the matrix model

$$K(t+1) = \mathbf{G}K(t) \,, \tag{12}$$

where the vector $K(t)$ sets out the multistate population disaggregated by age and

(1) A weakness of the 'option 1' method as applied in Rogers (1973b; 1975) was the assumption that multiple transitions could not occur during a unit age interval; for example, it was assumed that an individual could not migrate and die in the same age interval. Schoen (1975) put forward an improved estimation algorithm that dropped this assumption, and Rogers and Ledent (1976) then

state, and the matrix $\mathbf{G}$ is composed of zeros and elements that represent the various age-specific and state-specific components of population change in the manner described by the fourth paper in this issue.

To study the projection dynamics of the discrete model in equation (12), it is convenient to partition the matrix $\mathbf{G}$ at the point of the highest age of reproduction, say at age $\beta = 50$:

$$\mathbf{G} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} . \tag{13}$$

In this partitioning, the upper right-hand submatrix remains zero for all positive integral powers of $\mathbf{G}$, and it can be shown that the $\mathbf{U}$ and $\mathbf{Z}$ matrices never affect the population aged less than β. Thus the mathematical analysis of equation (12) can be carried out largely in terms of the matrix $\mathbf{H}$ and the associated top half of the vector $K(t)$, which will not be distinguished notationally from its longer counterpart in equation (12).

The $n \times n$ matrix $\mathbf{H}$ in equation (13) is of such a character that it stabilizes when raised to successively higher powers, in the sense that each element of the matrix with the higher power is proportional to the corresponding element of the matrix with the lower power; that is,

$$\mathbf{H}^{t+1} = \lambda \mathbf{H}^{t} . \tag{14}$$

The value of λ may be found by solving for the roots of the characteristic equation

$$\mathrm{f}(\lambda) = |\mathbf{H} - \lambda \mathbf{I}| = 0 \tag{15}$$

and selecting the largest root, λ_1 say.

The n roots of characteristic equation (15) apparently are always distinct in demographic applications. Associated with each root λ is a characteristic *column* vector K_i that satisfies the equality

$$\mathbf{H}K_i = \lambda_i K_i , \tag{16}$$

and a characteristic *row* vector V_i^{T} such that

$$V_i^{\mathrm{T}}\mathbf{H} = \lambda_i V_i , \tag{17}$$

where T is the transpose operator. For analytical convenience it is common practice to scale the elements of K_i to sum to unity and to normalize V_i^{T}:

$$\bar{V}_i^{\mathrm{T}} = \frac{V_i^{\mathrm{T}}}{V_i^{\mathrm{T}}K_i} . \tag{18}$$

Note that $\bar{V}_i^{\mathrm{T}}K_i = 1$ and $\bar{V}_i^{\mathrm{T}}K_j = 0$.

An observed population, set out as the age-by-state vector $K(0)$, say, may be expressed as a weighted linear combination of the stable column vectors associated with the projection matrix:

$$K(0) = c_1K_1 + c_2K_2 + \ldots + c_nK_n . \tag{19}$$

To compute c_i we premultiply equation (19) by the normalized row vector $\bar{V}_i^{\mathrm{T}}$ and find

$$c_i = \bar{V}_i^{\mathrm{T}}K(0) = \frac{V_i^{\mathrm{T}}K(0)}{V_i^{\mathrm{T}}K_i} . \tag{20}$$

Since for a constant $\mathbf{H}$

$$\mathbf{K}(t) = \mathbf{H}^{t}\mathbf{K}(0) , \tag{21}$$

we may premultiply equation (19) by $\mathbf{H}^t$ to obtain

$$K(t) = \lambda_1^t c_1 K_1 + \lambda_2^t c_2 K_2 + ... + \lambda_n^t c_n K_n \ . \tag{22}$$

The asymptotic properties of the projection in equation (21) have been extensively studied in mathematical demography (for example, Keyfitz, 1968, chapter 3). This body of theory draws on the properties of matrices with nonnegative elements and, in particular, on the Perron–Frobenius theorem (Gantmacher, 1959). Its application to equation (21) establishes the existence of a unique, real, positive, dominant characteristic root, λ_1 say, and an associated positive characteristic vector, K_1 say. Inasmuch as λ_1 is greater in absolute value than any other λ_j, the effects of all components beyond the first in equation (22) ultimately disappear as the population converges to the stable distribution defined by K_1.

Since the sum of the elements of K_1 is unity, the total population added over all ages and states is $\lambda_1^t c_1$, for a large t and a constant projection matrix. This permits us to call c_1 the *stable equivalent population*. It is the total which, if distributed according to the stable vector K_1, would ultimately grow at the same rate as the observed $K(0)$ projected by the projection matrix as $\mathbf{H}^t K(0)$.

The dominant right characteristic column vector of the projection matrix, K_1, defines the stable population across ages and states. The dominant left characteristic row vector of the same matrix, V_1^{T}, also has a useful interpretation. It describes the reproductive potential of the multistate population. The product $V_1^{\mathrm{T}} K(0)$ is known as the *total reproductive value* of the initial population (Rogers and Willekens, 1978).

Implicit in every multistate projection matrix is a stable distribution across ages and states, expressible in terms of age compositions and state shares. Deviations from these compositions and shares, in the initial age-by-state distribution, ultimately disappear, but in the short to medium run they create fluctuations and disturbances in age profiles and in allocations over states.

The conventional single-state population projection model yields only a single positive root. The multiradix multistate projection model generates several positive roots. In both cases the first component in equation (22), the one associated with the dominant root, is generally referred to as the *dominant component*. It accounts for that part of $K(0)$ which is stable. The other components in equation (22) that are associated with positive roots may be called *subdominant components*. They transmit the redistributional effects of interstate transfers. Finally the remaining components, associated with the negative and complex roots of equation (22), are called *cyclical components*. They generate fluctuations in population totals and age profiles known as 'waves'. The fifth paper in this collection focuses on the dynamics by which the dominant, subdominant, and cyclical components interact during the process of convergence to stable population growth.

3 Four papers on multistate demography

The four papers that follow in this issue, and the sixth that comments on them, constitute the proceedings of the session on mathematical demography held at the 1979 Annual Meeting of the Population Association of America in Philadelphia, 26–28 April. They are representative examples of work currently being carried out in multistate demographic analysis. Together they span a broad spectrum, from data collection and refinement to measurement, model construction, projection, and analysis.

3.1 *Data and accounts*

Empirical studies in multistate demography often begin with data, set out in tabular form, which describe changes in *stocks* that have occurred over two or more points in time. These changes arise as a consequence of increments and decrements associated with *events*, such as births and deaths, and with *flows* of individuals between different states of existence.

When all of the appropriate elements in such tables have been filled in with numbers, they generally are referred to as *accounts*. And when, as is often the case, some data are unavailable, ingenuity and sophisticated fudging are used to supply the missing entries. Prominent among such techniques are various row and column balancing methods that have been successfully implemented in economics (input–output matrices), transportation planning (origin–destination traffic flows), and statistics (contingency tables). A brief exposition of these methods appears in Willekens et al (1979).

Philip Rees, in his contribution to the set of papers included in this issue, considers some of the problems of data collection and account construction. He argues that the now widely accepted concept of economic accounts should be extended to multistate demographic analysis.

The idea of arranging monetary transactions in a system of interlocking statements, in which total inflows are forced to equal total outflows, is a familiar habit of thought in economics. Building on the work of the British economist Richard Stone, Rees demonstrates the utility of imposing a similar habit to the inescapable accounting interrelationships that arise in demographic data.

Rees uses his detailed demographic accounts to estimate two kinds of probability matrices: the matrix of survival probabilities, $\mathbf{P}(x)$, that appears in equation (8) and the matrix of survivorship proportions, $\mathbf{S}(x)$, that appears in equation (9). The latter is estimated directly from the population accounts, and the former is then interpolated from it.

3.2 *Movements and transitions in multistate life tables*

Jacques Ledent begins where Rees leaves off and considers the fundamental problem of constructing multistate life tables from multistate data. He shows how the conventional core problem, in ordinary life tables, of converting observed age-specific death rates into probabilities of dying within stated age intervals is complicated enormously when 'resurrections', in the form of return movements, are allowed.

During the course of a year, or some such fixed interval of time, a number of individuals change their current state of existence. A *move* out of a state of existence is an *event*: a separation. A *mover* is an individual who has made a move at least once during a given interval of time. A *migrant*, on the other hand, is an individual who at the end of a given time interval no longer inhabits the same state of existence as at the start of the interval. He has made a *transition* from one state to the other. (The act of separation from one state is linked with an addition to another). Thus paradoxically a multiple mover may be a nonmigrant by this definition; if, for example, a particular mover returns to his initial state of existence before the end of the unit time interval, no 'migration' is registered.

Ledent focuses on the crux of the life-table construction problem: the estimation of age-specific survival probability transition matrices, $\mathbf{P}(x)$, by use of data either on interstate *moves* or on interstate *transitions*. Since the data on multistate flows can come in the form of *move* counts or *people* counts, the methods used must be specific to each kind of data. Irrespective of the form of the data, however, no statements about probabilities can be made without a conversion of 'moves' information to 'people' information at some point in the analysis.

3.3 *Tables of working life and labor-force projections*

Estimates of the expected remaining working lifetime of a person at each of several given ages are now regularly computed for a large number of countries. Such estimates appear as basic elements of working-life tables. Frans Willekens reviews much of this literature in his contribution to this issue and observes that the fundamental technique for constructing such tables, until very recently, remained relatively unchanged for about three decades. He then outlines the multistate life-table approach, which, unlike the conventional method, focuses on the actual flows of people between active and inactive states.

Because of their focus on changes in stocks rather than on flows, conventional tables of working life must adopt three restrictive assumptions:

(1) entry into the labor force occurs only before the peak age of active life;

(2) retirement occurs only after the peak age, and reentry into the labor force is not possible;

(3) active and inactive individuals of labor-force ages are exposed to identical mortality regimes.

Willekens demonstrates that all three assumptions may be dropped in calculating a multistate table of working life. He also applies his model to Danish data previously analyzed by Hoem and Fong (1976) and contrasts the two sets of findings.

The distribution of a multistate population across its constituent states and the age compositions of its state-specific subpopulations are determined by the interactions of fertility, mortality, and interstate transfers. Individuals are born, age with the passage of time, reproduce, move between different states of existence, and ultimately die. In connecting these *events* and *flows* to determine the growth rate of each state-specific *stock*, one also obtains a count of the number of individuals in each state and their age composition.

Willekens concludes his paper by illustrating how a multistate projection model that connects labor-force-related events, flows, and stocks, can be used to generate labor-force projections, and identifies the fundamental role played in this by the survivorship proportions produced by a table of working life.

3.4 *Dynamics of stable growth*

Earlier it was observed that a multistate population system which is closed to external migration and subject to an unchanging multistate schedule of mortality, fertility, and migration will ultimately converge to a stable constant age-by-state distribution that increases at a constant stable growth ratio, λ say. Knowledge of the asymptotic properties of such a population projection helps us understand the meaning of observed age-specific birth, death, and migration rates.

Kao-Lee Liaw sets out the analytic solution of a multistate population projection that is generated by a constant multistate growth regime. He then shows that such a projection tends toward a fixed stable distribution in two stages: first, a quick disappearance of cyclical behavior and a relatively rapid convergence toward stable age compositions and, second, a gradual convergence toward a stable interstate allocation.

Liaw applies the analytic solution to data on multiregional population growth in Canada. Focusing on the fourteen-age-group, eight-region, female population during the 1966–1971 period, he illustrates how population waves are transmitted through cyclical components at the same time as spatial redistribution is achieved through subdominant (spatial) components. Regional stability in age composition comes after a hundred years with these data; stability in the spatial allocation of the national population, on the other hand, takes a much longer time and is only half completed after a hundred years.

Liaw accounts for the differential speeds of convergence by referring to the differences in magnitude between complex and positive characteristic roots of the population system's projection matrix. It might be useful to go further and to examine the structure of that matrix, examining in particular its *decomposability* along the lines suggested long ago by Simon and Ando (1961).

"The crux of the Simon–Ando theorem is the assertion that the equilibrium of a nearly completely decomposable dynamic linear system may be viewed as a composite growth process which evolves in three temporal phases. During the first phase, the variables in each subsystem arrive at equilibrium positions determined by the completely decomposed system. After a longer time period the system enters its second phase, at which point the variables of each subsystem, maintaining their proportional relationships, move together *as a block* toward equilibrium values established by the third phase of the growth process. In this final phase all variables approach the rate of growth defined by the largest characteristic root of the matrix associated with the original nearly completely decomposable system" (Rogers, 1976, page 72).

4 Conclusion

Single-state life tables and models have served reasonably well as tools of measurement and projection in all of the topics of interest discussed in this issue. Why then introduce the more complicated multistate methods? The answer is two-fold. First, single-state methods cannot deal with interstate transfers differentiated by origins and destinations, and must therefore analyze changes in stocks by reference to *net* flows, for example, net migration. Second, single-state models cannot follow individuals across several changes of states of existence and therefore cannot disaggregate current or future stocks and flows of individuals by previous regions of residence or states of existence.

Our understanding of patterns and behavior of mortality, fertility, nuptiality, migration, education, and labor-force participation is enhanced by a focus on occurrences of events and transfers and on their association with the populations that are exposed to the risk of experiencing them. A multistate perspective permits such an association; a single-state perspective does not. For example, there is no such individual as a net migrant, and attempts to explain 'his' behavior as a response to spatial variations in socioeconomic conditions are bound to produce misspecified models.

The propensity to experience various events and transfers differs across subpopulations; analyses and projections that can take this inhomogeneity into account can identify the contribution made by each subcomponent to the total. Again a multistate perspective permits such an association; a single-state perspective does not. For example, our understanding of marital instability is enriched by information regarding the degree to which current marital dissolution occurs among those previously divorced. Such information would show how much of the current increase in levels of divorce could be attributed to 'repeaters' as opposed to 'first-timers'.

A unifying thread throughout the set of papers in this issue is the use of matrix algebra to express, in compact form, a number of relationships that would be very difficult to identify and study using scalar (nonmatrix) arguments. Conceptualizing a multidimensional demographic process in matrix form confers advantages that are both notational and analytical in character. Matrix notation often leads to insights that otherwise may have been obscured by the more complicated nonmatrix formulations. And formulating a demographic problem in matrix terms places at our disposal a large mathematical apparatus on matrices and their properties. As a result, what at first is introduced as a purely pragmatic and notationally elegant conceptualization can ultimately become the vehicle for insights that are not obtainable by conventional methods of analysis, as the papers in this collection demonstrate.

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Multistate demographic accounts: measurement and estimation procedures

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Abstract. Accounting frameworks developed in the field of economics are applied to the problem of measuring changes-in-state of populations. Examples of accounts in the educational sector, in the job market, and in a regional system are described. Proper estimation of multistate demographic accounts involves attention to data sources, much initial estimation of variables, construction of a model to estimate missing items, and use of the possible constraints on the accounts matrix. These steps in accounts estimation are illustrated for a set of accounts for British regions for 1970-1976. Data are assembled for a base period, 1970-1971. Alternative methods of constructing accounts are tested by running the estimation model in projective mode for 1971-1976. One method is selected and used to complete the set of accounts.

1 Why prepare multistate demographic accounts?

The process of population change has been studied for a long time through the use of many different mathematical models and techniques. The numerical application of these models has not usually involved the prior development of sets of demographic accounts, that is, systematic arrangements of statistics about population change. A number of reasons for the integration of information on human stocks and flows in terms of demographic accounts for use in such models can, however, be suggested.

First, accounts have served economists well in their national economic modelling activities. Without a system of national account statistics integrating information on economic and financial stocks and flows, the national economic models would be much poorer.

Second, accounts force the analyst to attempt the matching of available data and a conceptual model. The need for further survey, or for further estimation, is revealed.

Third, diverse sets of statistics are brought together in accounts and are subject to comparison and to checking for consistency.

In this paper an attempt is made to show how demographic accounts may be defined, how the elements of those accounts may be estimated, particularly when information is missing, and how the accounts may be used in further analysis such as population projection.

Emphasis will be placed on illustration of the techniques of accounts building in practice rather than on an explication of the detailed mathematics, which is described elsewhere (Rees and Wilson, 1977). The examples are for the British population; different techniques may be needed in other countries, given poorer or better data sources.

Section 2 defines and describes the principal species of accounts. Section 3 presents a series of diverse implementations of closed demographic accounts. The details of data assembly and data estimation for one of the examples presented in section 3 are outlined in section 4. Section 5 concludes the paper with some remarks on the uses of multistate demographic accounts.

2 What are multistate demographic accounts?

2.1 *General definitions*

Accounts are arrangements of statistics in matrix or tabular form. Demographic accounts are such matrices or tables that involve either people or events connected with them. The adjective 'multistate' implies that there is concern with the transition of people among many states. Those states might be ages, amongst which there is a well-ordered set of transitions, with most transitions being impossible. Or the states might be educational grades, closely related to, but not the same as, ages. At least some of the states in accounts involve geographical areas.

Accounts matrices have two dimensions. The first, say the rows, represents the states of the population initially, and the second, the columns, the states of the population finally, after the transitions or movements have occurred.

A variety of entities may be represented in accounts: people, pupils, periods of unemployment, houses, households, jobholders, migrations, marriages and divorces, and time are but a few. A selection of accounts containing some of these different entities is presented in the following subsections.

2.2 *Simple components-of-growth accounts*

The very simplest type of accounts involves the arrangement of the terms in the components-of-growth equation:

$$P^i(t+T) = P^i(t)+I^i(t, t+T)+M^i(t, t+T) , \tag{1}$$

where

$P^i(t)$ is the population in region i at the start of the period, time t;
$P^i(t+T)$ is the population in region i at the end of the period, time $t+T$;
T is the length of the period, in years;
$I^i(t, t+T)$ is the natural increase in population in region i in the period t to $t+T$; and
$M^i(t, t+T)$ is the net migration into region i in the period t to $t+T$.

The first portion of table 1 (subtable 1.1) shows such a set of accounts for sections of Great Britain in 1970-1971.

The natural-increase term in equation (1) is usually further decomposed into constituent birth and death terms:

$$P^i(t+T) = P^i(t)+B^i(t, t+T)-D^i(t, t+T)+M^i(t, t+T) , \tag{2}$$

where

$B^i(t, t+T)$ is the total number of births in region i in the period t to $t+T$, and
$D^i(t, t+T)$ is the total number of deaths in region i in the period t to $t+T$.

Subtable 1.2 of table 1 shows this decomposition for sections of Great Britain. Similar accounts are published for the countries of the UK (England, Wales, Scotland, and Northern Ireland) by the Office of Population Censuses and Surveys (1975-1979), and such tables have been estimated for 1965-1976 for the standard regions [post-April-1974 definitions given in Office of Population Censuses and Surveys (1975b)] in Rees (1978a).

Further deconsolidation of the net migration term into separate inflow and outflow components is often desirable:

$$P^i(t+T) = P^i(t)+B^i(t, t+T)-D^i(t, t+T)+M^{Ri}(t, t+T)-M^{iR}(t, t+T) , \tag{3}$$

where

$M^{Ri}(t, t+T)$ denotes the migrations from the rest of the world, R, into region i in the period t to $t+T$, and
$M^{iR}(t, t+T)$ denotes the migrations out of region i to the rest of the world, R, in the period t to $t+T$.

The terms in equation (3) are all counts of *moves* (made by persons) rather than counts of *persons*. In the case of births and deaths the counts of moves and the counts of persons are numerically identical, but this is not true for the migration terms. Equation (3) can be reexpressed in person terms by adopting a slightly different notation:

$$K^{*(*)\sigma(i)}(t+T) = K^{\epsilon(i)*(*)}(t)+K^{\beta(i)*(*)}(t,\ t+T)-K^{*(*)\delta(i)}(t,\ t+T) + K^{*(R)*(i)}(t,\ t+T)-K^{*(i)*(R)}(t,\ t+T)\ . \qquad (4)$$

The letter K denotes persons and each variable in the equation is a different kind of person count:

$K^{*(*)\sigma(i)}(t+T)$ is the total number of persons surviving, σ, in region i at time $t+T$;
$K^{\epsilon(i)*(*)}(t)$ is the total number of persons in existence, ϵ, in region i at time t;
$K^{\beta(i)*(*)}(t,\ t+T)$ is the total number of persons born in region i in the period t to $t+T$;

Table 1. Accounts based on the components-of-growth equations for sections of Great Britain, 1970–1971.

	North	Midlands	South	Celtic Fringe	Great Britain
Subtable 1.1					
Final population	14607·7	8756·1	22767·0	7941·0	54071·8
Initial population	14576·0	8700·0	22687·0	7930·7	53893·7
Natural increase	61·0	54·7	88·8	32·3	236·8
Net migration	−29·4	1·4	−8·9	−22·0	−58·9
Subtable 1.2					
Final population	14607·7	8756·1	22767·0	7941·0	54071·8
Initial population	14576·0	8700·0	22687·0	7930·7	53893·7
Births	241·5	149·2	350·1	129·8	870·6
Deaths	180·5	94·5	261·3	97·5	633·8
Net migration	−29·4	1·4	−8·9	−22·0	−58·9
Subtable 1.3					
Final population	14607·7	8756·1	22767·0	7941·0	54071·8
Initial population	14576·0	8700·0	22687·0	7930·7	53893·7
Births	241·5	149·2	350·1	129·8	870·6
Deaths	180·5	94·5	261·3	97·5	633·8
In-migrants	206·7	182·1	442·6	130·1	384·4
Out-migrants	236·1	180·7	451·5	152·1	443·3
Subtable 1.4					
Final population	14607·7	8756·1	22767·0	7941·0	54071·8
Initial population	14576·0	8700·0	22687·0	7930·7	53893·7
Births	241·5	149·2	350·1	129·8	870·6
Deaths	180·5	94·5	261·3	97·5	633·8
In-migrants					
internal	145·6	139·9	199·3	92·1	–
external	61·1	42·2	243·3	38·0	384·4 [a]
Out-migrants					
internal	153·0	128·6	203·4	92·2	–
external	83·1	52·1	248·1	59·9	443·3 [a]

Notes
1. The sections of Great Britain are defined in terms of the standard regions as follows: North = North + Yorkshire and Humberside + North West; Midlands = East Midlands + West Midlands; South = East Anglia + South East + South West; Celtic Fringe = Wales + Scotland. The standard regions are defined in Office and Population Censuses and Surveys (1975b).
2. All figures are in thousands.

[a] The discrepancies between these values and the corresponding row sums are due to rounding.

$K^{*(*)\delta(i)}(t, t+T)$ is the total number of persons dying in region i in the period t to $t+T$;
$K^{*(R)*(i)}(t, t+T)$ is the total number of persons initially located in the rest of the world and finally located in region i in the period t to $t+T$; and
$K^{*(i)*(R)}(t, t+T)$ is the total number of persons initially located in region i and finally located in the rest of the world in the period t to $t+T$.

The asterisks denote summation over the superscripts which they replace, so that

$$K^{*(R)*(i)} = K^{\epsilon(R)\sigma(i)} + K^{\epsilon(R)\delta(i)} + K^{\beta(R)\sigma(i)} + K^{\beta(R)\delta(i)} \tag{5}$$

and

$$K^{*(i)*(R)} = K^{\epsilon(i)\sigma(R)} + K^{\epsilon(i)\delta(R)} + K^{\beta(i)\sigma(R)} + K^{\beta(i)\delta(R)} . \tag{6}$$

Subtable 1.3 of table 1 illustrates accounts based on equation (4), with in-migrants and out-migrants distinguished. Note that small net migration figures, as in the South, can mask very large inflows and outflows.

The migration terms in equations (3) and (4) are explicitly related in the following way:

$$M^{Ri}(t, t+T) = K^{*(R)*(i)}(t, t+T) + M^{Ri}_{\text{sur}}(t, t+T) \tag{7}$$

and

$$M^{iR}(t, t+T) = K^{*(i)*(R)}(t, t+T) + M^{iR}_{\text{sur}}(t, t+T) , \tag{8}$$

where

$$M^{Ri}_{\text{sur}}(t, t+T) = M^{iR}_{\text{sur}}(t, t+T) . \tag{9}$$

The terms $M^{Ri}_{\text{sur}}(t, t+T)$ and $M^{iR}_{\text{sur}}(t, t+T)$ refer to migrations surplus to those required to accomplish the transition of persons from initial to final states within the period t to $t+T$. The equality of equation (9) only holds for the sum of surplus migrations to and from a region i rather than for surplus migrations between region i and any other region.

It is often crucial to distinguish those in-migrant and out-migrant streams originating or having their destination in the same country as the region of interest from those originating or having their destination in the outside world. If this is done then equation (4) becomes further disaggregated into

$$\begin{aligned} K^{*(*)\sigma(i)}(t+T) = {} & K^{\epsilon(i)*(*)}(t) + K^{\beta(i)*(*)}(t, t+T) - K^{*(*)\delta(i)}(t, t+T) \\ & + \sum_{j\in I} K^{*(j)*(i)}(t, t+T) - \sum_{j\in I} K^{*(i)*(j)}(t, t+T) \\ & + \sum_{j\in E} K^{*(j)*(i)}(t, t+T) - \sum_{j\in E} K^{*(i)*(j)}(t, t+T) , \end{aligned} \tag{10}$$

where I refers to the internal set of regions (those inside the country containing region i) and E to the external set of regions. Subtable 1.4 of table 1 shows the components-of-growth accounts with this added disaggregation. The importance of external migration flows is very clear, and in the case of the South (East Anglia, the South East, and the South West) the external flows exceed those from the rest of the country.

Simple components-of-growth accounts can be rearranged to show the inflows to and outflows from a region in a time period:

$$\begin{aligned} K^{*(*)\sigma(i)}(t+T) + K^{*(*)\delta(i)}(t, t+T) + K^{*(i)*(R)}(t, t+T) = {} & K^{\epsilon(i)*(*)}(t) \\ & + K^{\beta(i)*(*)}(t, t+T) + K^{*(R)*(i)}(t, t+T) . \end{aligned} \tag{11}$$

The left-hand side of equation (11) contains the outflow terms—final population, deaths, and out-migrants—and the right-hand side contains the inflow terms—initial population, births, and in-migrants. Table 2 shows the figures from table 1 rearranged in the form of equation (11). The inflow or outflow total for a region represents the total number of persons existing in, entering, or leaving a region over a period and is a more valid measure of the demands made by the population than are the initial or final stock figures, although it would be better to weight the various flows by the time they spend in the region.

Table 2. Inflow-outflow accounts for sections of Great Britain, 1970-1971.

	North	Midlands	South	Celtic Fringe	Great Britain
Inflows					
Initial population	14576·0	8700·0	22687·0	7930·7	53893·7
Births	241·5	149·2	350·1	129·8	870·6
In-migrants	206·7	182·1	442·6	130·1	384·4
Total [a]	15024·2	9031·3	23479·7	8190·6	55148·7
Outflows					
Final population	14607·7	8756·1	22767·0	7941·0	54071·8
Deaths	180·5	94·5	261·3	97·5	633·8
Out-migrants	236·1	180·7	451·5	152·1	443·3
Total [a]	15024·3	9031·3	23479·8	8190·6	55148·9

Source: table 1.
Note: all figures are in thousands.
[a] The slight discrepancies between the inflow and outflow totals are due to rounding.

2.3 *Open, period-to-period accounts*

Simple components-of-growth accounts were first extended by Stone (1965; 1966; 1971a; 1971b; 1975), for several periods taken together, in what he calls 'open' accounts. The elements of the components-of-growth equation, in the form of equation (4), for example, are arranged for a sequence of years. Table 3 shows how this can be done for the four sections of Great Britain. The diagonal terms represent the population stocks 'transferred' between periods. The births and in-migrants are listed in the first two rows of the table, and the deaths and out-migrants in the first two columns.

Table 3. Open, year-to-year, accounts for sections of Great Britain, 1970-1973.

Input	Output		Population								Total
	deaths	out-migrants	1971-1972				1972-1973				
			N	M	S	C	N	M	S	C	
Births			230·0	143·0	338·7	124·0	210·4	132·6	317·7	115·2	
In-migrants			202·3	184·6	447·0	129·8	201·1	183·2	449·1	139·6	
Population											
1970-1971											
N	180·5	236·1	14607·7								15024·3
M	94·5	180·7		8756·1							9031·3
S	261·3	451·5			22767·0						23479·8
C	97·5	152·1				7941·0					8190·6
1971-1972											
N	183·1	229·6					14627·2				15039·9
M	96·2	172·3						8815·2			9083·7
S	264·5	426·9							22861·3		23552·7
C	98·8	151·1								7945·0	8194·8 [a]
Total			15040·0	9083·7	23552·7	8194·8	15038·7	9131·0	23628·1	8199·8	

Key: N—North; M—Midlands; S—South; C—Celtic Fringe.
Note: all figures are in thousands.
[a] The discrepancy between this value and the corresponding row sum is due to rounding.

In his 1971 monograph Stone (1971a) introduces further terms in the central portion of the accounts matrix (see, for example, Stone, 1971a, page 34, table III.11) that represent a transfer from a state in one period to another state in the next period. However, strictly speaking, such transfers cannot occur in the open accounts framework, and Stone has himself recognized the difficulties of using such accounts by basing his 1975 exposition (Stone, 1975) on closed demographic accounts, which are described next.

The reason for the confusion is that the open accounts developed by Stone referred to the educational system, where transfers between states occur at the end of one school year and at the beginning of the next when pupils change classes, grades, or schools. It is probably best to represent such transfers as occurring over a period even if they are concentrated in a short portion of that period.

2.4 *Closed demographic accounts*

So far, although we have considered the transitions into and out of many states, the transitions between states have been neglected apart from those fundamental to any demographic system (birth, death, and immigration/emigration transitions). Accounts that display multistate transitions fully are constructed as two-dimensional matrices together with their row and column totals. Table 4 shows such a set of multistate demographic accounts for the British example for the period midyear 1970 to midyear 1971.

The rows represent the initial states from which people start in a period. These initial states may be the state at the start of a period or the state into which persons are born at some time during a period. The columns represent the states in which people end up—either at the end of the period when still alive or at the time of their death before the end of the period. The accounts matrix links the two sets of states. Consider, for example, the rows and columns for the South. Some 22 687 thousand people lived there at midyear 1970; of these 21 982 thousand survived and stayed in

Table 4. Closed accounts for sections of Great Britain, 1970-1971.

Initial state	Final state										Total [b]
	survival at midyear [a] 1971					deaths 1970-1971					
	N	M	S	C	A	N	M	S	C	A	
Existence at midyear [a] 1970											
N	14164·1	43·1	80·6	27·3	81·9	177·7	0·2	0·5	0·2	0·5	14576·0
M	41·3	8428·1	69·5	15·9	51·5	0·3	92·7	0·4	0·1	0·3	8700·0
S	75·8	77·2	21982·2	47·7	244·8	0·5	0·4	256·8	0·3	1·4	22687·0
C	26·7	17·8	46·5	7683·9	59·1	0·2	0·1	0·3	95·9	0·4	7930·7
A	60·1	41·6	240·1	37·5	0·0	0·4	0·2	1·4	0·2	0·0	381·4
Births 1970-1971											
N	238·2	0·4	0·7	0·2	0·7	1·5	0·0	0·0	0·0	0·0	241·5
M	0·4	146·8	0·6	0·1	0·4	0·0	0·8	0·0	0·0	0·0	149·2
S	0·6	0·6	344·7	0·4	1·9	0·0	0·0	2·0	0·0	0·0	350·1
C	0·2	0·1	0·4	127·8	0·5	0·0	0·0	0·0	0·8	0·0	129·8
A	0·5	0·4	1·8	0·4	0·0	0·0	0·0	0·0	0·0	0·0	3·0
Total [b]	14607·7	8756·1	22767·0	7941·0	440·7	180·5	94·5	261·3	97·5	2·6	55148·8

Key: N—North; M—Midlands; S—South; C—Celtic Fringe; A—abroad.
Note: all figures are in thousands.

[a] Midyear = 30 June/1 July.

[b] The discrepancies between some of the totals and the corresponding row or column sums are due to rounding.

the South, 76 thousand moved to the North, 77 thousand to the Midlands, 48 thousand to Scotland and Wales, and 245 thousand emigrated abroad. Just under 257 thousand died in the South, and small numbers died after migrating to the other sections. When the column for the South is examined we see that the section received 81, 70, 47, and 240 thousand migrants from the North, the Midlands, the Celtic Fringe, and abroad respectively, some 345 thousand babies who were born in the South, and 0·7, 0·6, 0·4, and 1·8 thousand infant migrants who were born in the other four sections.

Table 5 shows, for a two-region system, the way in which the K notation defined earlier relates to the accounts. The variable K, representing persons or transitions, is classified by superscripts, the first of which represents the initial life state—existence, ϵ, or birth, β—in a region, the identity of which is given in the brackets that follow immediately. The second superscript gives the final life state—survival, σ, or death, δ—in the region indicated in the brackets. In general, accounts contain four kinds of variables:

$K^{\epsilon(i)\sigma(j)}$ *survivors*, initially in existence in region i who survive in region j; when $i = j$ they are stayers and when $i \neq j$ they are migrants;

$K^{\epsilon(i)\delta(j)}$ *nonsurvivors*, initially in existence in region i who die in region j; when $i = j$ they die in their initial region and when $i \neq j$ they migrate before dying;

$K^{\beta(i)\sigma(j)}$ *infant survivors*, born in region i who survive in region j; when $i = j$ they are stayers and when $i \neq j$ they are migrants;

$K^{\beta(i)\delta(j)}$ *infant nonsurvivors*, born in region i who die in region j; when $i \neq j$ they are stayers and when $i \neq j$ they migrate before dying in another region.

When asterisks replace superscripts this indicates that the superscript has been summed over. For example,

$$K^{\epsilon(i)*(*)} = \sum_j K^{\epsilon(i)\sigma(j)} + \sum_j K^{\epsilon(i)\delta(j)} . \qquad (12)$$

The sum totals have particular interpretations in terms of items of available population data: the $K^{\epsilon(i)*(*)}$ and $K^{*(*)\sigma(i)}$ terms are initial and final population stocks; the $K^{\beta(i)*(*)}$ and $K^{*(*)\delta(i)}$ terms are counts of births and deaths in the regions.

The key feature of tables 4 and 5 and of closed demographic accounts in general is the inclusion of a region that closes the system, called 'abroad', 'the outside world', 'the rest of the world', or 'other countries'. Without this region we could not interpret the accounts-table sums in the useful way they have been here. And we could not compute transition rates, by dividing each element in a row of the accounts matrix

Table 5. Closed accounts for a two-region system of interest: symbolic representation.

Initial state	Final state						Total
	survival at midyear 1971			deaths 1970-1971			
	region 1	region 2	abroad (R)	region 1	region 2	abroad (R)	
Existence at midyear 1970							
region 1	$K^{\epsilon(1)\sigma(1)}$	$K^{\epsilon(1)\sigma(2)}$	$K^{\epsilon(1)\sigma(R)}$	$K^{\epsilon(1)\delta(1)}$	$K^{\epsilon(1)\delta(2)}$	$K^{\epsilon(1)\delta(R)}$	$K^{\epsilon(1)*(*)}$
region 2	$K^{\epsilon(2)\sigma(1)}$	$K^{\epsilon(2)\sigma(2)}$	$K^{\epsilon(2)\sigma(R)}$	$K^{\epsilon(2)\delta(1)}$	$K^{\epsilon(2)\delta(2)}$	$K^{\epsilon(2)\delta(R)}$	$K^{\epsilon(2)*(*)}$
abroad (R)	$K^{\epsilon(R)\sigma(1)}$	$K^{\epsilon(R)\sigma(2)}$	–	$K^{\epsilon(R)\delta(1)}$	$K^{\epsilon(R)\delta(2)}$	–	$K^{\epsilon(R)*(*)}$
Births 1970-1971							
region 1	$K^{\beta(1)\sigma(1)}$	$K^{\beta(1)\sigma(2)}$	$K^{\beta(1)\sigma(R)}$	$K^{\beta(1)\delta(1)}$	$K^{\beta(1)\delta(2)}$	$K^{\beta(1)\delta(R)}$	$K^{\beta(1)*(*)}$
region 2	$K^{\beta(2)\sigma(1)}$	$K^{\beta(2)\sigma(2)}$	$K^{\beta(2)\sigma(R)}$	$K^{\beta(2)\delta(1)}$	$K^{\beta(2)\delta(2)}$	$K^{\beta(2)\delta(R)}$	$K^{\beta(2)*(*)}$
abroad (R)	$K^{\beta(R)\sigma(1)}$	$K^{\beta(R)\sigma(2)}$	–	$K^{\beta(R)\delta(1)}$	$K^{\beta(R)\delta(2)}$	–	$K^{\beta(R)*(*)}$
Total	$K^{*(*)\sigma(1)}$	$K^{*(*)\sigma(2)}$	$K^{*(*)\sigma(R)}$	$K^{*(*)\delta(1)}$	$K^{*(*)\delta(2)}$	$K^{*(*)\delta(R)}$	$K^{*(*)*(*)}$

by the row total, that have the convenient property of summing to unity; the same point applies to the computation of admission rates through division of elements in a column by the column total.

However, it should be stressed that the framework of closed demographic accounts deals only with the change from initial to final state and not with multiple changes of state in between. Thus the closed accounts matrix does not contain the numbers of moves between states. Such movement accounts have been discussed by Rees (1977a), Illingworth (1976), and Jenkins (1976), and the differences between transitions and movements are discussed by Courgeau (1973) and Ledent (1978a; 1978b; 1978c).

Ideally one would like to match movements and transitions very precisely, but this is only possible with good population registers. For most multistate projection, life-table, and economic investigations (Stone, 1975, pages 45-46), accounts tables based on transitions are more appropriate, and severe difficulties are encountered in estimating the appropriate transition information from register counts of movements and international migration counts. The only convenient solution for these 'multiple-transition' problems is to work with a time period short enough for the surplus of movements over transitions not to be large enough to matter.

A whole variety of different population investigations can be based on the information contained in a matrix of closed demographic accounts or in a series of them. However, a description of such investigations is postponed to section 4 of the paper. In the next section a number of different examples of multistate demographic accounts using different state definitions are described.

3 Examples of multistate demographic accounts

3.1 *Educational accounts*

Stone (1971a; 1971b; 1972; 1975) has reviewed the application of accounting principles to the study of a variety of social and demographic systems. In particular, accounting and associated modelling techniques have been applied in the educational field. Table 6 shows a stocks-and-flows matrix, taken from Stone (1972, page 64), in which the transitions of pupils between various sectors of the educational hierarchy are charted. Fuller versions of such tables include a classification of pupils into single years of age and a more detailed description of the '21 Other employment' sector. The table reveals that relatively few of the secondary schoolboys in England and Wales proceed to further education compared with those who enter the labour market directly.

Transition proportions can be calculated for these transfers and this is done in table 7. The diagonal proportions are high, indicating the movement of people within the various sectors; these are reduced in larger versions of such accounts. From the matrix of transition proportions given in table 7, the fundamental matrix, $(\mathbf{I}-\mathbf{C})^{-1}$, can be computed. This is set out in table 8. This table yields estimates of the numbers of years people spend in subsequent states, given that they start in particular states and given that the matrix of transition proportions, $\mathbf{C}$, remains unaltered. In effect $(\mathbf{I}-\mathbf{C})^{-1}$ is a discrete version of a multistate life-expectancy matrix (Rees and Wilson, 1977, pages 259-270).

Tables 6, 7, and 8 contain a wealth of information about the educational system of England and Wales in 1965-1966 and the tendencies inherent in that system should the transition proportions remain unchanged. The fundamental matrix (Stone, 1972, pages 75-77) provides life expectancies subdivided by time spent in different states. The average time spent in full-time formal education for the whole male population is the sum of rows 2 to 18 of column 1 of table 8, that is 13·08 years (with ten being the legal minimum for most pupils in 1965-1966). Calculated for many periods such a statistic would provide a valuable addition to a set of national social indicators. The life expectancy in subsequent educational states is clearly

Table 6. The active sequence as a whole: England and Wales, male population (in thousands of males), 1965-1966.

State in 1966	State in 1965: 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	Total
0		10·9	1·1		0·5										0·1			0·1		0·9	0·3	89·8	178·6	282·3
1	435·6	1602·2																						2037·8
2	−4·0	411·8	2055·9																					2463·7
3		5·8	0·8	35·7																				42·3
4	−1·5	2·5	324·6	0·1	896·1																			1221·8
5					64·6	16·7																		81·3
6				5·1	187·3																			192·4
7					70·6																			70·6
8						8·8																		8·8
9						42·9																		42·9
10							2·5	3·3			5·8											5·9		17·5
11								2·7	1·0	1·3	5·0	6·7												16·7
12	0·1								0·1	3·2		0·3	4·9									0·5		9·1
13	0·1													0·4				0·9						1·4
14							4·8	3·1	0·8	1·9					20·6							37·2		68·4
15								0·9	1·5	2·0		0·7				13·5		0·2		1·3	0·1	5·5		25·7
16	0·1									1·8		0·2					7·7	0·3				0·4		10·5
17										22·0	0·3	1·4	0·3					51·5				5·1		80·6
18	1·4																0·5	6·2	9·6			6·8		24·5
19															0·3	6·9	0·1	0·4	2·0	134·3		0·5		144·5
20																0·7		0·2	1·4	1·3	54·2	0·1		57·9
21	18·2						197·8	62·2	5·3	9·5	6·4	4·8	2·8	0·4	42·3	1·2	1·5	14·5	7·5	0·5	0·1	14414·5		14789·5
22				0·1																1·5	0·6	162·4	2079·8	2244·4
Total	450·0	2033·2	2382·4	41·0	1219·1	68·4	205·1	72·2	8·7	41·7	17·5	14·1	8·0	0·8	63·3	22·3	9·8	74·3	20·5	139·8	55·3	14728·7	2258·4	

Source: Stone (1972, page 64, table 1).

Key:
0 Outside world
1 Preschool
2 Nursery and primary school
3 Special school
4 Secondary school: 1st level
5 Secondary school: 2nd level
6 Final school year: no certificates
7 Final school year: O-levels
8 Final school year: one A-level
9 Final school year: more than one A-level
10 Further education: GCE O-level/OND/ONC
11 Further education: GCE A-level/HND/HNC
12 Further education: external 1st degree
13 Further education: external 2nd degree
14 Further education: other courses
15 Teacher training college
16 University 1st degree: medical
17 University 1st degree: other
18 University 2nd degree
19 Schoolteacher
20 Other teacher
21 Other employment
22 Home and retirement

Table 7. The **C** matrix of transition probabilities based on the fully adjusted version of table 6: England and Wales, male population, 1965–1966.

State in 1966	State in 1965																					
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0·802																					
2	0·188	0·868																				
3	0·003		0·879																			
4	0·001	0·131	0·003	0·759																		
5				0·035	0·370																	
6			0·118	0·153																		
7				0·053																		
8					0·109																	
9					0·521																	
10						0·011	0·043			0·339												
11							0·030	0·112	0·031	0·249	0·430											
12								0·013	0·072		0·023	0·567										
13													0·370				0·009					
14						0·021	0·040	0·087	0·048					0·307							0·002	
15							0·008	0·127	0·038		0·037				0·623		0·002		0·011	0·002		
16									0·039		0·015					0·760	0·004					
17									0·504	0·018	0·099	0·036					0·669					
18																0·050	0·078	0·444				
19														0·003	0·269	0·007	0·004	0·068	0·939			
20															0·024		0·002	0·040	0·008	0·952		
21						0·967	0·879	0·662	0·267	0·394	0·396	0·396	0·630	0·688	0·085	0·182	0·231	0·448	0·006	0·004	0·975	
22																			0·022	0·027	0·013	0·910

Source: Stone (1972, page 70, table 2).

Key:
1 Preschool
2 Nursery and primary school
3 Special school
4 Secondary school: 1st level
5 Secondary school: 2nd level
6 Final school year: no certificates
7 Final school year: O-levels
8 Final school year: one A-level
9 Final school year: more than one A-level
10 Further education: GCE O-level/OND/ONC
11 Further education: GCE A-level/HND/HNC
12 Further education: external 1st degree
13 Further education: external 2nd degree
14 Further education: other courses
15 Teacher training college
16 University 1st degree: medical
17 University 1st degree: other
18 University 2nd degree
19 Schoolteacher
20 Other teacher
21 Other employment
22 Home and retirement

Table 8. The fundamental matrix, $(\mathbf{I}-\mathbf{C})^{-1}$, based on table 7: England and Wales, male population, 1965–1966.

Subsequent state	Starting state 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	5·05																					
2	7·16	7·55																				
3	0·13	0·02	8·29																			
4	3·93	4·11	0·09	4·14																		
5	0·22	0·23		0·23	1·59																	
6	0·61	0·63	0·99	0·63		1·00																
7	0·21	0·22		0·22			1·00															
8	0·02	0·02		0·03	0·17			1·00														
9	0·11	0·12		0·12	0·83				1·00													
10	0·05	0·05	0·04	0·05	0·02	0·04	0·09	0·02	0·02	1·54	0·02	0·03	0·03	0·03	0·01	0·02	0·02	0·02			0·03	
11	0·04	0·05	0·02	0·05	0·09	0·02	0·09	0·21	0·06	0·67	1·77	0·01	0·01	0·01		0·01	0·01	0·01			0·01	
12	0·02	0·03		0·03	0·15		0·01	0·04	0·17	0·04	0·10	2·32										
13					0·02				0·02		0·01		1·59				0·04					
14	0·19	0·19	0·18	0·20	0·22	0·18	0·21	0·27	0·21	0·15	0·15	0·15	0·15	1·60	0·05	0·14	0·14	0·13	0·03	0·01	0·15	
15	0·07	0·07	0·04	0·07	0·23	0·04	0·07	0·45	0·18	0·12	0·25	0·04	0·04	0·04	3·08	0·07	0·08	0·11	0·57	0·16	0·04	
16	0·03	0·03	0·01	0·03	0·17	0·01	0·01	0·02	0·20	0·05	0·12	0·01	0·01	0·01		4·18	0·06				0·01	
17	0·23	0·24	0·05	0·24	1·34	0·05	0·07	0·10	1·60	0·33	0·58	0·30	0·05	0·05	0·02	0·04	3·06	0·04	0·01		0·05	
18	0·06	0·07	0·04	0·07	0·23	0·04	0·04	0·04	0·27	0·08	0·12	0·07	0·04	0·04	0·01	0·41	0·46	1·83	0·01		0·04	
19	0·43	0·44	0·25	0·45	1·40	0·25	0·40	2·06	1·27	0·66	1·29	0·31	0·24	0·32	13·62	1·31	1·11	2·57	18·96	0·71	0·24	
20	0·17	0·17	0·10	0·17	0·58	0·09	0·14	0·59	0·57	0·24	0·45	0·14	0·09	0·11	3·70	0·59	0·71	1·99	3·32	20·86	0·09	
21	43·85	45·29	46·16	45·53	43·16	46·17	45·82	42·11	43·37	45·16	43·67	45·94	46·19	45·90	16·46	43·41	43·26	38·52	8·18	4·37	46·19	
22	6·59	6·80	6·86	6·84	6·84	6·86	6·86	6·85	6·83	6·85	6·85	6·85	6·86	6·84	6·79	6·85	6·82	6·85	6·75	6·97	6·86	11·05

Source: Stone (1972, page 71, table 3).

Key:
1 Preschool
2 Nursery and primary school
3 Special school
4 Secondary school: 1st level
5 Secondary school: 2nd level
6 Final school year: no certificates
7 Final school year: O-levels
8 Final school year: one A-level
9 Final school year: more than one A-level
10 Further education: GCE O-level/OND/ONC
11 Further education: GCE A-level/HND/HNC
12 Further education: external 1st degree
13 Further education: external 2nd degree
14 Further education: other courses
15 Teacher training college
16 University 1st degree: medical
17 University 1st degree: other
18 University 2nd degree
19 Schoolteacher
20 Other teacher
21 Other employment
22 Home and retirement

shown to be dependent on previous attainment. Thus boys attaining more than one A-level can expect to spend 2·07 years at university (the entries in rows 16, 17, and 18 of column 9 added up) whereas those attaining no certificates can expect to spend only 0·10 of a year at university. Note that the latter figure is not zero. Of course these are averages made up of some people spending two, three, four, five, or six years at university and many spending no years there.

