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# Optimal language policy for the preservation of a minority language

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## HIGHLIGHTS

- A dynamic control model for the competition between two languages is developed.
- Family formation and intergenerational language transmission are modeled explicitly.
- Investments into status planning can be used to support the minority language.
- If bilingualism is valued high enough, survival of both languages can be optimal.
- Optimal steady states depend on initial distribution of speakers.

## ABSTRACT

We develop a dynamic language competition model with dynamic state intervention. Parents choose the language(s) to raise their children based on the communicational value of each language as well as on their emotional attachment to the languages at hand. Languages are thus conceptualized as tools for communication as well as carriers of cultural identity. The model includes a high and a low status language, and children can be brought up as monolinguals or bilinguals. Through investment into language policies, the status of the minority language can be increased. The aim of the intervention is to preserve the minority language in a bilingual subpopulation at low costs. We investigate the dynamic structure of the optimally controlled system as well as the optimal policy, identify stable equilibria and provide numerical case studies.

## 1. Introduction

In many of the states in this world, one can find two or more larger language groups, often in form of a majority language and one or several minority languages. This is by no means a static situation, since "[a]ll over the world, people are stopping speaking minority languages and shifting to languages of wider communication" (Sallabank, 2012, p. 104). This often results in the displacement of the minority languages by the majority language. To some extend such processes are inevitable and can be observed throughout human history. Nevertheless, in the modern world the decline of minority languages appears to occur much faster than ever before. It is predicted that 90% of the currently 7000 spoken languages will not survive the end of the century (Krauss, 1992).

*Language shift and maintenance*

In response to this accelerated process of (minority) language decline, revitalizing and maintaining (endangered) minority languages is on the agenda of many of their speakers. Moreover, governments, non-governmental organizations as well as international organizations such as the European Union "are actively working to save and stabilize endangered languages" (Fernando et al., 2010, p. 49). In scientific discourses a large variety of arguments to support (minority) language rights or to save endangered languages were put forward over the past decades. In this paper we

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will not assess such arguments in detail or develop new ones,[1] but rather investigate in a formal model setting the possibilities, effects and costs of language policies aiming at saving endangered languages. To do so, we first have to identify causes of language shift as well as measures that are available to reverse language shift. Here again, we will not go into all the details and mostly refer to the extensive literature on this topics, see e.g. Fishman (1991), Crystal (2000), Nettle and Romaine (2000), and May (2011). Furthermore, we have to specify the target function: what is the desired state of affairs that language policies should aim at?

Referring to Nettle and Romaine (2000) and Crystal (2000), Sallabank groups cause for language shift in four often overlapping main categories: (a) natural catastrophes, famine, disease, (b) war and genocide, (c) overt repression and (d) cultural/political/economic dominance, where the last one is the most common, cf. Sallabank (2012, pp.103f). Since we are interested in such cases, where individuals *voluntarily choose* to change to the majority language or not to pass the minority language to the next generation, we concentrate on the last category. Especially in nation states with one official/national language (which is often but not necessarily the language of the majority) this language is dominant in education, politics, media and public life. In modern democratic states the result is "that the majority culture [...] is endemic and omnipresent; and minority cultures, having very little, if any, public legitimization and private space, thereby constantly decline in survival potential, the more their members participate in the 'greater general good'" (Fishman, 1991, p. 63). Here, uneven power relations between the *national majority* and minorities play a major role. Minorities are often underrepresented in politics and in the public sphere and socially disadvantaged, cf. May (2011). This, in turn, can lead to negative attitudes towards the minority language, which are also internalized by its speakers (Sallabank, 2012, p. 104). When the two main aspects of language are considered – language as a tool for communication and language as a carrier of cultural identity – it is no surprise, that a language that cannot be used in the majority of societal domains and that is furthermore stigmatized to some degree will not be learned, spoken or passed to the next generation.[2]

A language shift is a process that is typically comprised of three phases. In a first phase, called diglossia, formal language domains are dominated by the majority language which implies a loss of official and public functions of the minority language. This *forces* the speakers of the minority language to use the dominant one. In a second phase more and more speakers of the minority language become bilingual, while both languages are still used, at least in some domains. Especially among the younger generation one can observe a decreasing number of speakers. This causes a further decline of domains where the minority language can be or is used. The third phase finally is the replacement of the minority language: "For a generation or two, some bilingual arrangements may be observed, but often [...] these prove to be way-stations on the road to a new monolingualism in the larger language" (Edwards, 2010, p. 6).

The language shift process can be counteracted by language policies aiming at the survival of the minority language. Language planning can be divided into three categories: status planning, corpus planning and acquisition planning. All three can have a positive impact on the chances of survival of minority languages. Through status planning, e.g. giving some official status to the minority language, the prestige of the language can be increased for its speakers as well as for the other members of the society. Corpus planning, which aims at standardizing the orthography and grammar of a language, can also increase its prestige and at the same time can reduce learning costs. Teaching the minority language at school, which belongs to the category of acquisition planning, enables students to learn the language properly/in the first place and can also have a positive impact on its status and identity value. In general, (re)introducing and/or strengthening the minority language in at least some domains can enhance the chances that it stays vital.

In this paper we concentrate on the role of the state in language revitalization processes. We presuppose that the state is basically interested in supporting the minority language by guarantying minority language rights.[3] At the same time, we assume that the state aims at ensuring social cohesion by enabling wide communication possibilities. The existence of two linguistically segregated language groups can threaten the solidarity between the society members and hence social cohesion. Even without referring to a necessity of a shared national identity for solidarity and cohesion one can at least say that "a shared language contributes to democracy" (Robichaud and Schutter, 2012, p. 135). Enabling wide communication possibilities while guarantying minority rights can be achieved through widespread bilingualism. If the minority language can be preserved in form of a relatively large number of bilingual individuals, the language minority is able to pass cultural values linked to the minority language to the next generations while communication possibilities throughout the society are assured. As outlined earlier, bilingualism is often a step towards the death of the minority language. Thus, preservation of a vital bilingual community requires a continuous effort by the state. In our model – and this is operationalized into the target function – that the state tries to maximize the number of bilingual speakers at minimal expenditures.

*Language competition models*

In the past two decades a wide variety of language competition models were developed. One important point of departure for this new research on language competition was the work by Abrams and Strogatz (2003). There, a simple language competition model with two monolingual subpopulations is developed. The fraction of speakers of each language evolves according to a differential equation, which takes into account the size of the subpopulations and the prestige of both languages. Although the authors can fit their model to aggregated empirical data of endangered languages, it shows some weaknesses. In Abrams and Strogatz (2003) neither bilingual speakers nor the social structure of the population are considered. Moreover, it is predicted that always one of the two competing languages will die out in the long run. Due to such limitations, the model was revised and extended by many authors, especially from the field of (statistical) physics. Patriarca and Leppänen (2004) and Patriarca and Heinsalu (2009) include spatial components in their adoptions of the AS model. Taking geographical inhomogeneities into account they were able to show that it is possible that both languages survive in two

[1] For an overview of the current discussions concerning language rights see e.g. May (2011) or Sallabank (2012). See also Fishman 1991 for a popular work on reversing language shift.

[2] "The communicative value of languages is largely determined by the number of speakers it gives access to and by the status or social positions of these speakers" (Robichaud and Schutter, 2012, p. 127).

[3] As mentioned above, there are many arguments supporting such policies:

"Indeed, the dynamics of ethnic tension involving language, leading in some cases to political conflict, occur most often *not* when language compromises are made or language right are recognized, but where they have been historically avoided, suppressed or ignored" (May, 2011, p. 161).

"So people's self-respect and dignity are often affected by the esteem their language gets from others or from the state. We might then justify different language policies by appealing to the importance of language recognition for individuals dignity" (Robichaud and Schutter, 2012, p. 136).

different geographical regions. Mira and Paredes (2005) introduce the concept of similarity between competing languages and prove that both languages can survive if they are close to each other. Stauffer et al. (2007) propose microscopic or individual based versions of the AS model and apply simulation techniques instead of averaging over the whole population. Mira and Paredes (2005), Minett and Wang (2008), Heinsalu et al. (2014) and others extend the A–S model by additionally considering bilinguals. Pinasco and Romanelli (2006) propose a Lotka–Volterra type model inspired by population dynamics to model language competition and also show the possibility of coexistence. Spatial extensions of this model can be found in Kandler and Steele (2008), Kandler et al. (2010). A good review of the different approaches is given in Patriarca et al. (2012).

In the model of Abrams and Strogatz (A–S model) speakers of two language *A* and *B* are assumed. Speakers of *A* can convert to speakers of language *B* and vice versa, while the population size remains constant. Minett and Wang point out that "in practice, [...] typically a speaker does not suddenly give up one language completely in favor of an other" (Minett and Wang, 2008, p. 23). Therefore, they include bilingual speakers in their adoption of the A–S model. Furthermore, Abrams and Strogatz implicitly consider language transmission from one generation to the other when fitting their mathematical model to empirical data from more than a hundred years without theorizing this fact. Minett and Wang therefore consider two modes of language transmission: (1) vertical, i.e. transmission from parents to their children and (2) horizontal, i.e. (adults) learning the second language and becoming bilingual. For the vertical mode, a uniparental model of transmission is applied. In contrast, Wickström (2005) only considers vertical transmission, but explicitly models family formation. It is assumed that adults mate due to a random search and matching process with a success probability that is smaller for couples with an *A*-monolingual and a *B*-monolingual partner than for all the other possible couples. In the so formed families offspring is produced and raised in one – or in some cases both – of the parents' languages, depending on the communicational value of each language and their status/prestige. As Wickström (2005) we only consider the vertical mode, i.e. intergenerational language transmission.[4]

In Wickström (2014) it is illustrated that the A–S model and its extension by Minett and Wang (2008) can be reformulated in terms of the general model presented in Wickström (2005). Furthermore the spatial model in Patriarca and Leppänen (2004) can be interpreted as a version of the Wickstŕom framework with two subpopulations I and II, which value language *A* differently. It is shown that under some general assumptions on the nexus between transition probabilities and the size of the subpopulations stable steady states of the system are the same as derived by Patriarca and Leppänen (2004) in spatial terms. For this paper we build on the general model formulation presented in Wickström (2005) and Wickström (2014). Hence we consider speakers of the majority language *A*, speakers of the minority language *B* and bilingual speakers *C*.