Stone (1972, pages 75–77; 1975, pages 42–50) discusses various ways in which the accounts framework and derived models can be extended. The accounts themselves must be regarded as simply the starting steps in any investigation of a complex system. Simple projections forward of the tendencies observed in the system (as in table 8) will not on their own be satisfactory if the system exhibits supply constraints or bottlenecks. The monograph by Armitage et al (1969) considers in detail how such systems should be studied.

3.2 *Socioeconomic-group accounts*

Other major systems described at length in Stone (1975) are those involving "social class, stratification and mobility" (chapter 12) and "earning activities, employment services and the inactive" (chapter 18). Normally, social stratification and mobility are studied using elaborate social surveys, and attention is focussed on intergenerational mobility, say between father's and son's occupation at a given age, over an indeterminate time period. Illingworth (1976) has attempted to construct matrices of the flows between socioeconomic groups over a specified period by use of census as well as survey data. Table 9 is an aggregated version of his figure 9.2. The table is a set of

Table 9. Socioeconomic-group accounts for males in England and Wales and the rest of the world, 1961-1966 (constructed under the high-stay hypothesis [a]).

Initial state	Final state								Total [c]
	survival at 1966 census [b]						deaths 1961-1966		
	EW					RW	EW	RW	
	0-9	10-14	NM	M	EI				
Births 1961-1966									
EW	2160·5					9·1	20·0	0·1	2189·8
RW	24·7						0·1		24·8
Existence at 1961 census [b]									
EW									
0-9	1798·2	1633·4				19·9	64·6	0·6	3516·8
10-14			344·3	382·4	1069·4	26·0	84·4	0·8	1907·3
NM			3926·1	15·4	43·3	99·5	322·3	3·2	4409·8
M			34·8	8891·0	108·4	248·9	806·6	7·9	10097·6
EI			428·4	475·7	1330·1	32·8	104·3	1·0	2372·4
RW	35·6	32·3	94·2	104·2	291·4		11·8		569·6
Total [c]	4019·0	1665·7	4827·7	9868·8	2842·6	435·7	1414·7	13·7	25088·1

Source: aggregated and estimated from Illingworth (1976, page 294, figure 9.2).
Key: EW—England and Wales; RW—rest of the world; 0-9—boys aged 0-9; 10-14—boys aged 10-14; NM—nonmanual workers; M—manual workers; EI—economically inactive.
Note: net figures are in thousands.

[a] The high stay hypothesis is one in which the diagonal probabilities (of staying in the same state) are set to their highest possible values subject to the marginal constraints.

[b] Circa 23/24 April.

[c] The discrepancies between some of the totals and the corresponding row and column sums are due to rounding.

closed demographic accounts with some age categories and some socioeconomic categories as the states.

The table provides a rich set of observations on the changing character of the English and Welsh social system. Although the numbers of economically active males increased by 1·3% over the five-year period, this overall increase conceals a decrease of 2·3% in manual workers and an increase of 9·5% in nonmanual workers. Relatively little of the growth in nonmanual workers (417900) can be attributed (in this estimate) to (intragenerational) social mobility (net gain of 19400). There is a minor surplus of recruits (from the 10–14 age group) over persons dying (18800), with the main net inflow coming from the economically inactive (385100). The recruits from this category are, of course, persons still in education, and the people leaving the nonmanual occupations to economic inactivity are mainly those retiring. Thus a fairly rapid transformation of the social structure is being accomplished by differential entry and exit from the occupational system, predicated on a changing pattern of demand for different occupations. A more up-to-date version of these accounts would, however, reveal a slowing of the growth of white-collar occupations and an increase in the numbers of the economically inactive.

3.3 *Accounts classified by age and sex*

Age and sex have been variables of continuing interest to population researchers. What is surprising, perhaps, is that so much analysis has been undertaken without the benefit of the corresponding closed demographic accounts disaggregated by age and sex. The reason is probably that an alternative framework—that of the life table—was adopted much earlier and that national demographers developed vital and census statistics that they felt supplied the data needs of the life table and associated projection models adequately.

Such a framework may be adequate where the area being studied constitutes a closed entity, little influenced by outflows to or inflows. When a country like England and Wales is considered, or when a region within a country is studied, the assumption that the unit is a closed system is untenable. Table 10, containing a set of age-disaggregated demographic accounts for females for the intercensal period 1961–1966 in England and Wales, shows that out-migration removes 436000 women from the population compared with mortality's 1370000, and that in-migration adds 582000 compared with fertility's 2141000. Surviving in-migrants and out-migrants make up 29·0% of the sum of the vital flows. No attempt to 'fudge' the closure problem by using net-migrant concepts will do: the pattern of net migration by age in table 10 shows extraordinary variation between positive and negative values in successive age groups, which no net-migration model could hope to deal with.

The structure of table 10 is a familiar one and is a transposed version of the matrix version of the cohort-survival model proposed in the 1940s by Bernadelli (1941), Lewis (1942), and Leslie (1945). Survivors within England and Wales are entered in the diagonal one above the principal diagonal, indicating a complete transfer from one age group (of age interval five years) to the next, with the exception of the last, semi-closed, age group where there is an entry in the principal diagonal. Although this arrangement of the accounts is often inconvenient when elements are being estimated, it is essential if the accounts are to be used in projection. If there are entries in the diagonal (as in Rees and Wilson, 1977, pages 210–211, figure 13.33, or in Stone, 1975, page 45, equation VIII.20) then use of the transition proportions in projection leads to erroneous results: people survive longer in an age group than that age group is long.

The birth entries are placed in the last row of the matrix, although components of the births total, classified by age of mother at the start of the intercensal period, have been bracketed in the appropriate positions in the main body of the table.

Table 10. Age-disaggregated demographic accounts for females in England and Wales, 1961-1966.

Initial state	Final state																Total
	age group at 1966 census														out-migrants	deaths	
	0-4	5-9	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65+			
Age group at 1961 census:																	
0-4		1805856													29788	10496	1846141
5-9			1640313												26843	3465	1670622
10-14	(82422)			1837614											63474	6218	1907306
15-19	(465030)				1550516										62637	8806	1621960
20-24	(683804)					1366905									59972	7507	1434385
25-29	(490534)						1377991								60175	7811	1445979
30-34	(264753)							1462204							28237	11197	1501637
35-39	(105414)								1566536						30192	19466	1616195
40-44	(18669)									1454232					9640	29950	1493822
45-49	(624)										1518398				10166	55239	1583802
50-54												1470427			10175	94800	1575403
55-59													1254790		6475	146580	1407846
60-64														913564	2998	179749	1096311
65+														1313145	5756	783526	2102426
In-migrants	24443	55170	47093	44221	77620	91888	82747	44250	41290	19666	18052	16029	16339	8341	n.a.	4837	581986
Births	2111250														29305	280	2140835
Total	2135693	1861027	1687405	1881835	1628137	1458793	1460739	1506454	1607827	1473898	1536450	1486456	1261129	2235047	435833	1369927	25026650

Source: aggregated from Rees et al (1977, figure 3.18, pages 100-101).

Key: n.a.—not available.

Note: bracketed figures do not contribute to row and column sums. The slight discrepancies between row and column totals and the sums of interior elements in the table are due to rounding in the accounts estimation computer program itself used in Rees et al (1977).

Surviving in-migrants from the rest of the world fall in the penultimate row of the accounts matrix, and surviving out-migrants are placed in the penultimate column.

The accounts of table 10 should really have been set out, had space permitted, in fully expanded form, with out-migrants and deaths classified by final as well as initial location and age group, and in-migrants classified by initial as well as final location and age group. Births should be classified by region of birth and mother's age group at the start of the period as well as by region of survival, aged 0–4, at the end of the period. An alternative classification might be by mother's initial location at the start of the period, since this makes a multistate application of Leslie's matrix model more straightforward, but such a classification is rarely available. An expanded version of the table 10 accounts is given in appendix 3 of Rees (1979b).

3.4 *Multiregional demographic accounts, classified by age and sex, for a base period*
The table 10 accounts concern a national territory. However, this state should be broken down into its constituent regions if we are interested in the monitoring and projection of regional populations. In a report (Rees, 1977b; East Anglia Economic Planning Council, 1979) on the future population of East Anglia (Britain's fastest growing region) a set of multiregional demographic accounts, disaggregated by age and sex, for a four-region system consisting of East Anglia, the South East, the rest of Great Britain, and the rest of the world were prepared for the intercensal period 1966–1971. Presentation of such multistate accounts in explicit form would occupy a vast and largely empty matrix, so instead the accounts were presented in normal tabular form in Rees (1977b) and age cohort by age cohort in more compact tables in Rees (1978b; 1979b). In order to achieve compactness the death terms, which in appendix 3 of Rees (1979b) were classified by age group at the start of the period and by age group at death, were aggregated by adding together terms in each row of the full matrix.

These East Anglian accounts have been used as the base-period data in a multiregional projection of East Anglia's population (Rees, 1977b) and also in the development of multiregional life tables (Rees, 1979a), although in the latter case the information concerning flows to and from the rest of the world was ignored.

3.5 *A time series of multiregional demographic accounts*
Single or 'one-off' sets of accounts, such as those described in the preceding examples, are rarely satisfactory since the migration and fertility behaviour and mortality experience of any population are continuously changing. What is needed instead is a time series of such accounts, relevant to the problem in hand, over the recent past. Such a time series is presented in the appendix, covering the years from mid-1970 to mid-1976 for the four sections of Great Britain for which other accounts have already been presented in tables 1, 2, 3, and 4. The problems posed and procedures involved in the estimation of this time series of accounts are discussed in the next section of the paper. Although the discussion is largely specific to this accounts example, a number of general principles governing accounts estimation are proposed.

4 Estimation procedures for multistate demographic accounts

4.1 *General principles*
The main purpose of this section is not to give an exhaustive description of estimation procedures for accounts building or of the main estimation models involved (see Stone, 1971a; 1975; Rees and Wilson, 1977) but rather to illustrate how the procedures can be applied in a particular case and how they need to be adapted.

The main principles involved in accounts construction can be summarized in a series of instructions as follows.

(1) *The specific purpose* must be determined for which accounts are being constructed. This may involve the specification of the projection model for which the accounts will form the data base, though often the accounts framework will profoundly affect such a specification.
(2) *A theoretical specification* for the accounts must be designed in terms of 'entities' to be accounted for and states between which the 'entities' will transfer. This specification or disaggregation should not be too ambitious or problems of dimensionality will be encountered (cf Rogers, 1976). However, a state such as 'the outside world', as used in Stone's work (see, for example, table 6 in this paper), is probably too aggregated to be useful and should be broken down into separate births, deaths, and rest-of-the-world categories. Often more disaggregation may be needed at the accounts estimation stage than at the later application stage.
(3) *All tables of demographic and socioeconomic data relevant to the specification* must be assembled and the degree of mismatch between the accounts design and available data supply determined. An attempt should then be made to separate the resulting estimation problems into those in which reasonable data are available and those in which the data are unlikely to be forthcoming (usually involving parts of the accounts matrix such as the exist–die quadrant or the born–die quadrant).
(4) *A series of estimation procedures* must be designed to convert reasonable data into the form required for the accounts.
(5) *An accounts-based model* must in general be designed in order to solve this last kind of problem (for details of some of the alternatives, see Rees and Wilson, 1977; Illingworth, 1976; Jenkins and Rees, 1977).
(6) To the initial estimates of the accounts matrices must be applied any additional *constraints* that may be available by use of the well-known 'biproportional matrix' or 'balancing factor' methods (Bacharach, 1970; Macgill, 1975). When constraining row and column totals are used, they are often in conflict and judgment must be used in selecting the best set.

4.2 *The example of the four sections of Great Britain: general outline*
The accounts for sections of Great Britain set out in the appendix are aggregates of those for the British standard regions developed in a study of demographic change in Great Britain. The main purpose was to explore solutions to accounts-building problems at the aggregate scale before applying them to population accounts disaggregated by age and sex, to be used in population projection.

Closed demographic accounts for the all-ages-and-both-sexes population of the standard regions of Great Britain were to be developed. The definitions of the aggregate regions or 'sections' for which tables of statistics are presented are given in note 1 to table 1. Northern Ireland was not included in the internal set of regions because accurate data on migrants to Northern Ireland from the mainland regions were not available.

Three choices of single-year period were available for accounts constructions, as set out in figure 1: the calendar year between 1 January and 31 December; the 'midyear' from 30 June/1 July in one year to 30 June/1 July in the next; and the census year between the census date at the end of April (25/26 April in 1971) in one year and

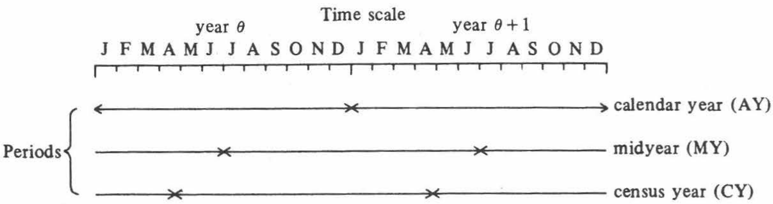

Figure 1. Alternative accounting years.

the same date in April in the next. Vital statistics (births, deaths, and international migrations) are most easily available for calendar years; official population estimates are prepared at midyear and official population projections start from a midyear base; internal migration tables are available only for the year (or five years) prior to the census, taken in late April.

If good time series were available for all demographic components, choice of the appropriate accounting year would not matter as interpolation from one type of year to another could be easily accomplished. However, since the internal migration statistics were available only for one census year (1970-1971), it was decided to build accounts initially for census years and, when the best methods of accounts building had been determined, to use the transition matrix for the 1970-1971 census year as the basis for the estimating midyear accounts for 1970-1976.

Figure 2 shows the strategy adopted. Four different ways (see subsection 4.4) are used to assemble accounts for the base period, 1970-1971 (the year prior to the

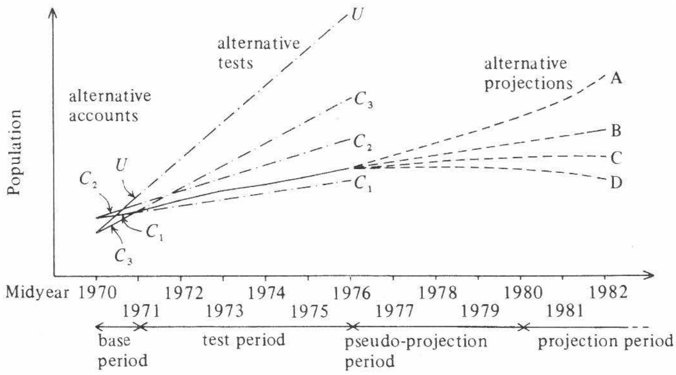

Figure 2. The structure of alternative accounts, tests, and projections.

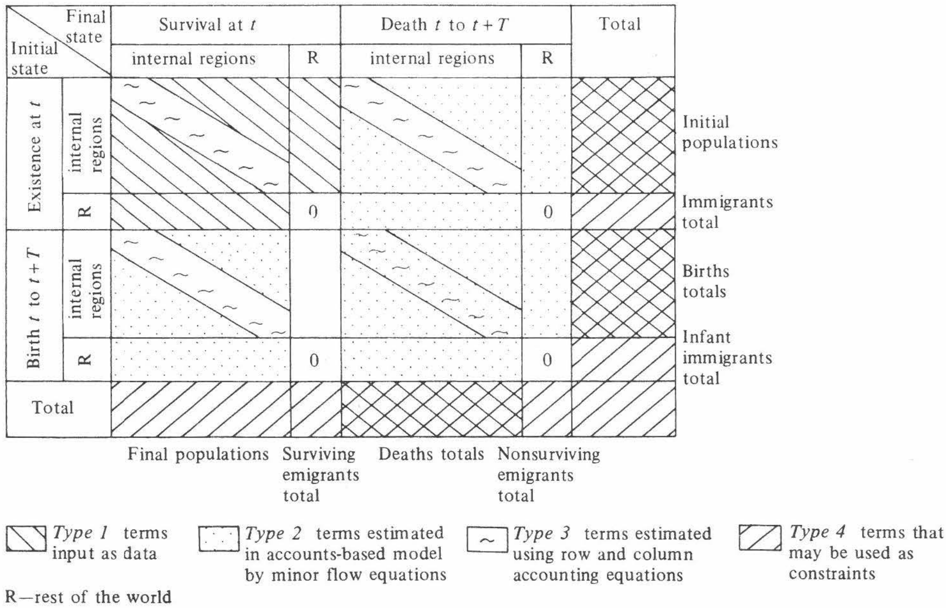

Figure 3. The terms in an accounts table classified by method of estimation.

census of 1971 in April of that year). Then the accounts are used as the base-period data in a series of projections through to 1976 (the latest year for which estimates were available at the time of computation, in early 1978) either with rates fixed at their 1970-1971 levels or with birth and death rates and external-migrant vectors allowed to take on their estimated values for the intervening years 1971-1976. In both cases the internal migration rates remain fixed at their 1970-1971 values. These internal migration rates are, however, rather sensitive to the method of accounts building adopted, and a comparison of 'projected' and estimated populations reveals which method gives the best-fit accounts. This method can then be adopted to construct accounts for the individual years 1971-1972, 1972-1973, 1973-1974, 1974-1975, and 1975-1976, as set out in the appendix.

In figure 3 are set out the items of an accounts table classified in terms of their origin. Type 1 terms are input as data to an accounts-based model; type 2 terms are

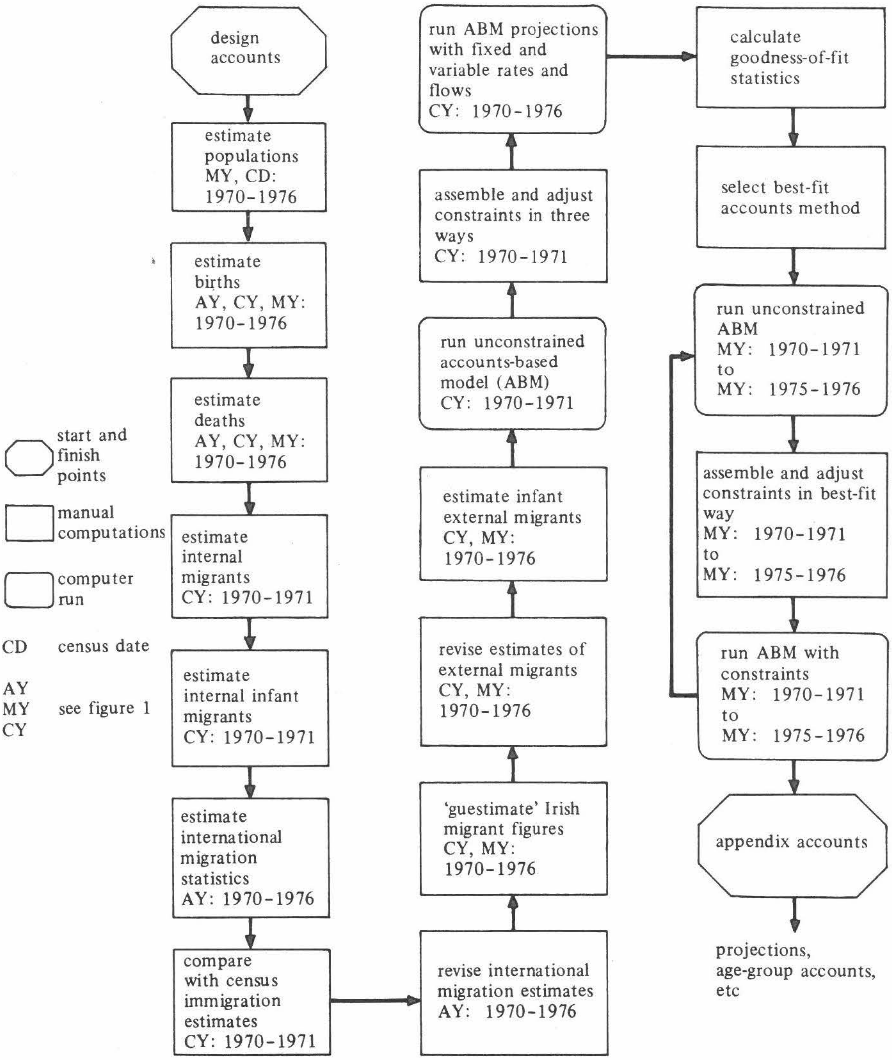

Figure 4. Steps in the development of a time series of multistate demographic accounts for sections of Great Britain, 1970-1976.

estimated by simple equations in the accounts-based model; type 3 terms are computed as residuals by use of the row and column equations. The exact scheme of equations differs according to whether aggregate, semiaggregate (the existence and birth parts of the aggregate accounts are treated separately), or age-disaggregated accounts are being constructed. Aggregate-model equations are set out in Rees and Wilson (1977, part 2), in Jenkins and Rees (1977), and in Illingworth (1976); the semiaggregate model is described in Jenkins and Rees (1977); the age-disaggregated model is set out in Rees and Wilson (1977, part 3), in Rees et al (1977), and in simpler form in Rees (1978b).

Type 4 elements in the accounts table (figure 3 are the row and column totals which may be used as constraints to which the initial estimate of the accounts matrix is adjusted (as spelled out in Rees, 1978b).

The steps undertaken in the development of the time series of multistate demographic accounts are set out in figure 4. Each step is described briefly and the outputs displayed in the remainder of section 4.

4.3 *Assembly of tables of demographic statistics and estimation of data required as input to the accounts-based model*

The step of assembling tables of demographic statistics relevant to the task at hand should be a relatively simple operation but rarely is.

Table 11 contains the first such set of statistics, for population, Midyear population estimates were used since the Office of Population Censuses and Surveys (OPCS, 1975a) had prepared such a series for the 'new' regions (post-April-1974 definitions). The most reliable estimates are those for 1971 (the year of the census) and the accuracy (unknown) of the estimates decays away from this date. These population estimates are for the home population definition, the most appropriate method for accounts building. Census-date (25/26 April) estimates were interpolated between midyears by means of an exponential interpolation function (table 11.2).

Table 11. Population estimates for sections of Great Britain, 1970–1976.

	North	Midlands	South	Celtic Fringe	Great Britain
11.1 Midyear estimates [a]					
1969	14563·0	8653·0	22613·0	7919·5	53748·5
1970	14576·0	8700·0	22687·0	7930·7	53893·7
1971	14607·7	8756·1	22767·0	7941·0	54071·8
1972	14627·2	8815·2	22861·3	7945·0	54248·7
1973	14632·3	8859·2	22933·8	7961·0	54386·3
1974	14619·0	8894·1	22930·9	7973·8	54417·8
1975	14599·2	8903·9	22932·0	7970·5	54405·6
1976	14568·5	8898·0	22950·6	7971·9	54389·5
11.2 Census-date estimates [b]					
1970	14573·6	8691·5	22673·6	7928·6	53867·3
1971	14602·0	8745·9	22752·5	7939·1	54039·5
1972	14623·7	8804·5	22844·2	7944·3	54216·7
1973	14631·4	8851·2	22920·6	7958·1	54361·3
1974	14621·4	8887·8	22931·4	7971·5	54412·1
1975	14602·8	8902·1	22931·8	7971·1	54407·8
1976	14574·0	8899·5	22947·2	7971·6	55492·3

Note: all figures are in thousands.

[a] Aggregated from OPCS (1975, page 40, table 8; 1977b, page 43, table 17).

[b] Interpolated from subtable 11.1 by use of an exponential interpolation function.

Birth estimates (table 12) were taken from the OPCS (1977b) publication *Population Trends 9*, principally because it provides estimates converted to the new-region basis for years prior to local government reorganization. Annual births (for the calendar year) had to be converted to census-year and midyear figures and this was done using simple apportionment fractions:

$$B^i(\theta, \theta+1) = F_2 B^i(\theta) + F_1 B^i(\theta+1) \ , \tag{13}$$

where B^i denotes the live births in region i and θ and $\theta+1$ are the labels attached to successive calendar years which are, in pairs, used to identify single census years or midyears. F_1 refers to the fraction of births falling in the first part of the year and F_2 that falling in the second part of the year. These are computed from national quarterly or monthly births figures.

A similar equation is used to produce the corresponding estimates of deaths in the regions (table 13).

Information on migration between British regions is, unfortunately, collected only at the periodic censuses, so that only one table, for the year prior to census date in 1971, can be presented (table 14). However, this table does give the right type of migration statistics: those for persons [migrants in Courgeau's (1973) typology; transitions in Ledent's (1978c) typology] rather than for moves (migrations or movement). Inclusion of move-type data in an accounts table (or derived set of projections or multiregional life tables) results in an overestimation of the amount of initial-state-final-state change occurring in the system.

Table 12. Birth estimates for sections of Great Britain, 1970-1976.

	North	Midlands	South	Celtic Fringe	Great Britain [a]
12.1 Calendar-year births [b]					
1970	242·5	149·2	350·2	129·8	871·7
1971	240·7	149·2	350·2	129·8	869·9
1972	219·9	137·2	327·9	118·6	804·0
1973	201·4	128·3	308·3	112·0	750·3
1974	190·4	120·1	292·5	106·3	710·0
1975	180·1	112·6	276·3	101·9	671·4
1976	174·4	109·0	267·5	101·7	652·6
12.2 Census-year births [c]					
1970-1971	241·9	149·2	350·2	129·8	871·1
1971-1972	233·9	145·3	342·9	126·1	848·2
1972-1973	213·8	134·3	321·5	116·4	786·0
1973-1974	197·8	125·6	303·1	110·1	736·6
1974-1975	187·0	117·6	287·2	104·9	696·7
1975-1976	178·2	111·4	273·4	101·8	664·9
12.3 Midyear births [d]					
1970-1971	241·5	149·2	350·1	129·8	870·6
1971-1972	230·0	143·0	338·7	124·0	835·7
1972-1973	210·4	132·6	317·7	115·2	775·9
1973-1974	195·7	124·1	300·1	109·1	728·9
1974-1975	185·1	116·2	284·1	104·0	689·5
1975-1976	177·1	110·7	271·8	101·8	661·4

Note: all figures are in thousands.

[a] Considerable rounding error is apparent in the original statistics (OPCS, 1977b) and the Great Britain totals do not, as a result, always add up to the total of sectional births.

[b] Aggregated from OPCS (1977b, page 48, table 22).

[c] Computed from subtable 12.1. $F_1 = 0{\cdot}327272$; $F_2 = 0{\cdot}672728$.

[d] Computed from subtable 12.1. $F_1 = 0{\cdot}513029$; $F_2 = 0{\cdot}486791$.

Table 13. Death estimates for sections of Great Britain, 1970-1976.

	North	Midlands	South	Celtic Fringe	Great Britain [a]
13.1 Calendar-year deaths [b]					
1970	182·4	95·4	262·5	98·6	638·9
1971	178·7	93·6	260·2	96·4	628·9
1972	187·2	98·6	268·5	101·0	656·9
1973	184·2	97·9	267·8	100·3	652·0
1974	183·1	98·3	266·5	100·3	650·0
1975	180·7	97·7	267·0	98·6	645·9
1976	180·2	101·7	282·6	101·2	665·6
13.2 Census-year deaths [c]					
1970-1971	181·1	94·8	261·7	97·8	635·4
1971-1972	181·7	95·3	263·1	98·0	638·1
1972-1973	186·2	98·4	268·3	100·8	653·5
1973-1974	183·8	98·0	267·3	100·3	649·5
1974-1975	182·3	98·1	266·7	99·7	646·7
1975-1976	180·5	99·1	272·4	99·5	651·5
13.3 Midyear deaths [d]					
1970-1971	180·5	94·5	261·3	97·4	633·7
1971-1972	183·1	96·2	264·5	98·8	642·7
1972-1973	185·6	98·2	268·1	100·6	652·6
1973-1974	183·6	98·1	267·1	100·3	649·2
1974-1975	181·8	98·0	266·8	99·4	646·0
1975-1976	180·4	99·8	275·1	100·0	655·3

Note: all figures are in thousands.

[a] Considerable rounding error is apparent in the original statistics (OPCS, 1977b) and the Great Britain totals do not, as a result, always add up to the total of sectional deaths.

[b] Aggregated from OPCS (1977b, page 62, table 31).

[c] Computed from subtable 13.1. $F_1 = 0{\cdot}347765$; $F_2 = 0{\cdot}652235$.

[d] Computed from subtable 13.1. $F_1 = 0{\cdot}522958$; $F_2 = 0{\cdot}477042$.

Table 14. Internal migrants for sections of Great Britain, 1970-1971.

Origin/ place of birth	Destination: section of residence, 25/26 April 1971				Total
	North	Midlands	South	Celtic Fringe	
14.1 Exist-survive migrants [a]					
North	–	44300	90800	27260	162360
Midlands	40010	–	76030	15520	131560
South	67030	70330	–	42020	179380
Celtic Fringe	26830	18230	52630	–	97690
Total	133870	132860	219460	84800	570990
14.2 Infant migrants [b]					
North	–	369	754	189	1312
Midlands	342	–	652	134	1128
South	517	544	–	324	1385
Celtic Fringe	304	148	430	–	882
Total	1163	1061	1836	647	4707

Note: all values are derived from 10% sample figures multiplied by ten.

[a] Aggregated from unpublished table DT4312 (OPCS, 1976), subsequently published in OPCS (1978a).

[b] Estimated from subtable 14.1 and table 12.

Data on infant migrants (subtable 14.2) should be readily available from the same source as the migrant data but are not. So the data in subtable 14.2 are estimated using the following equations:

$$K^{\beta(i)\sigma(j)} = \tfrac{1}{2}h^{\epsilon(i)\sigma(j)}K^{\beta(i)*(*)}\,, \tag{14}$$

$$h^{\epsilon(i)\sigma(j)} = \frac{K^{\epsilon(i)\sigma(j)}}{K^{\epsilon(i)*(*)}}\,, \tag{15}$$

where $h^{\epsilon(i)\sigma(j)}$ is the transition rate from region i to region j by persons in existence at the start of the accounting period and alive at the end of it. The assumption is

Table 15. International migration statistics for sections of Great Britain, 1970-1976.

	England				Celtic Fringe	Great Britain [a]	United Kingdom
	whole	North	Midlands	South			
15.1 Original data [b]							
Immigration							
1970	207·0	–	–	–	17·2	224·2	225·6
1971	183·0	32·8	20·0	130·1	15·2	198·2	199·7
1972	204·2	31·7	24·6	147·9	16·8	221·0	221·9
1973	178·6	30·2	23·0	125·3	16·1	194·7	195·7
1974	166·3	30·4	17·6	118·4	15·8	182·1	183·8
1975	–	–	–	–	–	–	197·2
1976	–	–	–	–	–	–	179·8
Emigration							
1970	252·7	–	–	–	31·9	284·6	290·7
1971	209·0	51·4	26·5	131·1	27·3	236·3	240·0
1972	199·4	44·2	20·5	134·7	27·4	226·9	233·2
1973	213·7	45·9	24·1	143·8	26·1	239·9	245·8
1974	231·8	53·6	27·3	150·8	32·2	264·1	269·0
1975	–	–	–	–	–	–	238·3
1976	–	–	–	–	–	–	210·4
15.2 Converted data [c]							
Immigration							
1970		36·5	23·3	147·2	17·2	224·2	
1971		32·3	20·5	130·1	15·2	198·2	
1972		31·2	25·1	147·3	16·8	221·0	
1973		29·7	23·5	125·3	16·1	194·7	
1974		30·4	17·6	118·4	15·8	182·1	
1975		32·7	18·9	127·0	17·0	195·5	
1976		29·7	17·2	115·8	15·5	178·2	
Emigration							
1970		61·2	33·0	158·5	31·9	284·6	
1971		50·6	27·3	131·1	27·3	236·3	
1972		43·5	21·2	134·7	27·4	226·8	
1973		45·2	24·8	143·8	26·1	239·9	
1974		53·6	27·3	150·8	32·4	264·1	
1975		47·3	24·2	133·6	28·7	234·0	
1976		41·9	21·4	117·9	25·3	206·6	

Note: all figures are in thousands.

[a] The discrepancies between some Great Britain totals and the sums of the corresponding rows are due to rounding.

[b] Sources: 1970—OPCS (1977a, table 2.7); 1971-1974—OPCS (1977a, tables 2.7 and 2.15); 1975-1976—OPCS (1977b, table 26). The 1970-1973 data in table 15.1 are for old regions; the 1974-1976 data are for new regions. The converted data in table 15.2 all refer to new regions.

[c] Estimated from data in table 15.1.

made that the infants migrate at the rate of the rest of the population but have only half of the period, on average, in which to accomplish the migration.

The matrix of transition rates for 1970–1971 is used to estimate the internal migratory behaviour of the population in the period 1971–1976. Estimates of this behaviour are prepared by the OPCS from National Health Service Register transfers as part of the process of producing net migration estimates as input to the population estimates themselves [see equation (2)], but they are not readily available. Good annual estimates of interregional migration could be very simply generated from *The General Household Survey* (OPCS, 1973, chapter 5), but OPCS is reluctant to disaggregate its sample spatially. However, a glance forward at the internal migrant figures in the appendix accounts shows that the methods adopted here result in estimates rather more invariant than is probably the case.

International migration statistics are available for calendar years between 1970 and 1976 but they (table 15) pose a number of difficult estimation problems.

The data subtable 15.1 are based on the International Passenger Survey (IPS) (OPCS, 1978b, pages 10–13), a 1–2% sample survey of passengers arriving at or leaving UK airports and seaports. No attempt is made to survey traffic (and migrants) between the UK and the Irish Republic, some ports are omitted from the survey, and no account is taken of military traffic. Migrants in the survey are respondents in indicating an intention to stay at least one year at their destination. Disaggregation by region of origin or destination was introduced only in 1971, and was unavailable at time of compilation for 1975 and 1976. To fill out the table and to produce subtable 15.2 the regional proportions of 1971 were used to break down the 1970 statistics, and the proportions for 1974 were used to break down the 1975 and 1976 proportions.

It was felt important to check the accuracy of the IPS statistics against equivalent statistics for 1970–1971 available in the migration tables from the 1971 census (OPCS; 1978a). This is done in table 16 for the immigration stream from 'outside the British Isles' to 'UK regions' (emigration figures are, of course, unavailable at the census). Row (5) contains the IPS estimates adjusted to the 1970–1971 census year

Table 16. Comparison of IPS and census immigration estimates for sections of Breat Britain, 1970–1971.

			North	Midlands	South	Celtic Fringe	Great Britain [a]
Census 1970–1971							
abroad	(A)	(1)	59·6	41·7	246·9	37·0	385·2
elsewhere in British Isles	(EBI)	(2)	8·0	4·6	21·0	3·2	36·8
outside British Isles	(A−EBI)	(3)	51·6	37·1	225·9	33·8	348·4
Census plus estimates of other accounts terms							
	(A−EBI)	(4)	52·4	37·8	228·9	34·5	353·5
IPS immigration estimates		(5)	35·5	22·6	143·2	16·7	218·1
Ratio of row (4) to row (5) x 100		(6)	148	167	160	207	162
Ratio of row (3) to row (4)		(7)	0·985	0·981	0·987	0·980	0·986
Combined ratio of row (3) to row (5)		(8)	1·45	1·64	1·58	2·02	1·60

Note: all immigration figures are in thousands.

Sources: rows (1) to (3)—aggregated from unpublished table DT4312 (OPCS, 1976), subsequently published in OPCS (1978a); row (4)—estimated using factors from accounts for 1970–1971 given in Rees (1976); row (5)—estimated from the 1970 and 1971 rows of subtable 15.2, immigration section.

[a] The discrepancies between some Great Britain totals and the sums of the corresponding rows are due to rounding.

by use of the equivalent of equation (13) (the F_1 and F_2 proportions are given in table 19). The census figures are given in row (3) of table 16, but these are inflated marginally in row (4) to include other kinds of migrants (nonsurvivors, infants, and nonsurviving infants) in order to make the match with the IPS statistics more exact. A comparison of rows (4) and (5) is disturbing. Row (6) of the table reveals that the census figures are 50 to 100% larger than the IPS estimates. Clearly one has either to believe the census figures or the IPS estimates, and in terms of relative reliability it must be the census that is chosen.

Therefore revised estimates of immigrants to and emigrants from the four sections of Great Britain (to and from the world outside the British Isles) were prepared (table 17) by multiplying the figures in subtable 15.2 by the ratios given in row (8) of table 16 of the census statistics in row (3) to the IPS estimates in row (5).

To these estimates must be added estimates of migrants to and from Northern Ireland and to and from the Irish Republic, the Isle of Man, and the Channel Islands (table 18). The immigrant figures for the 1970–1971 census year derive from the census migration tables. To the total immigration from Northern Ireland to Great Britain is added the net-migration estimate for Northern Ireland available in OPCS (1977b, tables 3 and 4), and the resulting emigrant total is distributed among the regions in the same proportion as immigrants. The net-migration estimates for successive years are related to immigrant and emigrant totals in the same ratio as in 1970–1971, and the totals are allocated to regions in the proportions observed in 1970–1971. For the other parts of the British Isles all that was available was an estimate of migration between the Irish Republic and the UK (Central Statistical Office, 1970, table 18): the ratio of emigrants to immigrants was applied to the 1971 census immigrants figure for the Irish Republic, the Isle of Man, and the Channel Islands (OPCS, 1978a). The resulting statistics are no more than 'guestimates': the figures for census years and midyears have been assumed to be approximately equal, and the flows to and from the Irish Republic are assumed to continue at their 'guestimated' 1970–1971 levels, in the absence of any other information.

The grand totals of external migrant flows to and from the sections of Great Britain are presented in table 19 in census-year form and midyear form. Table 19 is simply

Table 17. Revised estimates of immigrants and emigrants for sections of Great Britain, 1970–1976.

	North	Midlands	South	Celtic Fringe	Great Britain
Immigrants					
1970	53·1	38·2	232·2	34·4	357·9
1971	46·9	33·7	205·2	31·8	317·6
1972	44·7	42·4	233·7	35·4	356·2
1973	44·1	37·9	197·8	33·3	313·1
1974	44·9	29·1	190·9	32·2	297·1
1975	48·3	31·2	204·8	34·6	318·9
1976	44·0	28·4	186·7	31·5	290·6
Emigrants					
1970	88·3	55·3	257·5	64·4	465·5
1971	73·0	45·8	213·0	55·2	387·0
1972	64·5	35·1	218·0	56·3	373·9
1973	65·2	42·2	239·4	53·0	399·8
1974	79·2	45·9	249·3	66·1	440·5
1975	69·8	40·7	220·9	58·6	390·0
1976	61·9	35·9	195·0	51·7	344·5

Note: all figures are in thousands.
Source: estimated through application of table 16, row (8), ratios to the values in subtable 15.2.

a product of converting the figures in table 17 to census years and midyears and adding the figures in table 18. Note the high concentration of international migrants in the second half of the year (particularly the July-September quarter).

Finally, estimates (given in table 20) of the numbers of surviving infant external migrants are made using equations (14) and (15), directly for emigrant flows and in the following modified form for immigrant flows:

$$K^{\beta(R)\sigma(j)} = \tfrac{1}{2} b^j K^{\epsilon(R)\sigma(j)} , \qquad (16)$$

where b^j is the birth rate of region j.

We have now travelled down the first column of steps in figure 4 and half way up the second column. Some steps are rather more robust than others, and the creaking of some is positively deafening. However, none could be omitted without serious bias to the resulting accounts. The numbers in the accounts to be described have a large margin of error attached to them and have therefore been presented in all tables to

Table 18. Estimates of migrants to and from elsewhere in the British Isles for sections of Great Britain, 1970-1976.

	North	Midlands	South	Celtic Fringe	Great Britain [a]
Immigrants from Northern Ireland					
1970-1971	4·0	1·9	7·3	1·8	14·9
1971-1972	6·4	3·0	11·8	2·9	24·1
1972-1973	5·8	2·8	13·5	2·6	22·0
1973-1974	5·8	2·8	13·5	2·6	22·0
1974-1975	4·7	2·2	8·6	2·1	17·6
1975-1976	4·7	2·2	8·6	2·1	17·6
Emigrants to Northern Ireland					
1970-1971	2·9	1·4	5·3	1·3	10·9
1971-1972	4·7	2·2	8·6	2·1	17·6
1972-1973	4·3	2·0	9·9	1·9	16·1
1973-1974	4·3	2·0	9·9	1·9	16·1
1974-1975	3·4	1·6	6·3	1·5	12·9
1975-1976	3·4	1·6	6·3	1·5	12·9
Immigrants from the Irish Republic, the Isle of Man, and the Channel Islands					
1970-1971	4·1	2·7	13·7	1·5	22·0
1971-1972	4·1	2·7	13·7	1·5	22·0
1972-1973	4·1	2·7	13·7	1·5	22·0
1973-1974	4·1	2·7	13·7	1·5	22·0
1974-1975	4·1	2·7	13·7	1·5	22·0
1975-1976	4·1	2·7	13·7	1·5	22·0
Emigrants to the Irish Republic, the Isle of Man, and the Channel Islands					
1970-1971	1·8	1·2	6·2	0·7	10·0
1971-1972	1·8	1·2	6·2	0·7	10·0
1972-1973	1·8	1·2	6·2	0·7	10·0
1973-1974	1·8	1·2	6·2	0·7	10·0
1974-1975	1·8	1·2	6·2	0·7	10·0
1975-1976	1·8	1·2	6·2	0·7	10·0

Note: all figures are in thousands.
Sources: immigrants 1970-1971—aggregated from figures in unpublished table DT4312 (OPCS, 1976), later published in OPCS (1978a); immigrants 1971-1976, emigrants—the method of estimation is described in the text, the net-migration estimates used derive from OPCS (1977b), and the figures are assumed to apply both to census years and to midyears.

[a] The discrepancies between some Great Britain totals and the corresponding row sums is due to rounding.

the nearest hundred, though the level of accuracy is probably no more than to the nearest thousand. However, it would be relatively easy (and cheap) for official statistical bodies to improve on the accuracy of the accounts presented in this paper, should they adopt the framework.

4.4 *Application of the accounts-based model in the base period and subsequent tests*
Once the component demographic data have been assembled, the figures for census year 1970–1971 are selected and input to an unconstrained version of the accounts-based model (figure 4). This is done in order to yield estimates of the totals for immigrants, infant immigrants, surviving emigrants, and nonsurviving emigrants to use as constraints along with the population, births, and deaths totals.

The next step is then to examine the marginal totals and to check their consistency, that is, whether the sum of row marginal totals adds up to the sum of column marginal totals. Unless this condition is satisfied the adjustment of the initial estimate of the accounts matrix to the full set of marginal constraints will not be possible.

Table 19. Revised estimates of immigrants and emigrants for sections of Great Britain, 1970–1976, census years and midyears.

	North	Midlands	South	Celtic Fringe	Great Britain [a]
19.1 Census-year estimates [b]					
Immigrants					
1970–1971	59·6	41·7	246·9	37·0	385·2
1971–1972	56·9	41·5	237·4	37·0	372·8
1972–1973	54·5	46·8	249·7	39·0	390·0
1973–1974	54·2	41·3	220·7	37·1	353·3
1974–1975	54·5	34·5	216·5	36·4	341·9
1975–1976	56·0	35·5	222·9	37·4	351·9
Emigrants					
1970–1971	88·8	55·3	256·8	63·8	464·7
1971–1972	77·2	46·3	229·2	58·3	411·0
1972–1973	70·8	40·3	238·0	58·0	407·0
1973–1974	75·2	46·5	256·2	59·2	437·1
1974–1975	81·9	47·3	254·0	66·3	449·5
1975–1976	72·9	42·2	226·3	58·9	400·4
19.2 Midyear estimates [c]					
Immigrants					
1970–1971	58·7	41·0	242·9	36·6	379·2
1971–1972	56·5	42·8	241·7	37·5	378·5
1972–1973	54·4	46·2	244·4	38·7	383·6
1973–1974	48·0	37·8	193·2	32·5	311·5
1974–1975	55·0	34·8	218·6	36·7	345·1
1975–1976	55·4	35·1	220·2	37·0	347·7
Emigrants					
1970–1971	86·4	53·8	249·8	62·4	452·4
1971–1972	75·9	44·6	230·0	58·5	408·9
1972–1973	70·9	41·4	241·3	57·5	411·1
1973–1974	77·4	47·1	257·8	61·3	443·5
1974–1975	80·4	46·5	249·6	65·1	441·6
1975–1976	71·7	41·5	222·2	57·8	393·3

Note: all figures are in thousands.

[a] The discrepancies between some Great Britain totals and the corresponding row sums are due to rounding.

[b] Estimated from tables 17 and 18. $F_1 = 0{\cdot}235264$; $F_2 = 0{\cdot}764736$.

[c] Estimated from tables 17 and 18. $F_1 = 0{\cdot}383681$; $F_2 = 0{\cdot}616319$.

Table 21 shows that, when these initial constraints [columns (1) and (2) in the table] are added up, they rarely tally. There is a difference of 17197 between the row-total and column-total sums. It is then necessary to adjust some or all of the constraint figures in order to achieve a proper tally. There is clearly a very large number of ways in which this could be done, and the choice of which numbers to adjust will depend on assessment of the reliability of each constraint statistic.

Three different adjustments were used in the case of these British regional accounts.
(1) First the difference between the row-total and column-total sums was assigned entirely (and proportionately) to the two emigrant terms. The earlier discussion of prior-data estimation revealed these to be the least reliable demographic statistics. This is the adjustment shown in columns (3) and (4) in table 21. This method will be called the emigrant adjustment method and labelled C_1.
(2) A second method is to distribute the difference between the initial row-total sum and column-total sum amongst the initial populations of the sections proportionately to their size. The argument for this approach is that the 1970 population estimate is

Table 20. Estimates of infant immigrants and emigrants for sections of Great Britain, 1970-1976, census years and midyears.

	North	Midlands	South	Celtic Fringe	Great Britain [a]
20.1 Census-year estimates					
Infant immigrants					
1970-1971	0·5	0·4	1·9	0·3	3·1
1971-1972	0·5	0·3	1·8	0·3	2·9
1972-1973	0·4	0·4	1·8	0·3	2·8
1973-1974	0·4	0·3	1·5	0·3	2·4
1974-1975	0·3	0·2	1·4	0·2	2·2
1975-1976	0·3	0·2	1·3	0·2	2·2
Infant emigrants					
1970-1971	0·7	0·5	2·0	0·5	3·7
1971-1972	0·6	0·4	1·7	0·5	3·2
1972-1973	0·5	0·3	1·7	0·4	2·9
1973-1974	0·5	0·3	1·7	0·4	2·9
1974-1975	0·5	0·3	1·6	0·4	2·8
1975-1976	0·4	0·3	1·4	0·4	2·4
20.2 Midyear estimates					
Infant immigrants					
1970-1971	0·5	0·4	1·8	0·3	3·0
1971-1972	0·4	0·3	1·8	0·3	2·9
1972-1973	0·4	0·3	1·7	0·3	2·7
1973-1974	0·3	0·3	1·3	0·2	2·1
1974-1975	0·3	0·2	1·4	0·2	2·1
1975-1976	0·3	0·2	1·3	0·2	2·0
Infant emigrants					
1970-1971	0·7	0·5	1·9	0·5	3·6
1971-1972	0·6	0·4	1·7	0·5	3·1
1972-1973	0·5	0·3	1·7	0·4	2·9
1973-1974	0·5	0·3	1·7	0·4	2·9
1974-1975	0·5	0·3	1·5	0·4	2·7
1975-1976	0·4	0·3	1·3	0·4	2·4

Note: all figures are in thousands.
Source: estimated from tables 12 and 19.
[a] The discrepancies between some Great Britain totals and the corresponding row sums are due to rounding.

likely to be substantially in error as it is nine years after the previous full census (1961) and errors of estimation will be at their maximum. This will be called the initial-population adjustment method and labelled C_2.
(3) A third method is to work out the differences between the final populations produced by the unconstrained accounts and the census-based 1971 populations, and to add these differences to the initial population. This will be called the 'backcast' method and labelled C_3.