Only some of the language dynamics models outlined above deal with language revitalization policies. In terms of a mathematical model, such policies can be operationalized as a change of relevant model parameters that are related to the linguistic environment: "political, social and/or economic changes can lead to a change in the sociolinguistic environment and consequently to a change in the competition dynamics" (Kandler et al., 2010, p. 3859f). Yet, most often model parameters are assumed to be constant over time. To maintain a bilingual equilibrium Minett and Wang (2008) suggest a simple intervention strategy: whenever the amount of speakers of the minority language drops below some threshold value, then the status of the minority language or some other model parameters has to be increased. That such a "dramatic intervention" (Fernando et al., 2010, p. 51) is quite unrealistic, was already mentioned in Minett and Wang (2008). It can be seen as a theoretical approximation of a more sophisticated intervention, which starts to increase the minority language status when the numbers come close the threshold.

A greater effort to model language planning was undertaken in Fernando et al. (2010). They consider intergenerational language transmission as well as horizontal transmission. In contrast to Wickström (2005) parents do not just choose one or two languages to raise their children in. Instead, the probability that a child speaks a language *L* strongly depends on the amount of *L*-conversations it is exposed to. Within the family this amount only depends on the linguistic repertoires of the parents. Furthermore, Fernando et al. consider the influence of the community by taking into account conversations heard in the public sphere and languages taught at school. This is also reflected in three different kinds of interventions contemplated there: (1) increasing the status of the minority language,[5] (2) increase the amount of the minority language heard in public and (3) formal language teaching. In their simulations Fernando et al. illustrate the effect of different kinds of governmental interventions.

After 100 years simultaneously the status of the minority language as well as the amount of that language used in public is increased and the minority language is taught in formal education to some monolinguals of the high-status language. In the model this is realized by increasing three corresponding parameters at year 100. Citing Fernando et al. (2010, p. 51) when reviewing Minett and Wang (2008) one may ask: "How such a dramatic intervention could be achieved is not explained".

In Kandler et al. (2010) the authors fit their basic model with time-independent parameters ("shift coefficients") to data on language competition between Welsh and English in Wales. For the period from 1901 to 1971 the model captures the observed dynamics quite well. Yet, the basic model could not adequately account for maintenance interventions implemented in the past 40 years, which could be the cause of reduced decline of Welsh. Therefore, the authors extend their basic model "by incorporating a simplified concept of (extended) diglossia" (p. 3862). The high-status language is used in important domains as higher education or non-local businesses. This yields an incentive for speakers of the minority language to become bilingual. At the same time, political interventions might support the low status language in other domains such as local legislation. This, can create incentives for monolinguals of the dominant language to become bilingual and for bilingual parents to transmit both languages to the next generation. Kandler et al. introduce an additional term in their model that captures the demand of participation in domains where the low status language is used. This demand is reflected by the parameter $w_1$. Assuming that $w_1$ doubles after 1971, the extended model is able to approximate the empirical data. The increase of $w_1$ is a result of language planning incentives.

[4] Transmission in the family is the gold standard of language vitality and the most important factor in language survival (Fishman, 1991, p. 113).

[5] Unlike most of the models listed above, there is no explicit status parameter in Fernando et al. (2010). The status of the minority language is reflected by the parameter that "measures the effectiveness of hearing language [the minority language] in motivating its learning (i.e. the receptiveness of the child to [the minority language])" in an *AA* or *AC* family (p. 60). This parameter is not to be understood as an individual trait of the child. Among other things, it represents "the "status" of [the minority language], where status is used to mean the entire constellation of societal factors that motivate the learning of a given language" (p. 60, emphasis in original). This status related parameter functions as an amplifier for *B*-conversation heard by a child.

In the above three examples, language planning policies are modeled as a change in model parameters. These changes occur at some single point in time, i.e. at some point in time the value a parameter (or multiple parameters) jumps to another value. Depending on the parameter that is changed as well as on the size of the jump, such a "dramatic intervention" might be rather unrealistic. In their adoption of the model proposed in Minett and Wang (2008), Bernard and Martin (2012) also include the opportunity for policy makers to alter the status of the minority language. In contrast to the previous approaches, they assume that the variation of the status at each time step is bounded. Hence, the size of the jump is limited, which yields a potentially more realistic model for intervention. Setting up a dynamic control model, they were able to show that when starting in a given domain there exist adequate intervention strategies such that both monolingual subpopulations can be preserved.

In this paper we also propose a language competition model with dynamic intervention. A first difference to the model analyzed in Bernard and Martin (2012) is that we build on the general model formulation presented in Wickström (2005). Secondly, in our approach the status cannot be regulated directly. Instead, we assume that the state has a certain budget that can be used for status planning. To increase or even stabilize the status of a (minority) language continuous investments into status planning are necessary. Hence, we assume that whenever the state reduces its efforts to maintain the minority language beyond a certain value, then the status of that language decreases. This implies that without any intervention the status tends to zero in the long run. The investment strategy is denoted by a process $(s_t)_{t\geq 0}$. Since the budget is assumed to be finite, we can normalize the investment such that $s_t \in [0, 1]$. Thirdly, we propose an optimal control model. The aim of languages policies is not to maintain monolingual subpopulations of both languages, but to maintain both languages in a scenario with large communication possibilities throughout the society. Hence, the aim is to maximize the amount of bilingual speakers. Furthermore, investments into status planning are costly. Therefore, the objective here is to maximize the bilingual subpopulation at minimal costs.

The dynamic control model proposed below is a three-state system. The three states are: the fraction of speakers of language $A$ (denoted by $p_A$), the fraction of speakers of language $B$ (denoted by $p_B$) and the relative status of language $B$ (denoted by $S$). The fraction of bilingual speakers is simply given by $p_C = 1 - p_A - p_B$, and the relative status of the majority language $A$ is given by $1 - S$. In Fernando et al. (2010) the authors criticize such an assumption in the model of Minett and Wang because it implies "that it is impossible to make one language more attractive without making the other less so" (Minett and Wang, 2008, p. 50). However, in a language competition situation, where individuals have to decide for one language, the other or both, this assumption makes sense when we think of relative attractiveness instead of absolute attractiveness. Hence, instead of statements as 'language $A$ has an attractiveness value of 3.5' the model here only allows statements like 'language $A$ is three times as attractive as language $B$'.

The evolution of the system is described by three differential equations. The status can be affected by state intervention $s$, i.e. $\dot{S} = g(s, S)$, where $g$ is some function increasing in $s$. The evolution of the distribution of speakers depends only on the distribution itself and on the status $S$. Hence, the fractions $p_L$, $L = A, B, C$, can be influenced by state intervention, but only indirectly through the controlled status.

## 2. Model

We consider a (large) population consisting of individuals equipped with one of three different language repertoires $L$: monolingual speakers of the dominant language $A$, monolingual speakers of the minority language $B$ and bilinguals speakers $C$. The relative sizes (fractions of the population) of the respective language repertoire groups are denoted by $p_A$, $p_B$ and $p_C$. The fractions add up to 1, hence $p_C = 1 - p_A - p_B$. The variable $S$ represents the relative status of the minority language $B$ in the society.

**Table 1**
Distribution of families for a given distribution of adult speakers.

| $F$ | $\phi_F$ |
|---|---|
| $AA$ | $p_A^2 + p_A p_B$ |
| $AB$ | 0 |
| $AC$ | $2p_A p_C$ |
| $BB$ | $p_B^2 + p_A p_B$ |
| $BC$ | $2p_B p_C$ |
| $CC$ | $p_C^2$ |

### 2.1. Family formation

In every generation individuals form families. There are six family types $F$: $AA$ (two $A$ monolinguals), $AB$, $AC$, $BB$, $BC$ and $CC$. Family formation is assumed to be random but restricted by the condition that both adults should share a common language, i.e. they should be able to communicate with each other. Hence, couples with an $A$-monoglot and a $B$-monoglot are excluded. Given any distribution of speakers $p_A$, $p_B$, $p_C$, the expected distribution of family types is given in Table 1, where $\phi_F$ denotes the fraction of $F$-type families.[6]

### 2.2. Family behavior

Families bring up their children either as monolinguals in $A$ or $B$, or as bilinguals. The fraction of $F$-type families bringing up children with language repertoire $L$ is denoted by $\alpha_L(F; \cdot) \in [0, 1]$. Naturally, the $\alpha$'s add up to one: for every family type $F$

$$\sum_L \alpha_L(F; \cdot) = 1.$$

The $\alpha$-functions are one of the main ingredients of the model proposed here. Parents choose a language repertoire depending on their own languages, on their emotional attachment to those languages as well as on the communication values of all the languages at hand. Therefore, the fraction of families of type $F$ raising their children as $L$'s varies with the current distribution of speakers in the society as well as with the statuses of languages $A$ and $B$. Hence, $\alpha_L(F; \cdot) = \alpha_L(F; p_A, p_B, S)$. The dependence on the variables $p_L$ captures the practical advantage of belonging to a certain language group, since they measure the frequency with which an individual encounters another individual in group $A$, $B$ and $C$, respectively, and hence measure how many people one can communicate with. Following the individual utility maximization approach developed in Wickström (2005), we assume that $\alpha_A$ is non-decreasing in $p_A$ and $p_C$, and non-increasing in $p_B$, and vice versa for $\alpha_B$:

$$\frac{\partial \alpha_A(F; p_A, p_B, S)}{\partial p_A}, \ \frac{\partial \alpha_B(F; p_A, p_B, S)}{\partial p_B} \geq 0$$
$$\frac{\partial \alpha_C(F; p_A, p_B, S)}{\partial p_A}, \ \frac{\partial \alpha_C(F; p_A, p_B, S)}{\partial p_B} \geq 0$$
$$\frac{\partial \alpha_A(F; p_A, p_B, S)}{\partial p_B}, \ \frac{\partial \alpha_B(F; p_A, p_B, S)}{\partial p_A} \leq 0.$$

[6] See Appendix for a more detailed derivation of the expected distribution $\phi_F$.