Three slightly different sets of accounts result from using these different constraint adjustment procedures, all of which will differ from the unconstrained set of accounts, labelled U.

Table 21. The constraints adjustment procedure illustrated for 1970-1971.

Section	Initial constraints		Adjusted constraints	
	row totals (1)	column totals (2)	row totals (3)	column totals (4)
	Initial populations	*Final populations*	*Initial populations*	*Final populations*
North	14573648	14601962	14573648	14601962
Midlands	8691483	8745928	8691483	8745928
South	22673582	22752492	22673582	22752492
Celtic Fringe	7928566	7939136	7928566	7939136
	Immigrants	*Surviving emigrants*	*Immigrants*	*Surviving emigrants*
Abroad	387442	468422	387442	451325
	Births	*Deaths*	*Births*	*Deaths*
North	241911	181114	241911	181114
Midlands	149200	94774	149200	94774
South	350200	261701	350200	261701
Celtic Fringe	129799	97834	129799	97834
	Infant immigrants	*Nonsurviving emigrants*	*Infant immigrants*	*Nonsurviving emigrants*
Abroad	3071	2736	3071	2636
Total	55128902	55146099	55128902	55128902

Source: University of Leeds ICL 1906A file GEOPHRG.AUCP7071RES containing the unconstrained accounts based on a more detailed versions of the relevant statistics from tables 11, 12, 13, 14, 19, and 20. These constraints refer to census year accounts. Census year accounts were constructed first; midyear accounts only at a later stage (see figure 4).

Table 22. Goodness-of-fit calculations for the C_1 V projection, 1976.

Region	Estimate	Projection	Difference	|Difference|	|Difference| (%)
North	3122·6	3101·4	21·1	21·1	0·68
Yorkshire and Humberside	4893·7	4848·1	45·6	45·6	0·93
North West	6557·7	6582·6	−24·8	24·8	0·38
East Midlands	3732·4	3728·1	4·3	4·3	0·12
West Midlands	5167·1	5177·2	−10·2	10·2	0·20
East Anglia	1799·2	1783·8	15·4	15·4	0·86
South East	16898·6	16980·5	−81·9	81·9	0·48
South West	4249·5	4158·7	90·7	90·7	2·14
Wales	2766·3	2726·0	40·1	40·1	1·45
Scotland	5197·7	5147·7	50·0	50·0	0·96
Great Britain	54384·8	54234·2	150·6	384·4	8·20

Note: all figures are in thousands, except in the last column.

The accounts-based model is then used in projection mode in one of two ways. In the fixed-rate projections the birth rates, death rates, internal migration rates, internal infant migration rates, and external migrant and external infant migrant vectors associated with the 1971 census are used to project the regional populations forward to 1976 (census date). In the variable-rate projections the birth rates, death rates, and external migrant and external infant migrant vectors for the intervening years (derived from tables equivalent to those presented earlier) are used, and only the internal migration and internal infant migration rates remain fixed. Thus eight alternative projections of the populations of the British regions are produced.

The results of the projections are assessed at census date 1976 through the calculation of three goodness-of-fit statistics. Table 22 shows the calculations for the C_1 V (the emigrant adjustment method, variable rates) projection. The simple difference between estimated and projected population is calculated; the absolute difference is computed; and the absolute difference is computed as a percentage of the estimate. The sum totals of these statistics for Great Britain enable us to judge between projections. The simple difference alone may mask large cancelling deviations among the regions; the absolute difference measure corrects for this but may be unduly influenced by a large region; the percentage absolute difference measure gives equal weight to each region. Table 22 is presented in terms of the ten standard regions of the original analysis rather than in terms of the four sections of Great Britain because aggregation in this context does not make sense.

Table 23 displays the three overall goodness-of-fit statistics for the eight projections. The variable-rate projections are clearly better than the fixed, as one might have expected, and the constrained projections are better than the unconstrained. The backcast adjustment method appears to fare worst of the three procedures and there is little to choose between the migrant adjustment and initial-population adjustment methods. The former method was, on balance, chosen as more convenient since it involved retaining the official population estimates whereas the latter method would have involved their successive revision. We have now arrived at the third box in column three of figure 4.

Table 23. Calibration statistics for British regions, 1970-1976, for the 1976 population.

Model run	Difference in total	Sum of absolute differences	Sum of absolute % differences	Type of run	Status of birth and death rates and external migrants
UF	398·9	683·2	9·53	unconstrained	fixed
C_1 F	504·7	612·6	9·81	constrained 1	fixed
C_2 F	418·7	554·1	9·17	constrained 2	fixed
C_3 F	415·8	626·0	8·58	constrained 3	fixed
UV	−168·2	393·0	11·39	unconstrained	variable
C_1 V	−150·6	384·4	8·20	constrained 1	variable
C_2 V	−150·6	387·8	8·15	constrained 2	variable
C_3 V	−150·8	539·0	10·13	constrained 3	variable

4.5 *Estimation of the time series of accounts for 1971-1976*

The time series of accounts was then generated using the emigrant-adjusted constraints. The appropriate input data on births, deaths, populations, and external migrants was assembled for each year and a constrained set of accounts was produced using the internal migration rates and internal infant migration rates of the preceding year. The results are reproduced in the appendix.

5 Uses of multistate demographic accounts

Accounts are devices for displaying historical relationships in terms of population flows between demographic states. They enable us to understand better the pace and direction of demographic change. They have been used in carrying out educational projections (Stone, 1971a), multiregional population projections (Rees, 1976; 1977b), and in computing multiregional life tables (Rogers, 1975; Willekens and Rogers, 1978; although in the last application only the internal portion of the accounts matrix is used.

As yet demographic accounting has had little impact in either national statistical offices or at local or regional levels (see Baxter and Williams, 1978, for comments). The usual objection posed is that the preparation of accounts tables is too complex and time-consuming an exercise. It is hoped that this paper has served to dispel that view in part and that, with the improvement of computer packages for multistate demographic accounting, preparation of demographic accounts will become a common prior step in much future-oriented demographic analysis.

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APPENDIX: Best-fit midyear accounts for sections of Great Britain, 1970–1976

1970	1971										Total[a]
	survival					deaths					
	N	M	S	C	A	N	M	S	C	A	
Existence											
N	14164·1	43·1	80·6	27·3	81·9	177·7	0·2	0·5	0·2	0·5	14576·0
M	41·3	8428·1	69·5	15·9	51·5	0·3	92·7	0·4	0·4	0·3	8700·0
S	75·8	77·2	21982·2	47·7	244·8	0·5	0·4	256·8	0·3	1·4	22687·0
C	26·7	17·8	46·5	7683·9	59·1	0·2	0·1	0·3	95·9	0·4	7930·7
A	60·1	41·6	240·1	37·5	0·0	0·4	0·2	1·4	0·2	0·0	381·4
Births											
N	238·2	0·4	0·7	0·2	0·7	1·5	0·0	0·0	0·0	0·0	241·5
M	0·4	146·8	0·6	0·1	0·4	0·0	0·8	0·0	0·0	0·0	149·2
S	0·6	0·6	344·7	0·4	1·9	0·0	0·0	2·0	0·0	0·0	350·1
C	0·2	0·1	0·4	127·8	0·5	0·0	0·0	0·0	0·8	0·0	129·8
A	0·5	0·4	1·8	0·3	0·0	0·0	0·0	0·0	0·0	0·0	3·0
Total[a]	14607·7	8756·1	22767·0	7941·0	440·7	180·5	94·5	261·3	97·5	2·6	55148·8

1971	1972										Total[a]
	survival					deaths					
	N	M	S	C	A	N	M	S	C	A	
Existence											
N	14199·4	43·5	81·8	27·7	73·5	180·5	0·2	0·5	0·2	0·5	14607·7
M	41·3	8490·8	70·0	15·9	42·7	0·3	94·4	0·4	0·1	0·2	8756·1
S	75·3	77·5	22083·3	47·3	221·1	0·5	0·4	260·0	0·3	1·3	22767·0
C	26·7	18·0	47·2	7693·9	57·1	0·2	0·1	0·3	97·2	0·4	7941·0
A	56·2	43·2	242·0	37·2	0·0	0·4	0·2	1·4	0·2	0·0	380·0
Births											
N	226·7	0·4	0·7	0·2	0·6	1·4	0·0	0·0	0·0	0·0	230·0
M	0·3	140·8	0·6	0·1	0·3	0·0	0·8	0·0	0·0	0·0	143·0
S	0·6	0·6	333·6	0·4	1·6	0·0	0·0	1·9	0·0	0·0	338·7
C	0·2	0·1	0·4	122·1	0·4	0·0	0·0	0·0	0·8	0·0	124·0
A	0·4	0·4	1·8	0·3	0·0	0·0	0·0	0·0	0·0	0·0	2·9
Total[a]	14627·2	8815·2	22861·3	7945·0	397·4	183·1	96·2	264·5	98·8	2·3	55291·2

1972	1973										Total[a]
	survival					deaths					
	N	M	S	C	A	N	M	S	C	A	
Existence											
N	14225·1	42·7	81·9	29·3	63·8	183·0	0·2	0·5	0·2	0·5	14627·2
M	42·4	8546·6	73·0	17·5	38·2	0·3	96·5	0·4	0·1	0·2	8815·2
S	76·0	76·4	22174·5	50·2	218·1	0·5	0·4	263·7	0·3	1·3	22861·3
C	25·4	16·6	44·9	7708·9	49·3	0·2	0·1	0·3	99·0	0·3	7945·0
A	54·5	45·0	243·3	40·7	0·0	0·3	0·3	1·4	0·3	0·0	385·9
Births											
N	207·4	0·4	0·7	0·2	0·5	1·3	0·0	0·0	0·0	0·0	210·4
M	0·4	130·5	0·6	0·1	0·3	0·0	0·7	0·0	0·0	0·0	132·6
S	0·6	0·6	312·8	0·4	1·5	0·0	0·0	1·8	0·0	0·0	317·7
C	0·2	0·1	0·4	113·4	0·4	0·0	0·0	0·0	0·7	0·0	115·2
A	0·4	0·3	1·7	0·3	0·0	0·0	0·0	0·0	0·0	0·0	2·7
Total[a]	14632·3	8859·2	22933·8	7961·0	372·0	185·6	98·2	268·1	100·6	2·2	55413·2

Key: N—North; M—Midlands; S—South; C—Celtic Fringe; A—Abroad.

Note: all figures are in thousands.

[a] The discrepancies between some of the totals and the corresponding row or column sums are due to rounding.

APPENDIX (continued)

1973	1974 survival N	M	S	C	A	deaths N	M	S	C	A	Total[a]
Existence											
N	14236·6	44·4	83·0	29·6	56·2	181·2	0·2	0·5	0·2	0·4	14632·3
M	41·2	8591·4	70·5	16·7	42·1	0·3	96·4	0·4	0·1	0·2	8859·2
S	75·0	78·7	22241·9	50·2	222·4	0·5	0·4	263·1	0·3	1·3	22933·8
C	25·0	17·3	45·0	7736·8	37·3	0·2	0·1	0·3	98·8	0·2	7961·0
A	47·1	39·2	192·8	32·4	0·0	0·3	0·2	1·1	0·2	0·0	313·3
Births											
N	192·6	0·4	0·7	0·2	0·6	1·2	0·0	0·0	0·0	0·0	195·7
M	0·3	121·8	0·6	0·1	0·5	0·0	0·7	0·00	0·0	0·0	124·1
S	0·6	0·6	294·8	0·4	2·0	0·0	0·0	1·7	0·0	0·0	300·1
C	0·2	0·1	0·4	107·2	0·5	0·0	0·0	0·0	0·7	0·0	109·1
A	0·3	0·3	1·3	0·2	0·0	0·0	0·0	0·0	0·0	0·0	2·1
Total[a]	14619·0	8894·1	22930·9	7973·8	361·6	183·6	98·1	267·1	100·3	2·1	55430·7

1974	1975 survival N	M	S	C	A	deaths N	M	S	C	A	Total[a]
Existence											
N	14213·8	42·4	81·0	29·7	71·4	179·4	0·2	0·5	0·2	0·4	14619·0
M	43·3	8619·0	73·1	17·9	43·4	0·3	96·4	0·4	0·1	0·2	8894·1
S	77·1	76·9	22233·3	51·4	227·1	0·5	0·4	262·7	0·3	1·3	22930·9
C	25·2	16·4	44·2	7731·0	58·2	0·2	0·1	0·3	97·9	0·4	7973·8
A	56·1	33·7	217·9	37·3	0·0	0·3	1·9	1·3	0·2	0·0	347·1
Births											
N	182·3	0·4	0·7	0·2	0·5	1·1	0·0	0·0	0·0	0·0	185·1
M	0·4	114·2	0·6	0·2	0·3	0·0	0·6	0·0	0·0	0·0	116·2
S	0·6	0·6	279·5	0·4	1·4	0·0	0·0	1·6	0·0	0·0	284·1
C	0·2	0·1	0·4	102·3	0·4	0·0	0·0	0·0	0·6	0·0	104·0
A	0·4	0·2	1·4	0·2	0·0	0·0	0·0	0·0	0·0	0·0	2·2
Total[a]	14599·2	8903·9	22932·0	7970·5	402·6	181·8	98·0	266·8	99·4	2·4	55456·6

1975	1976 survival N	M	S	C	A	deaths N	M	S	C	A	Total[a]
Existence											
N	14195·5	41·7	83·5	30·1	68·9	178·1	0·2	0·5	0·2	0·4	14599·2
M	44·1	8624·4	77·0	18·5	40·6	0·3	98·3	0·5	0·1	0·2	8903·9
S	74·6	73·1	22252·9	51·2	206·8	0·5	0·4	271·0	0·3	1·2	22932·0
C	24·5	15·7	44·4	7733·1	53·4	0·2	0·1	0·3	98·5	0·3	7970·5
A	53·9	33·6	222·4	37·7	0·0	0·3	1·9	1·3	0·2	0·0	349·8
Births											
N	174·4	0·3	0·7	0·2	0·4	1·1	0·0	0·0	0·0	0·0	177·1
M	0·4	108·7	0·7	0·2	0·3	0·0	0·6	0·0	0·0	0·0	110·8
S	0·6	0·6	267·4	0·4	1·2	0·0	0·0	1·6	0·0	0·0	271·8
C	0·2	0·1	0·4	100·1	0·3	0·0	0·0	0·0	0·6	0·0	101·8
A	0·3	0·2	1·3	0·2	0·2	0·0	0·0	0·0	0·0	0·0	2·1
Total[a]	14568·5	8898·5	22950·6	7971·9	371·9	180·4	99·8	275·1	100·0	2·2	55418·9

Key: N—North; M—Midlands; S—South; C—Celtic Fringe; A—Abroad.

Note: all figures are in thousands.

[a] The discrepancies between some of the totals and the corresponding row or column sums are due to rounding.

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Multistate life tables: movement versus transition perspectives

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Abstract. This paper attempts to present a comprehensive view of the methodological and empirical aspects involved in the construction of increment-decrement life tables, that is life tables which allow entries into (increments) as well as withdrawals from (decrements) alternative states. The first principal part of the paper, section 2, presents a theoretical exposition of such tables, paralleling that of the ordinary life table, and discusses various issues raised by the conceptualization of multistate life-table functions. The second principal part, section 3, contrasts the two alternative approaches to the applied calculation of such tables. On the one hand, the movement approach, which views interstate transfers as events (such as deaths or births), requires data in the form of occurrence/exposure rates; on the other hand, the transition approach, which regards such transfers as the results of a change in an individual's state of presence between two points in time, uses data in the form of survivorship proportions.

1 Introduction

Among the various kinds of models used by mathematical demographers, one which seems to be fairly well represented builds on various extensions of some of the basic ideas underlying the life-table model. Models of this type in effect constitute helpful devices for following a group of people, born at the same moment, over time and age as they experience transitions between two or more states.

In the simplest situation, that of the *ordinary life table*, there are two states, alive and dead, and the emphasis is put on the transition, which is not reversible, from the former to the latter. A straightforward extension is the *multiple decrement life table*, which recognizes transitions to more than one final absorbing state (for example, various causes of death). However, in the case of recurrent events (for example, marriage), this model does not permit one to follow persons who have moved from one status category to another and to analyze their subsequent experiences.

Such a problem may be handled with the help of combined tables which allow for entries into (increments) as well as withdrawals from (decrements) different states. Because of their general nature, such *increment-decrement life tables* are valuable in analyses of marital status, labor-force participation, birth parity, interregional migration, etc. The simplest life tables of this type, called *hierarchical increment-decrement life tables*, deal with the case of advances through successive states with no reentry allowed. Reentries are permitted, however, in more complicated versions which, by contrast, are referred to as *nonhierarchical increment-decrement life tables*.

The fairly simple methodology underlying the simplest of these life tables is now firmly established [see Keyfitz (1968) for a discussion of the ordinary life-table case and Jordan (1967) for a similar discussion of the multiple decrement case]. The methodology of hierarchical increment-decrement life tables was set forth by Hoem (1970a; 1970b) in the context of nuptiality and fertility analyses. Some of the issues regarding the construction of nonhierarchical increment-decrement life tables have been examined in the past (Depoid, 1938; Mertens, 1965; Jordan, 1967); however, it was not until recently that a thorough and systematic discussion of the methodological and empirical problems raised by such construction has appeared in the literature

(Rogers, 1973; 1975; Rogers and Ledent, 1975; 1976; Schoen and Nelson, 1974; Schoen, 1975; Schoen and Land, 1976; 1977; Krishnamoorthy, 1979).

The key element responsible for the development of such increment-decrement life tables was the realization that such tables can be regarded as generalizations of ordinary life tables, in which multistate life-table functions in matrix format are substituted for the scalar life-table functions of the basic life table (Rogers and Ledent, 1975; 1976; Rogers, 1975). In most instances the matrix generalization turned out to be a relatively straightforward matter, but in some cases it did not. The crux of the matrix extension to the multistate case lay in the estimation of the age-specific survival probability matrices from which all the multistate life-table functions originate.

For the purpose of this estimation, two alternative approaches resulting from different conceptualizations of the 'passage' between alternative states have emerged: the movement approach devised by Schoen (1975) and the transition approach devised by Rogers (1973; 1975). The former, consistent with the approach taken in the ordinary life table, views 'passage' as an instantaneous event, similar to a birth or a death. It requires input data in the form of exposure/occurrence rates. By contrast, the latter conceives 'passage' as the result of change in an individual's state of presence between two points in time and uses data in the form of survivorship proportions.

In short, various contributions have in recent years led to the development of a formal mathematical treatment as well as precise construction methods which now give increment-decrement life tables a status comparable to that held by the ordinary life table. In the light of this, *the present paper reviews and summarizes the research on increment-decrement life tables* through a comprehensive exposition paralleling the classical exposition of the ordinary life table, such as that of Keyfitz (1968). Particular attention is devoted to the various methods used for their construction which, in the current stage of development, still raise methodological as well as empirical issues.

The paper consists of two principal parts. The first, section 2, sets out the generalization of ordinary life-table concepts to the case of increment-decrement life tables. It presents a theoretical derivation of the multistate stationary population and examines various issues regarding the conceptualization of the multistate life-table functions. The second principal part, section 3, explicitly discusses the construction of increment-decrement life tables, with a focus on the estimation of the age-specific transition probability matrices. The movement and transition perspectives are first examined and then contrasted.

The notation used throughout this paper differs from that used by previous students of increment-decrement life tables. It parallels and attempts to extend in a consistent manner the notation used by Keyfitz (1968) in dealing with the ordinary life table.

(a) Statistics relating to the multistate life-table population are denoted by lower-case letters (there are some exceptions, however, such as $\mathbf{L}_x$ and $\mathbf{T}_x$), whereas those referring to the observed population are denoted by capital letters.

(b) The functional notation $f(y)$ is used to denote a function of y considered as a continuous variable, whereas f_y is used whenever the function referred to is for a discrete set of values (y is in the position of a right subscript).

In the case of scalar quantities, the following rules have been adopted to account for the existence of intercommunicating states.

(a) State-specific values of a statistic f are denoted by a right superscript specific to the region, for example, f_y^i or $f^i(y)$.

(b) Interstate 'passages' are suggested by superscripts located on both sides of the variable concerned, for example, ${}^if_y^j$; the left superscript relates to the state of origin, the right one to the state of destination.

(c) If reference to the state of birth or the state of presence at any age less than the current age is necessary, it is indicated by two left subscripts, respectively denoting the relevant region and age; for example, ${}_{iy}f_x^j$ represents the value of the function f characteristic of those present at age x in state j who were in state i at age y, $y \leqslant x$.

2 Toward a generalization of ordinary life-table concepts

The ordinary life table is a stationary population model in which an individual born in a unique state (the state of being alive) moves sooner or later to an absorbing state (the state of death). Expressing the facts of mortality in probability terms allows one to ascertain the number of individuals out of a given cohort who survive to any age.

In many circumstances, however, the knowledge of such numbers may not be enough and one would like to know whether the survivors are married or divorced, within or outside the labor force, living in region 1 or in region 2, etc. This leads to the notion of an increment-decrement life table, in which, besides the state of death, there exist several nonabsorbing states. Like the multiple decrement life table, such a life table is a multistate life table, but it sharply differs from the former in that it allows entries into (increments) as well as withdrawals from (decrements) the various states. It is a device enabling one to examine the mortality and mobility history of a cohort of people born in one or several of the nonabsorbing states.

It is useful to distinguish between uniradix increment-decrement life tables—in which the initial cohort is concentrated in one state (such as the marital-status and working-status life tables)—and multiradix increment-decrement life tables—in which the initial cohort is allocated among several or all of the intercommunicating states (such as multiregional life tables, that is, increment-decrement life tables applied to interregional migration). In fact, however, there are no real fundamental differences between the two types of tables—this will become clear to the reader later on—and thus the remainder of this paper applies to both of them (except when specified otherwise).

This section mainly attempts to extend the concepts of the ordinary life table to the case of increment-decrement life tables by paralleling the conventional methodology of the ordinary life table. First a theoretical derivation of the multistate stationary population is presented and then various issues relating to the conceptualization of the multistate life-table functions are discussed.

2.1 *The multistate stationary population: a theoretical derivation*

In order to facilitate the reader's understanding, first a brief review of the principal derivations in the ordinary life table is presented, followed then by the corresponding multistate generalizations.

In the basic life table the main problem is to estimate the curve of survivors, $l(y)$, at any age y, out of a cohort of, for example, l_0 babies. The estimation starts with the definition of the instantaneous mortality rate (or force of mortality), $\mu(y)$, associated with age y. This definition appears in equation (1) in table 1, in which $d(y)$ denotes the number of deaths occurring to the group of survivors $l(y)$ between ages y and $y+\mathrm{d}y$, where $\mathrm{d}y$ is small.

Observing that the cohort is closed, that is, is not subject to any type of exit other than by death, it is easy to show, in equation (2), the decrement occurring to $l(y)$ between y and $y+\mathrm{d}y$. Then substitution of equation (2) into equation (1) gives the basic differential equation (3) expressing the relationship between $\mu(y)$ and $l(y)$ (Keyfitz, 1968, page 5).

The integral solution of equation (3) is given by equation (4), in which $\Omega(y)$ can be expressed as shown in equation (5). This permits one to define the number of survivors, l_x, at fixed ages $x = 0, n, 2n, ..., z$ by applying—as shown in equation (6)—a set of age-specific survival probabilities, p_x, defined by equation (7). Note that traditionally all age intervals considered are equal in length except for the last one, which is half open: z years and over.

We now turn to the increment-decrement case, whose presentation starts with the derivation of the multistate stationary population first studied by Schoen and Land (1976) and Krishnamoorthy (1979)(1). The present exposition of such a derivation, which borrows heavily from their work, is, however, guided by the desire to clarify the connection between ordinary and increment-decrement life tables.

Suppose we have a system of nonabsorbing states (denoted by $i = 1, ..., r$) and suppose that the initial cohort is allocated among s of them ($1 \leqslant s \leqslant r$). The main problem here is to estimate the state-specific curves of survivors, $l^i(y)$, at each age y. Let us consider an individual born in any state of the system and present in state i at age y and let us examine his possible location $\mathrm{d}y$ years later, where $\mathrm{d}y$ is small. He may be alive in the same state, or alive in another of the nonabsorbing states, or he may have died in between. Now let us shorten $\mathrm{d}y$ enough so that multiple events—such as a move to state j followed by a death in that state—can be ruled out. Then three alternatives are possible, for the individual considered over a short period of time $\mathrm{d}t = \mathrm{d}y$: a nonevent, namely to have stayed in state i, and two events, namely death in state i or a move to one of the other nonabsorbing states of the system. In other words, the 'passage' from one state to another is viewed here as an instantaneous event similar to a death in conventional demography. Thus there is no fundamental difference between a transfer to state $j = 1, ..., r$ and a transfer to the state of death, δ: deaths in state i can thus be considered as moves from state i to the state of death,

Table 1. A tabular comparison of the theoretical exposition of the ordinary and increment-decrement life tables.

Ordinary life table		Increment-decrement life table	
$\mu(y) = \lim\limits_{\mathrm{d}y \to 0} \dfrac{d(y)}{l(y)\mathrm{d}y}$	(1)	${}^i\mu^j(y) = \lim\limits_{\mathrm{d}y \to 0} \dfrac{{}^id^j(y)}{l^i(y)\mathrm{d}y}$	(8)
$l(y+\mathrm{d}y) = l(y) - d(y)$	(2)	$l^i(y+\mathrm{d}y) = l^i(y) - \sum\limits_{\substack{j=1\\j\neq i}}^{r+1} {}^id^j(y) + \sum\limits_{\substack{j=1\\j\neq i}}^{r} {}^jd^i(y)$	(12)
$\dfrac{\mathrm{d}}{\mathrm{d}y} l(y) = -\mu(y)l(y)$	(3)	$\dfrac{\mathrm{d}}{\mathrm{d}y} \mathbf{l}(y) = -\boldsymbol{\mu}(y)\mathbf{l}(y)$	(16)
$l(y) = \Omega(y)l(0)$	(4)	$\mathbf{l}(y) = \boldsymbol{\Omega}(y)\mathbf{l}(0)$	(20)
$\Omega(y) = \exp\left[-\int_0^y \mu(t)\mathrm{d}t\right]$	(5)	$\boldsymbol{\Omega}(y) = [\mathbf{I} - \boldsymbol{\mu}(\tau_n)\Delta y_n] \ldots [\mathbf{I} - \boldsymbol{\mu}(\tau_1)\Delta y_1] + \epsilon$	(22)
$l_{x+n} = p_x l_x$	(6)	$\mathbf{l}_{x+n} = \mathbf{p}_x \mathbf{l}_x$	(25)
$p_x = \dfrac{\Omega(x+n)}{\Omega(x)}$	(7)	$\mathbf{p}_x = \boldsymbol{\Omega}(x+n)\boldsymbol{\Omega}(x)^{-1}$	(24)

(1) A slightly different but equivalent presentation of increment-decrement tables starts with the specification of the Kolmogorov forward differential equations of the underlying Markov chain (Schoen and Land, 1977; Willekens, 1978).

referred to as state $r+1$. Consequently, in what follows, instantaneous and group-specific mortality rates are not distinguished from the other relevant mobility rates.

Let ${}^{i}d^{j}(y)$ denote the number of moves from state i ($i = 1, ..., r$) to each state j ($j = 1, ..., r+1$, $j \neq i$) made over the short period of time $dt = dy$ by those individuals who were members of the group of people surviving in state i at age y, $l^{i}(y)$. Since the exposure of these individuals to the risk of moving before reaching $y+dy$ is $l^{i}(y)dy$, the result is that ${}^{i}d^{j}(y)/l^{i}(y)dy$ is the corresponding mobility rate from state i to state j ($j \neq i$) associated with age y. Thus one can define the instantaneous mobility rate (or force of mobility), ${}^{i}\mu^{j}(y)$, as the limiting value of this mobility rate when $dy \to 0$,

$$ {}^{i}\mu^{j}(y) = \lim_{dy \to 0} \frac{{}^{i}d^{j}(y)}{l^{i}(y)dy} , \qquad i = 1, ..., r , \quad j = 1, ..., r+1 , \quad j \neq i . \tag{8} $$

(Such a limit is assumed to exist for all states of the system and all ages y.) Definition (8) is indeed a straightforward generalization of the corresponding definition (1) in the ordinary life table. Note that, because of the assumptions made, we have

$$ \sum_{\substack{j=1 \\ j \neq i}}^{r+1} {}^{i}d^{j}(y) \leqslant l^{i}(y) , \qquad i = 1, ..., r , \tag{9} $$

or equivalently

$$ \sum_{\substack{j=1 \\ j \neq i}}^{r+1} {}^{i}\mu^{j}(y) \leqslant \frac{1}{dy} , \qquad i = 1, ..., r . \tag{10} $$

As far as the two states i and k—the latter combining all states except i and the state of death—are concerned, an accounting framework linking the aforementioned d and l statistics can be set up as shown in table 2. It is then easy to write the following equation indicating the decrements and increments to the survivors, at age y in region i, of the initial cohort:

$$ l^{i}(y+dy) = l^{i}(y) - {}^{i}d^{\delta}(y) - {}^{i}d^{k}(y) + {}^{k}d^{i}(y) , \qquad i = 1, ..., r . \tag{11} $$

Recalling that k stands for all states excluding i and the state of death and that the state of death can be viewed as the $(r+1)$th state of the system, we can rewrite equation (11) as

$$ l^{i}(y+dy) = l^{i}(y) - \sum_{\substack{j=1 \\ j \neq i}}^{r+1} {}^{i}d^{j}(y) + \sum_{\substack{j=1 \\ j \neq i}}^{r} {}^{j}d^{i}(y) , \qquad i = 1, ..., r . \tag{12} $$

These equations form the multistate extension of equation (2), which shows the decrement to $l(y)$ in the ordinary life table.

Table 2. Demographic events in a two-region system.

Location at time $t+y+dy$	Location at time $t+y$		
	present in state i	present in state k	
Alive in state i	–	${}^{k}d^{i}(y)$	$l^{i}(y+dy)$
Alive in state k	${}^{i}d^{k}(y)$	–	$l^{k}(y+dy)$
Dead	${}^{i}d^{\delta}(y)$	${}^{k}d^{\delta}(y)$	
	$l^{i}(y)$	$l^{k}(y)$	

Substituting equations (8) into equations (12) leads to (Schoen and Land, 1976)

$$l^i(y+\mathrm{d}y) = l^i(y) - \sum_{\substack{j=1\\ j\neq i}}^{r+1} {}^i\mu^j(y) l^i(y)\mathrm{d}y + \sum_{\substack{j=1\\ j\neq i}}^{r} {}^j\mu^i(y) l^j(y)\mathrm{d}y\ , \qquad i = 1, ..., r\ , \tag{13}$$

or more compactly

$$l(y+\mathrm{d}y) = l(y) - \mu(y) l(y)\mathrm{d}y\ , \tag{14}$$

where $l(y)$ is a vector whose typical element is $l^i(y)$ and

$$\mu(y) = \begin{bmatrix} \sum\limits_{\substack{j=1\\ j\neq 1}}^{r+1} {}^1\mu^j(y) & -{}^2\mu^1(y) & \dots & -{}^r\mu^1(y) \\ -{}^1\mu^2(y) & \sum\limits_{\substack{j=1\\ j\neq 2}}^{r+1} {}^2\mu^j(y) & \dots & -{}^r\mu^2(y) \\ \vdots & \vdots & \ddots & \vdots \\ -{}^1\mu^r(y) & -{}^2\mu^r(y) & \dots & \sum\limits_{\substack{j=1\\ j\neq r}}^{r+1} {}^r\mu^j(y) \end{bmatrix} . \tag{15}$$

The ith diagonal element of $\mu(y)$ consists of the total force of mobility out of state i; the (i,j)th off-diagonal element contains minus the force of mobility from state j to state i. Defining $\mathrm{d}l(y)$ as the vector difference $l(y+\mathrm{d}y) - l(y)$ leads one to rewrite equation (14) as (Krishnamoorthy, 1979)

$$\frac{\mathrm{d}}{\mathrm{d}y} l(y) = -\mu(y) l(y)\ , \tag{16}$$

which appears as a straightforward multistate extension of equation (3).

The system defined by equation (16) admits r linearly independent solutions $l(y)_k$ ($k = 1, ..., r$), which when set side by side as the columns of a square matrix yield the integral matrix of the system, $\mathrm{l}(y) = [l(y)_1, ..., l(y)_r]$ (Gantmacher, 1959). Since every column of $l(y)$ satisfies equation (16), the integral matrix $\mathrm{l}(y)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}y} \mathrm{l}(y) = -\mu(y) \mathrm{l}(y)\ . \tag{17}$$

From the theorem on the existence and uniqueness of the solution of a system of linear differential equations, it follows that $\mathrm{l}(y)$ is uniquely determined when the value of $\mathrm{l}(y)$ for some initial value y_0 is known (Gantmacher, 1959):

$$\mathrm{l}(y) = {}_{y_0}\Omega(y) \mathrm{l}(y_0)\ . \tag{18}$$

The matrix ${}_{y_0}\Omega(y)$, uniquely defined as the normalized solution of equation (16), is called the *matricant* (Gantmacher, 1959). Its main property is one of transitivity:

$${}_{y_1}\Omega(y_3) = {}_{y_2}\Omega(y_3)\ {}_{y_1}\Omega(y_2)\ . \tag{19}$$

In the present case, letting $y_0 = 0$ and omitting the zero subscript in ${}_0\Omega(y)$, we have

$$\mathrm{l}(y) = \Omega(y) \mathrm{l}(0)\ , \tag{20}$$

which is the multistate extension of equation (4).

Note that $\Omega(y)$ cannot be simply expressed as a function of the $\mu(y)$. The straightforward generalization of equation (5) into

$$\Omega(y) = \exp\left[-\int_0^y \mu(t)\mathrm{d}t\right] \tag{21}$$

simply does not hold if $\boldsymbol{\mu}(t)$ is not a constant matrix. However, as indicated by Schoen and Land (1976) and Krishnamoorthy (1979), $\boldsymbol{\Omega}(y)$ can be determined by using the infinitesimal calculus of Volterra (see Gantmacher, 1959, chapter XIV):

$$\boldsymbol{\Omega}(y) = [\mathbf{I}-\boldsymbol{\mu}(\tau_n)\Delta y_n][\mathbf{I}-\boldsymbol{\mu}(\tau_{n-1})\Delta y_{n-1}] \ldots [\mathbf{I}-\boldsymbol{\mu}(\tau_1)\Delta y_1]+\epsilon , \tag{22}$$

where ϵ denotes the sum of terms of order two or greater. Basically the calculation of $\boldsymbol{\Omega}(y)$ from equation (22) requires decomposing the basic interval $(0 = y_0, y = y_n)$ into n parts by introducing intermediate points $y_1, y_2, ..., y_{n-1}$, setting $\Delta y_k = y_k - y_{k-1}$ $(k = 1, ..., n)$, and taking τ_k as an intermediate point in the interval (y_{k-1}, y_k).

Having derived an integral matrix of equation (16), we now need to interpret it. First let us suppose that the initial cohort is allocated among all states $(s = r)$. Clearly $\mathbf{l}(0) = \mathbf{l}_0$ is a diagonal matrix which denotes the state-specific allocation of the initial cohort: its typical diagonal element is l_0^i. Furthermore the ith column of $\mathbf{l}(y)$ is a vector representing the state-specific allocation of the survivors of l_0^i at age y. From equation (20) it follows that its (j, i)th element is the product of the (j, i)th element ${}_{i0}\Omega^j(y)$ of $\boldsymbol{\Omega}(y)$ with l_0^i. Hence ${}_{i0}\Omega^j(y)$ denotes the probability of a person born in state i surviving in state j at age y. To summarize, (a) the r independent solutions of equation (16) are the r multistate stationary populations that are generated by each of the radices, that is, each of the state-specific shares of the initial cohort, and (b) $\boldsymbol{\Omega}(y)$ is the matrix showing the state-specific survival probabilities—at age y—of the members of each radix.

Suppose now that the initial cohort is not allocated among all states $(s < r)$. In such circumstances $r-s$ columns of $\mathbf{l}(y)$ are simply zero vectors, but this affects the interpretation of $\boldsymbol{\Omega}(y)$ only slightly: the $r-s$ columns of $\boldsymbol{\Omega}(y)$ corresponding to the states initially empty appear as the hypothetical probabilities that would prevail if they were not initially empty. Thus, if $s < r$, equation (16) still admits r independent solutions, but only s of them correspond to real multistate stationary populations.

Note that $\boldsymbol{\Omega}(y)$ is a proper transition probability matrix: it can be shown that, as a consequence of (a) the definitions (8) regarding the instantaneous forces of mobility and (b) the inequalities (9) linking them, all elements of $\boldsymbol{\Omega}(y)$ are nonnegative and that the elements of each column add up to a number less than or equal to one. Such a property holds even in the less favorable cases, that is, when $s < r$. For instance, when state k is initially empty and is not entered by any individual aged less than y_a, the kth column of $\boldsymbol{\Omega}(y)$ for $y \leqslant y_a$ is not a zero vector: as a consequence of the definitions regarding the ${}^i\mu^j(y)$, its kth element is equal to one. It follows that $\boldsymbol{\Omega}(y)$ is never a singular matrix and always has an inverse.

Recalling our interpretation of the matricant, it follows that the probability ${}^ip_x^j$ that an individual present at age x in state i will survive in stage j n years later is simply the (j, i)th element of the matrix $\mathbf{p}_x = {}_x\boldsymbol{\Omega}(x+n)$. Thus, as a consequence of the transitivity property (19) of the matricant, we have

$$\mathbf{p}_x \boldsymbol{\Omega}(x) = \boldsymbol{\Omega}(x+n) \tag{23}$$

or, since $\boldsymbol{\Omega}(x)$ has an inverse,

$$\mathbf{p}_x = \boldsymbol{\Omega}(x+n)\boldsymbol{\Omega}(x)^{-1} , \tag{24}$$

a relation which is the multistate analog of equation (7). [As with $\boldsymbol{\Omega}(y)$, it is not possible to express $\mathbf{p}_x$ as an exact function of the $\boldsymbol{\mu}(y)$; again one must use the infinitesimal calculus of Volterra.]

Can we characterize $\mathbf{p}_x$ further? Since $\boldsymbol{\Omega}(y)$ is a proper transition probability matrix, it immediately follows from equation (24) that the $\mathbf{p}_x$ matrices determine a Markov transition probability model (Schoen and Land, 1976): all of the $\mathbf{p}_x$ matrices satisfy the three standard conditions specified in Cox and Miller (1965), namely

(1) $0 \leqslant {}^{i}p_x^j\ , \quad i, j = 1, ..., r\ ;$

(2) $0 \leqslant \sum_{j=1}^{r} {}^{i}p_x^j \leqslant 1\ , \quad i = 1, ..., r\ ;$

(3) the transitivity condition, which holds as a consequence of the transitivity property (19) displayed by the matricant of equation (16).

Then, letting $\mathbf{l}_x$ denote the set of $\mathbf{l}(y)$ matrices for $x = 0, n, 2n, ..., z$, we obtain the multistate analog of equation (6) as

$$\mathbf{l}_{x+n} = \mathbf{p}_x \mathbf{l}_x\ , \tag{25}$$

where $\mathbf{p}_x$ is obtained from equation (24).

From the preceding exposition, one may contrast the input in equation (16)—which contains r scalar equations showing how the sizes of each state-specific group of survivors evolve over time—with the end product [equation (20) or (24)]—which consists of r^2 scalar equations showing the changes in individuals' states of residence between each pair of possible origins and destinations. The reason for this apparent gain of information lies of course in the introduction of the Markovian assumption when defining the instantaneous mobility rates in equation (8). It is precisely the independence of these rates from the past mobility history of the individuals concerned, and thus from their state of birth (in the multiradix case), which makes it possible to view the evolution of the initial cohort as the compound evolution of the r radices (real or not) subjected to the same state-specific mobility and mortality patterns.

To summarize, the Markovian assumption makes the extension of the concepts of the ordinary life table to the increment-decrement case relatively straightforward: it suffices to substitute appropriate vectors and matrices for scalars. This extension is illustrated in table 1, which presents a tabular comparison of the derivation of the stationary population in the ordinary and in the increment-decrement cases.

2.2 *The multistate life-table functions*

We now return to the ordinary life table, to review the definitions of the life-table functions, and then go on to its multistate generalization.

In the ordinary life table, if one thinks of $l(y)$ as representing the evolution over time of a cohort l_0, the number of person-years, L_x, lived by the survivors of this cohort between ages x and $x+n$ is obtained by integrating $l(y)$ between those two ages [see equation (26) in table 3]. Similarly the expected total number of years, T_x, remaining to the l_x survivors of l_0 is found by integrating $l(y)$ from x to infinity(2) [see equation (27)]. Thus, for each of the l_x individuals, the average life expectancy at age x is given by equation (29).

We can also think of the ordinary life table as a distribution of individuals alive at a given time (stationary population). In this population the death rate, m_x, relating to the individuals aged x to $x+n$ is the ratio of the number of deaths, d_x, to the exposed population, L_x. Since changes in the cohort's membership can occur only in the form of a decrement due to death, m_x can be simply expressed as a function of the l and L statistics as shown in equation (30). Another life-table statistic of interest—needed for the numerical evaluation of the Leslie and Lotka models—is the proportion of those in age group x to $x+n$ who survive into age group $x+n$ to $x+2n$. Known as a survivorship proportion, it is defined as a function of the L statistics [see equation (31)].

(2) The maximum age to which any individual can live is infinite since the last interval is half-open; note that in practice one obtains T_x from

$$T_x = \sum_k L_{x+kn}\ . \tag{28}$$

Two different generalizations of the basic life-table functions are possible: one appears as a vector extension, the other as a matrix extension which subsumes the vector extension.

The first generalization, introduced by Schoen and Nelson (1974), consists of multistate life-table functions which are derived from the state-specific age distributions $l^i(y)$ considered in their entirety. For example, Schoen and Nelson define

$$L_x^i = \int_0^n l^i(x+t)\,\mathrm{d}t\,, \qquad i = 1, ..., r\,, \tag{32}$$

as functions which, like the L_x variable in the ordinary life table, have dual meanings. For a given i, it represents, first, the number of people alive in state i between ages x and $x+n$ and, second, the number of person-years lived by the initial cohort l_0 in state i between those ages. Equations (32) can be rewritten in vector form as

$$\mathbf{L}_x = \int_0^n \mathbf{l}(x+t)\,\mathrm{d}t\,. \tag{33}$$

Similarly the total numbers of years that the survivors of the initial cohort can be expected to live (in each state) before dying is found by integrating $\mathbf{l}(y)$ from x to infinity,

$$\mathbf{T}_x = \int_0^\infty \mathbf{l}(x+t)\,\mathrm{d}t\,. \tag{34}$$

With the idea of extending the definition (29) of expectations of life at exact ages, Schoen and Land (1976) define the mean duration of stay in state i beyond age x for all survivors in the system at age x as the quotient of the ith element of $\mathbf{T}_x$ by the sum of the elements of $\mathbf{l}_x$. This statistic is, however, not too informative and one would like to qualify it further by state of presence at age x. Unfortunately this is not easily done since one needs to distinguish the contribution to T_x^i due to the members of l_x^i from those due to the members of l_x^j ($j = 1, ..., n$; $j \neq i$). Thus we need variables such as ${}^ie_x^j$, denoting the number of years that a member of l_x^i can expect to spend in region j before his death. We then have the following equation

Table 3. The vector generalization of the ordinary life-table functions.

Ordinary life table		Increment-decrement life table	
$L_x = \int_0^n l(x+t)\,\mathrm{d}t$	(26)	$\mathbf{L}_x = \int_0^n \mathbf{l}(x+t)\,\mathrm{d}t$	(33)
$T_x = \int_0^\infty l(x+t)\,\mathrm{d}t$	(27)	$\mathbf{T}_x = \int_0^\infty \mathbf{l}(x+t)\,\mathrm{d}t$	(34)
$e_x = \dfrac{T_x}{l_x}$	(29)	$\mathbf{e}_x\mathbf{l}_x = \mathbf{T}_x$	(36)
$m_x = \dfrac{l_x - l_{x+n}}{L_x}$	(30)	$\mathbf{m}_x\mathbf{L}_x = \mathbf{l}_x - \mathbf{l}_{x+n}$	(53)
$s_x = \dfrac{L_{x+n}}{L_x}$	(31)	$\mathbf{s}_x\mathbf{L}_x = \mathbf{L}_{x+n}$	(56)

linking the e, l, and T statistics:

$$\sum_{j=1}^{r} {}^{j}e_x^i l_x^j = T_x^i , \qquad i = 1, ..., r , \tag{35}$$

or more compactly

$$\mathbf{e}_x \boldsymbol{l}_x = T_x , \tag{36}$$

where the (i,j)th element of $\mathbf{e}_x$ is ${}^{j}e_x^i$.

Vector equation (36) is clearly insufficient to draw $\mathbf{e}_x$ from knowledge of $\boldsymbol{l}_x$ and T_x. However, it suggests that the generation of r linearly independent $\boldsymbol{l}(y)$ distributions makes it possible to obtain $\mathbf{e}_x$ (since $\mathbf{e}_x$ is independent of $\boldsymbol{l}_x$). In fact, since the differential equation (16) underlying an increment-decrement life table admits r linearly independent solutions corresponding to r radices (real or not), it suffices to attach an additional subscript referring to the state of birth in order to define multistate life-table functions leading to the derivation of $\mathbf{e}_x$. This is the *matrix* generalization introduced by Rogers (1973; 1975).

The ensuing second generalization of the ordinary life-table functions thus starts with the definition of

$$_{j0}L_x^i = \int_0^n {}_{j0}l^i(x+t)\,\mathrm{d}t , \qquad i = 1, ..., r , \quad j = 1, ..., s , \tag{37}$$

where $_{j0}l^i(x+t)$ is the (i,j)th element of $\mathbf{l}(x+t)$. This statistic represents the number of people born in j and alive in state i of the life table between ages x and $x+n$—which is also the number of person-years lived in state i between those ages by the members of the jth radix. Equations (37) can be written more compactly in $r \times r$ matrix form:

$$\mathbf{L}_x = \int_0^n \mathbf{l}(x+t)\,\mathrm{d}t . \tag{38}$$

The total number of person-years lived in state i in prospect for the group born in state j and currently at age x may be taken as

$$_{j0}T_x^i = \int_0^\infty {}_{j0}l^i(x+t)\,\mathrm{d}t , \qquad i = 1, ..., r , \quad j = 1, ..., s . \tag{39}$$

Table 4. The vector and matrix generalizations of the ordinary life-table functions contrasted.

Vector generalization			Matrix generalization	
$L_x = \int_0^n l(x+t)\,\mathrm{d}t$	(33)		$\mathbf{L}_x = \int_0^n \mathbf{l}(x+t)\,\mathrm{d}t$	(38)
$T_x = \int_0^\infty l(x+t)\,\mathrm{d}t$	(34)		$\mathbf{T}_x = \int_0^\infty \mathbf{l}(x+t)\,\mathrm{d}t$	(40)
$\mathbf{e}_x l_x = T_x$	(36)		$\mathbf{e}_x = \mathbf{T}_x \mathbf{l}_x^{-1}$	(44)
$\mathbf{m}_x L_x = l_x - l_{x+n}$	(53)		$\mathbf{m}_x = (\mathbf{l}_x - \mathbf{l}_{x+n})\mathbf{L}_x^{-1}$	(61)
$\mathbf{s}_x L_x = L_{x+n}$	(56)		$\mathbf{s}_x = \mathbf{L}_{x+n}\mathbf{L}_x^{-1}$	(62)

These can be collected into an $r \times r$ matrix as follows:

$$\mathbf{T}_x = \int_0^\infty \mathbf{l}(x+t)\mathrm{d}t \ . \tag{40}$$

In practice $\mathbf{T}_x$ can be obtained from

$$\mathbf{T}_x = \sum_k \mathbf{L}_{x+n} \ . \tag{41}$$

In fact $\mathbf{L}_x$ and $\mathbf{T}_x$ have as many nonzero columns as $\mathbf{l}(x+t)$, that is, $r-s$, and therefore the matrix generalization conveys more information only in the case of a multiradix increment-decrement life table, that is, when $s \geqslant 2$.