This reflects the first aspect of language mentioned in the introduction: language as a tool for communication. The second aspect – language as a carrier for cultural identity – is reflected in the dependence of the $\alpha$'s on the family type $F$ and the relative status of the minority language $S$. It is hypothesized that the emotional attachment in the family to a certain language, and hence the frequency of its transmission to the next generation, depends on its strength in the family. The stronger the position of a language $L$ in the family, the higher is the fraction $\alpha_L$:

$$1 \geq \alpha_A(AA;\cdot) \geq \alpha_A(AC;\cdot) \geq \alpha_A(CC;\cdot) \geq \alpha_A(BC;\cdot) \geq \alpha_A(BB;\cdot) \geq 0$$

$$0 \leq \alpha_B(AA;\cdot) \leq \alpha_B(AC;\cdot) \leq \alpha_B(CC;\cdot) \leq \alpha_B(BC;\cdot) \leq \alpha_B(BB;\cdot) \leq 1.$$

It is furthermore assumed that both parents shall be able to communicate with their children, cf. Fernando et al. (2010). Hence,

$$\alpha_A(BC;\cdot) = \alpha_A(BB;\cdot) = 0$$
$$\alpha_B(AC;\cdot) = \alpha_B(AA;\cdot) = 0.$$

The average emotional attachment to a language $L$ also depends on the general prestige or cultural status of the language in the society. The higher the status, the higher is the willingness of its speakers to pass their language to the next generation. We therefore assume that $\alpha_A$ is non-increasing in $S$, while $\alpha_B$ is non-decreasing in $S$:

$$\frac{\partial \alpha_A(F;p_A,p_B,S)}{\partial S} \leq 0$$
$$\frac{\partial \alpha_B(F;p_A,p_B,S)}{\partial S} \geq 0.$$

From the assumptions made above two properties of the $\alpha$ functions can be concluded. Since $\alpha_B(AA)$ $\alpha_A(BB)$ are equal to zero, we get

$$\frac{\partial \alpha_A(AA;p_A,p_B,S)}{\partial p_A} = \frac{\partial \alpha_B(BB;p_A,p_B,S)}{\partial p_B} = 0.$$

Furthermore, $\alpha_B(AC;\cdot) = \alpha_A(BC;\cdot) = 0$ yield

$$\frac{\partial \alpha_A(AC;p_A,p_B,S)}{\partial p_A} = \frac{\partial \alpha_B(BC;p_A,p_B,S)}{\partial p_B} = 0.$$

### 2.3. Dynamics

While in Abrams and Strogatz (2003) a constant population size is assumed, other researches explicitly model *logistic* population growth, see e.g. Pinasco and Romanelli (2006) or Kandler et al. (2010). If growth rates and carrying capacities vary between the language repertoire groups, then the population dynamics can have a major impact on possible steady states. Yet, if growth is homogeneous throughout all the groups and a common carrying capacity is assumed, then population dynamics do not affect the steady states, cf. Heinsalu et al. (2014). In this paper we also assume homogeneous growth at rate $\theta$ and a common carrying capacity $K$. Since the number of children born in a family and thus the overall population dynamics are independent of the status,[7] considering status planning does not violate the homogeneity assumption. Therefore, the model proposed here "could describe the interaction between linguistic groups that have already reached a state in which reproduction and access to resources takes place in similar ways" (Heinsalu et al., 2014, p. 5), and cannot account for situations in which one language repertoire group has much less access to resources than the other language repertoire groups.

[7] The relative status $S$ only influences parents decisions on the language repertoires of their children.

Let $N$ denote the size of the population, and $N_L$, $L = A, B, C$, denote the sizes of the language repertoire groups. The dynamics of the overall population size is described by the logistic differential equation

$$\dot{N} = \theta N\left(1 - \frac{N}{K}\right) = \theta N\left(\sum_L\left[\sum_F \alpha_L(F;\cdot)\phi_F - \frac{1}{K}N_L\right]\right).$$

The size of language repertoire group $L$ changes according to

$$\dot{N}_L = \theta N \sum_F \alpha_L(F;\cdot)\phi_F - \theta\frac{N}{K}N_L.$$

Therefore, the relative size of language repertoire group $L$, $p_L = N_L/N$, evolves according to

$$\dot{p}_L = \theta\left(\sum_F \alpha_L(F;\cdot)\phi_F - p_L\right).$$

For languages $A$ and $B$ this reads as

$$\frac{\dot{p}_A}{\theta} = (p_A^2 + p_A p_B)\alpha_A(AA) + 2p_A p_C \alpha_A(AC) + p_C^2\alpha_A(CC) - p_A \tag{2.1}$$

$$\frac{\dot{p}_B}{\theta} = (p_B^2 + p_A p_B)\alpha_B(BB) + 2p_B p_C \alpha_B(BC) + p_C^2\alpha_B(CC) - p_B, \tag{2.2}$$

where $\alpha_L(F) = \alpha_L(F;p_A,p_B,S)$.