Now, with the addition of a subscript relating to the place of birth, we can rewrite equations (35) as

$$\sum_{j=1}^{r} {}^{j}e_x^i \, {}_{k0}l_x^j = {}_{k0}T_x^i \ , \qquad i = 1, ..., r \ , \quad k = 1, ..., s \ , \tag{42}$$

or more compactly as

$$\mathbf{e}_x \mathbf{l}_x = \mathbf{T}_x \ . \tag{43}$$

Thus we have, if $s = r$,

$$\mathbf{e}_x = \mathbf{T}_x \mathbf{l}_x^{-1} \ , \tag{44}$$

a relationship expressing the matrix generalization of equation (29). Equation (44) defines a matrix of expectations of life *by place of residence at age x*; however, one can also define a matrix of expectations of life by *place of birth* (see Rogers, 1975).

Note that substituting equation (40) into equation (44) and replacing $\mathbf{l}(x+t)\mathbf{l}(x)^{-1}$ by ${}_x\boldsymbol{\Omega}(x+t)$ yields

$$\mathbf{e}_x = \int_0^\infty {}_x\boldsymbol{\Omega}(x+t)\mathrm{d}t \ , \tag{45}$$

an expression that shows the independence of $\mathbf{e}_x$ vis-à-vis the state allocation of the initial cohort.

Now, if $s < r$, $\mathbf{l}_x$ has $r-s$ columns of zeroes and cannot be inverted. However, one can surmount this difficulty in two alternative ways. The first way is to remark that regardless of whether s is equal to r or not, equation (45) holds. But in fact when calculating applied increment-decrement life tables, equation (45) is difficult to use. Thus an alternative and computationally more feasible way is to extend further the matrix generalization of the l, L, and T statistics of the basic life table. The idea is to attach to these functions an index relating to the state of presence at any age y ($0 \leqslant y \leqslant x$) rather than to the state of birth (Ledent, 1978): this leads to the definition of such functions as ${}_y\mathbf{l}_x$, ${}_y\mathbf{L}_x$, and ${}_y\mathbf{T}_x$: for example, let ${}_y\mathbf{l}_x$ denote a matrix whose typical element ${}_{jy}l_x^i$ is the number of those who, in the group of people l_y^j present at age y in state j, survive to age x in state i; the definitions of ${}_y\mathbf{L}_x$ and ${}_y\mathbf{T}_x$ follow immediately from that of ${}_y\mathbf{l}_x$. Then taking $y = x$ we obtain

$$\mathbf{e}_x = {}_x\mathbf{T}_x \, {}_x\mathbf{l}_x^{-1} \ . \tag{46}$$

[An appropriate truncation of the matrices involved in equation (46) is necessary if ${}_x\mathbf{l}_x$ has zero columns and thus does not admit an inverse.] More generally, such generalized functions (with $y = x$) allow one to validate most of the matrix relationships, derived previously or to be derived, in the case when $s < r$.

So far our discussion of the multistate life-table functions has evolved around the generalization of the l, L, and e statistics of the ordinary life table (for a summary of this discussion, see the first three lines of tables 3 and 4). Yet anticipating the need for the applied calculation of increment–decrement life tables, we would like to discuss the generalization of two other statistics of the ordinary life table: the age-specific death rates, m_x, and survivorship proportions, s_x, whose multistate counterparts (in matrix format) form the basis for the implementation of the movement and transition approaches to the estimation of the age-specific transition probabilities.

Let ${}^im_x^j$ $(j \neq i)$ be the age-specific mobility rate analogous to the instantaneous force ${}^i\mu^j(y)$. It can be simply defined as the number of decrements ${}^id_x^j$, occurring in state i between ages x and $x+n$ (regardless of the state of presence at age x of those who move from i to j), to the exposed population L_x^i (Schoen and Nelson, 1974; Schoen, 1975):

$$ {}^im_x^j = \frac{{}^id_x^j}{L_x^i} , \qquad i = 1, ..., r , \quad j = 1, ..., r+1 , \quad j \neq i . \tag{47} $$

From the definition (8) of the instantaneous mobility rate ${}^i\mu^j(y)$, it follows that the number of moves ${}^id_x^j$ is given by

$$ {}^id_x^j = \int_0^n {}^i\mu^j(x+t) l^i(x+t) \mathrm{d}t , \qquad i = 1, ..., r , \quad j = 1, ..., r+1 , \quad j \neq i . \tag{48} $$

Then recalling the definition of L_x^i and substituting it into definition (47) gives

$$ {}^im_x^j = \frac{\displaystyle\int_0^n {}^i\mu^j(x+t) l^i(x+t) \mathrm{d}t}{\displaystyle\int_0^n l^i(x+t) \mathrm{d}t} , \qquad i = 1, ..., r , \quad j = 1, ..., r+1 , \quad j \neq i . \tag{49} $$

Noting that $l^i(x+t)$ is a weighted sum of the various survival probabilities ${}_{k0}\Omega^i(x+t)$ (where the weights are the radices l_0^k), we can rewrite equation (49) as

$$ {}^im_x^j = \frac{\displaystyle\sum_{k=1}^{s} \left[\int_0^n {}_{k0}\Omega^i(x+t) {}^i\mu^j(x+t) \mathrm{d}t \right] l_0^k}{\displaystyle\sum_{k=1}^{s} \left[\int_0^n {}_{k0}\Omega^i(x+t) \mathrm{d}t \right] l_0^k} , $$

$$ i = 1, ..., r , \quad j = 1, ..., r+1 , \quad j \neq i , \tag{50} $$

an expression which shows that, in the multiradix case, the value of ${}^im_x^j$ is affected by the state allocation of the initial cohort unless ${}^i\mu^j(x+t)$ is constant over the interval $(x, x+n)$.

Thus, *in the case of a multiradix increment-decrement life table, the discrete mobility rates*, unlike their instantaneous analogs, *are in general not independent of each other between states.*

Now, can we establish a relationship, analogous to the single-state equation (30), expressing mobility rates in terms of the l and L statistics? Integration of the elementary flow equations (12) leads to (Schoen and Nelson, 1974; Schoen, 1975)

$$ l_{x+n}^i = l_x^i - \sum_{\substack{j=1 \\ j \neq i}}^{r+1} {}^id_x^j + \sum_{\substack{j=1 \\ j \neq i}}^{r} {}^jd_x^i , \qquad i = 1, ..., r . \tag{51} $$

Substituting definitional equations (47) into equations (51) then yields

$$ l_{x+n}^i = l_x^i - \sum_{\substack{j=1 \\ j \neq i}}^{r+1} {}^im_x^j L_x^i + \sum_{\substack{j=1 \\ j \neq i}}^{r} {}^jm_x^i L_x^j , \qquad i = 1, ..., r , \tag{52} $$

which can be rewritten as

$$l_{x+n} = l_x - \mathbf{m}_x L_x \tag{53}$$

where $\mathbf{m}_x$ is the discrete counterpart of $\mu(y)$ defined by equation (15),

$$\mathbf{m}_x = \begin{bmatrix} \sum_{\substack{j=1\\ j\neq 1}}^{r+1} {}^1m_x^j & -{}^2m_x^1 & \dots & -{}^rm_x^1 \\ -{}^1m_x^2 & \sum_{\substack{j=1\\ j\neq 2}}^{r+1} {}^2m_x^j & \dots & -{}^rm_x^2 \\ \vdots & \vdots & \ddots & \vdots \\ -{}^1m_x^r & -{}^2m_x^r & \dots & \sum_{\substack{j=1\\ j\neq r}}^{r+1} {}^rm_x^j \end{bmatrix} . \tag{54}$$

Clearly vector equation (53) is insufficient to draw $\mathbf{m}_x$ from the availability of l_x, l_{x+n}, and L_x. Therefore it is rather tempting to expand equation (53) and write it in matrix format as

$$\mathbf{l}_{x+n} = \mathbf{l}_x - \mathbf{m}_x \mathbf{L}_x . \tag{55}$$

However, such a relationship does not generally hold because, as shown earlier, $\mathbf{m}_x$ is not a constant matrix (it depends on the radix allocation). In addition this suggests the constancy of life-table mobility rates by place of birth.

Thus *the existence of a predetermined pattern of mobility, as defined in continuous terms by equations (8), does not lead to the constancy of age-specific mobility rates but to the constancy of such rates further indexed by place of birth.* Of course, in the uniradix case the age-specific mobility rates do not bear any ambiguity since there exists only a single state of birth.

We now turn to the generalization in the multistate case of the survivorship proportion s_x, denoting the proportion of individuals aged x to $x+n$ who survive to be $x+n$ to $x+2n$ n years later.

Let ${}^is_x^j$ denote the proportion of individuals present in state i between ages x and $x+n$ who move to state j and survive to be included in that state's population aged $x+n$ to $x+2n$ years n years later. Then it is easy to show that equation (31) can be generalized into the following:

$$\mathbf{s}_x L_x = L_{x+n} , \tag{56}$$

where $\mathbf{s}_x$ is a matrix whose (i, j)th element is ${}^is_x^j$.

It is of course tempting to expand equation (56) and write it in matrix format as

$$\mathbf{s}_x \mathbf{L}_x = \mathbf{L}_{x+n} . \tag{57}$$

Again such an extension does not generally hold because the elements of $\mathbf{s}_x$ depend on the initial radix allocation. To see this, let us write ${}^is_x^j$ as follows:

$${}^is_x^j = \frac{\sum_{k=1}^{s} {}_k\alpha_i \, {}_{k0}L_{x+n}^j}{\sum_{k=1}^{s} {}_{k0}L_x^i} , \qquad i, j = 1, \dots, r , \tag{58}$$

where the numerator ${}_k\alpha_i \, {}_{k0}L_{x+n}^j$ represents the fraction of the total number of years lived in state j between ages $x+n$ and $x+2n$ by the k-born individuals who were also living in state i between ages x and $x+n$. Recalling the notation ${}_{iy_1}\Omega^j(y_2)$, it follows that

$$ {}^{i}s_x^j = \frac{\int_0^n {}_{i\,x+t}\Omega^j(x+t+n)\,{}_{k0}l^i(x+t)\,\mathrm{d}t}{\int_0^n {}_{k0}l^i(x+t)\,\mathrm{d}t} . \tag{59} $$

Substituting equation (59) into equation (58) yields, after noting that ${}_{k0}l^i(x+t)$ is simply equal to the product ${}_{k0}\Omega^i(x+t)l_0^k$,

$$ {}^{i}s_x^j = \frac{\sum_{k=1}^{s}\left[\int_0^n {}_{i\,x+t}\Omega^j(x+t+n)\,{}_{k0}\Omega^i(x+t)\,\mathrm{d}t\right]l_0^k}{\sum_{k=1}^{s}\left[\int_0^n {}_{k0}\Omega^i(x+t)\,\mathrm{d}t\right]l_0^k} , \qquad i,j = 1, ..., r , \tag{60} $$

an expression which shows that, in the multiradix case, the value of ${}^{i}s_x^j$ is affected by the state allocation of the initial cohort. In addition this suggests the constancy of life-table survivorship proportions by place of birth.

Thus *the existence of a predetermined mobility pattern, as defined in continuous terms by equations (8), does not lead to the constancy of age-specific survivorship proportions but to the constancy of such proportions further indexed by place of birth.* Of course in the uniradix case there is again no ambiguity since there exists a unique state of birth.

To summarize, the key element of this generalization of the ordinary life table resides in the definition (8) of the instantaneous forces of mobility: an individual's instantaneous propensities to move are independent of his past mobility history. This gives a Markovian-process interpretation to such tables and guarantees the *independence*, vis-à-vis the initial cohort, of the multistate life-table functions characteristic of an exact age. Formulas (21), (24), and (45) show that such functions as $\Omega(x)$, $\mathbf{p}_x$, and $\mathbf{e}_x$ depend only on the curves $\mu(y)$ and are in no circumstances affected by the state allocation of the initial cohort.

In contrast to these multistate life-table functions relating to exact ages, those functions which relate to age intervals, such as $\mathbf{m}_x$ and $\mathbf{s}_x$, are affected by the state allocation of the initial cohort (except in the uniradix case). Nevertheless the pattern of mobility is such that constant mobility rates and survivorship proportions *originating from each radix* can be found in the multistate stationary population.

In fact carrying further the reasoning which led to the preceding results, one obtains the more general results that, in any increment–decrement life table (uniradix or multiradix), the age-specific matrices of mobility rates, $\mathbf{m}_x$, and survivorship proportions, $\mathbf{s}_x$, depend on the state allocation of the multistate stationary population at any age y lying between 0 and x.

3 The applied calculation of increment–decrement life tables

In section 2 a theoretical exposition of increment–decrement life tables was presented, stressing the importance of the Markovian assumption introduced in the definition of the instantaneous forces of mobility. But how does one calculate such tables in practice? The problem here is one of estimating the multistate life-table functions, especially the transition probability matrices from which all the other functions can be generated.

There exist two main approaches to such estimation depending on the type of data available: the movement approach and the transition approach (Ledent, 1978). The first approach, similar to the classical estimation of the ordinary life table, relies on the availability of occurrence/exposure rates. The alternative approach, germane to the estimation of the ordinary life table from census information (United Nations, 1967), relies on information on survivorship proportions. Rogers (1975) refers to these as the 'option 1' and 'option 2' methods respectively.

3.1 *The movement and transition approaches: their characteristics contrasted*
Before turning to the presentation of the two estimation approaches, their essence will be characterized and compared with reference to a Lexis diagram.

The single-state Lexis diagram, found in the actuarial demographic literature, is easily generalizable to include the multistate case (Rogers, 1973; 1975). For illustrative purposes figure 1 presents the two-state Lexis diagram which simply consists of two separate Lexis diagrams, one directly beneath the other. The life lines of the migrants between the two regions connect the two diagrams. Note the three classes of life lines, represented by A, B, and C. Life line A refers to a survivor in state i who does not migrate. Life line B represents an individual in state i who out-migrates and survives the unit age interval in state j. Life line C refers to an individual who moves from state i to state j and returns before the end of the interval.

A first possibility for conceptualizing the 'passage' from state i to state j consists of observing the system at a given time: thus we see from figure 1 that one individual moves from state i to state j at time t_1, another at time t_2, and we observe a move from state j to state i at time t_3.

An alternative conceptualization of the interstate 'passage' follows from adopting a time reference based on an observation of the system at two points in time rather than at a single one. Thus we see from figure 1 that the individual corresponding to life line B lives at time $t+1$ in a state different from his state of presence at time t; by contrast the individual depicted by life line C lives in the same state at both times, although he has made two moves during the time interval.

In brief, the first conceptualization (that of the movement approach) views the 'passage' or transfer as an *event* (such as a death or a birth) taking place at a *given point in time*; by contrast the alternative conceptualization (that of the transition approach) regards such a transfer as the result of a *change in an individual's state of presence between two points in time.*

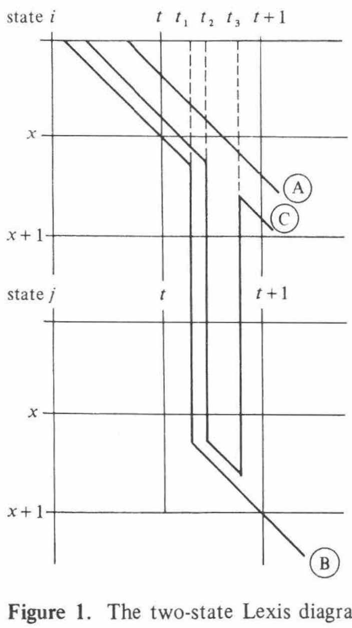

Figure 1. The two-state Lexis diagram (source: Rogers, 1973).

It is clear that the transition approach conveys less information than the movement approach, because the simple observation of the location of individuals at the beginning and end of an interval rather than over the whole interval ignores multiple moves and return migration within a unit time interval. This, however, does not hinder the reliability of the transition approach vis-à-vis the movement approach. *The fact is that any model of the life-table type is a transition model: that is, in either case moves have to be transformed into transitions.* Such a transformation occurs either within the model when estimating $\mathbf{p}_x$ (movement approach) or outside the model when the raw data come in the form of transitions (transition approach).

As briefly mentioned earlier, these two approaches correspond to similar approaches used in the applied calculation of ordinary life tables. The movement approach, because it focuses on events occurring during a given time interval, allows one to calculate, from the raw data, age-specific occurrence/exposure rates analogous to the classical evaluation of age-specific death rates. Then the problem is one of linking the $\mathbf{p}_x$ matrices to the observed mobility (death) rates in the same way that one links survival probabilities to observed death rates in the ordinary life-table case. By contrast the alternative approach, since it emphasizes the transitions occurring to individuals between two points in time, allows one to calculate, for each state, age-specific survivorship proportions (by state of destination). The problem then is one of linking the $\mathbf{p}_x$ matrices to such observed survivorship proportions in the same way that one links survival probabilities to observed survivorship proportions in the ordinary life-table case.

Note that in most of the applications of increment-decrement life tables to real situations, data come in a way consistent with the movement approach: this is the case in the analysis of marital status, as well as of labor-force participation—two of the fields to which increment-decrement life tables have been widely applied (Schoen and Nelson, 1974; Schoen, 1975; Krishnamoorthy, 1979; Schoen and Land, 1977—in the case of marital status; Hoem and Fong, 1976; Willekens, 1978—in the case of labor-force participation).

Another field of application for increment-decrement life tables is interregional migration (Rogers, 1975). Although the raw data can come in the form of moves observed between regions over a given period of time (when nations, such as Sweden or the Netherlands, maintain population registers with indications of individuals' places of residences), they also come from population censuses in the form of transitions, that is, numbers of persons of given categories who were in a specified other region one or five years earlier. Such transition migration data have been used by Rogers (1975), Rees and Wilson (1977), and Ledent (1978).

In fact the estimation of a life table from survivorship proportions appears to be a much more useful method in the multistate case (especially in the field of interregional migration) than in the single-state case, in which it is usually used only in the case of missing data (for a comparative summary of the characteristics of movement and transition approaches to the construction of increment-decrement life tables, see table 5).

In brief, the implementation of the movement (transition) approach requires a linking of the mobility patterns of the observed and stationary populations which, as in the ordinary life table, is performed by positing a simple equality of the mobility rates (survivorship proportions) with their observed counterparts.

Recall, however, that in section 2.2 we obtained the result that, in any increment-decrement life table, the age-specific matrices of mobility rates and survivorship proportions depend on the state allocation of the multistate stationary population. It follows that the estimation of an increment-decrement life table, that is, the transition from the continuous model as described in section 2.1 to its discrete equivalent, raises an aggregation problem. Apparently the only satisfactory solution to this problem is to assume that the age-specific mobility rates (survivorship proportions) are independent of the state allocation of the multistate stationary population. Upon this assumption the matrix relationship (55) or (57), whatever the case might be, holds and the estimation procedure is then a straightforward one. Accordingly we have (Rogers and Ledent, 1976)

$$\mathbf{m}_x = (\mathbf{l}_x - \mathbf{l}_{x+n})\mathbf{L}_x^{-1} \tag{61}$$

or (Rogers and Ledent, 1975; Rogers, 1975)

$$\mathbf{s}_x = \mathbf{L}_{x+n}\mathbf{L}_x^{-1} \ , \tag{62}$$

assuming that $\mathbf{L}_x$ and thus $\mathbf{l}_x$ have inverses (an appropriate truncation of the matrices involved in these two relationships is necessary if ${}_x\mathbf{l}_x$ has zero columns and thus does not admit an inverse). More generally, taking advantage of the multistate life-table functions by place of residence at any earlier age, we have that

$$\mathbf{m}_y = ({}_x\mathbf{l}_y - {}_x\mathbf{l}_{y+n})\,{}_x\mathbf{L}_y^{-1} \tag{63}$$

or

$$\mathbf{s}_y = {}_x\mathbf{L}_{y+n}\,{}_x\mathbf{L}_y^{-1} \ . \tag{64}$$

Note that the assumption here, made to facilitate the applied calculation of increment-decrement life tables, is Markovian in that the age-specific mobility rates or survivorship proportions are hypothesized to be independent of the mobility history of the survivors of the initial cohort. This assumption is, however, not equivalent to the Markovian assumption introduced in the theoretical model of section 2 when defining the instantaneous forces of mobility(3). Whereas the latter only relates to intrinsic characteristics, the former denies a role to the state allocation of the stationary population and is indeed a much stronger assumption.

It is concluded that the possibility of calculating an increment-decrement life table from either the movement or the transition approach requires use of an underlying assumption differing slightly with respect to the one underlying the theoretical model. Thus, *as is often noted in mathematical demography, the discrete model does not behave like the parallel continuous model.*

Note, however, that if the observed mobility rates and survivorship proportions are not independent of the place of birth, equating life-table and observed rates is likely

Table 5. Characteristics of the movement and transition approaches contrasted.

Characteristic	Movement approach	Transition approach
Conceptualization of interstate 'passage'	*Event* taking place at a *given time*, as in the case of a birth or a death	*Change in* an individual's *state or presence* between *two points in time*
Type of data required	Moves ↓ Occurrence/exposure rates	Movers (transitions) ↓ Survivorship proportions
Fields of application	Marital status Labor-force participation Interregional migration (if data come from population registers)	Interregional migration (if data come from census information)

(3) The assumptions relating to the two alternative approaches are also not equivalent.

to lead to rather inaccurate estimates of the transition probabilities $\mathbf{p}_x$ and the various multistate life-table functions. Thus, if adequate data by place of birth are available, it is advisable to perform the calculation of separate increment–decrement life tables for each possible radix. In the case of interregional migration, in which the dependence of migration propensities vis-à-vis the place of birth (Long and Hansen, 1975), such an alternative has been proposed and implemented by Ledent (1980a) from the transition perspective.

3.2 *Estimation from occurrence/exposure rates: the movement approach*

As earlier, first a brief review is presented of the corresponding estimation in the ordinary life table and then the multistate generalization is undertaken. The classical calculation of an ordinary life table is based on a linking of the mortality patterns of the observed and the life-table populations, a linkage which is performed at a discrete rather than a continuous level and which usually involves equating the age-specific mortality rates in the observed and life-table populations (denoted by M_x and m_x respectively).

The simplest construction method follows from assuming that $\mu(y)$ is constant within each age interval $(x, x+n)$ and thus equal to M_x [see equation (65) in table 6]. In those circumstances the substitution of M_x for m_x in equation (7) leads to the survival probability defined by equation (66) in table 6. This allows one to determine the set of survival quantities, l_x, by application of equation (6), and then the number of person-years lived, L_x, from equation (30) rewritten as equation (67). In the case of the last age group, L_z can be obtained from equation (67) by putting l_{z+n} equal to zero.

An alternative method for estimating the survival probability, p_x, from the observed death rates, M_x, follows from the linear calculation of L_x as indicated in equation (68); the corresponding formula appears as equation (69) in table 6.

In any case, regardless of the method used, once the L quantities have been obtained, one estimates the T, e, and s statistics from equations (28), (29), and (31) respectively.

These two construction methods are easily extended to the increment–decrement case because the resulting generalizations are consistent with the requirement that age-specific mobility rates be independent of the state allocation of the multistate stationary population.

The assumption of constant-force functions within each interval—suggested by Hoem (1970a; 1970b) and Hoem and Fong (1976)—leads, as pointed out during examination of formulas (50), to an ambiguous matrix of mobility rates. In such circumstances

$$\mu(y) = \mathbf{m}_x\ , \qquad \text{for } x \leqslant y < x+n\ , \tag{70}$$

so that the equality of $\mathbf{m}_x$ with its counterpart $\mathbf{M}_x$ leads to (Krishnamoorthy, 1979)

$$\mathbf{p}_x = \exp(-n\mathbf{M}_x)\ . \tag{71}$$

The scalar formulas pertaining to the two-state case appear as equations (72) to (76) in table 7; in practice, if $r > 2$, one would calculate $\mathbf{p}_x$ from

$$\mathbf{p}_x = \mathbf{I}+n\mathbf{M}_x+\frac{n^2}{n}\mathbf{M}_x^2+\ldots\ . \tag{77}$$

Once the transition probability matrices have been obtained by application of formula (71), the set of the survival quantities $\mathbf{l}_x$ is then easily obtained by a repetitive application of equation (25).

By contrast the estimation of the number of person-years lived, needed for calculating the other multistate life-table functions, is not so straightforward as in the

Table 6. Alternative methods for constructing ordinary and increment-decrement life tables.

Method	Ordinary life table	Increment-decrement life table
Movement approach (estimation from occurrence/exposure rates)		
	$m_x = M_x$	$\mathbf{m}_x = \mathbf{M}_x$
1	(65) $\left.\begin{cases} \mu(y) = m_x \\ x \leqslant y < x+n \end{cases}\right\} \rightarrow \begin{cases} p_x = \exp(-nM_x) & (66) \\ L_x = \dfrac{l_x - l_{x+n}}{M_x} & (67) \end{cases}$	(70) $\left.\begin{cases} \mu(y) = \mathbf{m}_x \\ x \leqslant y < x+n \end{cases}\right\} \rightarrow \begin{cases} \mathbf{p}_x = \exp(-n\mathbf{M}_x) & (71) \\ \mathbf{L}_x = \mathbf{M}_x^{-1}(\mathbf{l}_x - \mathbf{l}_{x+n}) & (78) \end{cases}$
2	(68) $L_x = \frac{n}{2}(l_x + l_{x+n}) \rightarrow p_x = \dfrac{1 - \frac{n}{2}M_x}{1 + \frac{n}{2}M_x}$ (69)	(83) $\mathbf{L}_x = \frac{n}{2}(\mathbf{l}_x + \mathbf{l}_{x+n}) \rightarrow \mathbf{p}_x = \left(\mathbf{I} + \frac{n}{2}\mathbf{M}_x\right)^{-1}\left(\mathbf{I} - \frac{n}{2}\mathbf{M}_x\right)$ (85)
Transition approach (estimation from survivorship proportions)		
	$s_x = S_x$	$\mathbf{s}_x = \mathbf{S}_x$
1	(96) $S_x = \dfrac{(1 + p_{x+n})p_x}{1 + p_x} \rightarrow p_x$	(97) $\mathbf{S}_x = (\mathbf{I} + \mathbf{p}_{x+n})\mathbf{p}_x(\mathbf{I} + \mathbf{p}_x)^{-1} \rightarrow \mathbf{p}_x$
2		(98) $\mathbf{p}_x \approx \frac{1}{2}(\mathbf{S}_{x-n} + \mathbf{S}_x)$
3		$\mathbf{p}_x$ obtained by interpolation between the survivorship proportions $\mathbf{S}_x$

Table 7. The two-state case in the movement approach.

Exponential assumption		Linear assumption	
$\mu(y) = \mathrm{m}_x\,, \qquad x \leqslant y < x+n$	(70)	$\mathrm{L}_x = \frac{n}{2}(\mathrm{l}_x + \mathrm{l}_{x+n})$	(83)
${}^ip_x^i = \frac{(r_2 + {}^iM_x^j + {}^iM_x^\delta)\exp(r_1 n) - (r_1 + {}^iM_x^j + {}^iM_x^\delta)\exp(r_2 n)}{r_2 - r_1}$	(72)	${}^ip_x^i = \frac{1 - \frac{n}{2}{}^iM_x^\delta - \frac{n}{2}{}^iM_x^j(U_x^j/V_x^j)}{1 + \frac{n}{2}{}^iM_x^\delta + \frac{n}{2}{}^iM_x^j(U_x^j/V_x^j)}$	(86)
${}^ip_x^j = \frac{{}^iM_x^j[\exp(r_2 n) - \exp(r_1 n)]}{r_2 - r_1}$	(73)	${}^ip_x^j = \frac{n({}^iM_x^j/V_x^j)}{1 + \frac{n}{2}{}^iM_x^\delta + \frac{n}{2}{}^iM_x^j(U_x^j/V_x^j)}$	(87)
${}^iq_x^\delta = \frac{r_2 - r_1 - (r_2 + {}^iM_x^\delta)\exp(r_1 n) + (r_1 + {}^iM_x^\delta)\exp(r_2 n)}{r_2 - r_1}$	(74)	${}^iq_x^\delta = \frac{n\,{}^iM_x^\delta + \frac{n^2}{2}{}^iM_x^j({}^jM_x^\delta/V_x^j)}{1 + \frac{n}{2}{}^iM_x^\delta + \frac{n}{2}{}^iM_x^j(U_x^j/V_x^j)}$	(88)
where		where	
$r_1 = \frac{-({}^iM_x^j + {}^iM_x^\delta + {}^jM_x^i + {}^jM_x^\delta) + [({}^iM_x^j + {}^iM_x^\delta - {}^jM_x^i - {}^jM_x^\delta)^2 + 4\,{}^iM_x^j\,{}^jM_x^i]^{1/2}}{2}$	(75)	$U_x^j = 1 + \frac{n}{2}{}^jM_x^\delta$	(89)
$r_2 = \frac{-({}^iM_x^j + {}^iM_x^\delta + {}^jM_x^i + {}^jM_x^\delta) - [({}^iM_x^j + {}^iM_x^\delta - {}^jM_x^i - {}^jM_x^\delta)^2 + 4\,{}^iM_x^j\,{}^jM_x^i]^{1/2}}{2}$	(76)	$V_x^j = 1 + \frac{n}{2}{}^jM_x^\delta + \frac{n}{2}{}^jM_x^i$	(90)

Note: ${}^iq_x^\delta$ is the probability that a person present in state i at age x dies within the next n years.

Source: Schoen and Land (1977).

ordinary life-table case. The calculation of the number of person-years lived follows from the matrix generalization of equation (53) into equation (55), which now holds. But, unless each state is initially nonempty, equation (55) is insufficient to generate all the other multistate life-table functions. Thus two cases are distinguished:

(1) If each state is initially nonempty (that is, if $s = r$), the number of person-years lived can be calculated from equation (55) rewritten as

$$\mathbf{L}_x = \mathbf{M}_x^{-1}(\mathbf{l}_x - \mathbf{l}_{x+n}) \, . \tag{78}$$

[Note that, in the case of the last age group, equation (78) remains valid if one sets $\mathbf{l}_{z+n} = 0$; then

$$\mathbf{L}_z = \mathbf{M}_z^{-1}\mathbf{l}_z \, , \tag{79}$$

which offers a simple matrix notation of the scalar solution proposed by Schoen (1975, appendix).] In some circumstances, especially for the younger age groups, which exhibit small death rates, the determinant of $\mathbf{M}_x$ might take on a value close to zero and thus the calculation of its inverse might prove to be inaccurate. Then one might have to calculate $\mathbf{L}_x$ from an alternative formula which could be inconsistent with the set of probabilities $\mathbf{p}_x$.

Then the T and e statistics can be estimated from equations (41) and (44). As for the survivorship proportions, regardless of whether s is less than or equal to r, one must either calculate such statistics by place of birth (Ledent, 1978) or obtain approximate values from equation (62).

(2) If at least one state is initially empty (that is, if $s < r$), one must resort to multistate life-table functions further indexed by place of residence at age x, that is, ${}_x\mathbf{l}_y$ and ${}_x\mathbf{L}_y$ (see section 2.2). In this case the calculation of the various expectations-of-life statistics is performed using equation (63) in which $\mathbf{M}_y$ is substituted for $\mathbf{m}_y$:

$${}_x\mathbf{L}_y = \mathbf{M}_y^{-1}({}_x\mathbf{l}_y - {}_x\mathbf{l}_{y+n}) \, , \qquad \text{for } y \geqslant x \, . \tag{80}$$

Then $\mathbf{e}_x$ is obtained from equation (46), where ${}_x\mathbf{T}_x$ is estimated as $\sum_y {}_x\mathbf{L}_y$.

Many alternatives to this method based on constant-force functions are possible which also suppose $\mathbf{m}_x$ to be independent of the state allocation of the stationary population. Fundamentally they depend on the choice of an explicit method for calculating $\mathbf{L}_x$. To see this, note that as a consequence of equation (55) we have

$$(\mathbf{I} - \mathbf{p}_x)\mathbf{l}_x = \mathbf{m}_x\mathbf{L}_x \tag{81}$$

or, if $\mathbf{l}_x$ has an inverse,

$$\mathbf{p}_x = \mathbf{I} - \mathbf{m}_x\mathbf{L}_x\mathbf{l}_x^{-1} \, , \tag{82}$$

an equation which is identical to the equation obtained by Willekens (1978). One possibility is to assume a linear calculation of $\mathbf{L}_x$ (Rogers, 1973; 1975):

$$\mathbf{L}_x = \frac{n}{2}(\mathbf{l}_x + \mathbf{l}_{x+n}) \, . \tag{83}$$

Combining equation (83) with equation (61) leads, after substitution of $\mathbf{M}_x$ for $\mathbf{m}_x$, to (Rogers and Ledent, 1976)

$$\mathbf{p}_x = \left(\mathbf{I} - \frac{n}{2}\mathbf{M}_x\right)\left(\mathbf{I} + \frac{n}{2}\mathbf{M}_x\right)^{-1} \tag{84}$$

or, since the two matrices $\mathbf{I} - \frac{n}{2}\mathbf{M}_x$ and $\mathbf{I} + \frac{n}{2}\mathbf{M}_x$ commute,

$$\mathbf{p}_x = \left(\mathbf{I} + \frac{n}{2}\mathbf{M}_x\right)^{-1}\left(\mathbf{I} - \frac{n}{2}\mathbf{M}_x\right) \, . \tag{85}$$

Unlike formula (61), formula (85) is valid regardless of whether $s \leqslant r$. The scalar formulas pertaining to the two-state case appear as equations (86) to (90) in table 7.

Note that, if the linear calculation is implemented, in a less restrictive way, at the level of the vector generalization, that is,

$$L_x = \frac{n}{2}(l_x + l_{x+n}) , \tag{91}$$

one obtains

$$l_{x+n} = \left(\mathbf{I} - \frac{n}{2}\mathbf{m}_x\right)\left(\mathbf{I} + \frac{n}{2}\mathbf{m}_x\right)^{-1} l_x \tag{92}$$

by combining equation (91) with equation (61). Note also that, when considering a matrix of average transfer durations, which generalizes Chiang's (1960) and Rogers's (1973; 1975) a's, Ledent (1978) derives a more general relationship linking l_x and l_{x+n} which subsumes relationship (92) of the linear integration method.

Thus $\mathbf{p}_x$ is given by equation (85) regardless of whether the linear calculation is performed at the level of the vector or the matrix generalization. In fact the linear integration assumption in vector format [relationship (91)] is sufficient to estimate $\mathbf{p}_x$, but its matrix expansion [relationship (83)] is needed to calculate the other multistate functions.

Further, equation (85) can be rearranged to yield

$$\mathbf{M}_x = \frac{2}{n}[\mathbf{I} + \mathbf{p}_x]^{-1}[\mathbf{I} - \mathbf{p}_x] , \tag{93}$$

a relationship which shows that the hypothesis that age-specific mobility rates are independent of the state allocation of the initial cohort is implicit in the linear integration assumption.

To date, both of the construction methods presented have been used in the applied calculation of uniradix increment-decrement life tables. The exponential (or constant-forces-of-transition) assumption has been employed by Hoem and Fong (1976) in an analysis of labor-force participation and by Krishnamoorthy (1979) and Schoen and Land (1977) in analyses of marital status. As for the linear integration assumption, it has been utilized in an analysis of labor-force participation (Willekens, 1978) as well as in one of the marital-status studies cited (Schoen and Land, 1977). Note that in all instances these methods have been applied in reference to single-year age groups.

By contrast, the linear assumption has been used to calculate multiregional life-tables—multiradix increment-decrement life tables applied to interregional migration—for Finland (Manninen, 1979) and other countries maintaining a compulsory system of registration, with reference to five-year age groups.

A rather curious result obtained in the Manninen study for Finland is a negative retention probability ${}^{i}p_{20}^{i}$ for the Uuden region, a region of high out-migration. Such a result calls our attention to the fact that the probability matrices $\mathbf{p}_x$ in equations (71) and (85) derived by application of the exponential and linear assumptions respectively, are not, unlike the probability matrices $\mathbf{p}_x$ obtained in our theoretical derivation of section 2, proper transition probability matrices. This fact can be easily shown to hold in the two-state case by use of the formulas displayed in table 7. It is clear, both with the exponential and the linear assumptions, that (a) the migration and death probabilities can be greater than one and (b) the retention probabilities can be negative. (Recall that, in the ordinary life table, p_x always lies between zero and one in the exponential case but can be greater than one in the linear case if $M_x > 2/n$, that is, if M_x is large and/or n is large.)

For example, in a two-state system it follows from equations (72) and (86) respectively that ${}^{i}p_x^{i}$ is negative in the case of the exponential assumption if

$$ {}^{i}M_x^{\delta} + {}^{i}M_x^{j} > \frac{r_2 - r_1}{\exp(r_2 n) - \exp(r_1 n)} \approx \frac{1}{n} \tag{94} $$

and in the case of linear integration if

$$ {}^{i}M_x^{\delta} + {}^{i}M_x^{j} \frac{U_x^{j}}{V_x^{j}} > \frac{2}{n} \, . \tag{95} $$

A close look at inequalities (94) and (95) indicates that, as expected, a negative retention probability is more likely to occur (a) the higher the mortality rate in state i or the mobility rate from state i to state j and (b) the higher the value of the typical age interval n. In brief, ${}^{i}p_x^{i}$ may be negative because the discrete mobility rates are not bounded: in theory anyone in the system can move constantly back and forth between states so that mobility rates, estimated over an n-year period, can be quite large, causing inequality (94) or (95) to hold. This is in sharp contrast with the theoretical approach, in which the instantaneous forces of mortality and mobility are 'constrained' by equation (10), causing the $\mathbf{p}_x$ matrices to be proper transition probabilities.

In practice, if n is kept small (for example, $n = 1$), improper transition probabilities do not occur when calculating an applied increment-decrement life table because most of the situations with which we are confronted involve relatively low age-specific mobility rates. However, although we may obtain matrices $\mathbf{p}_x$ with elements that lie between zero and one, the accuracy of the elements of $\mathbf{p}_x$ obtained from equations (71) and (85) is questionable, especially for age groups in which mobility rates are not negligible. The precision of the estimates obtained diminishes as n increases.

To summarize, the applied calculation of increment-decrement life tables from occurrence/exposure rates (movement approach) can be performed by generalizing the corresponding methods used in the case of the ordinary life table. However, it has been shown that the possibility of such a generalization requires the adoption of an assumption that is inconsistent with the theoretical underpinning of increment-decrement life tables described in section 2, namely the independence of the age-specific mobility rates from the state allocation of the multistate stationary population.

Two of the most widely used methods are based on exponential and linear assumptions, which lead to formulas, relating the transition probabilities to the mobility rates, similar to those used in the corresponding cases of the ordinary life table: the only difference is that matrices replace scalars.

From a practical point of view it seems that the numerical evaluation of increment-decrement life tables from occurrence/exposure rates raises some questions of accuracy with regard to the specific data at hand. Also, when performing such an evaluation, one must pay special attention to the values of the various mobility rates—especially if n is large (for example, equal to five, as is often the case for interregional migration data)—so as to decide the reasonableness of the exponential and linear assumptions.

3.3 *Estimation from survivorship proportions: the transition approach*

In some circumstances the data concerning phenomena of social mobility come in the form of transitions (movers) rather than moves. This is often the case in the field of interregional migration, where data on changes of residence from population censuses are common.

Let us suppose that census information allows the calculation of age-specific survivorship proportions summarized in a matrix $\mathbf{S}_x$, the observed analog of the life table $\mathbf{s}_x$ defined in section 2.2; the problem of estimating $\mathbf{S}_x$ has been treated elsewhere [see Rogers and von Rabenau (1971) for the estimation of $\mathbf{S}_x$ from lifetime migration data and Ledent (1978) for an estimation from migration data relating to a fixed time period] and so is not discussed here. Then the logical idea is to calculate an increment-decrement life table by generalizing the method sometimes used by demographers to calculate ordinary life tables which rely on census information alone. According to this method (United Nations, 1967), because the survival probabilities and survivorship proportions of the ordinary life table are related to each other by relation (96) in table 6 (if one adopts the linear integration method for calculating L_x), it is possible to estimate a set of p_x by setting the life-table survivorship proportions s_x equal to their observed counterparts.

In theory this method can be generalized to the multistate case, in which the approximate survivorship proportion matrices $\mathbf{s}_x$ are linked to the survival probability matrices $\mathbf{p}_x$ by the matrix analog of equation (96) (Rogers, 1975):

$$\mathbf{s}_x = (\mathbf{I}+\mathbf{p}_{x+n})\mathbf{p}_x(\mathbf{I}+\mathbf{p}_x)^{-1} \ . \tag{97}$$

However, as pointed out by Ledent (1978), this procedure is much less effective here than in the case of the ordinary life table. The main reason for this is that the observed survivorship proportions reflect the consolidation of migratory moves whose pattern may have varied over the observation period; thus the survival probabilities observed by such a method are average values, which are likely to be highly inaccurate owing to the particular averaging method implied by equation (97); this contrasts with the case of the ordinary life table, where, provided that the survivorship proportions have been accurately observed, the average mortality rates implies by equation (96) reflect acceptable average values (since mortality patterns are much less volatile than migration patterns). In addition, because the estimation procedure relies on a formula linking statistics of two consecutive groups, estimation errors made in a given age group are passed on to the next: the 'noise' thus introduced is likely to increase as one carries the estimation procedure over the whole set of age groups.

Because of this, Rees and Wilson (1977) instead estimate the set of survival probabilities from the following approximation:

$$\mathbf{p}_x \approx \tfrac{1}{2}(\mathbf{S}_{x-n}+\mathbf{S}_x) \ . \tag{98}$$

[Note that, alternatively, one could think of estimating $\mathbf{p}_x$ as the geometric (rather than arithmetic) average of the two consecutive survivorship proportion matrices $\mathbf{S}_{x-n}$ and $\mathbf{S}_x$.] Formula (98) has to be amended in the case of the first age group: let $\mathbf{s}_{-n}$ denote the probability

$$\mathbf{s}_{-n} = \frac{1}{n}\mathbf{L}_0\mathbf{l}_0^{-1} \tag{99}$$

that newborns will survive to the end of the interval (Ledent, 1978); then we can estimate $\mathbf{p}_0$ from

$$\mathbf{p}_0 \approx \tfrac{1}{2}(\mathbf{S}_{-n}^2+\mathbf{S}_0) \ . \tag{100}$$

It turns out that no matter how crude the method defined by equation (98) may appear, it provides better results than those obtained by the procedure based on equation (97), especially in the case of the older age groups.

In view of this I have elsewhere proposed (Ledent, 1980b) an estimation method that improves upon the one suggested by Rees and Wilson (1977). The improved method begins by noting that $\mathbf{s}_x$ is in fact a weighted average of the survival probabilities $\mathbf{p}_{x+kn}$, where k takes on all values between zero and one. Thus it is suggested that an adequate estimation of $\mathbf{p}_x$ can be obtained by interpolating between the observed survivorship proportions $\mathbf{S}_x$ but in a less crude fashion than set out in approximation (98). For each pair of nonabsorbing states i and j, such an interpolation can be easily performed using cubic spline functions, which are increasingly used in the field of demography (McNeil et al, 1977). The ordinate of the continuous curves thus obtained represents the probability that an individual present at age y in state i is present in state j n years later. The required probabilities then may be found as the estimates of these curves at ages $x = 0, n, 2n, ..., z$.

An advantage of this interpolative procedure with respect to the methods based on equations (97) and (98) is that it does not necessarily require survivorship proportions calculated for a period T equal to the length, n, of the typical age group. If $T \neq n$ the interpolative procedure also allows one to estimate a set of transition probabilities $\mathbf{p}_x$ for $x = 0, T, 2T, ...$.

Note that, in the case of the age groups characterized by low mortality, the straightforward application of the aforementioned interpolation procedure could be troublesome in that it may be that $\sum_j {}^ip_x^j > 1$. Therefore, rather than interpolate between the survivorship proportions ${}^is_x^i$, it is preferable to interpolate between the mortality proportions $1 - \sum_j {}^is_x^j$, so as to obtain the quotient of mortality ${}^iq_n^\delta$, and then to derive the retention probability p_x^i as $1 - {}^iq_x^\delta - \sum_{j \neq i} {}^ip_x^j$.

In practice, however, especially in the case of interregional migration, the calculation of survivorship proportions relating to a current period is not possible because census data do not provide estimates of the populations concerned at the beginning of the observation period. This is, for example, the case with the US Census migration data; changes of residence are available over a five-year period (for example, 1965-1970) but cannot be related to the beginning-of-the-period populations (in 1965), which are unknown. In such circumstances the interpolation method can be amended first by estimating a set of transition probabilities $\bar{\mathbf{p}}_x$ conditional on survival and second by transforming them into the required transition probabilities by introducing independent mortality information. Then $\mathbf{p}_x$ is obtained from

$$\mathbf{p}_x = \bar{\mathbf{p}}_x \hat{\mathbf{p}}_x^\delta , \tag{101}$$

where $\bar{\mathbf{p}}_x$ is the corresponding transition probability matrix conditional on survival and $\hat{\mathbf{p}}_x^\delta$ is a diagonal matrix whose elements are similar to the survival probabilities of ordinary life tables. (This procedure assumes that mortality rates are dependent on the place of residence at age x rather than on the place of occurrence.) The interpolation procedure allows one to calculate the various migration probabilities ${}^i\bar{p}_x^j$ ($j \neq i$) conditional on survival; the retention probabilities conditional on survival follow immediately from ${}^i\bar{p}_x^i = 1 - \sum_{j \neq i} {}^i\bar{p}_x^j$.

Note that the accuracy of this interpolation procedure for calculating the migration probabilities, conditional on survival or not, greatly depends on the type of mobility phenomena at hand. Clearly, if the mobility schedule concerned displays rapid variations and sharp turns over age, the interpolation procedure may improve, if only slightly, the arithmetic-average method proposed by Rees and Wilson (1977).

In most instances, however, mobility schedules exhibit very stable regularities. Then it is possible to improve the interpolation procedure by using simultaneously a graduation of a typical mobility schedule. For example, in the case of interregional migration, such age-specific regularities have been substantiated over space (Long, 1973) and time (Rogers and Castro, 1976). This has recently led to the notion of model migration schedules similar to that of model fertility schedules (Rogers et al, 1978). Thus the estimates of the migration probabilities, especially those conditional on survival, can be greatly improved by fitting, for each pair of origin and destination regions, a model migration schedule to the set of survivorship proportions.

The estimation of $\hat{\mathbf{p}}_x^\delta$ is relatively straightforward and is not discussed here.

The interest of this improved method is that, in contrast to the methods used to estimate $\mathbf{p}_x$ in the movement approach, it avoids empirical problems such as the occurrence of negative retention probabilities. In fact

(a) the various transition probabilities ${}^{i}\bar{p}_x^{j}$ conditional on survival are constrained to be between zero and one as a consequence of the interpolation method;

(b) the survival probabilities ${}^{i}\hat{p}_x^{\delta}$ are obtained as the survival probabilities of an ordinary life table—they lie between zero and one if the exponential assumption is chosen and may be negative if n is large and/or ${}^{i}\hat{M}_x^{\delta}$ is large (it is, however, easy to get away from that problem by consolidating survival probabilities for single-year age groups estimated from mortality rates by single-year age groups).

Now, how does one complete the calculation of an increment-decrement life table based on the transition approach once transition probabilities have been observed? The issue here is one of calculating $\mathbf{L}_x$ adequately, since this calculation immediately allows the estimation of the remaining multistate life-table fractions from equations (41), (44), and (62). As an alternative to the linear integration method (Rogers, 1973; 1975), Ledent (1980b) suggests a more realistic method, whereby the greater the length, n, of the age intervals the larger the improvement with respect to the linear integration method.