#### 2.3.1. The status variable

The status of the minority language $B$ is expressed in the variable $S$, $0 \leq S \leq 1$. Investments in status planning $s$ can increase the status of the minority language:

$$\dot{S} = f(S,s) - \mu S. \tag{2.3}$$

It is assumed that the function $f$ is non-increasing in $S$ and non-decreasing in $s$. Furthermore, for $s = 0$ the function $f$ should be zero. This implies, that without any state intervention the relative status of the minority language $B$ converges to zero at rate $\mu$.

### 2.4. The objective function

The aim of state intervention is a large bilingual subpopulation. At the same time, state interventions to increase the status of the minority language are costly. Hence, the decision maker is looking for an investment policy $(s(t))_{t\geq 0}$, $s_t \in [0,1]$, that yields a high level of individual bilingualism (benefit) at low costs. By $w(p_A(t), p_B(t), s(t))$ we denote the value of the system at time $t$, i.e. benefits minus costs at time $t$. We require $w$ to be increasing in $p_C = 1 - p_A - p_B$, non-increasing in $p_A$ and $p_B$, and decreasing in $s$. The total discounted value is given by

$$\int_0^\infty e^{-rt} w(p_A(t), p_B(t), s(t))dt,$$

where $r \in (0,1)$ denotes the discount rate. The problem of finding the best investment strategy for language maintenance can now be formulated as a maximization problem:

$$\max_{(s_t)_{t\geq 0}} \int_0^\infty e^{-rt} w(p_A(t), p_B(t), s(t))dt.$$

Note, that $S(t)$ and therefore $p_A(t)$ and $p_B(t)$ depend on the size of $s$ prior to time $t$, cf. (2.3), (2.1) and (2.2).

## 3. Specific functional forms

In this section we provide specifications of the $\alpha$-functions, the status dynamics and the objective function that satisfy the general assumptions made above.

For parameters $0 \leq \eta < \beta < \delta$ and $\varepsilon + \gamma < \zeta < 1$ let

$$\alpha_A(AA; p_A, p_B, S) = 1 - \eta S p_B$$
$$\alpha_A(AC; p_A, p_B, S) = \max\{0, \zeta(1-S) - \beta S p_B\}$$
$$\alpha_A(CC; p_A, p_B, S) = \max\{0, \varepsilon(1-S) + \gamma(1-S)p_A - \delta S p_B\}$$

and

$$\alpha_B(BB; p_A, p_B, S) = 1 - \eta(1-S)p_A$$
$$\alpha_B(BC; p_A, p_B, S) = \max\{0, \zeta S - \beta(1-S)p_A\}$$
$$\alpha_B(CC; p_A, p_B, S) = \max\{0, \varepsilon S + \gamma S p_B - \delta(1-S)p_A\}.$$

These constructions imply, that given a sufficiently high fraction of $A$ speakers in the society and a sufficiently low status of the minority language $B$, bilingual or even mixed couples ($BC$) will not raise their children as monolinguals in $B$, since in this scenario neither $B$ is a very useful communication tool in this society nor can the prestige of this language really compensate the communication disadvantage.

Throughout the paper we will assume $\eta$ to be zero. In this case the system dynamics simplify to

$$\frac{\dot{p}_A}{\theta} = p_C\left[2p_A\alpha_A(AC; p_A, p_B, S) + p_C\alpha_A(CC; p_A, p_B, S) - p_A\right] \quad (3.4)$$
$$\frac{\dot{p}_B}{\theta} = p_C\left[2p_B\alpha_B(BC; p_A, p_B, S) + p_C\alpha_B(CC; p_A, p_B, S) - p_B\right]. \quad (3.5)$$

### 3.1. Dynamics for fixed status

For the moment let $S$ be fixed. The essential dynamics of $p_A$ and $p_B$ can each be described by two parameters, cf. Wickström (2005). These parameters are introduced in the following. Let $p_B^{\Delta}(S)$ denote the fraction of $B$ speakers where $p_A = 0$ and $\dot{p}_A = 0$. Hence,

$$\alpha_A(CC; p_A, p_B, S) = 0 \Rightarrow \varepsilon(1-S) - \delta S p_B = 0$$
$$\Leftrightarrow p_B^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{1-S}{S}.$$

For $p_A^{\Delta}$ respectively we get

$$p_A^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{S}{1-S}.$$

Next we look for $p_A^*$ and $p_B^*$. $p_A^*$ is the fraction when $\dot{p}_A = 0$ given $p_B = 0$. Hence, $p_A^*$ is a solution to

$$0 = 2p_A\alpha_A(AC; p_A, p_B, S) + (1-p_A)\alpha_A(CC; p_A, p_B, S) - p_A,$$

or, with the above specifications, $p_A^*$ is the unique positive solution to the quadratic equation

$$0 = \gamma p_A^2 - \left[2\zeta + \gamma - \varepsilon - \frac{1}{1-S}\right]p_A - \varepsilon. \quad (3.6)$$

Note, $p_A^* < 1$ iff $S > 1/2\zeta$. From this, we easily conclude that $p_A^*$ is increasing in $\zeta$, $\varepsilon$ and $\gamma$, and decreases with an increase of $S$. On the other hand, $p_A^{\Delta}$ increases in $\varepsilon$ and $S$ and decreases with an increase in $\gamma$. It is unaffected by a change of $\zeta$.