3.4 *The movement and transition approaches: further contrasts*

To summarize, there are two alternative approaches to the construction of increment-decrement life tables, depending on the type of data available. One approach relies on data in the form of moves recorded over a given interval of time (movement approach) and transformed into occurrence/exposure rates; the other (transition approach) uses data in the form of changes in individuals' states of presence or residence between two points in time, translating these into survivorship proportions.

As shown in section 3.2, the movement approach leads to formulas expressing the age-specific survival probabilities in terms of the corresponding mobility rates, which are not characteristic of a proper transition probability matrix: in the case when the typical age interval, n, is large, high migration propensities can lead to numerically negative retention probabilities, independently of whether one chooses the exponential or linear integration methods.

By contrast, the implementation of the transition approach as proposed in section 3.3 is little affected by such a problem. The transition probability matrix is the product of a transition probability matrix conditional on survival—which is unlikely to be improper—and a matrix of survival proportions—whose nonzero elements are calculated as in the ordinary life table.

This result, that the transition approach is less prone to implementation problems than the movement approach, is not surprising if one recalls that any life table is basically a transition model. In effect the basic issue of a life table is one of estimating transition probabilities from the actual moves. In the ordinary life table this is not a problem: because death is not a recurrent event, the consolidation of moves (that is, deaths) into transitions is unambiguous. In the multistate case, however, because mobility is a recurrent event, such a consolidation is less straightforward and can be conceived in two different ways: either such consolidation is undertaken with the help of an underlying process of mobility or it is performed at the level of the data in order to obtain the equivalent data expressed in terms of transitions (that is, changes in the state of presence).

Relating this to the features of the movement and transition approaches leads one to conclude that
(a) in the former, the consolidation of moves into transitions is implemented within the model underlying an increment-decrement life table: it is precisely where the Markovian assumption—or more exactly its discrete equivalent (${}^{i}m_x^j$ independent of the state allocation of the multistate stationary population)—comes in;
(b) in the latter, the consolidation occurs outside the model.

Leaving aside empirical considerations—such as those relating to the use of a proper estimation method for calculating the transition probabilities $\mathbf{p}_x$—what can we infer from such a conclusion with regard to the essence of the alternative approaches?

Clearly, if the two approaches could be applied to an identical mobility situation, different results would be obtained. By construction the transition approach leads to transition probabilities which more or less match the observed ones. By contrast the movement approach cannot replicate observed transition probabilities as well, especially if the age interval, n, is large: in effect the consolidation of moves into transitions within the model supposes population homogeneity and independence vis-à-vis the past, and thus the movement approach cannot adequately account for multiple moves and return migrations within the unit time interval. The validity of such a statement is suggested by the work of Singer and Spilerman (1976; 1978) and, to a lesser degree, Rees (1977), and can be shown by comparing—for each pair of states i and j—the number of moves per person transferring, calculated from the model with the corresponding observed ratio. Thus the movement approach usually leads to transition probability matrices with inflated off-diagonal elements.

The conclusion here is that, for a given choice of n, the transition approach is preferable to the movement approach because it attenuates the stringent consequences of the population homogeneity and Markovian assumptions.

Since the possibility of multiple moves within a given interval increases with the length, n, of that interval, the larger the value of n, the larger the overestimation of the off-diagonal elements of $\mathbf{p}_x$ obtained with the movement approach; that is, the more preferable is the transition approach with respect to the movement approach.

In practice, since the data available come in the form either of moves or of transitions, the main issue is not one of choosing between the two alternative approaches but one of choosing the optimal value of n for implementing each alternative approach. Thus, let us suppose that we can calculate a five-year transition probability matrix $\mathbf{p}_x$—both from the movement and from the transition perspectives—in two different ways:
(a) directly from data for five-year age groups;
(b) from the product of single-year matrices.

In the case of the *movement* approach, the results are not expected to be very much different because both methods implicitly adopt the population homogeneity and Markovian assumptions. Thus, since the estimation of $\mathbf{p}_x$ raises fewer empirical problems as n decreases, the best calculation of an increment-decrement life table is performed when n is small, that is, when data come in single-year age groups.

By contrast, in the *transition* approach, the second method is likely to exhibit inflated off-diagonal elements with respect to the first method because it does not adequately account for multiple moves and return migrants (Singer and Spilerman, 1976; 1978). Thus it is concluded that, with this approach, the best calculation of an increment-decrement life table is performed when n is large. (A larger n yields better estimates of not only the transition probabilities but also the numbers of person-years lived, $\mathbf{L}_x$, if one uses the original method described in section 3.2.) In practice a value of n around five appears to be optimal.

4 Conclusion

The generalization of ordinary life-table concepts to the increment–decrement case is a relatively straightforward matter in which matrix algebra is used as a computing device. The various combinations of moves between the alternative states are captured by formulas as simple as those relating to an ordinary life table, the only difference being that vectors and/or matrices replace scalars. In the case of life-table functions associated with groups of individuals of a given age, there is a direct correspondence between those of the ordinary life tables and those of the increment–decrement life tables, so that one can go from the former to the latter by substituting appropriate matrices for scalars. By contrast, in the case of the life-table functions associated with groups of individuals in a given age bracket (age-specific mobility rates and survivorship proportions), there is no such direct correspondence because of the dependence of these functions on the state allocation of the initial cohort.

From a practical point of view the key element is the estimation of the age-specific survival probabilities from which the calculation of all the multistate life-table functions originates. Two alternative approaches to such estimation are available. One possibility consists of duplicating the methodology commonly used to make an ordinary life table: this is known as the movement approach, which views interstate passages as instantaneous events and thus relies on inputs in the form of occurrence/exposure rates. One important feature of this methodology is its requirement of an additional assumption with respect to the theoretical model, mainly the independence of the age-specific mobility rates from the state allocation of the multistate stationary population. The other possibility is germane to the construction of the ordinary life table from survivorship proportions: this is known as the transition approach, which focuses attention on changes in individuals' states of presence between two points in time. The implementation of this approach, as proposed by Ledent (1980b), looks to the calculation of age-specific probabilities by interpolating between observed survivorship proportions conditioned on survival.

In practice the application of the movement approach raises a few empirical problems:

(a) the possibility of negative probabilities, especially if n is large;

(b) inaccurate results arising from the underlying Markovian and population homogeneity assumptions.

By contrast the transition approach is much less affected by such problems, especially when implemented with n taking on a value of around five years.

The conclusion here is that, ceteris paribus, the construction of an increment–decrement life table should be performed preferably from the transition approach rather than from the movement approach. But since mobility data generally come under the form of moves (occurrence/exposure rates) *or* (not and) transitions (survivorship proportions), the approach to be used is imposed by the form in which the data are collected and tabulated.

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Multistate analysis: tables of working life

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Abstract. The demographer's approach to the study of the labor force includes the calculation of life tables for the active population. Although this tool is extensively used and many countries publish working-life tables, and although they rely on very restrictive assumptions, the technique for constructing such tables has not improved since its development about thirty years ago. This paper reviews the conventional method for the construction of working-life tables and proposes a multistate approach which does not rely on restrictive assumptions such as the unimodality of the labor-force participation curve. Instead of focusing on changes in stocks, the actual flows of people between active and inactive life are considered. The technique is compared with a similar procedure recently developed by Hoem and Fong. The increment-decrement table of working life serves as a basis for a multistate model for labor-force projection. The proposed model is compared with conventional approaches. The methods presented in this paper are illustrated using Danish data.

1 Introduction

At the St Louis meeting of the Population Association of America in 1977, Dr S J Smith of the US Bureau of Labor Statistics described an interesting example of the use of working-life tables, namely the estimation of lost earnings in the case of death before or during active life (Smith, 1977). This 'money-value of man' is of particular relevance in liability estimations. With the increase in the number of liability claims over the years, working-life tables now are used more than ever. Liability attorneys, and others interested in determining foregone earnings, rely on the concept of working-life expectancy as an input to their determinations.

On the other hand, working-life tables are used in macroeconomic analysis to describe the supply of labor and to study the impact of structural changes in the labor force, both in terms of changing activity rates and in terms of changing age compositions (Durand and Miller, 1968, page 19). Today working-life tables are being published for a large number of nations. An illustrative list is given by the United Nations (1973, page 318) and by Hoem and Fong (1976a, pages 6-7).

The expanding interest in the table of working life as a powerful analytical tool for studying the worklife potential of an individual or of a population forces us to view such tables in a new light and to rethink the way they are calculated. Smith concluded her paper at the St Louis meeting as follows:

> "It is urged that the user of working life tables give careful consideration to the assumptions on which they rest, and the validity of those assumptions for his particular problem. These tables can be useful in many circumstances, but should never be thoughtlessly applied" (Smith, 1977, page 23).

The technique for constructing such a life table has not improved since the design of the first tables in the late 1940s (Wolfbein, 1949). Several authors, however, refer to the unrealistic assumptions on which this technique is based and to how they may affect the results. It is only recently that one observes methodological innovations. Hoem and Fong (1976a) have developed a working-life table which does not rely on the unrealistic assumptions required in conventional working-life tables.

It is a multistate increment–decrement life table which explicitly incorporates the gross flows of people between active and inactive life.

It is the purpose of this paper to reformulate the approach of Hoem and Fong in simpler terms and with less restrictive assumptions by applying recent findings of multiregional demography (Rogers, 1975). This requires some theoretical considerations on the construction of multistate life tables. First, however, I shall review the conventional method for calculating working-life tables.

2 The conventional table of working life

Besides the usual assumptions underlying a classical life table, conventional tables of working life generally adopt three additional general assumptions (Kpedekpo, 1974, page 292).

(1) Persons who enter the labor force do so prior to the age at which the activity rate reaches its maximum. This implies that the labor-force participation (LFP) schedule has a maximum, that is, is unimodal.

(2) Prior to the age of maximum labor-force participation, no survivors retire from the labor force and become members of the inactive population. Retirement only occurs at ages beyond the age of the maximum activity rate. Once a person has left the labor force, he can never return.

(3) The rates of mortality at each age are the same for economically active and inactive persons.

The first two assumptions are usually satisfied for males but not for females. The male LFP curve has a very regular pattern: labor-force participation starts at about the age of fifteen and reaches a peak around the age of thirty. Between the mid-thirties and the mid-forties the proportion of the male population in the labor force drops gradually, and then declines rapidly due to retirement. There is much less regularity in the female LFP schedules. They are usually bimodal, particularly in North America and in European countries. This is due to the patterns of entry into the labor force and of withdrawal or retirement, which are related to the life cycle of marriage and fertility. Women may drop out of the labor force to marry and have children and enter it again later when the children no longer need their close attention. Divorce and widowhood are also events determining the age of (re)entry into the labor force [for a discussion of LFP curves by demographic and social categories, and of their patterns of change, see Bowen and Finegan (1969) and Durand (1975)]. The problem of bimodality of female LFP curves may be eliminated by constructing working-life tables for women by marital status (Garfinkle, 1967; Smith, 1977, page 9). Such tables are regularly calculated by the US Bureau of Labor Statistics.

The third assumption is generally not true, since the age-specific mortality of an active population generally exceeds that of an inactive population. All three assumptions may be dropped by constructing multistate tables of working life, which will be the task of the next section.

A typical working-life table represents the life history of a hypothetical population or cohort. In addition to the mortality experience, it describes the pattern of labor-force participation. All the columns are derived by applying the mortality rates and LFP rates to a hypothetical population or cohort. The LFP rates may be replaced by rates of labor-force entry and of separation by use of assumptions (1) and (2) (Fullerton, 1971, pages 51–52). The computational procedure of constructing a working-life table has been given by Wolfbein (1949), by Durand and Miller (1968, annex A), and by Fullerton (1971, pages 52–54), among others. The procedure is summarized in the remainder of this section.

2.1 *Number of persons living,* $L(x)$

Let $L(x)$ denote the number of persons aged x to $x+1$ in the stationary life-table population. They are computed from the age-specific mortality rates only and appear in all standard life tables. Usually $L(x)$ is expressed per 100000 births (that is, the radix is 100000). Besides the number-of-persons interpretation of $L(x)$, $L(x)$ also may be interpreted as the number of years expected to be lived between ages x and $x+1$ by the cohort. The two interpretations, number of people and person-years, are used interchangeably in demography.

2.2 *Number of persons in the labor force,* $L^{w}(x)$

The age composition of the labor force in the stationary population is derived as the product

$$L^{w}(x) = w(x)L(x) \;, \tag{1}$$

where $w(x)$ is the age-specific activity rate. Note that $L^{w}(x)$ also represents the expected time spent in active life between ages x and $x+1$ by the cohort. It should be remarked that, although most authors derive the working-life table from the $L(x)$ column of the life table, some use the $l(x)$ column, the number of people at *exact* ages x (Fullerton, 1971; 1972; Kpedekpo, 1969).

Summing the expected time spent in active life between all pairs of ages beyond age x gives

$$T^{w}(x) = \sum_{y=x}^{z} L^{w}(y) \;, \tag{2}$$

where z denotes the final age interval. It is the total expected time spent in active life beyond age x by the cohort of 100000 people. The variable $T^{w}(x)$ is analogous to the total person-years lived beyond age x in the conventional life table. For ages below $\bar{\alpha}$, the minimum age for working activity, $T^{w}(x)$ is identical and equal to $T^{w}(0)$. The index $T^{w}(\bar{\alpha})$, when expressed per unit cohort, has been labeled the *net years of active life* (Farooq, 1975, page 45). An index of the time spent in active life in the absence of mortality is the *gross years of active life* (GYA) index (Farooq, 1975, page 44; Durand, 1975, page 226) and is given by $\sum_{x} w(x)$. The area under the LFP curve gives the GYA and this area is analogous to the gross reproduction rate in fertility analysis. The difference between GYA and $T^{w}(\bar{\alpha})$ represents the loss in working life due to mortality. It has been estimated for males as 4·8 years in industrialized countries, 8·5 years in semi-industrialized countries, and 11·4 years in agricultural countries (United Nations, 1973, page 319).

2.3 *Expectation of working life,* $e^{w}(x)$

The average remaining number of years of working life or the expectation of working life beyond age x is

$$e^{w}(x) = \left[\sum_{y=x}^{z} L^{w}(y)\right][l^{w}(x)]^{-1} \;. \tag{3}$$

It defines the average number of years of working life remaining to a person *in the labor force* at exact age x. The value of $l^{w}(x)$ is computed as follows (Wolfbein, 1949, page 291; Fullerton, 1971, page 54): if $w(k)$ is the maximum labor-force participation rate, attained at age k, then the value of $l^{w}(x)$ is given by

$$l^{w}(x) = \tfrac{1}{2}[L^{w}(x-1)+L^{w}(x)] \;, \qquad \text{for } x > k \;, \tag{4}$$

and by

$$l^{w}(x) = \tfrac{1}{2}[L(x-1)+L(x)]w(k) \;, \qquad \text{for } x \leqslant k \;. \tag{5}$$

The consideration of the maximum activity rate for ages below or equal to k is made to eliminate the effect of entries into the labor force in the years following age x. The working-life expectancy at age x refers to the cohort of active population $l^{w}(x)$. Therefore additional entries into the labor force after x may not be considered. The implicit assumption is that all work is done by a distinct cohort of workers. The application of $w(k)$ for $x \leqslant k$ reflects the assumption that all entries into the labor force occur at the youngest labor-force age, $\bar{\alpha}$ (sixteen years say). Between ages $\bar{\alpha}$ and k, no person is supposed to leave the labor force [assumption (2)]. Hence the active population of exact age k is smaller than at age $\bar{\alpha}$ only because of mortality in the intervening years.

Several authors use two measures of working life in their analysis (for example, Wolfbein, 1949, page 293; Fullerton and Byrne, 1976, page 35; Farooq, 1975, page 46). The first, the so-called *labor-force-based measure*, is identical to $e^{w}(x)$ and is sometimes called the *average remaining number of years of active life*. The second, the *population-based measure*, is the ratio $\left[\sum_{y=x}^{z} L^{w}(y)\right][l(x)]^{-1}$, where $l(x)$ is the total number of people of exact age x in the life table, and is known as the *expectation of active life*. The second measure assumes that all persons in the population, currently active or inactive, have an equal probability of participating in the labor force. The two measures serve different purposes. If one is interested in the working-life expectancy of a person not yet in the labor force, or of a person regardless of his current labor-force status, the population-based measure is appropriate. This approach has been used by Durand (1948, pages 259–265) in estimating what he calls "the average number of years in the labor force". However, if one is interested in the remaining years of work of a currently active person, the labor-force-based measure will be more accurate (for example, see Durand and Miller, 1968, pages 24–27). For example, the expected length of working life at birth is

$$e^{w}(0) = \left[\sum_{y=0}^{z} L^{w}(y)\right][l(0)]^{-1} . \tag{6}$$

2.4 *Rate of accession to the labor force, $A(x)$*

This measure shows the net accessions to the life-table labor force between ages x to $x+1$ as a ratio of the life-table population, $L(x)$. It gives the proportion of the population aged x to $x+1$ in a life-table cohort who are not currently in the labor force but who will engage in labor activity in the next year. The ratio of the net accessions to the life-table labor force is (Wolfbein, 1949, page 288)

$$A(x) = \frac{L^{w}(x+1) - L^{w}(x)[1-M(x)]}{L(x)} , \tag{7}$$

where $M(x)$ is the age-specific mortality rate, and the product $L^{w}(x)M(x)$ is the mortality in the labor force between ages x and $x+1$. The quantity $A(x)$ is not computed for ages above k, since it has been assumed that people enter the labor force only up to age k, at which age the LFP rate is at its maximum.

2.5 *Rate of separation from the labor force, $M^{w}(x)$*

The rate of separation due to all causes (mortality and retirement) is defined as the ratio of the net separation from the labor force between ages x and $x+1$ to the stationary labor force, $L^{w}(x)$:

$$M^{w}(x) = \frac{l^{w}(x) - l^{w}(x+1)}{L^{w}(x)} . \tag{8}$$

This measure is very similar to the death rate in a conventional life table. Before age k it is assumed that withdrawal from the labor force is due to mortality only. After age k, two types of separation occur: mortality and retirement. The rate of separation due to mortality is

$$M_{\delta}^{w}(x) = \frac{l^{w}(x)q(x)}{L^{w}(x)} , \tag{9}$$

where $q(x)$ is the probability of dying between ages x and $x+1$ (identical for an active and an inactive population). The rate of separation due to retirement is a residual:

$$M_{r}^{w}(x) = M^{w}(x) - M_{\delta}^{w}(x) . \tag{10}$$

Separation rates are important for manpower planning because they permit the calculation of expected losses from active life due to death and retirement (see, for example, Garfinkle, 1967).

2.6 *Replacement*

The difference between the total accession rate and the total separation rate is the *labor-force replacement rate* (United Nations, 1973, page 319; Farooq, 1975, page 52). The ratio of accessions to separations is the *replacement ratio*, a measure of pressure on the labor market. A ratio of less than one shows that not all vacancies caused by death and retirement are being filled.

3 The increment-decrement table of working life: methodology

The conventional method for constructing tables of working life is based on a number of unrealistic assumptions, such as the unimodality of the LFP curve. Data availability and existent methodology may have required these assumptions. The method presented here introduces more realism into the working-life table, but at the same time it increases the data requirements. Today most countries do not publish all the necessary data; therefore, parallel to the development of the life-table methodology, attention should be devoted to procedures for estimating missing data.

The working-life table proposed in this study is an increment-decrement life table. Recently there has been renewed interest in increment-decrement life tables (for example, Schoen, 1975) and in their formulation by the application of the theory of multiregional mathematical demography (Rogers, 1973; 1975; Rogers and Ledent, 1976; Ledent, 1978; 1980). Also in this paper, multiregional demographic techniques are used for deriving increment-decrement tables of working life. The method presented has a number of conceptual similarities with a technique developed recently by Hoem and Fong (1976a); however, the mathematics are somewhat simpler and regional differences in mortality can be easily handled.

One begins with a population, disaggregated by age and sex, that is partitioned into two groups: active population (labor force) and inactive population. The analysis here will be performed for a single sex only, although the extension to a two-sex model is straightforward. These are the three states of the system. Everyone starts out as a member of the inactive population, state 1. Labor-force membership is denoted by state 2. State 3, denoted by δ, is the state of death. In each age group, people may move between state 1 and state 2 (transient states), and into state δ (absorbing state). The age-specific *gross flows* in and out of the labor force are explicitly taken into account. The focus on gross flows represents a fundamental difference from the conventional technique of constructing working-life tables, with its emphasis on the size (stock) of the labor force and on changes in the stock through *net* flows. The assumption, made in conventional working-life tables, that everyone enters the labor force before age k and retires at ages over k, is therefore no longer necessary. Consequently the need for a unimodal LFP curve no longer exists.

The increment-decrement table of working life represents the mortality and mobility experience of a cohort. The mobility experience represents movement into and out of the labor force. All life-table statistics are derived from a set of age-specific mobility and mortality probabilities.

The usual procedure of life-table construction is followed, beginning with the expression of probabilities in terms of instantaneous rates of mobility and mortality (also known as intensities or forces of mobility and mortality). Let $\mu_{12}(x)$ denote the instantaneous rate of accession to the labor force at age x: the rate at which people of age x enter the labor force. The instantaneous rate is the limiting value of the accession rate as the age interval becomes infinitesimal. The instantaneous rate of separation from the labor force at age x is $\mu_{21}(x)$, and the forces of mortality are given by $\mu_{i\delta}(x)$, where $i = 1, 2$. The force of mortality of the active population may be different from that of the inactive population. The assumption of equal mortality rates, which is implicit in conventional tables of working life, is therefore no longer needed.

3.1 *The transition probabilities*

The transition from one state to another is defined by the Kolmogorov equation. Consider the cohort of people who are now of exact age y. Denote the probability that an individual of age y in state 1 will be in state 2 n years later by ${}_{1y}l_2(y+n)$. Denoting $y+n$ by x ($x \geqslant y$), this probability may be written as ${}_{1y}l_2(x)$. The probability that a person in state 1 at age y will be in state 2 at age $x+\mathrm{d}x$ is

$$ {}_{1y}l_2(x+\mathrm{d}x) = {}_{1y}l_1(x)\mu_{12}(x)\mathrm{d}x + {}_{1y}l_2(x)\mu_{22}(x)\mathrm{d}x \,, \tag{11}$$

where $\mu_{12}(x)\mathrm{d}x$ is the probability that an x-year-old member of the inactive population will become active within time $\mathrm{d}x$ and $\mu_{22}(x)\mathrm{d}x$ is the probability that an x-year-old member of the labor force will still be active $\mathrm{d}x$ years (or time units) later. It is assumed that only one interstate 'passage' is possible in the small interval $\mathrm{d}x$.

The transition probabilities are independent of the status of the individual at age y, but do depend on his status at exact age x, that is, at the beginning of the interval $(x, x+\mathrm{d}x)$. This is the Markovian assumption. Let

$$ \mu_{22}(x)\mathrm{d}x = 1-[\mu_{2\delta}(x)+\mu_{21}(x)]\mathrm{d}x \,. \tag{12}$$

Then equation (11) may be written as

$$ {}_{1y}l_2(x+\mathrm{d}x) = {}_{1y}l_1(x)\mu_{12}(x)\mathrm{d}x + {}_{1y}l_2(x) - {}_{1y}l_2(x)[\mu_{2\delta}(x)+\mu_{21}(x)]\mathrm{d}x \tag{13}$$

or as

$$ \frac{{}_{1y}l_2(x+\mathrm{d}x) - {}_{1y}l_2(x)}{\mathrm{d}x} = {}_{1y}l_1(x)\mu_{12}(x) - {}_{1y}l_2(x)[\mu_{2\delta}(x)+\mu_{21}(x)] \,, \tag{14}$$

which is the Kolmogorov differential equation

$$ \frac{\mathrm{d}}{\mathrm{d}x}\, {}_{1y}l_2(x) = {}_{1y}l_1(x)\mu_{12}(x) - {}_{1y}l_2(x)[\mu_{2\delta}(x)+\mu_{21}(x)] \,. \tag{15}$$

Equation (15), which may also be found in Hoem and Fong (1976a, page 69), describes the changes in ${}_{1y}l_2(x)$ as a function of the instantaneous rates of mobility and mortality and of the initial condition, namely the state of a person at exact age x.

Paralleling the derivation of equation (15), one may obtain expressions for changes in ${}_{1y}l_1(x)$, ${}_{2y}l_1(x)$, and ${}_{2y}l_2(x)$. The result may be expressed in matrix form:

$$ \frac{\mathrm{d}}{\mathrm{d}x}\, {}_{y}\mathbf{l}(x) = -\boldsymbol{\mu}(x)\, {}_{y}\mathbf{l}(x) \,, \tag{16}$$

where $\boldsymbol{\mu}(x)$ is the matrix of instantaneous rates,

$$\boldsymbol{\mu}(x) = \begin{bmatrix} \mu_{1\delta}(x)+\mu_{12}(x) & -\mu_{21}(x) \\ -\mu_{12}(x) & \mu_{2\delta}(x)+\mu_{21}(x) \end{bmatrix}, \tag{17}$$

and

$$_y\mathbf{l}(x) = \begin{bmatrix} _{1y}l_1(x) & _{2y}l_1(x) \\ _{1y}l_2(x) & _{2y}l_2(x) \end{bmatrix}, \tag{18}$$

where $_{iy}l_j(x)$ is the probability that a person aged y in state i will be in state j n years later, that is, at exact age x ($n = x-y$). The age y denotes the cohort. If $y = 0$ the cohort considered is the birth cohort or radix. Another interesting cohort is $y = \bar{\alpha}$, the lowest age of labor-force participation.

The solution to equation (16) is of the form

$$_y\mathbf{l}(x) = {_y\boldsymbol{\Omega}(x)}\,{_y\mathbf{l}(y)}\ , \tag{19}$$

where $_y\boldsymbol{\Omega}(x)$ is the transition matrix, which was known in early literature as the *matricant* (Gantmacher, 1959).

The determination of the transition matrix of a linear time-variant system of differential equations, such as system (19) has received considerable attention in systems-theory literature, from which the necessary analytic results can be drawn (for example, Wolovich, 1974). The principal problem is to relate the transition matrix to the instantaneous rates. One may generalize the approach adopted in single-region life-table construction and try the following 'solution' to equation (16):

$$_y\mathbf{l}(x) = \exp\left[\int_y^x -\boldsymbol{\mu}(t)\,\mathrm{d}t\right]{_y\mathbf{l}(y)}\ . \tag{20}$$

However, equation (20) is a solution to equation (16) only if $\boldsymbol{\mu}(t)$ commutes for all t, a restriction which in general does not hold. Two approaches will be considered here in order to get around this problem.

The first approach is to introduce the assumption that $\boldsymbol{\mu}(t)$ *is constant in the interval* $x-h$ *to* x, that is, that $\boldsymbol{\mu}(t) = \boldsymbol{\mu}(x-h)$ for $x-h \leqslant t < x$ (Hoem and Fong, 1976a; Krishnamoorthy, 1979; Ledent, 1980). This implies that the continuous matrix function $\boldsymbol{\mu}(t)$ is approximated by a step function. The heights of these steps between ages $x-h$ and x are given by the elements of $\boldsymbol{\mu}(x-h)$. Under this assumption equation (20) becomes

$$_y\mathbf{l}(x) = \exp[-h\,\boldsymbol{\mu}(x-h)]\,{_y\mathbf{l}(x-h)}\ . \tag{21}$$

The problem of solving equation (16) is now reduced to evaluating the exponential of the matrix $-h\boldsymbol{\mu}(x-h)$. The evaluation of a function of the form $\exp(\mathbf{A})$ is a major issue in mathematical systems theory, and a number of procedures have been developed (Wolovich, 1974). One is to use the expansion

$$\exp(\mathbf{A}) = \mathbf{I}+\mathbf{A}+\frac{1}{2!}\mathbf{A}^2+\frac{1}{3!}\mathbf{A}^3+\ldots\ , \tag{22}$$

which applied to equation (21) gives

$$_y\mathbf{l}(x) = \left\{\mathbf{I}-h\boldsymbol{\mu}(x-h)+\frac{1}{2!}h^2[\boldsymbol{\mu}(x-h)]^2-\ldots\right\}{_y\mathbf{l}(x-h)}\ . \tag{23}$$

This procedure is adopted by Krishnamoorthy (1979) and Ledent (1980).

A second approach to solving equation (16), however, is followed in this paper. The following theorem is used. It is stated here without proof, but a proof can be found in Brauer et al (1970, pages 312–313). A simplified solution to equations (15) and (16) may be derived if one assumes no mortality or equal mortality in both states (see the appendix).

Theorem: *Solving the system of differential equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Y}(t) = \mathbf{A}(t)\mathbf{Y}(t)\,, \qquad \mathbf{Y}(t_0)\ \textit{given,} \tag{24}$$

is equivalent to finding a continuous matrix function $\mathbf{Z}(t)$ *such that*

$$\mathbf{Z}(t) = \mathbf{Y}(t_0) + \int_{t_0}^{t} \mathbf{A}(\tau)\mathbf{Z}(\tau)\mathrm{d}\tau\,. \tag{25}$$

Hence equation (16) may be replaced by the integral equation

$${}_y\mathbf{l}(x) = {}_y\mathbf{l}(y) - \int_0^n \boldsymbol{\mu}(y+t)\,{}_y\mathbf{l}(y+t)\mathrm{d}t\,, \tag{26}$$

where $n = x-y$. The problem of solving the system of Kolmogorov differential equations (16) has now been replaced by the problem of evaluating the integral $\int_0^n \boldsymbol{\mu}(y+t)\,{}_y\mathbf{l}(y+t)\mathrm{d}t$. This may be done in steps of length h, say:

$${}_y\mathbf{l}(y+h) = {}_y\mathbf{l}(y) - \int_0^h \boldsymbol{\mu}(y+t)\,{}_y\mathbf{l}(y+t)\mathrm{d}t\,, \tag{27}$$

$${}_y\mathbf{l}(y+2h) = {}_y\mathbf{l}(y+h) - \int_0^h \boldsymbol{\mu}(y+h+t)\,{}_y\mathbf{l}(y+h+t)\mathrm{d}t\,, \tag{28}$$

$$\vdots$$

$${}_y\mathbf{l}(x) = {}_y\mathbf{l}(x-h) - \int_0^h \boldsymbol{\mu}(x-h+t)\,{}_y\mathbf{l}(x-h+t)\mathrm{d}t\,, \tag{29}$$

$${}_y\mathbf{l}(x+h) = {}_y\mathbf{l}(x) - \int_0^h \boldsymbol{\mu}(x+t)\,{}_y\mathbf{l}(x+t)\mathrm{d}t\,. \tag{30}$$

Define the matrix of conditional probabilities

$$\mathbf{P}(x) = \begin{bmatrix} p_{11}(x) & p_{21}(x) \\ p_{12}(x) & p_{22}(x) \end{bmatrix}, \tag{31}$$

where an element $p_{ij}(x)$ denotes the probability that a person alive in state i at age x will be in state j h years later, that is, at age $x+h$. The probabilities $p_{ij}(x)$ are assumed to be independent of the state at age y and only depend on the state at the beginning of the interval h. $\mathbf{P}(x)$ may also be written as ${}_x\boldsymbol{\Omega}(x+h)$. The probability of dying is obtained by subtraction:

$$q_i(x) = 1 - p_{i1}(x) - p_{i2}(x)\,. \tag{32}$$

For a nonsingular ${}_y\mathbf{l}(x)$, $\mathbf{P}(x)$ is given by the following expression:

$$\mathbf{P}(x) = {}_y\mathbf{l}(x+h)\,[{}_y\mathbf{l}(x)]^{-1}\,. \tag{33}$$

The nonsingularity condition is generally satisfied for ages $x \geqslant \bar{\alpha}$. Since no persons are in the labor force at the younger ages, whenever $x < \bar{\alpha}$ we have $p_{21}(x) = p_{12}(x) = p_{22}(x) = 0$ and $p_{11}(x)$ is simply the probability of surviving from age x to $x+h$. This probability is equal to

$$p_{11}(x) = \frac{{}_{y1}l_1(x+h)}{{}_{y1}l_1(x)}\,, \qquad \text{for } x < \bar{\alpha}\,. \tag{34}$$

In other words matrix expression (33) reduces to a scalar expression when $x < \bar{\alpha}$.

Equation (33) states that ${}_y\mathbf{l}(x+h) = \mathbf{P}(x)_y\mathbf{l}(x)$. A comparison of this expression with equation (30) yields the following for $\mathbf{P}(x)$:

$$\mathbf{P}(x) = \mathbf{I} - \int_0^h \boldsymbol{\mu}(x+t)_y\mathbf{l}(x+t)[{}_y\mathbf{l}(x)]^{-1}\mathrm{d}t \ . \qquad (35)$$

Note the difference between equations (35) and (21), in which $\mathbf{P}(x) = \exp[-h\boldsymbol{\mu}(x-h)]$. Which of the two estimations yields the best result is difficult to say without some additional information about the underlying mortality and mobility schedules.

From equation (30) one may also derive an expression for the annual age-specific rates in the increment-decrement life table. Note that

$${}_y\mathbf{l}(x+h) - {}_y\mathbf{l}(x) = -\left[\int_0^h \boldsymbol{\mu}(x+t)_y\mathbf{l}(x+t)\mathrm{d}t\right]\left[\int_0^h {}_y\mathbf{l}(x+t)\mathrm{d}t\right]^{-1}\left[\int_0^h {}_y\mathbf{l}(x+t)\mathrm{d}t\right] . \qquad (36)$$

The expression $\int_0^h {}_y\mathbf{l}(x+t)\mathrm{d}t$ denotes the number of years lived in each state between ages x and $x+h$ per person in each state at age y. Denote this by ${}_y\mathbf{L}(x)$,

$${}_y\mathbf{L}(x) = \int_0^h {}_y\mathbf{l}(x+t)\mathrm{d}t \ . \qquad (37)$$

The matrix ${}_y\mathbf{L}(x)$ may also be looked upon as representing the number of people by state in age group x to $x+h$ per person in each state at age y. Hence it gives the age distribution of the stationary life-table population. The expression

$$\left[\int_0^h \boldsymbol{\mu}(x+t)_y\mathbf{l}(x+t)\mathrm{d}t\right]\left[\int_0^h {}_y\mathbf{l}(x+t)\mathrm{d}t\right]^{-1} = \mathbf{m}(x) \qquad (38)$$

is then the matrix of age-specific life-table rates [note the difference in the expression obtained by Ledent (1980), which is derived not from the Kolmogorov equation but from his particular definition of the elements of $\mathbf{m}(x)$]. Equation (36) becomes therefore

$${}_y\mathbf{l}(x+h) - {}_y\mathbf{l}(x) = -\mathbf{m}(x)_y\mathbf{L}(x) \ . \qquad (39)$$

To derive an expression for $\mathbf{P}(x)$ in terms of life-table rates, it should be recalled that ${}_y\mathbf{l}(x+h) = \mathbf{P}(x)_y\mathbf{l}(x)$. Hence

$$[\mathbf{P}(x) - \mathbf{I}]_y\mathbf{l}(x) = -\mathbf{m}(x)_y\mathbf{L}(x) \ , \qquad (40)$$

and so

$$\mathbf{P}(x) = \mathbf{I} - \mathbf{m}(x)_y\mathbf{L}(x)[{}_y\mathbf{l}(x)]^{-1} \ . \qquad (41)$$

This expression may also be derived directly from equation (36) by applying equations (37) and (38).

3.2 *Numerical approximation of* $\mathbf{P}(x)$

All life-table statistics may be derived from the transition probability matrices $\mathbf{P}(x)$. The expression of $\mathbf{P}(x)$ in terms of instantaneous rates is given in equations (35) and (41). On the other hand, Rogers and Ledent (1976) have shown that $\mathbf{P}(x)$ may be written as follows:

$$\mathbf{P}(x) = \left[\mathbf{I} + \frac{h}{2}\mathbf{M}(x)\right]^{-1}\left[\mathbf{I} - \frac{h}{2}\mathbf{M}(x)\right] , \qquad (42)$$

where $\mathbf{M}(x)$ is the matrix of observed (empirical) rates, associated with age group x to $x+h$, set out in the same format as the $\boldsymbol{\mu}(x)$ matrix.

The transition from equation (41) to equation (42) involves a number of assumptions. The Markovian assumption, introduced earlier, states that the transition probability during a certain interval depends only on the state in which the person is at the beginning of the interval and is independent of previous states. Another assumption is that transitions and deaths are distributed uniformly over the interval x to $x+h$. The mean duration of transfers is therefore $\frac{1}{2}h$. The uniform distribution implies that the integral $\int_0^h {}_y\mathbf{l}(x+t)\,\mathrm{d}t = {}_y\mathbf{L}(x)$ may be approximated by linear integration (*the linear integration hypothesis*):

$$ {}_y\mathbf{L}(x) = \frac{h}{2}[{}_y\mathbf{l}(x) + {}_y\mathbf{l}(x+h)] \ . \qquad (43)$$

Introducing equation (43) in the expression for $\mathbf{P}(x)$ yields

$$\mathbf{P}(x) = \mathbf{I} - \frac{h}{2}\mathbf{m}(x)[{}_y\mathbf{l}(x) + {}_y\mathbf{l}(x+h)][{}_y\mathbf{l}(x)]^{-1} = \mathbf{I} - \frac{h}{2}\mathbf{m}(x)[\mathbf{I} + \mathbf{P}(x)] \ , \qquad (44)$$

whence

$$\mathbf{P}(x) + \frac{h}{2}\mathbf{m}(x)\mathbf{P}(x) = \mathbf{I} - \frac{h}{2}\mathbf{m}(x) \qquad (45)$$

and

$$\mathbf{P}(x) = \left[\mathbf{I} + \frac{h}{2}\mathbf{m}(x)\right]^{-1}\left[\mathbf{I} - \frac{h}{2}\mathbf{m}(x)\right] \ . \qquad (46)$$

To compute $\mathbf{P}(x)$ from observed data, a third assumption is introduced: that annual age-specific life-table rates are equal to the annual age-specific rates of the observed population, that is, $\mathbf{m}(x) = \mathbf{M}(x)$. This assumption implies that cross-sectional data (observed rates) provide adequate estimates of the life-table rates to be used in longitudinal analysis. It makes equation (46) equal to equation (42). According to Ledent (1980), equation (46) only holds for the movement approach. In the numerical illustration in section 4, the time interval of one year is short enough to assume that everyone makes only one move during the interval, and hence the transition approach coincides with the movement approach.

3.3 *Other life-table statistics*

Several useful statistics of the working-life table may be derived from the matrices of probabilities ${}_y\mathbf{l}(x)$ and $\mathbf{P}(x)$. The procedures are analogous to the construction of a multiregional life table in which states refer to regions (Rogers, 1975, chapter 3). The matrix ${}_y\mathbf{L}(x)$ has already been derived. For example, the average number of years spent in the labor force between ages x and $x+h$ by an active person of exact age y is

$$ {}_{2y}L_2(x) = \int_0^h {}_{2y}l_2(x+t)\,\mathrm{d}t \ , \qquad (47)$$

which may be approximated by the linear interpolation

$$\frac{h}{2}[{}_{2y}l_2(x) + {}_{2y}l_2(x+h)] \ . \qquad (48)$$

The total number of years expected to be spent in the labor force beyond age y by a person of age y already in the labor force is

$$ {}_{2y}e_2(y) = \int_0^{\beta-y} {}_{2y}l_2(y+t)\,\mathrm{d}t \ , \qquad (49)$$

where β is the highest age of active life (about seventy years). The working-life expectancy of a person not active at age y is

$$ {}_{1y}e_2(y) = \int_0^{\beta-y} {}_{1y}l_2(y+t)\,\mathrm{d}t \ . \qquad (50)$$

In an analogous way one may formulate the expected time spent outside the labor force before age $\bar{\beta}$ by a person who is active (inactive) at age y. The results may be set out as the matrix expression

$$ {}_y\mathbf{e}(y) = \int_0^{\bar{\beta}-y} {}_y\mathbf{l}(y+t)\,\mathrm{d}t \;, \tag{51}$$

where

$$ {}_y\mathbf{e}(y) = \begin{bmatrix} {}_{1y}e_1(y) & {}_{2y}e_1(y) \\ {}_{1y}e_2(y) & {}_{2y}e_2(y) \end{bmatrix} \; . \tag{52}$$
$$ \qquad\qquad {}_{1y}e_{.}(y) \qquad {}_{2y}e_{.}(y) $$

The column sum ${}_{1y}e_{.}(y)$ denotes the total life expectancy of a person outside the labor force at age y. A part of this person's remaining lifetime will be spent in inactivity $[{}_{1y}e_1(y)]$. The column sum ${}_{2y}e_{.}(y)$ refers to the life expectancy of a person in the labor force at age y. He will spend ${}_{2y}e_1(y)$ and ${}_{2y}e_2(y)$ years in active and inactive life respectively. The measure of ${}_{2y}e_2(x)$ is analogous to the labor-force-based measure of active life developed in the conventional tables of working life.

The population-based measure of working life is defined as follows:

$$ {}_{.y}e_2(y) = \int_0^{\bar{\beta}-y} [{}_{1y}l_2(y+t)+{}_{2y}l_2(y+t)]\,\mathrm{d}t \;. \tag{53}$$

It is clear that, for $y < \bar{\alpha}$, ${}_{.y}e_2(y) = {}_{1y}\dot{e}_2(y)$.

Formulas (51) and (53) refer to the life expectancy beyond age y of people aged y years. What is the life expectancy beyond age x of people in a given state at age y? Since the state at age x is of no importance, one first computes the distribution of the x-year-old people by their state n years ago, that is, at age y. This is a vector ${}_y\boldsymbol{l}_{.}(x)$ computed as follows:

$$ {}_y\boldsymbol{l}_{.}(x) = [{}_y\mathbf{l}(x)]^{\mathrm{T}}\boldsymbol{1} \;, \tag{54}$$

where T denotes transpose and $\boldsymbol{1}$ is a vector of ones. The life expectancy beyond age x by future state and by state at age y is

$$ {}_y\hat{\mathbf{e}}(x) = \left[\int_0^{\bar{\beta}-x} {}_y\mathbf{l}(x+t)\,\mathrm{d}t\right][{}_y\hat{\mathbf{l}}(x)]^{-1} \;, \tag{55}$$

where ${}_y\hat{\mathbf{l}}(x)$ is a diagonal matrix with the elements of ${}_y\boldsymbol{l}_{.}(x)$ along the diagonal. It is analogous to the matrix formula of life expectancy by place of birth in the multiregional life table (Willekens, 1977b, page 656). Instead of regions and birth cohorts, we are considering here states and cohorts of people aged y at a given point in time. This shows how the state-specific life expectancy of a given cohort changes as age increases.

Knowing the values ${}_y\mathbf{l}(y+t)$, $t \geqslant 0$, for a given cohort, one may derive the life expectancy of a person in a given state at age x (and not y). This measure is analogous to the expectation of life by place of residence in multiregional demography. In working-life tables, the place of residence at age $y+n$ is replaced by the state at age x. The life expectancy by state at age x is

$$ {}_x\mathbf{e}(x) = \left[\int_0^{\bar{\beta}-x} {}_y\mathbf{l}(x+t)\,\mathrm{d}t\right][{}_y\mathbf{l}(x)]^{-1} \;. \tag{56}$$

This expression enables one to derive an interesting recursive expression for the expectation of life (Hoem and Fong, 1976a). One may rewrite equation (56) as follows:

$$
\begin{aligned}
{}_x\mathbf{e}(x) &= \left[\int_0^h {}_y\mathbf{l}(x+t)\,\mathrm{d}t + \int_0^{\beta-x-h} {}_y\mathbf{l}(x+h+t)\,\mathrm{d}t\right][{}_y\mathbf{l}(x)]^{-1} \\
&= {}_x\mathbf{L}(x) + \left[\int_0^{\beta-x-h} {}_y\mathbf{l}(x+h+t)\,\mathrm{d}t\right][{}_y\mathbf{l}(x+h)]^{-1}[{}_y\mathbf{l}(x+h)][{}_y\mathbf{l}(x)]^{-1} , \qquad (57)
\end{aligned}
$$

or equivalently

$$ {}_x\mathbf{e}(x) = {}_x\mathbf{L}(x) + {}_{x+h}\mathbf{e}(x+h)\mathbf{P}(x) . \qquad (58) $$

For the last age group, z,

$$ {}_z\mathbf{e}(z) = {}_z\mathbf{L}(z) = [\mathbf{M}(z)]^{-1} . \qquad (59) $$

4 The increment–decrement table of working life: numerical illustrations

The increment–decrement table of working life may be derived from a set of age-specific accession rates, separation rates, and mortality rates. The required data for such a life table have been provided by Hoem and Fong (1976b, pages 7 and 12) and are based on Danish labor-force panel surveys during the period 1972–1974. Although sex-specific data are provided, the analysis is performed for the male population only. The data are repeated in the first three columns of table 1. They are not the observed rates but are computed by a step function and smoothed by a moving-average technique (Hoem and Fong, 1976a, pages 44–48). No mortality differentials have been assumed between the active and the inactive population. Although this assumption is not required, it has been made to allow comparison of these results with those published by Hoem and Fong (1976b).

Transition probabilities are shown in the last four columns of table 1[(1)]. They are computed using equation (42). The transition probabilities associated with age group x depend only on the mobility and mortality rates of the same age group. For example, the probability matrix $\mathbf{P}(x)$ for age group 20 to 21 is

$$ \mathbf{P}(20) = [\mathbf{I}+\tfrac{1}{2}\mathbf{M}(20)]^{-1}[\mathbf{I}-\tfrac{1}{2}\mathbf{M}(20)] , $$

where

$$ \mathbf{M}(20) = \begin{bmatrix} 0{\cdot}001221+0{\cdot}457690 & -0{\cdot}092260 \\ -0{\cdot}457690 & 0{\cdot}001221+0{\cdot}092260 \end{bmatrix} . $$

Hence

$$ \mathbf{P}(20) = \begin{bmatrix} 0{\cdot}6402 & 0{\cdot}0723 \\ 0{\cdot}3586 & 0{\cdot}9265 \end{bmatrix} . $$

The probability that a person aged 20 will enter the labor force in the next year is 0·3586; the probability that he will survive but remain inactive is 0·6402.

For the present calculations it is important to note two points. First, the probability of separating from the labor force for 16 year olds is not zero. This is caused by the fact that people enter the labor force and drop out in the same year. The drop-out rate of young people is relatively high (figure 1). Second, for some age groups the transition probabilities $p_{12}(x)$ and $p_{21}(x)$ differ considerably from those obtained by Hoem and Fong. However, the probabilities of dying, obtained as residuals, are similar.

(1) The computer program used for the calculation of the increment-decrement table of working life is a slightly modified version of the multiregional life-table program listed in Willekens and Rogers (1978).

Table 1. Age-specific rates of mortality, accession, and separation, and transition probabilities.