From the relations between $p_A^{\Delta}$, $p_B^{\Delta}$ and $p_A^*$, $p_B^*$ we can identify possible bilingual equilibria for the fixed status $S$:

**Lemma 3.1.** *Let $\eta = 0$.*

(a) *If $p_A^{\Delta} \leq p_A^* < 1$ there exists a stable equilibrium with $0 < p_A < 1$ and $p_B = 0$; the fraction of A-speakers equals $p_A^*$*
(b) *If $p_B^{\Delta} \leq p_B^* < 1$ there exists a stable equilibrium with $0 < p_B < 1$ and $p_A = 0$; the fraction of B-speakers equals $p_B^*$*
(c) *If $1 \geq p_A^{\Delta} > p_A^*$ and $1 \geq p_B^{\Delta} > p_B^*$, we have a stable equilibrium with bilinguals and monolinguals in both languages ($p_A, p_B, p_C > 0$).*

**Table 2**
Possible stable equilibria for the fixed status problem for different values of $S$. The first line contains intervals for $S$, while the second one shows the corresponding potential stable equilibria. "A, AB" means that a pure $A$-monolingual steady state as well as a steady state with monolingual speakers of $A$ and $B$ is possible.

| $S \in$ | $[0, \underline{S}]$ | $(\underline{S}, \tilde{S} \wedge 1-\tilde{S})$ | $(\underline{S} \vee 1-\tilde{S}, \tilde{S}]$ | $(\underline{S} \vee \tilde{S}, 1-\tilde{S})$ |
|---|---|---|---|---|
| Steady state | *A*, *AB* | *AC* | *AC*, *BC* | *ABC* |

**Lemma 3.2.** *Let $\eta = 0$. For monolingual stable equilibria the following statements hold true*
(a) *$p_A = 1$ is a stable equilibrium if and only if $S \leq 1 - 1/2\zeta$.*
(b) *$p_B = 1$ is no stable equilibrium*
(c) *$p_A, p_B \in (0, 1)$ with $p_A + p_B = 1$ is stable iff*

$$p_A\alpha_A(AC; p_A, p_B, S) + p_B\alpha_B(BC; p_A, p_B, S) \geq \frac{1}{2}. \quad (3.7)$$

*A necessary condition for this last inequality is $S \leq 1 - 1/2\zeta$.*

Lemma 3.1 can be established using a phase diagram, cf. Wickström (2005). The proof of Lemma 3.2 is found in the Appendix.

### 3.2. Variable status and status control

Now we specify the dynamics of the minority language status $S$, which is increasing as a result of investments into language policies and decreasing due to a general negative trend. We assume the following functional form:

$$\dot{S} = f(S, s) - \mu S = \nu(1-2S)\sqrt{s} - \mu S, \quad (3.8)$$

where $\nu > 0$ is a model parameter correlated to the effectiveness of intervention. Here two assumptions are made: (a) for a low status language the necessary effort to increase its status is low, while for a high status language it takes more effort. (b) Language $B$ stays the minority language. This assumption is expressed in the term $(1-2S)$. The status cannot exceed $1/2$, while the $(1-S)$, which can be interpreted as the status of $A$, does not fall below $1/2$. $A$ can be thought as the first official language.

The control variable $s$ is bounded ($s \leq 1$). Thus, any steady state status $S$ ($\dot{S} = 0$) has an upper bound:

$$S \leq \frac{\nu}{2\nu + \mu}.$$

Since $p_A^*$ is decreasing in $S$, while $p_A^{\Delta}$ increases in $S$, Lemma 3.1(a) yields a second upper bound for $S$, which is relevant for equilibria with $0 < p_A < 1$ and $p_B = 0$. A third one results from Lemma 3.1(b), see below. A minimal value for this kind of equilibrium is given by $p_A^*(S) < 1$, where $p_A^*$ is the unique positive solution to (3.6).