Age	Age-specific rate			Transition probability			
	mortality	1 to 2	2 to 1	1 to 1	1 to 2	2 to 1	2 to 2
16	0·000733	0·339080	0·525690	0·7627	0·2366	0·3668	0·6325
17	0·000934	0·605350	0·171700	0·5635	0·4356	0·1236	0·8755
18	0·001202	0·635730	0·135850	0·5405	0·4583	0·0979	0·9009
19	0·001327	0·557140	0·121990	0·5832	0·4154	0·0910	0·9077
20	0·001221	0·457690	0·092260	0·6402	0·3586	0·0723	0·9265
21	0·001016	0·373170	0·085200	0·6957	0·3033	0·0693	0·9297
22	0·000944	0·345440	0·080600	0·7145	0·2845	0·0664	0·9327
23	0·000997	0·341970	0·071500	0·7159	0·2831	0·0592	0·9398
24	0·001014	0·355140	0·057370	0·7048	0·2941	0·0475	0·9515
25	0·000932	0·393230	0·041280	0·6763	0·3228	0·0339	0·9652
26	0·000901	0·423520	0·032250	0·6545	0·3446	0·0262	0·9729
27	0·000980	0·449790	0·024180	0·6357	0·3633	0·0195	0·9795
28	0·001033	0·474040	0·019460	0·6191	0·3799	0·0156	0·9834
29	0·001039	0·492260	0·015090	0·6067	0·3923	0·0120	0·9870
30	0·001066	0·499390	0·012050	0·6016	0·3973	0·0096	0·9893
31	0·001198	0·495120	0·012120	0·6043	0·3945	0·0097	0·9891
32	0·001319	0·478300	0·012530	0·6151	0·3836	0·0100	0·9886
33	0·001284	0·451480	0·012780	0·6327	0·3660	0·0104	0·9884
34	0·001260	0·416880	0·010740	0·6557	0·3431	0·0088	0·9899
35	0·001377	0·379300	0·009370	0·6814	0·3172	0·0078	0·9908
36	0·001604	0·340640	0·008730	0·7088	0·2896	0·0074	0·9910
37	0·001854	0·304750	0·009660	0·7353	0·2629	0·0083	0·9898
38	0·002089	0·274410	0·010370	0·7582	0·2397	0·0091	0·9889
39	0·002220	0·251820	0·010570	0·7756	0·2221	0·0093	0·9885
40	0·002331	0·237500	0·008190	0·7866	0·2111	0·0073	0·9904
41	0·002559	0·230650	0·008670	0·7919	0·2055	0·0077	0·9897
42	0·002869	0·229460	0·009600	0·7927	0·2044	0·0086	0·9886
43	0·003341	0·231850	0·010500	0·7905	0·2061	0·0093	0·9873
44	0·003679	0·236230	0·012440	0·7870	0·2094	0·0110	0·9853
45	0·003875	0·240450	0·011710	0·7834	0·2127	0·0104	0·9858
46	0·004171	0·242860	0·012200	0·7813	0·2145	0·0108	0·9851
47	0·004482	0·241200	0·013880	0·7825	0·2130	0·0123	0·9833
48	0·004900	0·234270	0·015370	0·7878	0·2073	0·0136	0·9815
49	0·005483	0·221660	0·017530	0·7976	0·1970	0·0156	0·9790
50	0·006406	0·204470	0·019970	0·8109	0·1827	0·0178	0·9758
51	0·007386	0·185040	0·019070	0·8260	0·1667	0·0172	0·9755
52	0·008007	0·165660	0·020620	0·8416	0·1504	0·0187	0·9733
53	0·008411	0·147960	0·023110	0·8564	0·1352	0·0211	0·9705
54	0·008972	0·132980	0·026080	0·8689	0·1221	0·0240	0·9671
55	0·009876	0·120890	0·028230	0·8787	0·1114	0·0260	0·9642
56	0·010748	0·111010	0·032840	0·8868	0·1025	0·0303	0·9590
57	0·011871	0·103250	0·036300	0·8928	0·0954	0·0335	0·9547
58	0·013511	0·097030	0·040470	0·8970	0·0896	0·0374	0·9492
59	0·015083	0·091630	0·046700	0·9006	0·0845	0·0430	0·9420
60	0·016280	0·086830	0·051160	0·9039	0·0800	0·0471	0·9367
61	0·017890	0·082600	0·052510	0·9062	0·0761	0·0483	0·9339
62	0·020671	0·078570	0·057910	0·9074	0·0721	0·0531	0·9264
63	0·023411	0·074720	0·066760	0·9086	0·0682	0·0610	0·9159
64	0·025597	0·070650	0·086570	0·9108	0·0639	0·0783	0·8964
65	0·028280	0·065750	0·109690	0·9133	0·0588	0·0982	0·8740
66	0·031012	0·063300	0·152780	0·9140	0·0555	0·1339	0·8356
67	0·033397	0·062050	0·592000	0·9217	0·0454	0·4333	0·5338
68	0·036293	0·053820	0·310740	0·9203	0·0440	0·2542	0·7101
69	0·039273	0·043810	0·387380	0·9267	0·0348	0·3076	0·6539
70	0·042486	0·044910	0·388590	0·9229	0·0355	0·3074	0·6510
71	0·046803	0·031620	0·293400	0·9282	0·0261	0·2417	0·7125
72	0·051322	0·039470	0·398030	0·9190	0·0309	0·3119	0·6381
73	0·055770	0·023390	0·356970	0·9271	0·0187	0·2851	0·6606
74	0·061222	0·030290	0·440420	0·9174	0·0232	0·3376	0·6030

Key: 1—inactive (outside labor force); 2—active (inside labor force).

Source: the rates are from Hoem and Fong (1976b, page 7).

The life history of the initial cohort is given in table 2. From a cohort of 100000 births, 97562 children survive to age 16. Up to this age they are all inactive. The transitions during single years of age $(x, x+1)$ are computed by applying transition probabilities to the numbers of people at exact age x. For example, the number of people entering the labor force at age 20 is

$$ {}_{1\,20}l_2(21) = p_{12}(20)\,{}_{..}l_1(20) \tag{60}$$

or

$$ 8355 = 0{\cdot}3586 \times 23301\ . \tag{61}$$

The total number of people in the labor force at age 21 is

$$ {}_{..}l_2(21) = {}_{1\,20}l_2(21) + {}_{2\,20}l_2(21) \tag{62}$$

or

$$ 76780 = 8355 + 68424\ . \tag{63}$$

Table 3 gives the number of years spent in active and inactive life within the age interval x to $x+1$ per unit cohort of age 16. For example,

$$ {}_{1\,16}L_2(20) = \frac{\frac{1}{2}[{}_{1\,16}l_2(20) + {}_{1\,16}l_2(21)]}{{}_{1\,16}l_1(16)} \tag{64}$$

or

$$ 0{\cdot}7720 = \frac{\frac{1}{2}(73853 + 76780)}{97562}\ . \tag{65}$$

The fraction of a year spent in the labor force between ages 20 and 21 per unit cohort born is

$$ 0{\cdot}7720\,\frac{97562}{100000} = 0{\cdot}7532\ . \tag{66}$$

The difference between 0·7720 and 0·7532 is due to mortality before the active ages.

An important statistic of the working-life table is the number of years in active life. Two approaches have been followed: the population-based measure of active life and the labor-force-based measure. The population-based measure is given in table 3. Since at age 16 everyone is inactive, the population-based measure is identical to the expression of the life expectancy by state at age 16. In other words, computing an increment-decrement life table by state at age 16 yields a population-based measure of the expected number of years in active and inactive life. The same measure is

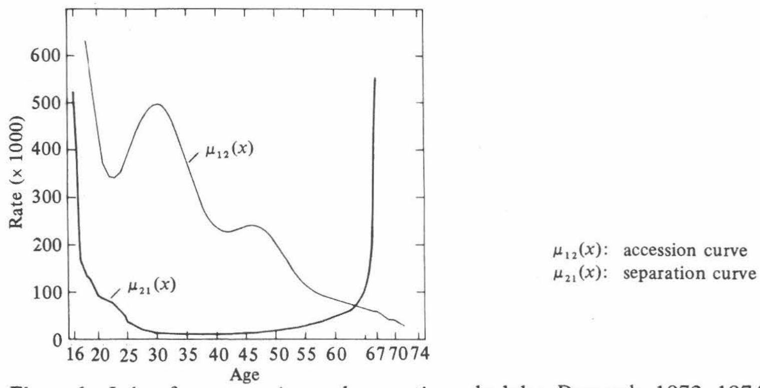

Figure 1. Labor-force accession and separation schedules, Denmark, 1972-1974, males (adapted from Hoem and Fong, 1976a, page 24).

Table 2. Life history of the initial cohort.

Age	Deaths		Transitions				Number of people at exact age x		
	inactive	active	1 to 1	2 to 1	1 to 2	2 to 2	inactive	active	total
16	71	0	74410	0	23081	0	97562	0	97562
17	69	22	41926	2852	32414	20208	74410	23081	97491
18	54	63	24204	5153	20521	47405	44778	52622	97399
19	39	90	17122	6179	12196	61657	29357	67926	97282
20	28	90	14917	5338	8355	68424	23301	73853	97153
21	21	78	14091	5317	6144	71385	20255	76780	97035
22	18	73	13868	5147	5522	72308	19408	77528	96936
23	19	78	13612	4607	5384	73146	19015	77830	96845
24	18	80	12842	3731	5359	74718	18219	78529	96748
25	15	75	11208	2713	5349	77289	16573	80077	96650
26	13	74	9111	2169	4798	80395	13922	82639	96560
27	11	83	7171	1664	4098	83446	11280	85193	96473
28	9	90	5470	1365	3356	86089	8835	87544	96379
29	7	93	4146	1076	2681	88276	6835	89445	96279
30	6	97	3142	872	2075	89988	5222	90957	96179
31	5	110	2425	889	1584	91064	4014	92063	96077
32	4	122	2039	931	1271	91594	3315	92647	95962
33	4	119	1879	962	1087	91784	2970	92866	95835
34	4	117	1863	821	975	91934	2841	92871	95712
35	4	128	1829	728	851	92052	2684	92908	95592
36	4	149	1812	689	740	92065	2557	92904	95460
37	5	172	1839	773	658	91860	2502	92806	95307
38	5	193	1981	838	626	91487	2613	92518	95131
39	6	204	2187	859	626	91050	2819	92113	94932
40	7	213	2396	667	643	90796	3046	91676	94722
41	8	234	2426	706	629	90498	3063	91438	94501
42	9	261	2483	779	640	90087	3132	91128	94260
43	11	303	2579	847	672	89578	3262	90728	93990
44	13	331	2696	995	717	88924	3426	90250	93676
45	14	347	2891	929	785	88366	3691	89641	93332
46	16	371	2985	961	820	87819	3820	89151	92971
47	18	396	3087	1087	840	87156	3946	88639	92584
48	20	430	3288	1197	865	86369	4174	87996	92170
49	25	477	3577	1359	883	85399	4485	87235	91720
50	32	551	4003	1540	902	84191	4936	86282	91218
51	41	626	4578	1462	924	83005	5542	85093	90636
52	48	669	5083	1571	908	81689	6040	83929	89969
53	56	692	5699	1744	900	80161	6654	82597	89251
54	66	724	6468	1942	909	78395	7443	81061	88504
55	83	779	7389	2064	937	76461	8409	79304	87713
56	101	827	8383	2347	969	74224	9453	77398	86851
57	127	887	9580	2522	1024	71783	10730	75193	85923
58	162	977	10855	2721	1084	69108	12102	72807	84909
59	203	1051	12226	3022	1147	66120	13576	70193	83769
60	246	1086	13783	3169	1219	63012	15248	67267	82515
61	301	1139	15362	3105	1289	59987	16952	64231	81183
62	378	1254	16758	3256	1332	56766	18467	61276	79743
63	463	1344	18186	3542	1366	53211	20014	58097	78112
64	549	1379	19789	4274	1389	48923	21727	54577	76304
65	671	1403	21977	4939	1416	43970	24064	50312	74376
66	822	1386	24600	6077	1493	37923	26916	45386	72302
67	1008	1295	28277	17080	1393	21041	30678	39416	70094
68	1617	800	41743	5704	1997	15931	45357	22434	67791
69	1827	691	43969	5514	1650	11723	47447	17928	65375
70	2059	556	45667	4111	1758	8707	49483	13374	62857
71	2276	479	46204	2530	1297	7456	49777	10465	60242
72	2439	438	44788	2730	1507	5585	48734	8753	57487
73	2578	385	44052	2022	888	4686	47518	7093	54610
74	2737	331	42267	1881	1070	3361	46074	5573	51647

Key: 1—inactive; 2—active.

Table 3. Expected number of years lived within age interval x to $x+1$ and beyond age x.

Age	Expected number of years lived by member of initial cohort					
	within age interval x to $x+1$			beyond age x		
	inactive	active	total	inactive	active	total
16	0·8813	0·1189	0·9996	10·6390	42·0028	52·6414
17	0·6108	0·3880	0·9988	9·7648	41·9148	51·6796
18	0·3799	0·6178	0·9977	9·1621	41·5654	50·7274
19	0·2699	0·7266	0·9965	8·7921	40·9958	49·7879
20	0·2232	0·7720	0·9952	8·5327	40·3206	48·8533
21	0·2033	0·7908	0·9941	8·3187	39·5936	47·9124
22	0·1969	0·7962	0·9931	8·1226	38·8380	46·9606
23	0·1908	0·8013	0·9922	7·9320	38·0725	46·0044
24	0·1783	0·8128	0·9912	7·7474	37·3024	45·0498
25	0·1563	0·8339	0·9902	7·5753	36·5198	44·0950
26	0·1292	0·8601	0·9893	7·4244	35·7113	43·1357
27	0·1031	0·8853	0·9884	7·3005	34·8736	42·1741
28	0·0803	0·9071	0·9874	7·2033	34·0117	41·2150
29	0·0618	0·9246	0·9863	7·1294	33·1277	40·2571
30	0·0473	0·9380	0·9853	7·0741	32·2243	39·2984
31	0·0376	0·9466	0·9842	7·0336	31·3062	38·3398
32	0·0322	0·9507	0·9830	7·0038	30·3813	37·3851
33	0·0298	0·9519	0·9817	6·9803	29·4535	36·4338
34	0·0283	0·9521	0·9804	6·9589	28·5211	35·4800
35	0·0269	0·9523	0·9791	6·9388	27·5853	34·5241
36	0·0259	0·9518	0·9777	6·9209	26·6501	33·5710
37	0·0262	0·9498	0·9760	6·9055	25·7186	32·6240
38	0·0278	0·9462	0·9741	6·8914	24·7923	31·6837
39	0·0301	0·9419	0·9720	6·8772	23·8717	30·7489
40	0·0313	0·9385	0·9698	6·8615	22·9546	29·8161
41	0·0317	0·9356	0·9674	6·8452	22·0393	28·8845
42	0·0328	0·9320	0·9648	6·8299	21·1273	27·9572
43	0·0343	0·9275	0·9618	6·8155	20·2206	27·0361
44	0·0365	0·9219	0·9584	6·8026	19·3223	26·1250
45	0·0385	0·9163	0·9548	6·7896	18·4298	25·2194
46	0·0398	0·9112	0·9510	6·7755	17·5398	24·3154
47	0·0416	0·9052	0·9469	6·7619	16·6530	23·4149
48	0·0444	0·8980	0·9424	6·7482	15·7696	22·5178
49	0·0483	0·8893	0·9375	6·7342	14·8918	21·6260
50	0·0537	0·8783	0·9320	6·7196	14·0226	20·7421
51	0·0594	0·8662	0·9256	6·7049	13·1673	19·8722
52	0·0651	0·8534	0·9185	6·6903	12·3256	19·0158
53	0·0722	0·8387	0·9110	6·6730	11·4917	18·1647
54	0·0812	0·8219	0·9031	6·6497	10·6642	17·3139
55	0·0915	0·8031	0·8946	6·6192	9·8462	16·4654
56	0·1034	0·7820	0·8855	6·5821	9·0418	15·6239
57	0·1170	0·7585	0·8755	6·5358	8·2515	14·7873
58	0·1316	0·7329	0·8645	6·4794	7·4786	13·9580
59	0·1477	0·7045	0·8522	6·4142	6·7267	13·1410
60	0·1650	0·6739	0·8389	6·3371	5·9960	12·3331
61	0·1815	0·6432	0·8247	6·2428	5·2846	11·5273
62	0·1972	0·6118	0·8090	6·1334	4·5930	10·7264
63	0·2139	0·5775	0·7914	6·0152	3·9248	9·9400
64	0·2347	0·5376	0·7722	5·8841	3·2795	9·1636
65	0·2613	0·4904	0·7517	5·7289	2·6594	8·3882
66	0·2952	0·4346	0·7298	5·5406	2·0739	7·6145
67	0·3897	0·3170	0·7067	5·3044	1·5343	6·8386
68	0·4756	0·2069	0·6825	4·9237	1·1302	6·0540
69	0·4968	0·1604	0·6572	4·3959	0·8633	5·2592
70	0·5087	0·1222	0·6309	3·8010	0·6489	4·4499
71	0·5049	0·0985	0·6034	3·1421	0·4792	3·6213
72	0·4933	0·0812	0·5745	2·4359	0·3350	2·7709
73	0·4797	0·0649	0·5446	1·6830	0·2076	1·8905
74	0·4624	0·0513	0·5137	0·8735	0·0968	0·9703

Table 4. Number of years lived in each state between ages x and $x+1$.

Age	State at age x					
	inactive			active		
	inactive	active	total	inactive	active	total
17	0·7817	0·2178	0·9995	0·0618	0·9378	0·9995
18	0·7703	0·2291	0·9994	0·0490	0·9504	0·9994
19	0·7916	0·2077	0·9993	0·0455	0·9538	0·9993
20	0·8201	0·1793	0·9994	0·0361	0·9632	0·9994
21	0·8478	0·1516	0·9995	0·0346	0·9649	0·9995
22	0·8573	0·1423	0·9995	0·0332	0·9663	0·9995
23	0·8579	0·1416	0·9995	0·0296	0·9699	0·9995
24	0·8524	0·1471	0·9995	0·0238	0·9757	0·9995
25	0·8381	0·1614	0·9995	0·0169	0·9826	0·9995
26	0·8272	0·1723	0·9996	0·0131	0·9864	0·9996
27	0·8179	0·1816	0·9995	0·0098	0·9897	0·9995
28	0·8096	0·1899	0·9995	0·0078	0·9917	0·9995
29	0·8033	0·1961	0·9995	0·0060	0·9935	0·9995
30	0·8008	0·1986	0·9995	0·0048	0·9947	0·9995
31	0·8021	0·1973	0·9994	0·0048	0·9946	0·9994
32	0·8075	0·1918	0·9993	0·0050	0·9943	0·9993
33	0·8164	0·1830	0·9994	0·0052	0·9942	0·9994
34	0·8278	0·1715	0·9994	0·0044	0·9950	0·9994
35	0·8407	0·1586	0·9993	0·0039	0·9954	0·9993
36	0·8544	0·1448	0·9992	0·0037	0·9955	0·9992
37	0·8676	0·1314	0·9991	0·0042	0·9949	0·9991
38	0·8791	0·1199	0·9990	0·0045	0·9944	0·9990
39	0·8878	0·1111	0·9989	0·0047	0·9942	0·9989
40	0·8933	0·1055	0·9988	0·0036	0·9952	0·9988
41	0·8960	0·1028	0·9987	0·0039	0·9949	0·9987
42	0·8964	0·1022	0·9986	0·0043	0·9943	0·9986
43	0·8953	0·1031	0·9983	0·0047	0·9937	0·9983
44	0·8935	0·1047	0·9982	0·0055	0·9926	0·9982
45	0·8917	0·1064	0·9981	0·0052	0·9929	0·9981
46	0·8906	0·1073	0·9979	0·0054	0·9925	0·9979
47	0·8912	0·1065	0·9978	0·0061	0·9916	0·9978
48	0·8939	0·1036	0·9976	0·0068	0·9908	0·9976
49	0·8988	0·0985	0·9973	0·0078	0·9895	0·9973
50	0·9054	0·0914	0·9968	0·0089	0·9879	0·9968
51	0·9130	0·0834	0·9963	0·0086	0·9877	0·9963
52	0·9208	0·0752	0·9960	0·0094	0·9866	0·9960
53	0·9282	0·0676	0·9958	0·0106	0·9852	0·9958
54	0·9345	0·0611	0·9955	0·0120	0·9836	0·9955
55	0·9394	0·0557	0·9951	0·0130	0·9821	0·9951
56	0·9434	0·0512	0·9946	0·0152	0·9795	0·9946
57	0·9464	0·0477	0·9941	0·0168	0·9773	0·9941
58	0·9485	0·0448	0·9933	0·0187	0·9746	0·9933
59	0·9503	0·0422	0·9925	0·0215	0·9710	0·9925
60	0·9519	0·0400	0·9919	0·0236	0·9684	0·9919
61	0·9531	0·0380	0·9911	0·0242	0·9670	0·9911
62	0·9537	0·0360	0·9898	0·0266	0·9632	0·9898
63	0·9543	0·0341	0·9884	0·0305	0·9580	0·9884
64	0·9554	0·0320	0·9874	0·0392	0·9482	0·9874
65	0·9566	0·0294	0·9860	0·0491	0·9370	0·9860
66	0·9570	0·0277	0·9847	0·0670	0·9178	0·9847
67	0·9609	0·0227	0·9836	0·2167	0·7669	0·9836
68	0·9602	0·0220	0·9822	0·1271	0·8550	0·9822
69	0·9634	0·0174	0·9807	0·1538	0·8270	0·9807
70	0·9614	0·0178	0·9792	0·1537	0·8255	0·9792
71	0·9641	0·0130	0·9771	0·1209	0·8563	0·9771
72	0·9595	0·0155	0·9750	0·1559	0·8190	0·9750
73	0·9635	0·0093	0·9729	0·1426	0·8303	0·9729
74	0·9587	0·0116	0·9703	0·1688	0·8015	0·9703

Table 5. Expectations of inactive and active life by state at age x.

Age	State at age x					
	inactive			active		
	inactive	active	total	inactive	active	total
17	10·092	41·588	51·680	8·711	42·969	51·680
18	9·974	40·753	50·728	8·471	42·256	50·728
19	10·025	39·763	49·788	8·259	41·529	49·788
20	10·108	38·745	48·853	8·036	40·818	48·853
21	10·113	37·799	47·912	7·845	40·067	47·912
22	9·980	36·980	46·961	7·658	39·303	46·960
23	9·790	36·214	46·004	7·478	38·526	46·004
24	9·582	35·468	45·050	7·322	37·726	45·050
25	9·379	34·716	44·095	7·202	36·593	44·095
26	9·231	33·905	43·136	7·120	36·016	43·136
27	9·123	33·051	42·174	7·059	35·115	42·174
28	9·054	32·161	41·215	7·016	34·198	41·215
29	9·032	31·225	40·257	6·984	33·273	40·257
30	9·063	30·235	39·298	6·960	32·338	39·298
31	9·150	29·190	38·340	6·941	31·398	38·340
32	9·295	28·090	37·385	6·922	30·463	37·385
33	9·495	26·938	36·434	6·900	29·534	36·434
34	9·741	25·739	35·480	6·874	28·606	35·480
35	10·009	24·515	34·524	6·850	27·674	34·524
36	10·276	23·294	33·571	6·828	26·742	33·571
37	10·511	22·113	32·624	6·808	25·816	32·624
38	10·690	20·993	31·684	6·784	24·900	31·684
39	10·804	19·945	30·749	6·757	23·992	30·749
40	10·857	18·959	29·816	6·729	23·387	29·816
41	10·866	18·018	28·884	6·710	22·174	28·884
42	10·853	17·104	27·957	6·692	21·266	27·957
43	10·840	16·196	27·036	6·671	20·365	27·036
44	10·846	15·278	26·125	6·649	19·476	26·125
45	10·886	14·333	25·219	6·621	18·548	25·219
46	10·967	13·349	24·315	6·596	17·719	24·315
47	11·093	12·322	23·415	6·569	16·846	23·415
48	11·258	11·260	22·518	6·534	15·984	22·518
49	11·447	10·178	21·626	6·492	15·134	21·626
50	11·636	9·106	20·742	6·438	14·304	20·742
51	11·797	8·075	19·872	6·373	13·499	19·872
52	11·903	7·113	19·016	6·315	12·702	19·016
53	11·932	6·232	18·165	6·249	11·915	18·165
54	11·875	5·439	17·314	6·170	11·144	17·314
55	11·736	4·729	16·465	6·077	10·389	16·465
56	11·529	4·095	15·624	5·978	9·646	15·624
57	11·259	3·528	14·787	5·862	8·926	14·787
58	10·938	3·020	13·958	5·738	8·220	13·958
59	10·576	2·564	13·141	5·609	7·532	13·141
60	10·176	2·157	12·333	5·467	6·866	12·333
61	9·734	1·793	11·527	5·321	6·206	11·527
62	9·254	1·472	10·726	5·193	5·533	10·726
63	8·744	1·196	9·940	5·075	4·865	9·940
64	8·200	0·963	9·164	4·962	4·202	9·164
65	7·616	0·772	8·388	4·826	3·562	8·388
66	6·990	0·625	7·614	4·681	2·933	7·614
67	6·327	0·512	6·839	4·508	2·331	6·839
68	5·652	0·402	6·054	3·452	2·602	6·054
69	4·959	0·300	5·259	2·906	2·352	5·259
70	4·228	0·222	4·450	2·221	2·229	4·450
71	3·481	0·141	3·621	1·532	2·089	3·621
72	2·681	0·090	2·771	1·270	1·700	2·771
73	1·855	0·035	1·890	0·527	1·363	1·890
74	0·959	0·012	0·970	0·169	0·801	0·970

obtained by computing the life expectancy at birth. A 16 year old is expected to spend about 42 years in active life and about 11 years in inactive life. At birth a baby is expected to spend $42 \times 0{\cdot}97562 = 40{\cdot}98$ years in the labor force. As before, the difference is due to mortality at ages below 16 years. The population-based measure of the expected remaining numbers of active years of a 20 year old is

$$_{..}e_2(20) = \sum_{y=20}^{74} \frac{_{..}L_2(y)}{_{..}\hat{l}_{.}(20)} , \tag{67}$$

where $_{..}\hat{l}_{.}(20) = {}_{..}l_{.}(20)/{}_{..}l_{.}(16) = \dfrac{97\,153}{97\,562} = 0{\cdot}99581$. Hence $_{.}e_2(20)$ is

$$40{\cdot}321 = \frac{40{\cdot}1518}{0{\cdot}99581} . \tag{68}$$

To obtain numerical values for the labor-force-based measure of active life, the 'multiregional' life table 'by place of residence' or current state was calculated. The life history of the population by current state is identical to table 2. This is as expected since table 2 also does not distinguish between different cohorts (taking the cohort of the inactive population of age 16 implies the total population, because at age 16 everyone is inactive). The fraction of a year spent inside and outside the labor force between ages x and $x+1$ by a person who is in (or outside) the labor force *at exact age* x is given in table 4. For example, a male inactive person aged 20 is expected to spend $_{1\ 20}L_2(20) = 0{\cdot}1793$ years in the labor force and to spend an average of $_{1\ 20}L_1(20) = 0{\cdot}8201$ years in inactive life before reaching the age of 21. On the other hand, a person already in the labor force at age 20 spends an average of $_{2\ 20}L_2(20) = 0{\cdot}9632$ years in the labor force and $_{2\ 20}L_1(20) = 0{\cdot}0361$ years in inactive life before reaching 21 years. Table 4 gives the average time spent in each state during the following year per active and inactive person of exact age x.

Instead of considering the average time spent in each state during one-year intervals, one may calculate the time spent in each state beyond age x. The result is represented in table 5. It gives the remaining lifetime inside and outside the labor force per active and inactive person of exact age x. The right-hand side of the table shows the labor-force-based measures of active and inactive life. For example, an active person aged 20 years may expect to spend 40·818 years in the labor force and 8·036 years in inactive life. The left-hand side of the table contains the inactive-life-based measures of active and inactive life. A 20-year-old person who is inactive is expected to spend 38·745 and 10·108 years in active and inactive life respectively. Table 5 begins with age 17 because no person of age 16 is in the labor force and therefore a labor-force-based measure of active life below this age is not meaningful.

5 Application: labor-force projections

Traditionally, labor-force projections are made by applying trends of LFP rates to population projections by sex and age. The conventional procedure focuses on the *stock* of the labor force. In order to include people who enter and leave the labor force at each age, Cohn et al (1974) developed a model denoted here as the CNN model. However, new entrants into the labor force are not generated by the model but must be projected separately. A model that projects the active and inactive population simultaneously and that accounts for the differences and the interactions between these two states is proposed in subsection 5.3. It is similar to the multiregional demographic growth model (Rogers, 1975, chapter 5).

5.1 *Traditional techniques*

The general approach to labor-force projections is to project the population by age and sex and to apply trend-adjusted LFP rates to the projected population. Let $K(t)$

denote the age composition of the population (by sex), $\mathbf{G}(t)$ the growth matrix, and $\mathbf{W}(t)$ a diagonal matrix of age-specific LFP rates. The labor force by age group at time $t+1$ then becomes

$$E(t+1) = \mathbf{W}(t+1)K(t+1) = \mathbf{W}(t+1)\mathbf{G}(t)K(t) . \qquad (69)$$

If $\mathbf{W}(t)$ and $\mathbf{G}(t)$ remain constant, we have

$$E(t+1) = \mathbf{WG}K(t) = \mathbf{H}K(t) . \qquad (70)$$

The matrix $\mathbf{H}$ is of the same form as $\mathbf{G}$.

If the rate of change of labor-force participation rates is fixed, then $\mathbf{W}(t) = \mathbf{D}^t\mathbf{W}(0)$, where $\mathbf{D}$ is a diagonal matrix whose nonzero elements are equal to one plus the rate of change of the LFP rate, and $\mathbf{W}(0)$ is the matrix of LFP rates in the base year (Fullerton and Prescott, 1975, page 69). A logistic curve has been used by Im and Ramachandran (1961).

This approach is also being used by the US Bureau of Labor Statistics (see, for example, Johnston, 1973a, page 11; 1973b, page 15) and by the Organisation for European Economic Cooperation (OEEC, 1961; 1966). Current labor-force projections reflect changes in the age composition of the population and changes in the age-specific LFP rates. Johnston clearly states which of the two components is more important:

> "The predominant factor in these projections is the anticipated changes in the size and age-sex distribution of the population; projected changes in participation rates play a relatively minor role" (Johnston, 1973b, page 17).

As far as the LFP rates are concerned, the most frequent assumptions made are (Johnston, 1973a, page 11; Rosenblum, 1972):

(a) full employment (a generally favorable demand situation, or the demand for labor is completely elastic);
(b) no significant change in the size of the armed forces;
(c) social and political stability;
(d) continuation of education trends; and
(e) no change in the definition of 'labor force', 'employment', or 'unemployment'.

5.2 *The CNN model*

The method of applying age-specific labor-force participation rates to the projected population gives an estimate of the future population in active life. It does not distinguish between new entrants and people leaving the labor force. A projection procedure which explicitly considers new entrants and leavers was proposed by Cohn et al (1974) for the forecasting of the aggregate supply of coalminers. The model considers five-year age groups and projection intervals of five years. The procedure consists of two parts. The first part is the estimation of the expected labor force for each of the projection years, disregarding new entrants, and the second part is the estimation of the expected number of entrants into the labor force by age.

To estimate the expected labor force, disregarding new entrants, they use the concept of a labor-force retention rate, which is one minus the total separation rate. The expected number of people active at time t and aged x to $x+4$ who will still be active five years later is

$$E_s^{(t+1)}(x+5) = [1-M^w(x)]E^{(t)}(x) , \qquad (71)$$

where $E_s^{(t)}(x)$ is the surviving active population in age group x to $x+4$ at time t, $E^{(t)}(x)$ is the total labor force aged x to $x+4$ at time t, and $[1-M^w(x)]$ is the labor-force retention rate. Writing equation (70) for several age groups, we have the matrix expression

$$\begin{bmatrix} E^{(t+1)}(\bar\alpha) \\ E^{(t+1)}(\bar\alpha+5) \\ \vdots \\ E^{(t+1)}(x-5) \\ E^{(t+1)}(x) \\ \vdots \\ E^{(t+1)}(\bar\beta-5) \\ E^{(t+1)}(\bar\beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ R^{\mathrm{w}}(\bar\alpha) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & R^{\mathrm{w}}(x-5) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & R^{\mathrm{w}}(\bar\beta-5) & 0 \end{bmatrix} \begin{bmatrix} E^{(t)}(\bar\alpha) \\ E^{(t)}(\bar\alpha+5) \\ \vdots \\ E^{(t)}(x-5) \\ E^{(t)}(x) \\ \vdots \\ E^{(t)}(\bar\beta-5) \\ E^{(t)}(\bar\beta) \end{bmatrix}, \tag{72}$$

where $R^{\mathrm{w}}(x) = [1-M^{\mathrm{w}}(x)]$. Equation (72) may be written more concisely as

$$E_{\mathrm{s}}^{(t+1)} = \mathbf{R}E^{(t)} . \tag{73}$$

The proposed method for estimating the expected number of entrants into the labor force by age is based on two premises. First, the number of new entries in the five-year period around the base year is assumed to be equal to the difference between the actual labor force at the end of the interval and the expected labor force at the end of the interval (calculated by the retention-rate method). Second, it is assumed that these entry levels by age will remain constant; that is, at each projection interval the same number of people enter the labor force and their age composition is fixed. The number of new entrants by age are computed as follows:

$$E_{\mathrm{n}}^{(t+1)}(x) = E^{(t+1)}(x) , \qquad \text{for } x = \bar\alpha , \tag{74}$$

$$E_{\mathrm{n}}^{(t+1)}(x) = E^{(t+1)}(x)-[1-M^{\mathrm{w}}(x-5)]E^{(t)}(x-5), \quad \text{for } x > \bar\alpha . \tag{75}$$

In other words, the labor force in age group $\bar\alpha$ at time $t+1$ is completely made up of new entrants (between t and $t+1$). The labor force in age group $(x, x+4)$ for $x > \bar\alpha$ at time t is composed of people who were in the labor force at t and survived, plus new entrants between t and $t+1$:

$$E^{(t+1)}(x) = E_{\mathrm{s}}^{(t+1)}(x)+E_{\mathrm{n}}^{(t+1)}(x) = [1-M^{\mathrm{w}}(x-5)]E^{(t)}(x-5)+E_{\mathrm{n}}^{(t+1)}(x) , \qquad \text{for } x > \bar\alpha . \tag{76}$$

Note that, if $E^{(t+1)}(x)$ is less than $E^{(t)}(x)$, then $E^{(t+1)}(x) = 0$. The total labor force at time $t+1$ therefore is

$$E^{(t+1)} = \mathbf{R}E^{(t)}+E_{\mathrm{n}}^{(t+1)} . \tag{77}$$

where $E_{\mathrm{n}}^{(t+1)}$ is the vector of new entrants during the time interval t to $t+1$.

5.3 *The two-state projection model*

The labor-force projection model proposed in this section proceeds from the analysis of increment-decrement tables of working life and uses findings of multiregional demography (Rogers, 1975). The active and inactive populations by age (and sex) are projected simultaneously. This method enables one to consider mortality and fertility differences between the active and inactive population and to examine the interactions between these subsets of the population.

The necessary data for labor-force projection by this method consist of:

(a) the active and inactive population by age in the base year (all ages);
(b) the age-specific rates of mortality both of the active and of the inactive populations, and rates of accession and separation;
(c) the age-specific birth rates for the active and inactive populations.

The model is presented for the case of a single-sex population: the extension to a two-sex population is straightforward. The active and inactive population by age at time t is denoted by the vector $K(t)$.

$$K(t) = \begin{bmatrix} K^{(t)}(0) \\ K^{(t)}(1) \\ \vdots \\ K^{(t)}(x) \\ \vdots \\ K^{(t)}(z) \end{bmatrix} , \qquad \text{where } K^{(t)}(x) = \begin{bmatrix} K_1^{(t)}(x) \\ K_2^{(t)}(x) \end{bmatrix} . \tag{78}$$

An element $K_i^{(t)}(x)$ denotes the number of people aged x to $x+1$ in state i at time t ($i = 1$ for the inactive population, $i = 2$ for the active population). At ages below $\bar{\alpha}$ and above $\bar{\beta}$, $K_2^{(t)}(x)$ will be zero. Of the people aged x to $x+1$ in the labor force at time t, some may remain in the labor force during the time period $(t, t+1)$, some may become inactive, and some may die. On the other hand, the people aged $x+1$ to $x+2$ in the labor force at time $t+1$ are composed of active-population survivors at time t and of new entrants into the labor force:

$$K_2^{(t+1)}(x+1) = s_{22}(x)K_2^{(t)}(x) + s_{12}(x)K_1^{(t)}(x) , \tag{79}$$

where $s_{22}(x)$ and $s_{12}(x)$ are survivorship proportions, which may be derived as part of the working-life table or computed directly from age-specific rates of mortality, accession, and separation:

$$\mathbf{S}(x) = [\mathbf{I} + \tfrac{1}{2}\mathbf{M}(x+1)]^{-1}[\mathbf{I} - \tfrac{1}{2}\mathbf{M}(x)] . \tag{80}$$

The inactive population aged $x+1$ to $x+2$ at time $t+1$ is

$$K_1^{(t+1)}(x+1) = s_{11}(x)K_1^{(t)}(x) + s_{21}(x)K_2^{(t)}(x) . \tag{81}$$

Equations (79) and (81) may be combined into the following matrix expression

$$K^{(t+1)}(x+1) = \mathbf{S}(x)K^{(t)}(x) . \tag{82}$$

The number of children in the first age group (from 0 to 1) at time $t+1$ is given by the surviving births during the interval $(t, t+1)$. Let $F_1(x)$ and $F_2(x)$ be the age-specific annual birth rates of a member of the labor force and of the inactive population respectively. It is realistic to allow for different fertility schedules for the active and the inactive populations. Assuming a uniform distribution of births over the time interval, the number of births between t and $t+1$ to mothers aged x to $x+1$ at time t is equal to

$$\sum_{i=1}^{2} \tfrac{1}{2}[F_i(x)K_i^{(t)}(x) + F_i(x+1)K_i^{(t+1)}(x+1)] . \tag{83}$$

Not all of these babies will survive to become members of the first age group at time $t+1$. (However, all the survivors will be in state 1, that is, inactive.) Assuming that infant mortality is independent of the activity status of the parent, the number of children aged 0 to 1 years at time $t+1$ is given by

$$K_1^{(t+1)}(0) = \sum_x \tfrac{1}{2}\hat{p} \sum_{i=1}^{2} [F_i(x)K_i^{(t)}(x) + F_i(x+1)K_i^{(t+1)}(x+1)] \tag{84}$$

and

$$K_2^{(t+1)}(0) = 0 , \tag{85}$$

where $\hat{p}$ denotes the proportion of babies born during the interval $(t, t+1)$ that will survive to time $t+1$. In matrix notation we may write

$$K^{(t+1)}(0) = \sum_x \tfrac{1}{2}\hat{\mathbf{P}}[\mathbf{F}(x)K^{(t)}(x) + \mathbf{F}(x+1)K^{(t+1)}(x+1)] , \tag{86}$$

where

$$\mathbf{F}(x) = \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{P}} = \begin{bmatrix} \hat{p} & \hat{p} \\ 0 & 0 \end{bmatrix} . \tag{87}$$

Equation (86) may be reformulated as follows:

$$K^{(t+1)}(0) = \sum_x \tfrac{1}{2}\hat{\mathbf{P}}[\mathbf{F}(x) + \mathbf{F}(x+1)\mathbf{S}(x)]K^{(t)}(x) = \sum_x \mathbf{B}(x)K^{(t)}(x) . \tag{88}$$

Equations (82) and (88) combined constitute the complete growth model of the active and inactive populations:

$$\begin{bmatrix} K^{(t+1)}(0) \\ K^{(t+1)}(1) \\ K^{(t+1)}(2) \\ \vdots \\ K^{(t+1)}(z) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbf{B}(\hat{\alpha}) & \cdots & \mathbf{B}(\hat{\beta}) & 0 & \cdots & 0 & 0 \\ \mathbf{S}(0) & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{S}(1) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \mathbf{S}(z-1) & 0 \end{bmatrix} \begin{bmatrix} K^{(t)}(0) \\ K^{(t)}(1) \\ K^{(t)}(2) \\ \vdots \\ K^{(t)}(z) \end{bmatrix} \tag{89}$$

or

$$K(t+1) = \mathbf{G}K(t) . \tag{90}$$

The ages $\hat{\alpha}$ and $\hat{\beta}$ are respectively the lowest and highest ages of the reproductive period.

6 Conclusion

The purpose of this paper was to illustrate how the mathematical theory of multiregional or multistate demography may be applied to calculate tables of working life *which do not rely on restrictive assumptions* such as the unimodality of the LFP curve, entrance into the labor force only at ages before the peak and retirement only at ages after the peak, and independence of mortality from the labor-force status. These assumptions are implicit in working-life tables that have been published to date in a large number of countries.

The fundamental difference between the new technique and the conventional method is its focus on *flows* instead of *stocks*. In the conventional method, only changes in the stock (net flows) of the active population by age are considered. The new technique focuses on gross flows in and out of the labor force at each age.

A second objective of the paper was to contribute to a theoretical analysis of multistate life-table construction without going into the details of movement versus transition approaches. All life-table functions, including the life-table rates, were derived directly from the Kolmogorov equation, and an attempt was made to state all the necessary assumptions explicitly.

A drawback of the new method for the calculation of working-life tables is the large data requirement. Today most countries do not publish the data required to build increment–decrement tables of working life. Therefore the design of better methods for labor-force analysis should be accompanied by greater attention to techniques for estimating missing data. Here, too, multiregional demography and migration research may contribute. Procedures developed to estimate age-specific migration flows, by origin and destination, from incomplete data may be applied to infer age-specific gross flows in and out of the labor force, that is, between the inactive and active states (Rogers et al, 1977; Rees, 1977; Willekens et al, 1979).

It has been an important recent observation that the applicability of multiregional demography is not limited to the study of a system of regions but extends to the analysis of any system of states or categories of age-specific populations (marital status, health status, etc) for which increment-decrement life tables may be developed. And, in addition to life-table construction, multiregional demography also may be used for the development of better models of labor-force projection, such as the two-state projection model proposed in section 5.3 of this paper.

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APPENDIX

This appendix gives the solution to equations (15) and (16), derived by Hoem and Fong (1976a, pages 68-70), assuming no mortality or equal mortality in both states.

No mortality

Without mortality $[\mu_{i\delta}(x) = 0,\ i = 1, 2]$, all people in state i at age y will be either in state i or in state j at age x $(x = y+n);\ i, j = 1, 2)$. Therefore ${}_{iy}\bar{l}_i(x) = 1 - {}_{iy}\bar{l}_j(x)$, $i \neq j$, for a unit cohort, where the bar denotes the absence of mortality. Introducing this in equation (15) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{1y}\bar{l}_2(x) = [1 - {}_{1y}\bar{l}_2(x)]\mu_{12}(x) - {}_{1y}\bar{l}_2(x)\mu_{21}(x)\ , \tag{A1}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{1y}\bar{l}_2(x) = \mu_{12}(x) - \gamma(x)\,{}_{1y}\bar{l}_2(x)\ , \tag{A2}$$

where $\gamma(x) = \mu_{21}(x) + \mu_{12}(x)$. The solution to equation (A2) is

$${}_{1y}\bar{l}_2(x) = \int_0^n \mu_{21}(y+t)\exp\left[-\int_t^n \gamma(y+u)\mathrm{d}u\right]\mathrm{d}t\ . \tag{A3}$$

This formula is numerically evaluated by replacing the continuous functions $\mu_{21}(y+t)$ and $\gamma(y+t)$ by step functions. If $\mu_{21}(y+t) = \bar{\mu}_{21}(y)$ and $\gamma(y+t) = \bar{\gamma}(y)$ in the interval from y to $x = y+n$, then, since

$$\int_0^n \exp\left[-a\int_s^n \mathrm{d}u\right]\mathrm{d}s = \int_0^n \exp[-a(n-s)]\mathrm{d}s = \int_0^n \exp(-an)\exp(as)\mathrm{d}s$$
$$= \exp(-an)\int_0^n \exp(as)\mathrm{d}s = \exp(-an)\left[\frac{1}{a}\exp(an)-\frac{1}{a}\exp(0)\right]$$
$$= \frac{1}{a}[1-\exp(-an)] \;, \tag{A4}$$

one gets

$$_{1y}\bar{l}_2(x) = \frac{\bar{\mu}_{21}(y)}{\bar{\gamma}(y)}\left\{1-\exp[-n\bar{\gamma}(y)]\right\} \;. \tag{A5}$$

Equal mortality

In the case of equal mortality, the differential equation describing the dynamics of mortality is, for example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{1y}l_\delta(x) = [1-{}_{1y}l_\delta(x)]\mu_\delta(x) \;. \tag{A6}$$

Note that for differential mortality in both states, equation (A6) does not hold; if mortality depends on the state (active/inactive), one must consider the state of the person at exact age x. The variable $_{1y}l_\delta(x)$ measures the probability that a person who is inactive at age y will die before reaching the age x. Since mortality is equal in both states, it is not important where the person will be at the time of death.

The solution to equation (A6) is

$$_{1y}l_\delta(x) = 1-\exp\left[-\int_0^n \mu_\delta(y+t)\mathrm{d}t\right] = {}_yq(x) \;. \tag{A7}$$

Note that the solution is independent of the state at age y. The quantity $_yq(x)$ measures the probability of dying between ages y and $x = y+n$.

To derive an expression for $_{1y}l_2(x)$ in the case of equal mortality, the equation

$$_{1y}l_1(x) = 1-{}_yq(x)-{}_{1y}l_2(x) \tag{A8}$$

is introduced into equation (15), which is then solved for $_{1y}l_2(x)$:

$$_{1y}l_2(x) = {}_{1y}\bar{l}_2(x)[1-{}_yq(x)] \;. \tag{A9}$$

Hence the probability of changing states in the case of equal mortality is the product of the probability in the case of no mortality and the survivorship probability, $1-{}_yq(x)$. In general we may write

$$_{iy}l_j(x) = {}_{iy}\bar{l}_j(x)[1-{}_yq(x)] \;. \tag{A10}$$

This formula separates the mortality and the mobility effects.

Note that this approach to solving equations (15) and (16) differs from that in the main body of the paper, when constant instantaneous rates in the interval $(y, y+n)$ were not assumed. These differences partly explain the deviations between Hoem and Fong's (1976a) results and the ones listed in this paper. The deviations are not, however, substantial.

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Multistate dynamics: the convergence of an age-by-region population system

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Abstract. This paper uses Rogers's (1975) discrete model of multiregional demographic growth to study the convergence properties of a fourteen-age-group, eight-region, female Canadian population system, which is subject to the 1966-1971 age-specific rates of births, deaths, and interregional migration. The focus is on (1) the changes in the age-by-region population distribution and (2) the evolution of regional growth and component rates.

For the youngest age group, the fluctuations in regional population size are almost entirely determined by a small number of low-frequency and long-durability cyclical components. As age increases, the high-frequency and short-durability cyclical components tend to play a more important role. The half-lives of major cyclical components are related to the shapes of regional fertility schedules. The legacy of the postwar baby boom is nationwide, so that the population waves in different regions tend to have similar phases and periodicities. The slow and persistent spatial convergence is noncyclical, but one region overshoots and two others undershoot their respective long-run regional shares. Over a hundred years the convergence toward the long-run regional age profiles is practically completed, while the spatial convergence has gone only halfway.

The difference in sensitivity among regional birth, death, in-migration, and out-migration rates to moving population waves and changing slopes of regional age profiles depends on the characteristics of regional schedules of fertility, mortality, and interregional migration. The persistent interregional contrast in growth rates is mainly determined by the relative competitiveness in the interregional migration transaction. The fluctuations of regional growth rates are dominated by the cyclical pattern of regional birth rates in early stages and by the fluctuations of regional death rates in later stages. The interpretation of long-run (intrinsic) regional in-migration or net migration rates as indices of regional attractiveness or competitiveness can be more misleading than the interpretation of long-run regional death rates as indices of regional mortality.

1 Introduction

In studying the dynamic properties of an age-by-region population system with a time-invariant structural matrix, this paper takes two viewpoints: one focuses on the changing age-by-region *population distribution* and the other on the evolution of *regional rates* of population change. Mathematically,

$$\begin{matrix}\text{population distribution}\\ \text{at a time point}\end{matrix} = \begin{matrix}\text{a time-dependent linear combination of the right}\\ \text{eigenvectors of the system's structural matrix}\end{matrix} + \begin{matrix}\text{an 'error'}\\ \text{term}\end{matrix}\;; \qquad (1)$$

and for a given region in each time period

$$\text{growth rate} = \text{birth rate} - \text{death rate} + \text{in-migration rate} - \text{out-migration rate}\;. \qquad (2)$$

It is hoped that by taking a close look at the quantities on the right-hand sides of these two equations the behaviors of the quantities on the left-hand sides might be easier to understand. Whereas equation (2) holds by definition, equation (1), the so-called analytic solution, will be specified more precisely later.

In terms of the Rogers model adopted in this study (Rogers, 1975), the population distribution will eventually converge to a fixed long-run age-by-region pattern which is completely determined by the dominant right eigenvector of the system's time-invariant structural matrix, and all regional growth rates will also converge to a fixed long-run growth rate, which is the corresponding dominant eigenvalue minus one.

Furthermore all regional *component* rates (those of birth, death, in-migration, and out-migration) will sooner or later become time-invariant. Of particular interest is the fact that some types of convergence proceed much faster than others. It has been suggested by Rogers (1976) and shown by Liaw (1978b) that the convergence toward the relatively smooth long-run regional age profiles is accomplished much faster than the convergence toward the long-run regional shares, provided of course that the initial and long-run distributions differ significantly in regional shares as well as in regional age profiles. One of the major purposes of this paper is to elaborate on these two types of convergence in terms of different groups of eigenvalues and eigenvectors. Besides the brief discussion on decomposing long-run regional growth rates by Rogers (1975, pages 129-132), the behavioral characteristics of regional growth and component rates have so far not been studied. Therefore, in addition to providing more insight into an old finding, this paper will break some new ground. For one thing, by showing how the regional component rates are differentially affected by changes in the age-by-region population distribution, I will be able to show that to interpret regional long-run (intrinsic) in-migration or net migration rates as indices of regional attractiveness or competitiveness may be more misleading than to interpret regional intrinsic death rates as indices of regional mortality.

This study is based on a fourteen-age-group, eight-region, female Canadian population system. The age groups are 0-4, 5-9, ..., 65+, and the regions are the Atlantic region, Quebec, Ontario, Manitoba, Saskatchewan, Alberta, British Columbia, and the North. The data are the same as those used in Liaw (1978b) except that the male part is now omitted for presentational convenience and that immigration and emigration rates are now set to zero so that the impacts of interregional migration may be seen more clearly. The effects of eliminating foreign migration are discussed in Liaw (1977; 1978b; 1979). The time interval of the data base is between 1 June 1966 and 31 May 1971.

Multiregional mathematical demography is useful if it helps one obtain a comprehensive understanding of the behavior of a *real-world* interregional population system. My interest is as strong in the Canadian population as in mathematical demography. Instead of using selected regions to demonstrate the main points, I have chosen to cover all regions with comparable thoroughness.

This paper is organized into three parts. The first part, comprising sections 2 through 7, deals with the changing age-by-region population distribution. The second part, comprising sections 8 through 10, studies the evolution of regional growth and component rates. The third part, section 11, summarizes the main points.

Part 1 starts by describing the mathematical model of the population system and its analytic solution (section 2). The spatial and cyclical components in the analytic solution are then classified into several groups according to the clustered pattern of the corresponding eigenvalues (section 3). The differential effects of the cyclical components are then studied (section 4). The characteristics of the convergence toward smooth regional time paths for the first and the tenth age groups are then described and explained in terms of the analytic components (sections 5 and 6). Finally the convergences toward the long-run regional age profiles and toward the long-run interregional distribution are contrasted in terms of an index of dissimilarity (section 7).

Part 2 starts by describing the regional indices of fertility, mortality, emissiveness, attractiveness, and competitiveness (section 8). Being unaffected by the changes in the age-by-region population distribution, these indices serve as benchmarks for studying the time paths of regional component rates (section 9). Finally the behavioral characteristics of regional growth rates are accounted for by those of the corresponding regional component rates (section 10).

Part 1

2 The Rogers model and its analytic solution

Let $K_a(t)$ be an 8×1 column vector whose jth element, denoted by $K_{aj}(t)$, represents the number of people in age group a and region j at time t. In other words $K_a(t)$ represents the interregional population distribution of the ath age group at time t. Let $K(t)$ be a 112×1 column vector whose ath block is $K_a(t)$. That is, $K(t)$ represents the age-by-region distribution of the population system at time t. With five years as the unit time interval, the Rogers model of the Canadian interregional population system is

$$K(t+1) = \mathbf{G}K(t)\ , \qquad t = 0, 1, 2, \ldots\ , \tag{3}$$

where the 112×112 structural matrix is of the form

$$\mathbf{G} = \left[\begin{array}{cccccc|cccc} 0 & \mathbf{B}_2 & \mathbf{B}_3 & \ldots & \mathbf{B}_9 & \mathbf{B}_{10} & 0 & 0 & 0 & 0 \\ \mathbf{S}_1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{S}_9 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & \mathbf{S}_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \mathbf{S}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \mathbf{S}_{12} & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \mathbf{S}_{13} & \mathbf{S}_{14} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H} & 0 \\ \hline \mathbf{U} & \mathbf{Z} \end{array}\right] . \tag{4}$$

The 8×8 submatrices **B** show how babies are born, survive, and are relocated. The small number of births beyond the tenth age group are lumped into the tenth age group; by use of the sensitivity formulas in Liaw (1978c), it can be shown that this simplification has negligible effects on the system's long-run properties. The 8×8 submatrices **S** indicate how existing individuals in specific age groups survive and migrate among the regions. The submatrices **H**, **U**, and **Z** are self-explanatory. We are interested in the convergence properties of the age-by-region population distribution, while assuming **G** to be time-invariant.

Since the nonzero eigenvalues of **G** turn out to be distinct, the analytic solution of equation (3) is, according to Liaw (1978a),

$$K(t) = \sum_{i=1}^{80} b_i \lambda_i^t Q_i + \sum_{i=1}^{8} c_i \tau_i^t R_i + D(t)\ , \tag{5}$$

where

- λ_i is the ith eigenvalue of **H**;
- Q_i is the right eigenvector of **G** associated with the eigenvalue λ_i;
- b_i is the inner product of $K(0)$ and the normalized left eigenvector[(1)] of **G** associated with λ_i;
- τ_i is the ith eigenvalue of $\mathbf{S}_{14}$;
- R_i is the right eigenvector of **G** associated with the eigenvalue τ_i;
- c_i is the inner product of $K(0)$ and the normalized left eigenvector of **G** associated with τ_i; and
- $D(t)$ is a 112×1 column vector whose first ten blocks of eight elements, corresponding to the first ten age groups, are zero, and whose blocks corresponding to the age groups beyond the reproductive range will all become zero (one block in each time period) by $t = 4$.

(1) A normalized left eigenvector is a left eigenvector whose inner product with the corresponding right eigenvector is unity.

Essentially the first eighty-eight terms of equation (5) represent the cumulative effect of the reproductive part of the initial population distribution, and the last term represents the temporary effect of the post-reproductive part of the initial population distribution. In twenty years $D(t)$ will disappear completely. It can be easily shown (Liaw, 1978a) that the R_i are filled with zeros except in the last age group. That is, the middle part of equation (5) only affects the oldest age group. Thus, for $a = 1, 2, \ldots, 10$, the analytic solution is

$$K_a(t) = \sum_{i=1}^{80} b_i \lambda_i^t Q_{ia} \,, \qquad t = 0, 1, 2, \ldots \,, \tag{6}$$

where Q_{ia} is the ath block of Q_i.

Cyclical waves are transmitted within the population system through the terms containing complex or negative eigenvalues. Since the eigenvalues of $\mathbf{S}_{14}$ are real and positive [based on analysis of many migration matrices by the author (US, Canadian, and Yugoslav data), it seems that a positive matrix with a dominant diagonal cannot have negative or complex eigenvalues], all complex and negative eigenvalues of G are found from the reproductive submatrix **H**. Of course, complex terms in the analytic solution occur in conjugate pairs. The ith pair can be written in real form as

$$2d_i \sigma_i^t [\cos(t\theta_i + e_i) X_i - \sin(t\theta_i + e_i) Y_i] \,, \tag{7}$$

where

σ_i is the magnitude of the ith complex eigenvalue of **G**;

θ_i is the amplitude of the ith complex eigenvalue of **G**;

X_i is the real part of the complex right eigenvector of **G** associated with (σ_i, θ_i);

Y_i is the imaginary part of the complex right eigenvector of **G** associated with (σ_i, θ_i); and

$$d_i = \{[U_i^{\mathrm{T}} K(0)]^2 + [V_i^{\mathrm{T}} K(0)]^2\}^{1/2} \,, \tag{8}$$

$$e_i = \tan^{-1} \left[\frac{V_i^{\mathrm{T}} K(0)}{U_i^{\mathrm{T}} K(0)} \right] \,, \tag{9}$$

where U_i^{T} and V_i^{T} are respectively the real and imaginary parts of the normalized left eigenvector of **G** associated with (σ_i, θ_i). The element in formula (7) corresponding to the ath age group and jth region can be further simplified into

$$2d_i \gamma_{iaj} \sigma_i^t \cos(t\theta_i + e_i + \phi_{iaj}) \,, \tag{10}$$

where

$$\gamma_{iaj} = (X_{iaj}^2 + Y_{iaj}^2)^{1/2} \tag{11}$$

and

$$\phi_{iaj} = \tan^{-1} \left[\frac{Y_{iaj}}{X_{iaj}} \right] \,, \tag{12}$$

where X_{iaj} and Y_{iaj} are respectively the elements in X_i and Y_i corresponding to the ath age group and jth region. It follows from formula (10) that the half-life and the period of the ith pair of complex terms in the analytic solution are respectively

$$t_i = -\frac{\ln 2}{\ln \sigma_i} \tag{13}$$

and

$$p_i = \frac{2\pi}{\theta_i} \,. \tag{14}$$

In summary, the analytic solution of the time path of the projected population size of the ath age group in region j is

$$K_{aj}(t) = \sum_{i=1}^{n_1} \overset{+}{b}_i \overset{+}{\lambda}{}_i^t \overset{+}{Q}_{iaj} + \sum_{i=1}^{n_2} \bar{b}_i \bar{\lambda}{}_i^t \bar{Q}_{iaj} + \sum_{i=1}^{n_3} 2d_i \gamma_{iaj} \sigma_i^t \cos(t\theta_i + e_i + \phi_{iaj})$$
$$+ \sum_{i=1}^{8} c_i \tau_i^t R_{iaj} + D_{aj}(t) \; ,$$
$$a = 1, 2, ..., 14 \, , \quad j = 1, 2, ..., 8 \, , \quad t = 0, 1, 2, ... \, , \qquad (15)$$

where + and − are used to distinguish terms associated with positive and negative real eigenvalues of **G** respectively. It should be clear how the elements Q_{iaj}, R_{iaj}, and $D_{aj}(t)$ are selected from the vectors $\boldsymbol{Q}_i$, $\boldsymbol{R}_i$, and $\boldsymbol{D}(t)$. Of course $n_1 + n_2 + 2n_3 = 80$. The term associated with the dominant eigenvalue is the *dominant component*, which determines the system's long-run growth rate and long-run age-by-region population distribution. The terms associated with the remaining positive eigenvalues are called *spatial components*, since the fact that the elements in each of these terms have a sign contrast (polarity) between regions rather than between age groups implies that they determine the spatial redistribution of the population. The terms associated with complex or negative eigenvalues are *cyclical components*, as they determine the transmission of population waves. Since the sign contrast of the elements of a cyclical component is mainly between age groups rather than between regions, population waves are transmitted mainly between age groups within individual regions. Exploiting the clustered nature of the eigenvalues, the analysis of the system's convergence properties is simplified by grouping spatial components into two sets and the cyclical components into five sets.

3 Grouping the spatial and cyclical components, based on the clustered pattern of the eigenvalues

The eigenvalues of the structural matrix of the Canadian age-by-region population system are shown as dots in figure 1. The overall pattern shows that components which fluctuate with relatively high frequencies tend to diminish at relatively fast rates. In other words the components which have short periods tend to have short half-lives.

The triangles in figure 1 are the eigenvalues of the Leslie model, which is consolidated from the Rogers model such that the system's long-run growth rate and long-run national age profile remain unchanged (Liaw, 1977, pages 58–61). It is interesting that the eigenvalues of the Rogers model tend to cluster in the vicinity of each of the eigenvalues of the consolidated Leslie model. Since the two models differ only in regional disaggregation, it is not surprising that the number of eigenvalues in most clusters is equal to eight (the number of regions). A similar pattern of clustering is found when foreign migration is not eliminated (Liaw, 1977; 1978b). In these earlier reports our cartographer added two nonexistent dots to the third cluster of complex eigenvalues.

There are sixteen positive real eigenvalues belonging to two distinct clusters. The first cluster of eight eigenvalues is obtained from the reproductive submatrix **H**, and the second cluster comes from $\mathbf{S}_{14}$. The half-life of the *least* durable eigenvalue in the first set (31·35 years) is more than twice the half-life of the *most* durable eigenvalue in the second set (13·00 years). Furthermore, although the components associated with the eigenvalues of **H** affect the behavior of all age groups, the components associated with the eigenvalues of $\mathbf{S}_{14}$ affect only the last age group. Thus the components associated with the first set of positive eigenvalues, excluding the dominant component, will be called the *major spatial components*, whereas those associated with the second set will be called the *minor spatial components*.

The complex eigenvalues fall mainly into four clusters. Among them, those in the first cluster have the longest half-lives (from 18·44 to 9·46 years) and periods (from 28·32 to 23·26 years). Those in the second cluster have half-lives of about five years and periods of about twenty-one years. Compared with those in the second cluster, those in the third cluster tend to have longer half-lives (about six years) but shorter periods (about thirteen years). The remaining complex eigenvalues all have very short half-lives (less than four years). The components associated with the first cluster of complex eigenvalues will be called *major cyclical components*, because they will undoubtedly determine the *generation effects* of the reproductive process upon the regional age profiles.

All negative real eigenvalues have very short half-lives, which range from 4·52 years to 1·30 years. Of course the components associated with negative eigenvalues have a period of ten years, which is the shortest possible period for a discrete-time Rogers model using five years as the unit time interval. For convenience all cyclical components with half-lives less than seven years are called *minor cyclical components.*

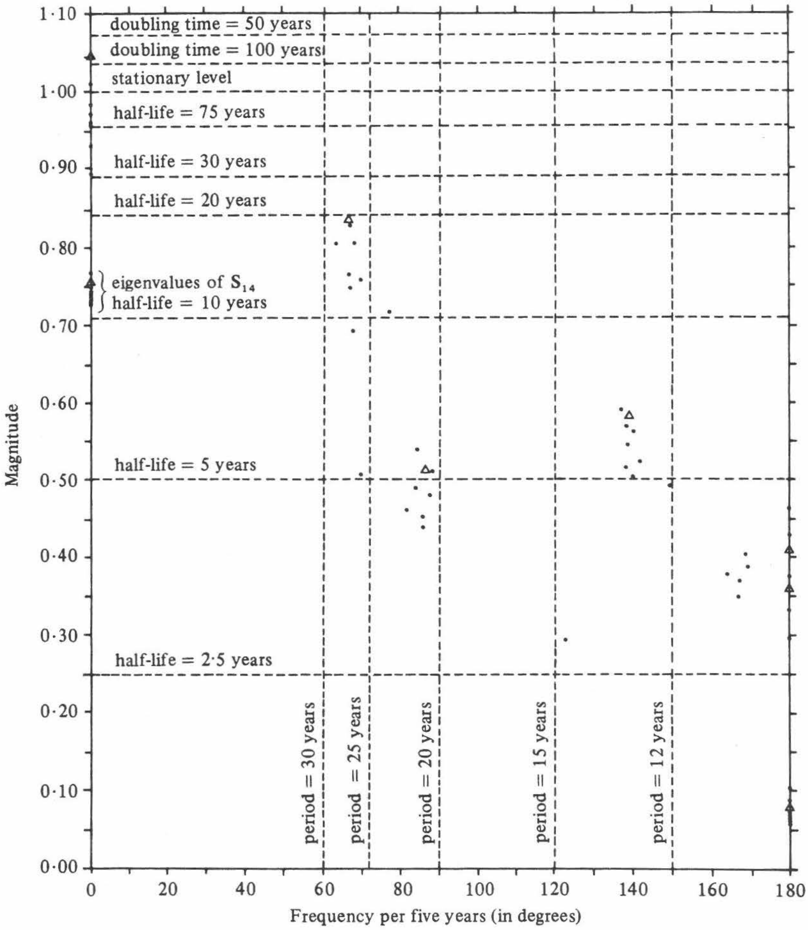

Figure 1. The clustered pattern of the eigenvalues of the structural matrix. Each conjugate pair of complex eigenvalues is represented by one dot.

4 The differential effects of the cyclical components

Since there are very few people migrating between Quebec and Saskatchewan, it seems reasonable to suspect that a cyclical component which has strong effect on Quebec (Saskatchewan) will have little effect on Saskatchewan (Quebec). We may also suspect that the nature of a cyclical component which has a strong effect on a particular region may be related to the fertility schedule of that region. Therefore it is meaningful to investigate the differential effects of the cyclical components.

We see in equation (15) that, for age group a and region j, the upper bound of fluctuations of the ith cyclical component is $2d_i\gamma_{iaj}$ or $|\bar{b}_i\bar{Q}_{iaj}|$. The regional upper bounds of all cyclical components for the first age group are shown in table 1. The cyclical components are grouped into five sets. The first three sets correspond to the first three clusters of eigenvalues in figure 1. The fourth set contains the eight most durable components among those which have periods between ten and eleven years. The remaining ten components make up the fifth set. Each component is identified by its half-life and period. Within each set the components are arranged in descending order of their half-lives.

Within each of the first four sets, each cyclical component tends to have its greatest effect (in terms of upper bounds of fluctuations) on a unique region. For example, within the first set the most durable cyclical component has its greatest effect on Ontario, whereas the least durable has its greatest effect on Saskatchewan. By assigning Ontario, British Columbia, Quebec, the Atlantic region, Alberta, Manitoba, the North, and Saskatchewan to the eight major cyclical components in descending order, we can obtain some insight about the durabilities of these cyclical components. After double log-transformations, the half-lives of the major cyclical components and the standard deviations of the corresponding regional fertility schedules turned out to have a moderately high correlation ($r = -0{\cdot}64$); when immigration and emigration rates are included in the structural matrix, this correlation is $-0{\cdot}72$ (Liaw et al, 1979). In other words, as far as the generation effects are concerned, regions whose fertility schedules are concentrated into a few age groups tend to be affected mainly by relatively more durable cyclical components.

It is difficult to see how the minor cyclical components are related to the characteristics of the corresponding regional schedules of births, deaths, and migration. However, it is quite clear that a cyclical component which has its greatest impact on a particular region tends to have strong effects on regions which are major destinations of that region's migrants. In other words, the transmission of population waves among regions tends to be positively related to the interregional out-migration rates.

Among the five sets of cyclical components in table 1, we see that the effects on the regional time paths of the first age group tend to decrease with the decrease in average half-life. Thus the fifth set has practically no effect on the regional time paths of the first age group. But it is important to realize that the upper bounds of cyclical components with small half-lives increase with age at extremely high rates. For example, for the tenth age group in Ontario the upper bound is 20100235 persons for the first component in the fifth set, and is only 351485 persons for the most durable major cyclical component. Thus, if the former component is dropped from the analytic solution, there will be extremely large errors in the analytic solution for older age groups at early time points. But in fifty years the error of 20100235 persons will be substantially reduced to about 350 persons because of the strong dampening effect of $\sigma_i^t = 0{\cdot}33433^t$. Therefore for relatively old age groups the components with very short half-lives can be treated as 'residuals' only when t is large.

Table 1. The regional upper bounds of fluctuations of each cyclical component for the first age group. Upper bounds are measured in numbers of persons; half-lives and periods are measured in years.

Half-life	Period	Atlantic	Quebec	Ontario	Manitoba	Saskat-chewan	Alberta	British Columbia	North
18·44	26·72	11258	12917	75733*	6839	3138	21754	39615	1110
16·14	26·39	2587	2040	21665	1544	963	8253	16390**	377
16·06	28·32	3535	56430*	18325	1099	353	1850	2997	92
13·02	26·82	12049*	550	10729	563	131	271	808	9
12·58	25·81	28	35	171	432	466	7658**	7714	53
11·96	26·70	483	102	1971	6523*	1068	3542	3354	68
10·30	23·26	6	2	17	6	4	45	46	199*
9·46	26·48	85	55	159	2121	5750*	3843	723	36
5·61	21·26	5417*	718	1586	118	226	249	294	57
5·19	20·43	1538	8853*	3063	254	398	461	705	88
5·12	25·82	17	8	31	16	15	77	72	261*
4·89	21·40	366	271	1153	606	2206*	1770	1543	187
4·76	20·57	581	1058	4290*	353	440	600	919	77
4·51	21·94	21	12	82	46	277	679*	300	30
4·38	20·87	16	11	397	222	154	377	1030*	44
4·24	20·89	3	4	63	260**	44	50	304	8
6·63	13·15	1980	21518*	8127	455	265	1224	1589	77
6·19	12·97	4698	4290	14242*	1005	715	3499	4199	188
6·04	12·81	1020	554	3997	601	678	4330	4948*	233
5·74	12·94	446**	58	975	46	22	119	135	6
5·38	12·68	25	17	93	17	118	1359**	1759	11
5·26	13·00	56	27	733	2309*	778	1339	818	35
5·04	12·87	36	23	130	875	1479*	1404	262	18
4·92	12·01	13	5	26	8	9	70	70	275*
4·52	10·00	63	23	458	15	12	30	199	645*
4·10	10·00	374	305	8420*	4	127	396	1151	548
3·85	10·67	3277*	691	1271	76	71	133	249	48
3·71	10·64	883	5902*	1154	72	51	51	166	32
3·58	10·97	38	42	160	856*	342	192	153	23
3·56	10·00	8	5	247	4	7	64	616*	59
3·51	10·76	32	40	150	374	1400*	771	614	92
3·32	10·79	9	8	37	17	209	658*	209	32
3·16	10·00	88	103	1556*	40	0	14	301	5
2·86	10·00	5	3	36	8	14	26	283*	18
2·85	14·69	0	0	0	0	0	2	2	8*
1·53	10·00	0	0	0	0	0	0	0	0
1·44	10·00	0	0	0	0	0	0	0	0
1·34	10·00	0	0	0	0	0	0	0	0
1·31	10·00	0	0	0	0	0	0	0	0
1·30	10·00	0	0	0	0	0	0	0	0
1·29	10·00	0	0	0	0	0	0	0	0
1·26	10·00	0	0	0	0	0	0	0	0

Note: for each component, the number with an asterisk is the greatest regional upper bound, whereas the number with two asterisks is the second greatest regional upper bound. The asterisks are used to emphasize that there *tends* to be a unique region of greatest effect for each cyclical component.

5 Convergence toward smooth regional time paths for the youngest age group

To study the regional time paths of the first age group, equation (15) will first be rewritten as

$$\begin{matrix}\text{projected} \\ \text{population}\end{matrix} = \begin{matrix}\text{dominant} \\ \text{component}\end{matrix} + \begin{matrix}\text{superimposed major} \\ \text{spatial component}\end{matrix} + \begin{matrix}\text{superimposed major} \\ \text{cyclical component}\end{matrix} + \begin{matrix}\text{superimposed minor} \\ \text{cyclical component}\end{matrix} \qquad (16)$$

In equation (16) the sum of the first two terms will be called the 'D&S' component and the sum of the last two terms the 'all-cyclical' component.

In figure 2 we see that the projected regional time paths of the first age group fluctuate around the smooth regional time paths of the D&S component. The relatively slow growth in Manitoba and negative growth in Quebec and Saskatchewan during the first fifty years (1966–2016) are due to the imbalance in interregional migration, which is clearly captured by the superimposed major spatial component. [The superimposed major spatial component also reflects the interregional differences in fertility (and even mortality) *levels*. On the one hand the high regional fertility level has prevented the time path of Atlantic's spatial component from declining, even though the region's out-migration exceeds its in-migration. On the other Quebec's low fertility level causes the time path of its spatial component to overstate the loss in interregional migration.] From the smoothness of the regional time paths of the D&S component, we realize that the fluctuations in projected regional time paths are completely captured by the all-cyclical component. Within the ten-year period beginning in 1966, the regional time paths of the all-cyclical component become practically identical to the regional time paths of the superimposed major cyclical component. This implies that population waves passing through the first age group can almost be completely explained by the generation effect of the reproductive process.

The fact that all regional population waves passing through the first age group have almost identical phases and periodicities indicates that the mean ages of regional fertility schedules are quite similar and that the shapes of the 1966 regional age profiles are also similar. To the extent that the periods of the eight major cyclical components may be related to the mean ages of the corresponding regional fertility schedules, it is worth mentioning that the periods of these components differ much less than the corresponding half-lives (see table 1). As far as the initial regional age profiles are concerned, all regions except the North have a deep hollow in the 25–34 age interval owing to the nationwide low fertility level of the 1930s, and a subsequent huge bulge due to the prolonged postwar baby boom and the sharp decline in fertility levels since the early 1960s (see figure 4).

As the sizeable population waves leave the first age group, they bring in numerous painful effects (for example, costly overexpanded schools, jobless college graduates, and overburdened social insurance systems). However, the severity of the population waves differ among regions. They are less severe in the North, British Columbia, and Alberta. These are regions with either a relatively smooth initial age profile or initially heavy interregional in-migration. The population waves are most severe in Saskatchewan and Quebec. The cushioning effect of positive natural growth is overwhelmed by heavy out-migration in Saskatchewan and is nullified by the very small in-migration into Quebec.

The fluctuations of the projected regional time paths for the first age group around the smooth time paths of the D&S component become very small over fifty years, and almost completely disappear over a hundred years.

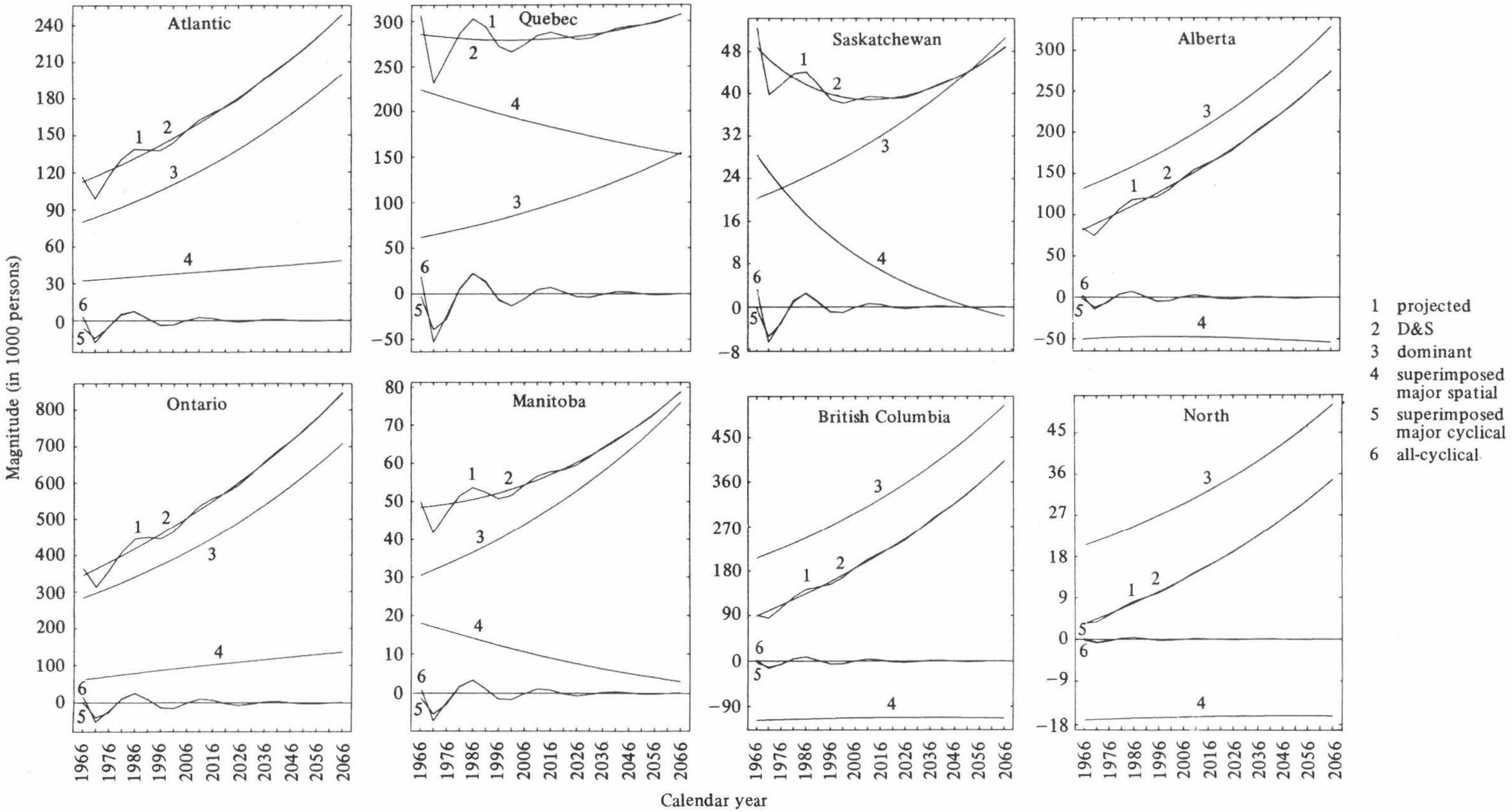

Figure 2. The regional time paths and their decompositions for the first (0–4) age group.

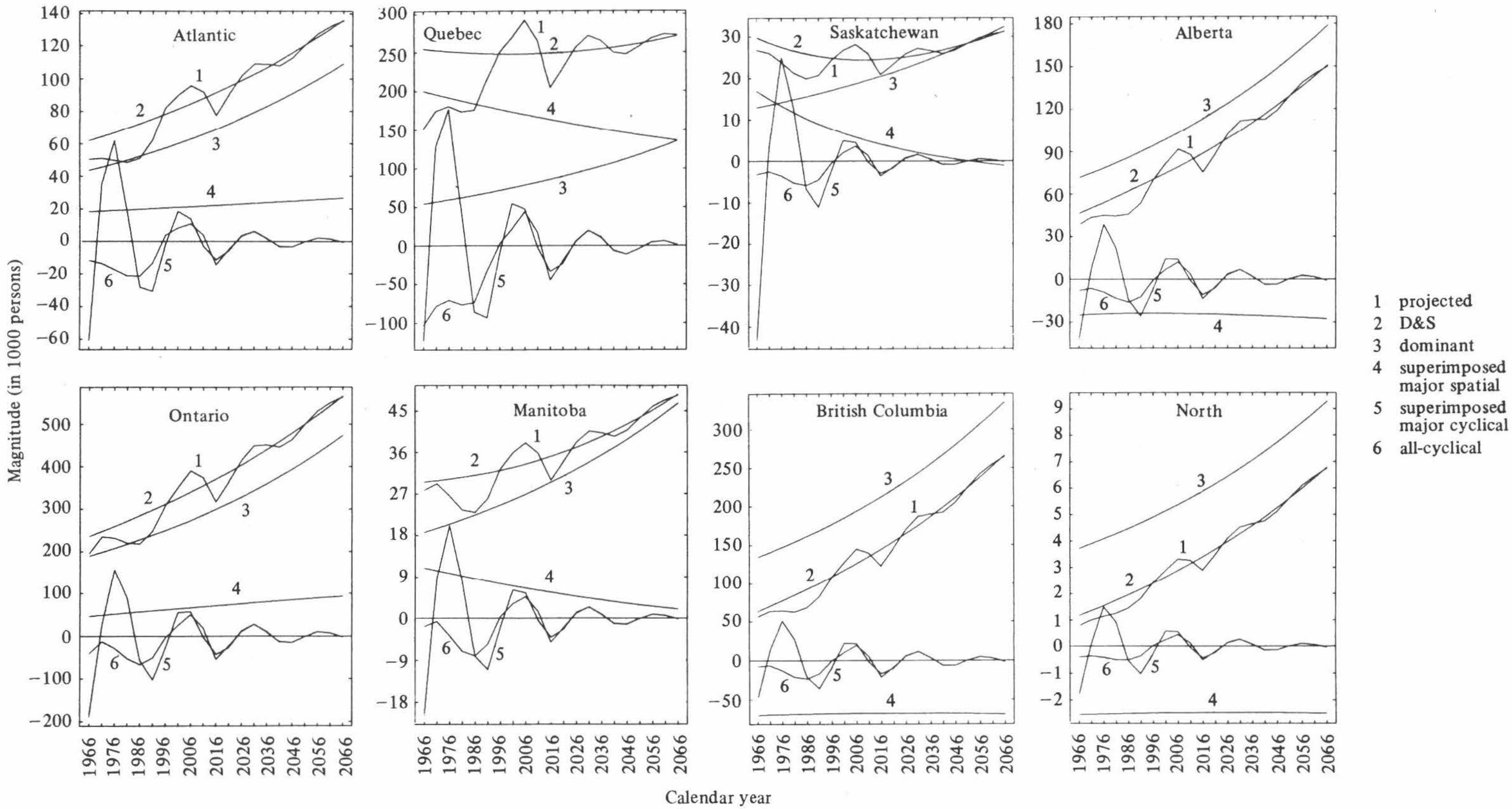

Figure 3. The regional time paths and their decompositions for the tenth (45–49) age group.

6 Convergence toward smooth regional time paths for the tenth age group

The projected regional time paths for the tenth age group fluctuate with greater magnitudes and take a longer time to converge to the smooth regional time paths of the D&S component than do those for the first age group (compare figures 2 and 3). This is reasonable because population waves generated by past changes in fertility levels are mainly dampened by the reproductive process rather than the death or migration process. Specifically it is mainly the variance of each regional fertility schedule that dampens the population waves.

The fact that in figure 3 the regional time paths of the superimposed major cyclical component and the all-cyclical component differ substantially in the first twenty-five years supports the previous argument that minor cyclical components cannot be omitted in accounting for the projected regional time paths for relatively old age groups. But after about sixty years the minor cyclical components lose their effects on the tenth age group. After a hundred years the generation effects represented by the superimposed major cyclical component are still visible but become relatively unimportant. The regional time paths of the all-cyclical component provide a clear contrast between the first-generation and second-generation effects of the sharp decrease in the number of babies born in the 1930s and the postwar baby boom.

Again the regional time paths of the superimposed major spatial component suggest that Saskatchewan, Manitoba, and Quebec are losing population through imbalanced interregional migration. The nearly horizontal time paths of the superimposed major spatial component in Alberta, British Columbia, and the North show that the historical westward shift in population redistribution will continue *slowly*. However, the fact that these nearly horizontal time paths are far below zero indicates that the westward shift is a very *persistent* trend.

7 Convergence toward the long-run regional age profiles and long-run interregional distribution

The 1966, 2066, and long-run age profiles in each region are shown in figure 4. Except in the North, the 1966 and long-run regional age profiles differ substantially in mean age as well as in smoothness. The differences in mean age are 4·45 in the Atlantic region, 8·42 in Quebec, 4·31 in Ontario, 3·34 in Manitoba, 5·79 in Saskatchewan, 4·36 in Alberta, 4·24 in British Columbia, and −0·30 in the North. Whereas the North is tending toward a slightly younger age profile, the remaining regions all tend to significantly older populations. For the whole nation the difference in mean age is 5·02 years. The regional indices of dissimilarity between the 1966 and the long-run age profiles are 10·02 for the Atlantic region, 15·95 for Quebec, 8·52 for Ontario, 8·26 for Manitoba, 10·93 for Saskatchewan, 8·85 for Alberta, 8·88 for British Columbia, and 5·57 for the North. For Canadian females as a whole, this dissimilarity index is 9·37. The lowest recorded value is 1·4 for England and Wales in 1881, and the record high is 26·3 for Japan in 1966 (Keyfitz and Flieger, 1971, page 30).

The dissimilarity indices of projected regional age profiles with respect to the corresponding regional long-run age profiles decrease sharply, with some fluctuations, to within the range between 1·40 (for the Atlantic region) and 3·32 (for Quebec) by the year 2026. Then the speed of convergence in regional age profiles becomes smaller. By the year 2066 the dissimilarity indices are reduced to the range between 0·47 (for Ontario) and 2·70 (for Quebec). Figure 4 shows that the 2066 projected regional age profiles are all very close to the corresponding long-run regional age profiles. Surprisingly the dissimilarity between the 2066 and the long-run age profiles in the North (which started with the smallest dissimilarity index) is more noticeable than in most of the other regions.

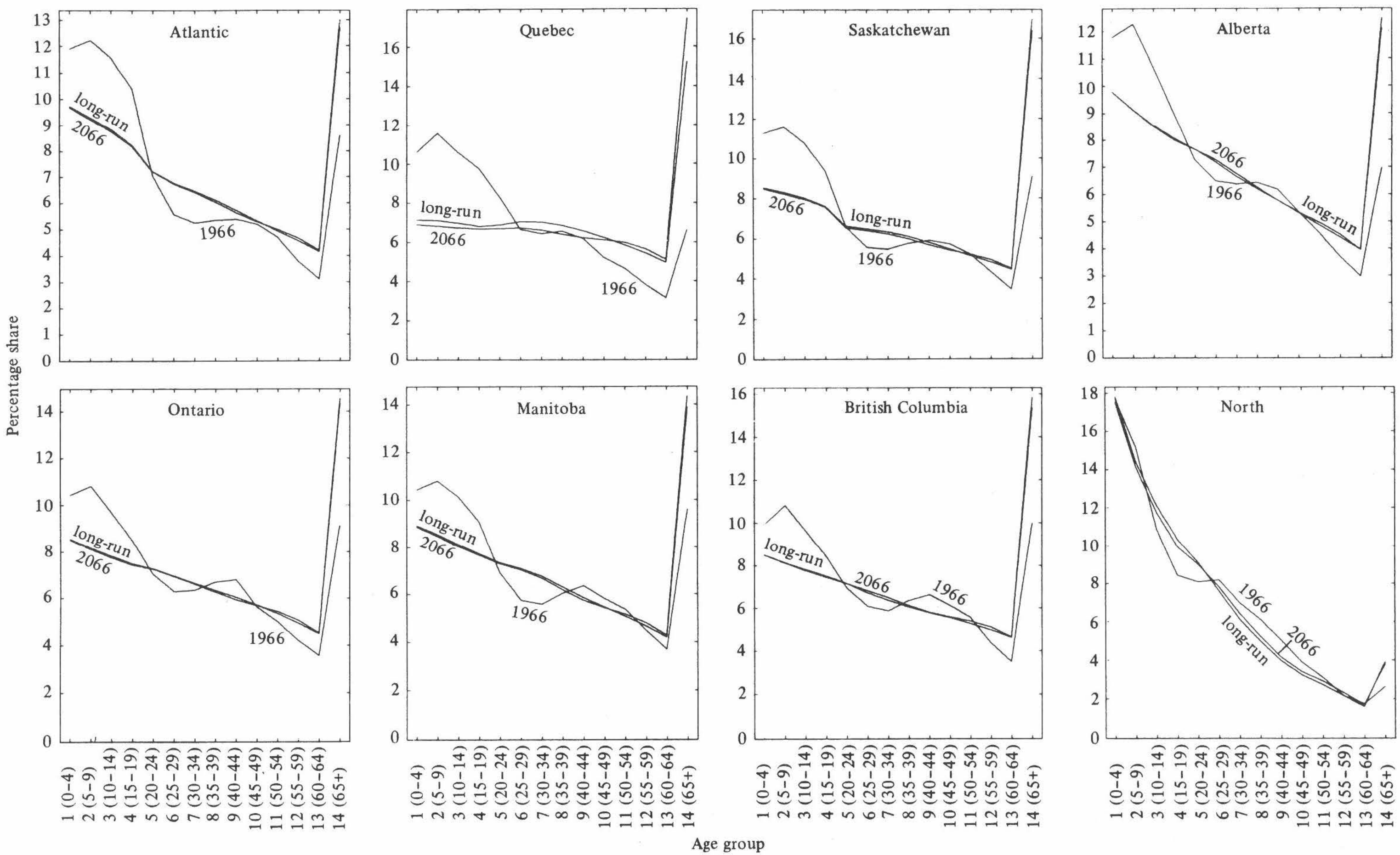

Figure 4. The 1966, 2066, and long-run regional age profiles.

The steepest and the flattest long-run regional age profiles are found in the North and Quebec respectively. This result is consistent with the fact that the long-run regional birth rates are highest in the North (20·3% per five years) and lowest in Quebec (7·2% per five years).

The regional long-run age profiles are less smooth than the long-run age profiles obtained from a nonspatial Leslie model. It is well-known that migration is a very selective process. Thus an imbalance in interregional migration may result in distorted long-run regional age profiles. The most noticeable distortions are found in the Atlantic region and in Saskatchewan. The dent in the 20–29 age groups can be explained by the heavy concentration of the regions' out-migrants in the 15–24 age groups.

Although the initial and long-run proportional shares of Ontario are almost identical (about 35%), the spatial dissimilarity index of the 1966 and the long-run regional shares is very large for the whole country, being 24·40 to be exact. Since the spatial components are associated with eigenvalues of relatively large magnitudes, the spatial dissimilarity index of the projected and long-run regional shares remain at the high level of 11·90 in 2066, when the convergence toward the regional long-run age profiles has practically been completed. Figure 5 shows that, although the time paths of projected regional shares are noncyclical, they may overshoot (in Ontario) or undershoot (in Manitoba and Saskatchewan) the corresponding long-run regional shares. The gaining regions are Alberta, British Columbia, and the North, and the losing regions are the Atlantic region, Quebec, Manitoba, and Saskatchewan. Ontario retains its dominant position all the time. The main loser is Quebec, and the main winners are British Columbia and Alberta. The spatial redistributional trend is clearly westward.

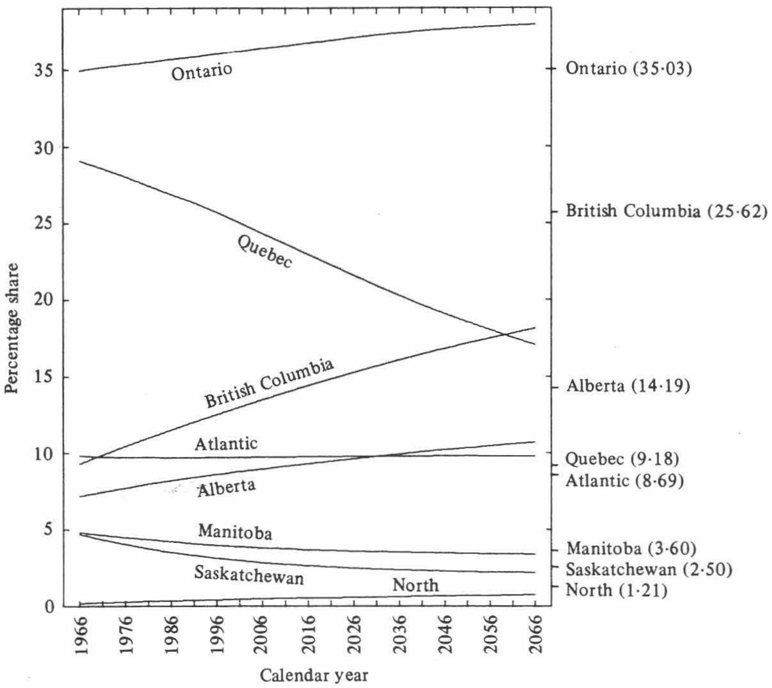

Figure 5. The time paths of regional shares. The labels on the right axis indicate the regional long-run percentage shares.

Part 2

8 The indices of fertility, mortality, emissiveness, attractiveness, and competitiveness

In equation (3) fertility and mortality are represented by a set of age- and region-specific birth and death rates in the system's structural matrix. For a given region, the regional (non-age-specific) birth and death rates during the time interval $(t, t+1)$ depend on the region's age profile at time t as well as on the region's age-specific birth and death rates. As the regional age profile changes through time, so do the corresponding regional birth and death rates. To assist explanation, it is essential that the regional age-specific birth (death) rates be combined into an index representing the regional level of fertility (mortality). Among several alternative sets of combining weights, the long-run age profile of Ontario is chosen to compute the *regional fertility* and *mortality indices*(2). As shown in figure 4, this long-run age profile is very smooth and has a gentle slope.

Regional emissiveness is represented by a set of age-specific origin–destination *out-migration* rates in the system's structural matrix. For a given region the regional out-migration rate during the time interval $(t, t+1)$ depends both on the region's age profile at time t and on its time-invariant age-specific out-migration rates. For example, if the regional population is heavily concentrated in the 20–29 age groups, its out-migration rate will tend to be large because of the high mobility of young adults. To eliminate the influence of interregional differences in age profiles, Ontario's long-run age profile is again used to combine the age-specific out-migration rates of each region into an *index of regional emissiveness.*

The structural matrix **G** in equation (3) does not contain age-specific origin–destination *in-migration* rates. Thus the assumption that **G** is time-invariant does not imply that the age-specific regional in-migration rates are time-invariant. The in-migration rate of region i during the time interval $(t, t+1)$ depends on (1) the time-invariant age-specific out-migration rates from all other regions into i, (2) the age profiles of all regions at time t, and, more important, (3) the relative population shares of all regions at time t (3). It becomes clear that to define an index of regional attractiveness in terms of in-migration we have to specify a standard interregional population distribution as well as a standard age profile. Since it will be seen later in this paper that the long-run interregional distribution implied by the system's structural matrix is not suitable for standardization, a standard age-by-region population distribution is constructed whose regional shares are identical to the initial (1966) observed regional shares and whose regional age profiles are the same as Ontario's long-run age profile. This standard distribution is then combined with the structural matrix **G** to generate regional in-migration rates which are interpreted as *indices of regional attractiveness.*

If the standardized regional in- and out-migration rates can be interpreted as indices of attractiveness and emissiveness respectively, then we may consider standardized regional net migration rates as *indices of competitiveness* in migration transactions. Table 2 contains the values of these indices computed for the Canadian regions from the 1966–1971 schedules of birth, death, and interregional out-migration.

(2) Use of the well-known gross reproduction rate or total fertility rate to measure fertility is equivalent to assuming a 'rectangular' age profile as the weighting scheme. The regional contrast in fertility levels to be described in this paper remains true when this alternative age profile is used for standardization. The GRR's are 1·45 (Atlantic), 1·06 (Quebec), 1·27 (Ontario), 1·31 (Manitoba), 1·35 (Saskatchewan), 1·41 (Alberta), 1·29 (British Columbia), and 2·58 (North).

(3) Since the numerator of the regional migration rate used in this paper includes surviving newborn migrants and excludes all nonsurviving migrants, each regional migration rate is to some extent affected by regional fertility and mortality levels. For the definitional equations of the regional component rates, see Rogers (1975, pages 129–132).

All discrete-time quinquennial rates are translated into continuous-time annual rates by the method which was introduced by Rogers (1975, page 132) in his study of intrinsic rates. When regional growth rates differ substantially, this or any other simplistic method may not be totally satisfactory.

The regional values of the fertility index in Canada in 1966–1971 are of four levels: Quebec has the lowest annual rate of about 15 births per 1000 people; British Columbia, Ontario, and Manitoba cluster around 18 per 1000; Saskatchewan, Alberta, and the Atlantic region are close to 20 per 1000; and finally the North is at the high level of 33 per 1000. Only Quebec's low fertility level seems to be worrysome, and hence has attracted much public attention.

The interregional difference in mortality in Canada is rather small. But the interregional ranking in mortality has remained largely unchanged in recent decades. Except for the North, Quebec's mortality index has been the highest in Canada, whereas Saskatchewan has had one of the lowest mortality levels in recent decades. Table 2 shows three levels of mortality in 1966–1971: all provinces between Ontario and British Columbia, inclusive, have annual rates of about 9 deaths per 1000 people; the Atlantic region and Quebec are close to 10 per 1000; and the North has the rate of 11 per 1000. (The mortality level of the North is suspiciously low; demographic data of this thinly populated region are relatively unreliable.) An interesting question to geographers is: why does the mortality level tend to be lower in the western provinces than in the rest of Canada?

The regional indices of emissiveness in Canada differ substantially. On the one hand there are the highly emissive North (36 out-migrants per 1000 people), Saskatchewan (23 per 1000), and Manitoba (19 per 1000); on the other we see the restraining Quebec (5 per 1000) and Ontario (slightly over 5 per 1000). British Columbia's index of emissiveness equals the national average of 8 per 1000, but the indices of the Atlantic region (10 per 1000) and Alberta (14 per 1000) are on the high side. Besides the cultural and linguistic barriers, can the trapping effect of a large population, described in Alonso (1971), account for the low emissiveness of Quebec and Ontario?

If the standardized in-migration rates can be interpreted as indices of attractiveness, we seen in table 2 the celebrated paradox that attractiveness is positively correlated with emissiveness. We also observe the well-known tendency that regions differ more in attractiveness than in emissiveness. At one extreme Quebec has an attractiveness index of less than 3 per 1000; at the other extreme the corresponding index of the

Table 2. Standardized regional rates of birth, death, in-migration, out-migration, and net migration for Canadian females, 1966–1971.

Region	Birth	Death	In-migration	Out-migration	Net migration
Atlantic	20·21	9·86	7·20	10·19	−2·99
Quebec	14·86	9·98	2·81	4·90	−2·09
Ontario	17·69	9·28	6·62	5·37	1·25
Manitoba	18·42	9·17	11·58	18·71	−7·13
Saskatchewan	19·38	8·86	7·92	22·78	−14·86
Alberta	19·39	8·60	17·50	14·31	3·19
British Columbia	17·50	8·71	20·41	8·01	12·40
North	33·00	11·02	53·64	35·94	17·70
Canada	17·37	9·41	8·10	8·10	0·00

Note: all rates are expressed as continuous-time annual rates (number per 1000 people). In the standard population the regional shares are identical to the 1966 observed regional shares, and every regional age profile is identical to Ontario's long-run age profile.

North is 54 per 1000. For those who know the regional geography of Canada, it seems natural that the attractiveness indices of British Columbia (20 per 1000) and Alberta (18 per 1000) are relatively high. But it may be a little surprising that on a per capita basis Ontario (6·6 per 1000) is less attractive than the Atlantic region (7·2 per 1000), Manitoba (11·6 per 1000), and Saskatchewan (7·9 per 1000).

Despite the strong positive correlation between attractiveness and emissiveness, the regional indices of competitiveness still vary substantially between the low value of Saskatchewan (−14·9 per 1000) and the high value of the North (17·7 per 1000). The competitive regions in migration transactions are the North, British Columbia (12·4 per 1000), Alberta (3·2 per 1000), and Ontario (1·3 per 1000), whereas the noncompetitive regions are Quebec (−2·1 per 1000), the Atlantic region (−3·0 per 1000), Manitoba (−7·1 per 1000), and Saskatchewan.

9 The evolution of regional component rates

9.1 *Regional birth rates*

As an approximate reflection of the four levels of fertility identified in section 8, the time paths of the regional birth rates shown in figure 6 are also at four distinct levels. The exceptionally high time path of the North results partly from the steep slope of the regional age profile, which is in turn the result of a prolonged exposure to a high level of fertility. It is interesting to note that although Saskatchewan has a fertility level most similar to those of Alberta and the Atlantic region, its time path of birth rates is closest to those of British Columbia, Ontario, and Manitoba. This is mainly due to Saskatchewan's very low level of mortality and its heavy concentration of out-migrants in the young adult age groups. Of course the causal relation works through the regional age profile.

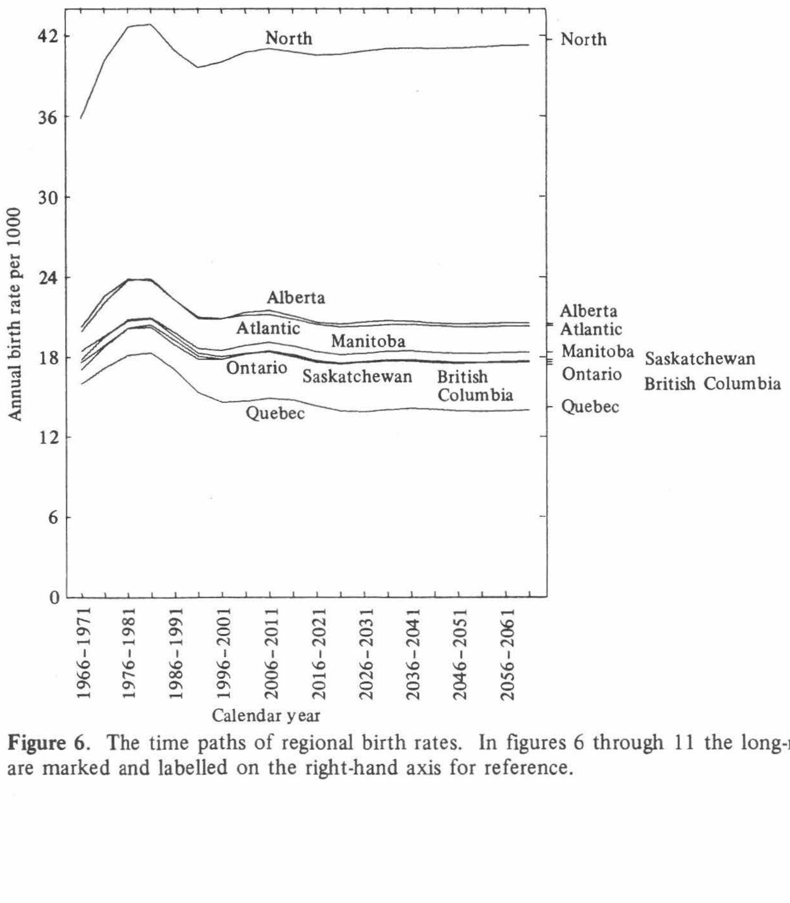

Figure 6. The time paths of regional birth rates. In figures 6 through 11 the long-run regional rates are marked and labelled on the right-hand axis for reference.

The fact that the time paths of regional birth rates fluctuate in similar ways shows that the legacy of the postwar baby boom is nationwide in Canada. The time paths stabilize in the second quarter of the twenty-first century to levels close to the respective long-run regional birth rates. The interregional differences in birth rates are actually greater in the long-run than 1966–1971. These increased differences are due to the fact that the slopes of the regional age profiles also differ more in the long-run than in the initial period—the initial and long-run age profiles are shown for all regions in figure 4. However, the interregional differences in long-run age profiles do not have much effect on the positive correlation between regional fertility levels and the corresponding long-run birth rates. Thus the interpretation of regional long-run birth rates as indices of regional fertility levels would not be too misleading.

9.2 *Regional death rates*

Whereas a typical fertility schedule is highly concentrated in a few young adult age groups, a typical mortality schedule will be concentrated predominantly in the oldest age groups. Thus regional death rates are much more sensitive to changes in the slope of regional age profiles than are regional birth rates. Figure 7 shows the dramatic divergence through time of the initially very similar death rates of Quebec and the North. The divergence is due to the increasing differences between the two regional age profiles. In fact the regional death rates are so much affected by the interregional differences in age profile that they can *never* be considered as indices of regional mortality levels in Canada, not even in the long-run.

Undoubtedly the upward trends of all regional death rates can be partly accounted for by the delayed deaths which are due to the significant improvement in life expectancy in Canada over the last few decades. But the major fluctuations in the

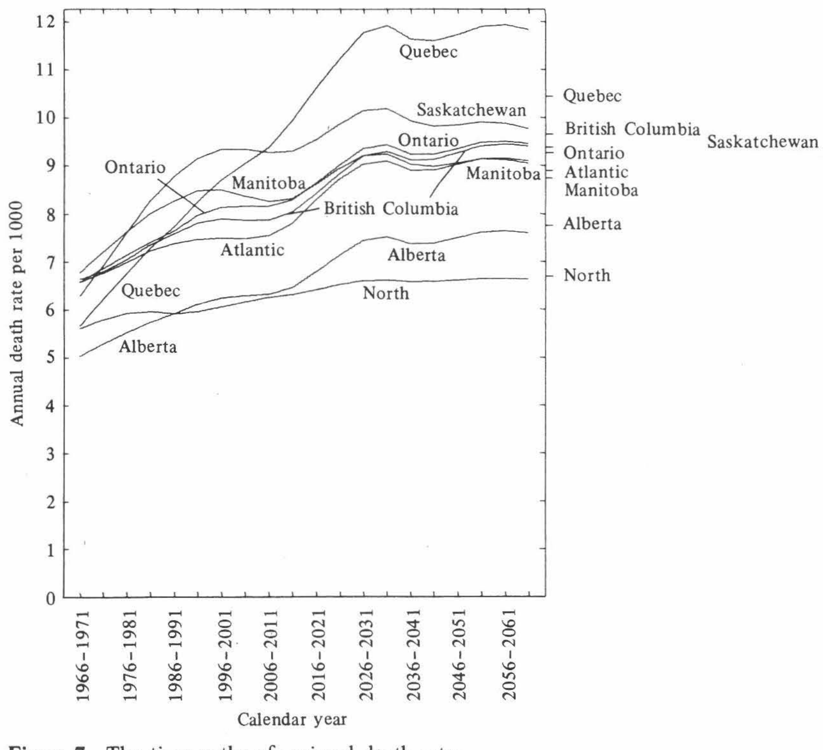

Figure 7. The time paths of regional death rates.

time paths of the regional death rates are caused by past changes in regional fertility levels. With a waiting time of about seventy years, the postwar baby boom raises the regional death rates to a peak in 2026–2031. Comparing figures 6 and 7 we see that the major fluctuations in the time paths of regional birth rates are repeated in the time paths of regional death rates after a fifty-year time lag.

9.3 Regional out-migration rates

Although all regional out-migration schedules have a high concentration in the young adult age groups (20–29), there are enough people in all age groups who migrate among regions in Canada to ensure that the time paths of regional out-migration rates are highly insensitive to changes in regional age profiles. Figure 8 shows the smoothness of the time paths of these rates. The passing effects of the postwar baby boom are barely visible for most regions. Even in Saskatchewan, where the out-migration schedule is most concentrated in the young adult age groups, the time path of the regional out-migration rate does not fluctuate much.

The relative insensitivity to changes in regional age profiles makes the regional long-run out-migration rates rather similar to the corresponding standardized out-migration rates. Thus the interpretation of the long-run regional out-migration rates as indices of emissiveness is generally acceptable.

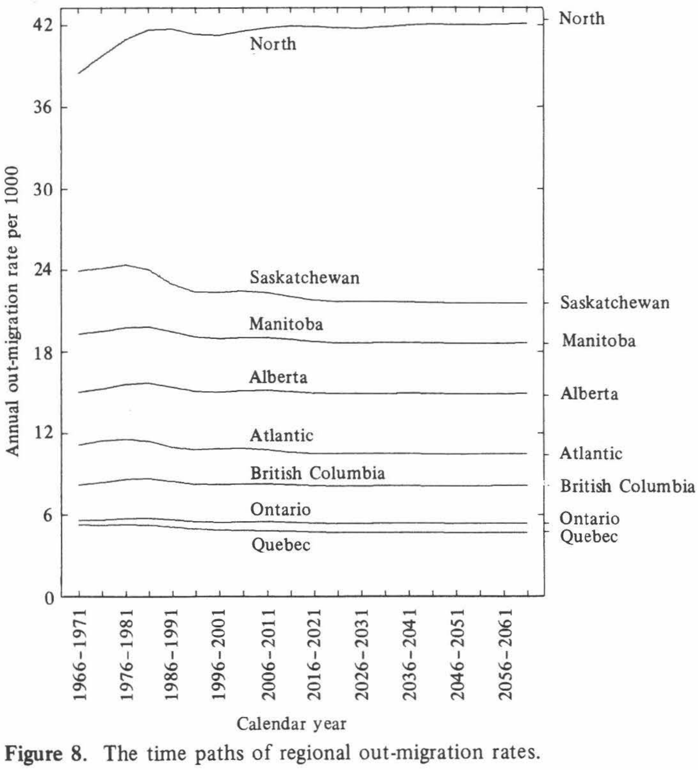

Figure 8. The time paths of regional out-migration rates.

9.4 Regional in-migration rates

Compared with a single-region-with-net-migration model and different versions of the gravity model, the Rogers interregional model shown in equation (3) is usually more useful in carrying out population projections or extracting interpretable long-run properties. Rogers has demonstrated how a single-region model with net migration can quickly exaggerate the importance of the migration component and lead to "serious errors into the process of projecting population" (Rogers, 1976, page 527). After extensive experiments, Ledent (1978) showed that the gravity model, under various specifications, has unfavorable dynamic properties (for example, the frequent occurrence of negative or zero regional populations, and an overdependence of long-run properties on the initial population distribution). In contrast the Rogers model avoids producing radical results by allowing in-migration rates of gaining (losing) regions to decrease (increase). The relative attractiveness or competitiveness of a region is gradually absorbed into the region's population share, and the regional in-migration or net migration rate eventually loses its function completely as a proxy of attractiveness or competitiveness.

The crisscrossed pattern of the time paths of regional in-migration rates in figure 9 shows that these rates are quite sensitive to changes in regional population shares. The fact that in the long run Quebec's in-migration rate (9·9 per 1000) becomes higher than that of British Columbia (9·3 per 1000) demonstrates that interpreting long-run regional in-migration rates as indices of attractiveness can be very misleading. Having absorbed the redistributive potential of the interregional differences in attractiveness, the long-run regional population shares can no longer be used in constructing a standard age-by-region population for computing regional in-migration rates as indices of attractiveness.

The smoothness of the time paths in figure 9 shows that regional in-migration rates are very insensitive to the transmission of major population waves through the regional age profiles. This is because migration transactions between regions are less concentrated into a few age groups than are the events of birth and death.

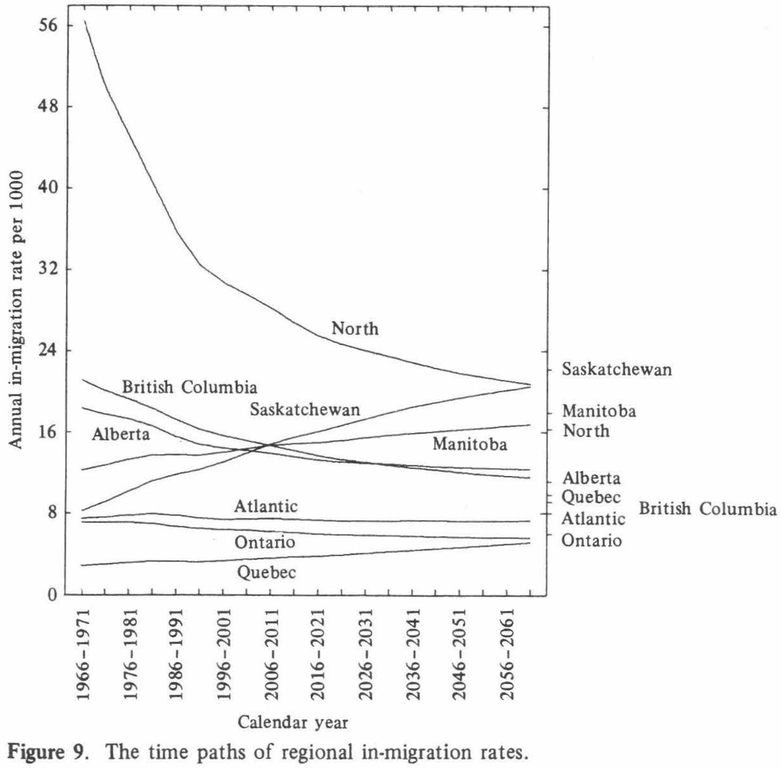

Figure 9. The time paths of regional in-migration rates.

It is interesting to note that, among the four kinds of regional component rates, only regional in-migration rates fail to achieve after a hundred years an interregional ranking which is close to the ranking of the corresponding long-run values. This reflects the previously described fact that spatial convergence takes more time to accomplish than does the convergence towards regional long-run age profiles.

9.5 *Regional net migration rates*

The near constancy of the time paths of regional out-migration rates dictates that the time paths of regional net migration rates be similar to those of regional in-migration rates—smooth but crisscrossed. In the long run a high regional net migration rate simply reflects the region's low natural growth rate and has nothing to do with the concept of competitiveness.

10 The evolution of regional growth rates

We see in figure 10 that in 1966–1971 the regional growth rates differ substantially in Canada. While the North was growing at an average annual rate of 48·2 per 1000, Saskatchewan was *declining* at the rate of 3·5 per 1000. The most populous region, Ontario, had a growth rate of 12·6 per 1000, which was somewhat higher than the national average of 11·4 per 1000.

The convergence of regional growth rates toward the nationwide long-run growth rate of 9·2 per 1000 proceeds quite rapidly in the first fifty years and then slows down. Except for Quebec the ranking of regional growth rates in 1966–1971 remains unchanged all the way to 2061–2066. The interregional contrast in growth rates is mainly determined by relative competitiveness; regions with large standardized net migration rates tend to have large growth rates. The fluctuations of all regional

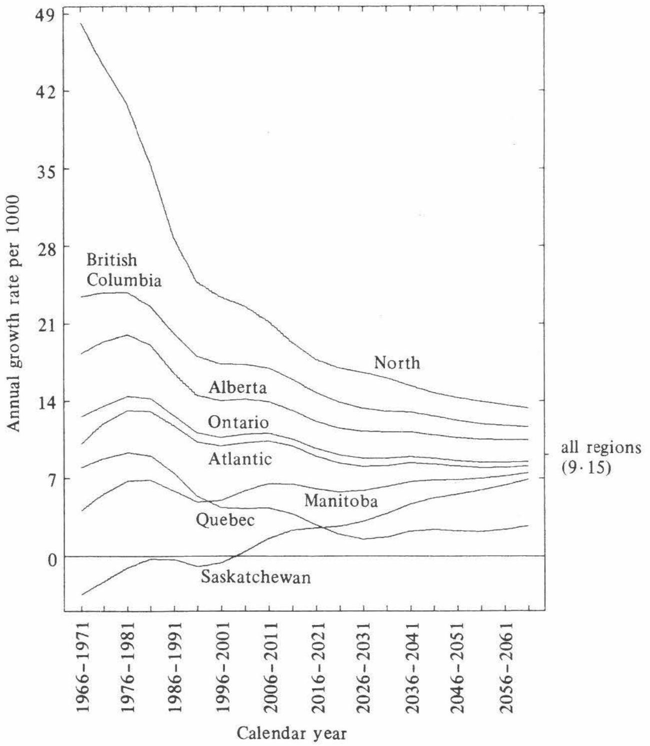

Figure 10. The time paths of regional growth rates.

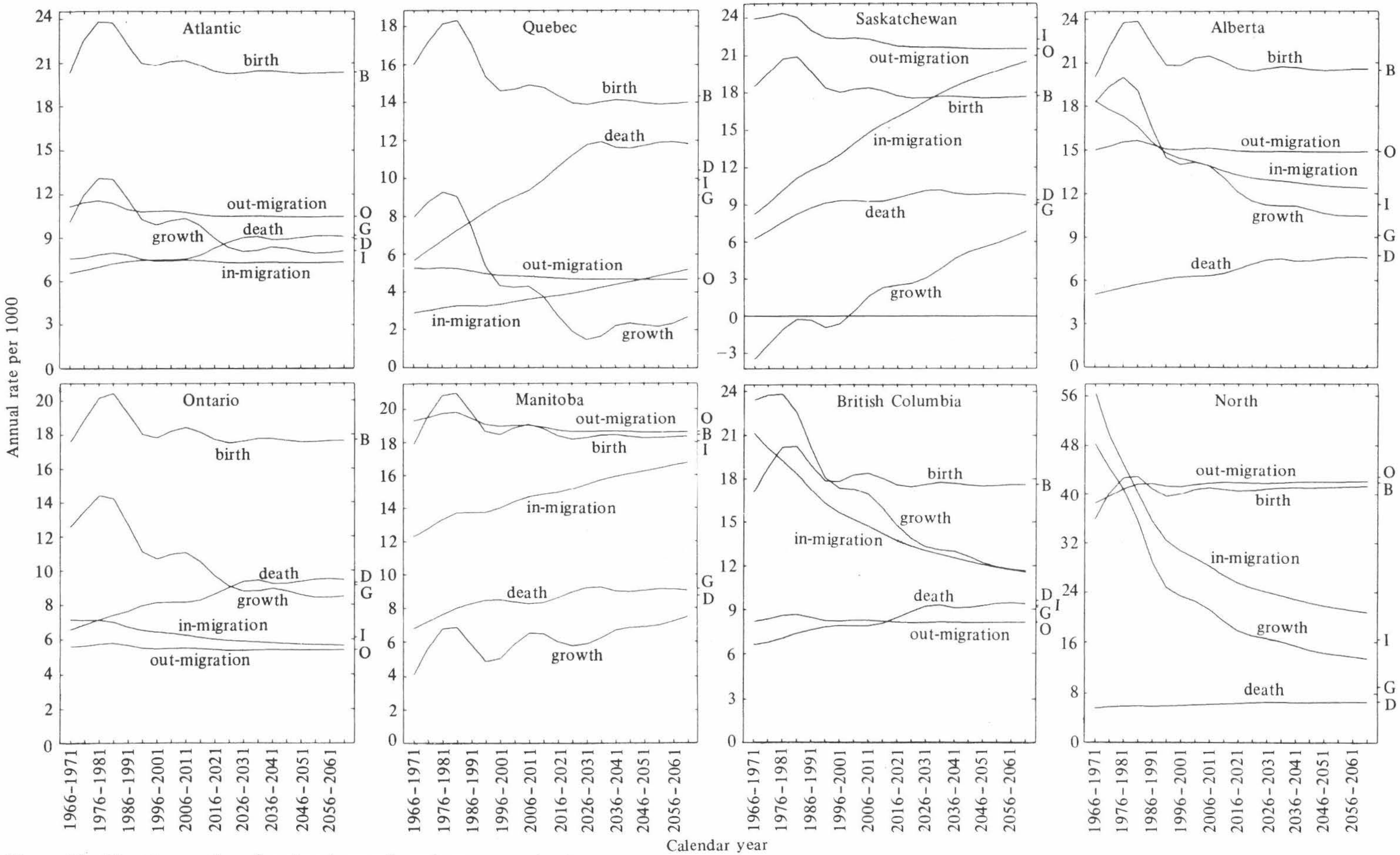

Figure 11. The time paths of regional growth and component rates.

growth rates are similar; they are dominated by the cyclical pattern of regional births in the first fifty years and by the fluctuations of regional death rates in the remaining fifty years.

Figure 11 makes it easier to see how the time path of each regional growth rate is determined jointly by the time paths of the corresponding component rates.

In the noncompetitive Atlantic region, the gap between in- and out-migration rates remains nearly constant at the level of −3 per 1000 over the hundred years. It is mainly the high birth rate of over 20 per 1000 that keeps the regional growth rate from falling below the level of 8 per 1000. The moderate decline in the regional growth rate is mainly due to an increase in the regional death rate.

In unattractive and noncompetitive Quebec, the initial gap of −2 per 1000 between in- and out-migration rates is *narrowed* continually and becomes positive in less than a hundred years. Thus the sharp *decline* in the regional growth rate from about 8 per 1000 to below 2 per 1000 is not due to the imbalance in migration transaction at all but rather to the combination of a sharp rise in the death rate and a significant drop in the birth rate. This joint influence disappears as the baby-boom cohort expires by the year 2031. The regional growth rate then starts the long march back to the long-run value of 9·2 per 1000.

In unattractive but quite competitive Ontario, the in-migration rate continues to be higher than the out-migration rate. But the gap is small (1·5 per 1000 in 1966–1971) and is shrinking gradually to less than 0·3 per 1000 in 2061–2066. Although the reduction in the gain in migration transaction is not totally negligible, the decline in Ontario's growth rate is mainly due to the increase in the region's death rate.

In highly emissive Manitoba, the initial gap of −7·0 per 1000 between in- and out-migration rates is continually reduced to −1·8 per 1000 in 2061–2066. This reduction is large enough to counter the opposite effect of a rising death rate, so that the regional growth rate shows an *upward* trend.

In Saskatchewan, the region with the lowest index of competitiveness in Canada, the gap between in- and out-migration is narrowed substantially from −15·7 per 1000 in 1966–1971 to −1·0 per 1000 in 2061–2066. This overwhelms the moderate increase in the death rate and causes the regional growth rate to increase sharply from −3·5 per 1000 in 1966–1971 to 6·9 per 1000 in 2061–2066. Since the age-specific out-migration rates directed into Saskatchewan from Alberta and British Columbia are relatively large, the cause of the rapid increase in Saskatchewan's growth rate can be traced further to the spillover effect of the increasing population shares in the two nearby regions.

In Alberta, a highly attractive region, the large positive net migration rate of 3·3 per 1000 in the initial period is reduced to less than zero in twenty-five years. By 2061–2066 the region's net migration rate reaches −2·5 per 1000. The significant decline in the net migration (in-migration) rate and a moderate rise in the death rate together cause a marked decline in the regional growth rate.

In British Columbia, another highly attractive region in Canada, the gain through migration transaction is substantially reduced from 13·0 per 1000 in 1966–1971 to 3·4 per 1000 in 2061–2066. Again the large decline in the in-migration rate and a moderate rise in the death rate lead to a sharp decline in the regional growth rate.

Finally, in the North, the region with the highest attractive and emissive indices in Canada, the decline in the in-migration rate is most drastic, whereas all other component rates show very small upward trends. The dramatic decline in the regional growth rate is almost completely determined by the drastic decline in the in-migration rate.

Part 3
11 Conclusion
On the one hand the changing age-by-region population distribution of the Canadian population system has been examined in terms of regional age profiles and regional shares. On the other each regional growth rate has been decomposed into regional component rates that are easier to comprehend.

Population waves are transmitted through regional age groups by the cyclical components of the system's analytic solution. The fluctuations of the projected regional time paths for the youngest age group can almost entirely be explained by the generation effect of the superimposed major cyclical component. The corresponding fluctuations for the tenth age group depend on the minor as well as the major cyclical components; but the importance of the minor cyclical components decreases rapidly so that in about fifty years the generation effect dominates the fluctuations.

The durability of a major cyclical component is negatively related to the standard deviation of the corresponding regional fertility schedule. The population wave transmitted from one region to another tends to be positively related to the corresponding interregional out-migration rate. For any age group, the population waves in different regions tend to have similar phases and periodicities—suggesting that past changes in fertility levels were similar among most regions, that the mean ages of most regional fertility schedules are quite similar, and perhaps that interregional migration tends to have a synchronizing effect.

Spatial redistribution is controlled by the spatial components of the analytic solution. The redistribution proceeds in a smooth and noncyclical fashion, although for several decades Ontario overshoots and Manitoba and Saskatchewan undershoot their respective long-run proportional shares. The fact that the major spatial components are associated with eigenvalues of relatively large magnitudes implies that the spatial convergence will proceed slowly. However, the fact that the regional time paths of the superimposed major spatial component are also of relatively large magnitudes indicates that this process must be very persistent as well. Over a hundred years the convergence toward the long-run regional age profiles is practically completed, while the spatial convergence has only gone halfway in terms of the dissimilarity index.

With respect to the behavioral characteristics of the regional component rates, the major findings are as follows. First, the transmission of a large population wave through regional age profiles has a great effect on the time paths of regional birth rates, a moderate effect on the time paths of regional death rates, and a small effect on the time paths of regional in- and out-migration rates. Second, major changes in the slopes of regional age profiles have a great effect on the time paths of regional death rates and a small effect on the time paths of regional birth and in- and out-migration rates. The differential effects are due to the different shapes of the schedules of fertility, mortality, and interregional out-migration. Third, large interregional differences in fertility levels make it highly risky to interpret current or long-run regional death rates as indices of regional mortality levels, because within each region there is a strong causal link going from the level of fertility through the slope of the age profile to the value of the death rate. Fourth, since the redistributional potential of interregional differences in attractiveness and competitiveness is gradually absorbed by changes in regional population shares, the regional in-migration and net migration rates become less meaningful through time. In the long run these regional rates cease to have anything to do with the concepts of attractiveness and competitiveness.

The persistent interregional contrast in growth rates in Canada is mainly determined by the relative competitiveness in the interregional migration transaction. As population waves pass through, the fluctuations of all regional growth rates are dominated by the cyclical pattern of regional birth rates in early stages and by the fluctuations of regional death rates in later stages. The regional growth rates of Quebec and Saskatchewan, both being regions with negative competitiveness, have radically different time paths. The precipitous decline in Quebec's growth rate is due to the 'double-crunch' of low fertility and high mortality levels, whereas the sharp rise in Saskatchewan's growth rate is due to the spillover effect of the increasing population shares in Alberta and British Columbia.

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Multistate demography and its data: a comment

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Abstract. Much of sociology and practically all of demography deal with transitions of people from one state at a certain moment to another state a year or more later. It is now clear that all such calculations of transition or movement are formally identical with what may be called the basic migration problem. The arithmetic for handling people moving from the single to the married state is identical with that for them moving from New York to Pennsylvania. The papers in this issue cover a wide range of substantive problems and exemplify several points concerning the methodology of demography. They provide a fair sample of multidimensional theory, of the means of application to data, and of the results of that application. They demonstrate the advantage of incorporating several demographic processes in a single model, even though they leave some questions unanswered.

1 Introduction

The traditional problems of demography include finding out how many years of working life are implied by the ages of entry and departure from the labor force that we see in a particular country for a particular year; how the future married population is affected by changes in the rates of marriage and divorce; what regional distribution of a population would result if present currents of interregional migration continued. Demographers have struggled with the calculations required, have attained partial solutions, have used arithmetic approximations that were often very crude. With greater or less patience each demographer somehow transformed raw data into diverse, more or less refined, measures of the entities of interest.

Now it turns out that they were in a certain sense all facing the same problem. Through the insights of Andrei Rogers, Robert Schoen, Kenneth Land, and others, it becomes clear that the various calculations of transition or movement are formally identical with what may be called the basic migration problem. The arithmetic for handling people moving from the single to the married state is identical with that for their moving from New York to Pennsylvania. Both cases present a set of transitions, and the question is what happens when they occur over successive periods of time and age.

2 Migration as the most general case

Migration is the one unrestricted case of the transitions with which demographers deal, in the sense that movement is usually possible among all the states recognized. All the other models are restricted in one way or another: a woman cannot go from parity two to parity one; a person can go from living to dead but not back; a never-married person can go to married but cannot subsequently revert to never-married. Whatever the groupings, age is precisely specified: n years from now one must be n years older, if one survives. Only in the migration case is the matrix specifying the transitions 'full', that is, has no elements that are necessarily zero according to the nature of the problem.

Where zero elements exist, those doing pencil-and-paper calculations have naturally taken advantage of the restrictions. Whelpton (1936) devised methods of population projection by age and sex by the use of survivorship and by applying fertility rates to the female population of reproductive ages, paying no attention to the age combinations between which transitions are not permitted. Whelpton's arithmetic was used by him and others for some decades, before Bernardelli (1941), Lewis (1942), and Leslie (1945) showed that it really amounted to premultiplying a vector by a matrix. Once the mass of numbers on a large worksheet are seen to be equivalent to matrix multiplication, a great freeing of the subject occurs. This was especially true with the advent of computers that could handle general forms of matrices. Demographers could be liberated from petty arithmetic and ad hoc approximations. The great power of matrix algebra, a field that had been under development from the time of Cayley and Sylvester in the mid-nineteenth century, was now at the disposal of demographers.

In another and quite separate development, Kolmogorov used matrices to express and solve a particular differential equation: $\mathrm{d}l(x)/\mathrm{d}y = -\mu(x)l(x)$. This is the equation that defines the main columns of the life table, with the difference that for Kolmogorov $\mu(x)$ and $l(x)$ are matrices. In terms of this differential equation, we can pull together the work of traditional demographers and mathematicians going back over a century, using mathematics that Willekens and Ledent have summarized in their papers in this issue. More than anyone else, Rogers (1975 and elsewhere), has worked towards the unification of population analysis that the equation implies. If he did not invent all the equipment *de novo*, he has at least spread the word of it. If an invention is the combining of previously existing elements in a new way to solve an old problem, then Rogers has been responsible for an invention of wide applicability. The elements that he brought together were implicit in the work of demographers, actuaries, and population biologists; his contribution is to economy of expression. He has made it possible for the contributors to this issue to deal more or less effectively with questions that would be impossible even to formulate by traditional methods.

3 How far from application should one go?

Admittedly not all of the problems that can be formulated are of substantive interest. Mathematics shows its great richness in many ways, of which one is its ability to state and solve problems that are many removes distant from the real world. How far it is useful to extend a piece of theory will always be a subject of contention. It was valuable to have Leslie's (1945) matrix formulation of the population projection; it was also valuable, given that formulation, to know that in the time-nonhomogeneous case a population forgets its past—two initially different age distributions converge to one and the same trajectory if they are subject to the same changes of mortality and fertility, a proposition familiar through the work of Coale (1972) and Lopez (1961). How useful is it to go further and prove this for an infinite number of age groups and for an infinite range of ages of childbearing, or for matrices that go through periodic cycles? And if all of these are of importance then there will be no difficulty in devising yet more remote problems and cleverly solving them. In some instances these more esoteric conditions do correspond to the real world, but it is up to the scholar who develops them to show this. If he takes the usefulness of anything that can be formulated for granted, he is no longer a demographer but a mathematician.

It is right and proper, and testifies to the liveliness of a discipline, that scholars work their way outwards from the clearly applicable to the not conceivably applicable. The question of applicability is not straightforward, since what is today an abstract theory is tomorrow a badly needed concrete method. We need a division of labor between method and application, and, once such a division of labor is set up and the methodologists come to be separated out, they would not be doing their part of the job if they did not at times go beyond present and probable future applications.

Our judgment on applicability can be aided by the use of numbers in a methodological paper. If we find only symbols and no numbers at all we must be suspicious; if the author did not show how to apply his method, then it is unlikely—though not impossible—that anyone else will take the trouble to apply it. If he does apply it, but to hypothetical numbers rather than to real data, then such an illustration helps explain what he is doing but falls short of proving its usefulness.

The set of papers in this issue covers a wide range of substantive problems, of methods for tackling them, and of closeness to and remoteness from the real world. They are varied enough to exemplify the several points I have been making, points concerning the methodology of demography.

4 The combining of sources

Of the four authors, Rees is by far the closest to data, working hard to use effectively independently obtained stock and flow statistics, following on the pioneer work of Stone (1966). He gives a good deal of attention to the amount of aggregation or consolidation, and by implication raises a basic question that is too rarely discussed, concerning the path from primary data to demographic conclusion. All demographic information ultimately concerns individuals: John Jones died aged fifty-six years, three months, and four days on 4 September 1978; Mary Jones gave birth to a baby boy on 19 February 1972; little Martha Jones, aged seven, was promoted from first to second grade in June 1979, etc, etc, in nearly infinite detail. Such detail has to be consolidated if the demographer is to cope with it. Sometimes the statistical agency shows too much detail, so that further consolidation has to be done by the demographer; sometimes the agency consolidates too much and loses important information, so that the statistics are only fit for incorporation into a very coarse model. A common example of the latter is the usual five-year age groups; 10-14, 15-19, ...; even if five-year age intervals suffice, these particular intervals have a built-in downward bias in the face of concentration of reporting on multiples of five.

When Rees assembles data from diverse sources he turns up inconsistencies. Estimating the true numbers when data are in conflict is a problem that needs all the resources of statistics. We must congratulate the author, who brings together information from diverse sources, and we expect him to make estimates more precise than any of the sources. Rees does not, however, take this problem as seriously as he might. He speaks of "massaging" the data and of using judgment in selecting the best set when there is conflict among sources. With the whole theory of statistical inference available, we should somehow be able to do better than this. Could one assign a prior trustworthiness to the material from each source, use this as a weight, and then find an estimate that minimizes the sum of the departures of the observations from the estimate? In a sense Rees approaches this when he tells us that he has tried each of four methods and then chosen the one that fits best, but such selection does not extract as much information as a suitable weighting would.

How to weight information that comes from several sources is a problem to which there is not, and probably cannot be, any general solution. The one case that is dealt with effectively by statisticians is where the several estimates are obtained by probability survey sampling or some other random process, in which case the several items are to be weighted by the reciprocal of their variances. Here a model of the data-generating process is available, and it is this model that makes optimal combination possible.

To be able to say as much for data subject to error, but error whose generating process is known only vaguely, is beyond the present state of the art. And yet the problem comes up as often as any in population analysis. If we know from a census the number of children under one year of age and the total population, and have even a rough estimate of infant mortality, then the birth rate is calculable. The same calculation using children who were five years of age at their last birthday gives another, and in general a different, estimate. This one refers to a different point of time, of course, but aside from that it will in general be higher than the estimate based on the population under one year of age because usually enumeration is more complete at five years than just after birth.

Given a census, and a willingness to make some assumptions (for instance on mortality), the number of estimates of the birth rate and the rate of natural increase is very large. How can one choose among them? More to the point, how can one obtain some measure of the accuracy of each, analogous to the reciprocal of the variance, by which they can be weighted? More attention could well be given to this problem, since multiple estimates are in the forefront of present preoccupations. Perhaps the problem is essentially unsolvable; perhaps there is no way of ascertaining the accuracy of the several estimates, that is, of finding the amount of information in R A Fisher's sense that each contains, and so weighting them for an optimum estimate.

Certainly the problem will remain unsolved until someone suggests a mechanism by which the several items of erroneous data are generated. Only such a mechanism will allow efficient averaging.

I repeat that it is an advantage of this collection that the several authors concentrate on different aspects of the multidimensional statistical-demographic problem. Rees spares us mathematics and mostly deals with consistency and reconciliation among data sources. These are important matters for the demographer who is working in our world of imperfect statistics, and one wishes he had pursued further the task he set himself.

5 The supply of data

Willekens, in his contribution to this issue, gives us the mature and elegant statement of the Kolmogorov-Rogers-Schoen-Land-Krishnamoorthy approach, as applied to working life. He himself, in other publications, has contributed no small amount to its development. It is a merit of the mathematics that he uses that he can dispense with the restrictive assumptions common to such work—for instance, that no one withdraws from the labor force prior to the age of maximum participation, that no one enters after this age, and that mortality is the same for those working and those not working at a given age. We all use assumptions that are contrary to fact, but these are grossly contrary to fact, especially remembering that women who leave the labor force to have a child and then return are becoming the majority. It is a real advantage of the new unified method that it makes these assumptions superfluous; the job can now be done just as easily without them.

It is true, as Willekens says, that the new and more flexible model increases the data requirements. Now that we have the mathematics for distinguishing the mortality of the married and of the single population, we find ourselves short of official tabulations that preserve that distinction. Does that make the effort vain? Not at all—one of the general principles that emerges from the history of our subject is that techniques somehow generate data. Official statistics gatherers and compilers become aware of the need for separating deaths according to marital status when they are badgered for such data by analysts, and they will not be badgered before the analysts have a convenient way of using such data. Here we have another aspect of the division of labor between official statisticians, methodologists, and analysts.

6 Movements and transitions

Willekens deals throughout with transitions, which is to say, he (or the agencies that supply the data) starts with a series of events, groups them into frequency distributions, and converts the distribution of events into probabilities of a change of state over the unit period, all as one does in making a life table out of deaths and exposed population.

One can then ask the question: what sort of events or movement could have given rise to the transitions? For marriage or birth data the transitions themselves are at second remove from the primary data, which consist of events or what Ledent, in his paper in this issue, calls movements. In case this is not clear, let me repeat the sequence through which the material passes: events or movements occur (for example, a birth); they are grouped into frequencies (so many births during a particular calendar year to women aged 25-29); the frequencies are then converted into probabilities (so much chance that a woman aged 25-29 will go from the second to the third parity during the next five years); from these probabilities of transition, the theory that we are here discussing takes off.

There is one common exception to this sequence, and that is data on migration, where as a rule the source is a question on where the person was living a year or five years ago, which directly provides a transition probability and eliminates the first three of the steps described for births.

If the events occur independently of one another, one can go from events to transitions, and equally from transitions to events. The relation between the two is elementary. If there are only two states and p is the probability of going from the first to the second, or equally from the second to the first, all in the unit time period, and μdt is the probability of the move occurring in the small time interval dt, then we can find p in terms of μ. For one move only we have the chance that no move occurs in the first t parts of the unit time interval, $\exp(-\mu t)$, multiplied by the chance that a move occurs in the time dt, which is μdt, multiplied by the chance that no move occurs in the remainder of the time interval, which is $\exp[-(1-t)\mu]$. The chance of the combination is $\exp(-\mu t)\mu dt \exp[-(1-t)\mu]$, and since the unique event occurring at one moment excludes the possibility that it occurs at some other moment, we find the chance of it occurring at any moment in the unit time interval to be the integral of the preceding product:

$$\int_0^1 \exp(-\mu t)\mu dt \exp[-(1-t)\mu] = \mu\exp(-\mu) \ . \tag{1}$$

Similarly the chance of exactly three moves occurring during the unit time interval is $\mu^3\exp(-\mu)/3!$, etc. The chance of a transition is equal to the chance of an odd number of moves, so we have

$$p = \mu\exp(-\mu) + \frac{\mu^3\exp(-\mu)}{3!} + \ldots \tag{2}$$

or

$$p = \frac{1-\exp(-2\mu)}{2} \ . \tag{3}$$

Solving this in the other direction shows that if the chance of a move at time dt is μdt then

$$\mu = -\tfrac{1}{2}\ln(1-2p) \ . \tag{4}$$

Some of this is generalizable to matrices, though the mathematics is not as simple.

The methods for relating moves to transitions are the main substance of Ledent's paper. His sophisticated treatment is not easy to follow, but this is due to some intrinsic difficulties of matrix analysis. In the multistate analogue the problems arise out of the fact that, if $\mathbf{A}$ and $\mathbf{B}$ are matrices, the exponential of the sum $\mathbf{A}+\mathbf{B}$ is not the product of the exponentials of the two matrices,

$$\exp(\mathbf{A}+\mathbf{B}) \neq \exp(\mathbf{A})\exp(\mathbf{B}) \ , \tag{5}$$

except for the uninteresting case in which the matrices $\mathbf{A}$ and $\mathbf{B}$ commute. If the exponent is an integral of the rates, as in the demographic case, no simple evaluation is possible.

These difficulties lead Ledent to a judgment at the core of his paper: "The conclusion here is that, ceteris paribus, the construction of an increment-decrement life table should be performed preferably from the transition approach rather than from the movement approach" (page 560). The ceteris paribus (other things being equal) furnishes protection to any such statement, but one can still object that the judgment is based on methodological convenience rather than on substance.

7 Heterogeneity

Ledent's discussion (like mine in section 6) supposes throughout the independence of moves from one another, the constancy of the probability of moving over a time period, and the homogeneity of the individuals in a given state at a given moment; in short, the Markovian condition. Yet those working on mobility have observed that the chance of moving at every moment is not the same for all individuals (Blumen et al, 1955; Singer and Spilerman, 1974); one way of accommodating this is to suppose that some people are movers and some are stayers. That raises questions that have drawn heated discussion for a generation, which unfortunately are overlooked in this issue. It is to be hoped that Ledent, with his superb mathematical training, will get around to issues of heterogeneity. If the Rogers approach can go on to analyze these, it will greatly expand its usefulness.

8 Time to convergence

Liaw's paper in this issue is an example of how the matrix formulation of a problem that was previously handled by arithmetic enables one to ask and answer questions that no one would have thought of earlier, or if raised would have been impossibly difficult to answer. To ask "What will happen if the 1979 rates (birth, death, marriage, migration, or other) continue?" is the demographer's way of interpreting what is happening in 1979. In a sense it is his microscope—it magnifies present tendencies and so makes them clearly visible. Liaw shows, for example, that on present trends Quebec drops from nearly 30% of Canada's population to less than 20%.

How long will it take an age-differentiated population evolving at fixed rates to stabilize after a disturbance is a question that Ansley Coale and I, as well as others, have dealt with (Coale, 1972; Keyfitz, 1977, pages 255-262; Trussel, 1977). Liaw apparently unaware of this literature, raises the question in relation to age and region. He finds that stabilization after an age disturbance is quick, arising from the fairly wide range of ages at which reproduction occurs. (The evolutionary advantage of this has been pointed out.) After a migration, disturbance convergence seems to be slower—it requires a hundred years for the amplitude to be reduced by half in the particular case he deals with, as against twenty years for a typical fertility pattern.

Liaw considers he has explained this when he says that the different speeds are "due to" the magnitudes of the eigenvalues. But this is really a restatement of the phenomenon itself. One would prefer a method of explanation in terms of non-mathematical entities. This is not the place to go into the nature of explanation, a subject that remains puzzling even after many papers that historians and philosophers have devoted to it. But our aspiration ought at least to be explanation in terms of some entity beyond the direct mathematical equivalent of what is being explained.

Aside from this, I have difficulty with what may be called sampling. How robust is the conclusion that age convergence is faster than region convergence? It depends after all on the circumstances of the particular time and place—in particular how far the initial distribution is from the stable condition implied by the rates of the initial moment of time. One can draw no general conclusion from this one case. That we have a poor sample here is suggested by the large migration out of Quebec. As a purely hypothetical example, suppose there is a category of persons that we might call 'irreconcilable English-speakers'. Many of these moved permanently out of the province in 1966–1971 and cannot continue to move out, and to project such a one-time movement forward is meaningless. It is the author's responsibility to examine this possibility in the course of choosing his model. One must not assume that a single five-year interval is representative of all such intervals. He would also help us by looking into the mathematics and finding conditions for convergence in terms of the pattern of transition probabilities, and, noting the difference between the nonzero elements of the age matrix and of the region matrix, explain why the latter produces quicker convergence than the former.

Apparently this has been studied in various contexts, and nearly decomposable or weakly connected matrices have been analyzed. If a matrix can be rearranged in blocks within which the connectivity is strong but between which the connections are relatively weak, then when it is taken to successively high powers the numbers within blocks will settle down quickly to constant ratios, and only some time after they have done so will the ratios of numbers in different blocks stabilize. The matrix is like a building with a good mixing of air within each room but little circulation between rooms; we can expect that after any disturbance the within-room variation will settle down to the stable form more quickly than the between-room variation.

9 Matrices unify and simplify demography

Just as one-dimensional man was declared to be obsolete—by Herbert Marcuse in a book now itself obsolete but of which this one phrase remains—so one-dimensional demography is now transcended by Andrei Rogers and his coworkers. The four main papers in this issue provide a fair sample of multidimensional theory, of the means of application to data, and of the results of that application. For many of us, including myself, multidimensionality is esoteric; it has taken me a good deal of hard work extending over many years to follow what has been going on at Berkeley, Northwestern, and now IIASA.

But we can be sure that what is strange and difficult to us will be natural and easy to our children. What is a narrow and almost closed sect within the profession today is going to be standard and obvious demographic technique in the next decade. A table of working life made the way Dublin et al (1949) made theirs, or a table of marriage similarly calculated, will be regarded as quaint; it will compare with the matrix method discussed in this issue as a cumbersome and inflexible desk calculator of the 1940s would compare with a hand-programmable calculator of the late 1970s.

To our children the matrix methods will seem inevitable because they enable demographers to fall back on a hundred years of development in mathematics—due to Sylvester, Frobenius, Kolmogorov and others—rather than to invent the mathematics independently as they go along. Does the ordinary population projection converge to stable form? We know from the Perron–Frobenius theory of matrices with nonnegative elements that it does—as long as there are two relatively prime ages of positive fertility. It would take a long time to find that out without matrix theory.

That we need to take account of more than one factor at a time is increasingly clear. Time in the married state is affected by mortality. Kingsley Davis points out that, because mortality has gone down, and despite the great increase in divorce, married couples stay together almost as long today as they did at the turn of the century. In short, couples seem able to endure one another only so long—if they do not die they get divorced. That can only appear in a multiple decrement table showing marriage and mortality simultaneously. Do marriages in which there are children hold together better than those without? To answer this requires a further decrement—parity.

And what is the effect on divorce of wives having jobs? A further set of states—working or not—has to be recognized to deal with this. Complicated? Not at all; the matrix theory is the same in four dimensions as in one, and even the methods of calculation are not much different.

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