We therefore introduce the following status thresholds

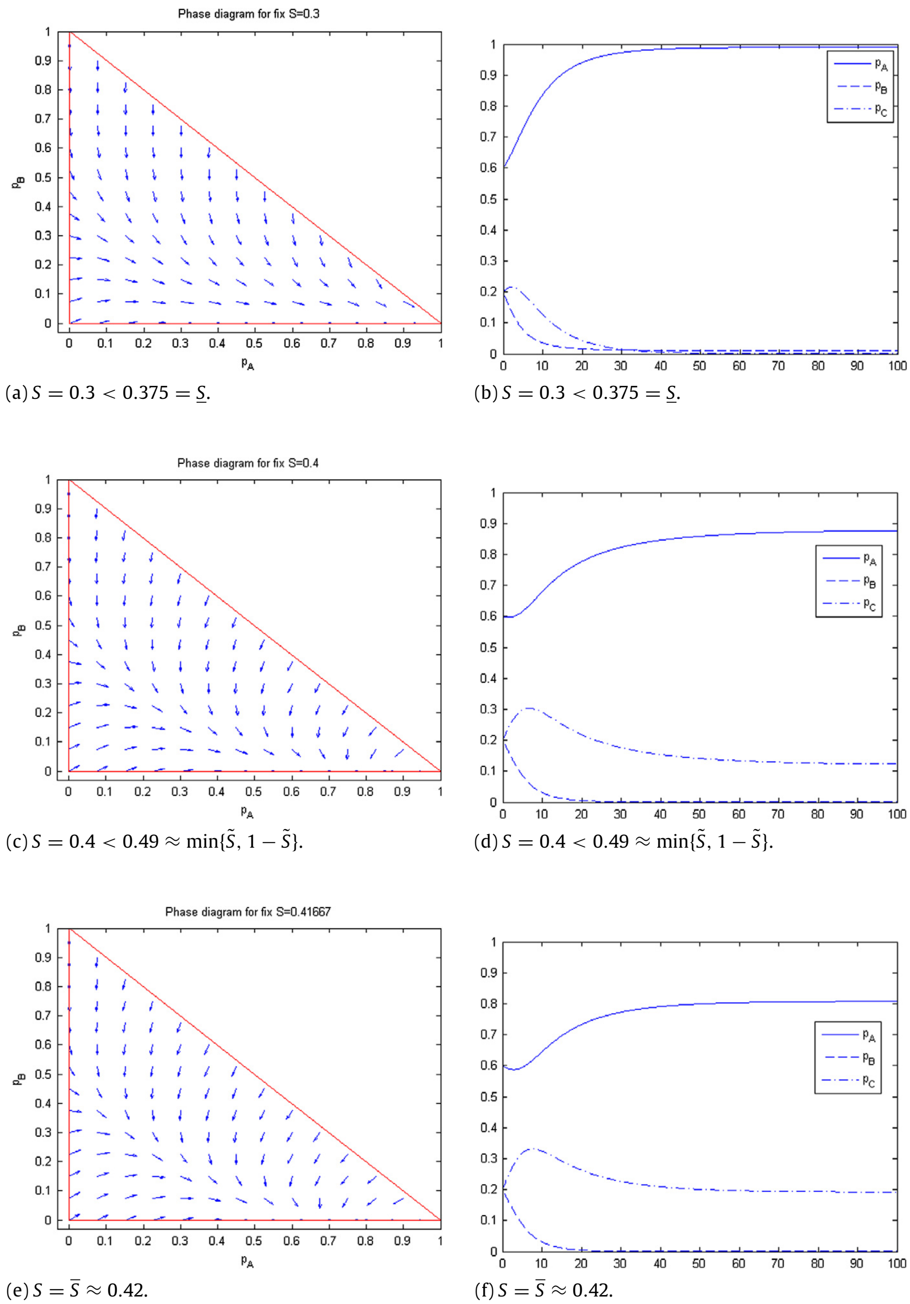
$$\bar{S} := \frac{\nu}{2\nu + \mu}$$
$$\tilde{S} : p_A^*(\tilde{S}) = p_A^{\Delta}(\tilde{S})$$
$$\underline{S} := 1 - \frac{1}{2\zeta}.$$

Note, that due to symmetry it holds $p_B^*(1-\tilde{S}) = p_B^{\Delta}(1-\tilde{S})$. Table 2 shows possible stable equilibria for the fixed status problem corresponding to these threshold values. Fig. 1 illustrates some of the cases listed in Table 2.

To find optimal state intervention strategies we need to consider the derivatives of the function $f(S, s) = \nu(1-2S)\sqrt{s}$:

$$\frac{\partial f}{\partial s}(S, s) = \frac{\nu}{2}\frac{1-2S}{\sqrt{s}}, \quad (3.9)$$

$$\frac{\partial f}{\partial S}(S, s) = -2\nu\sqrt{s}. \quad (3.10)$$




(a) $S = 0.3 < 0.375 = \underline{S}$.

(b) $S = 0.3 < 0.375 = \underline{S}$.

(c) $S = 0.4 < 0.49 \approx \min\{\tilde{S}, 1 - \tilde{S}\}$.

(d) $S = 0.4 < 0.49 \approx \min\{\tilde{S}, 1 - \tilde{S}\}$.

(e) $S = \overline{S} \approx 0.42$.

(f) $S = \overline{S} \approx 0.42$.

**Fig. 1.** Panels (a), (c) and (e) show phase diagrams for fixed $S$ for different values of $S$. Panels (b), (d) and (f) show trajectories for fixed $S$ for different values of $S$. For the trajectories the initial distribution is $p_A = 0.6$ and $p_B = 0.2$. Parameters are as in Example 5.1 in Section 5.

### 3.3. *Objective*

Departing at the initial state $p_A(0)$, $p_B(0)$ and $S(0)$ the aim of the optimization problem is to find the best investment policy $(s(t))_{t \geq 0}$ such that, $r \in (0, 1)$, $k > 0$, $\xi \in [0, 1]$,

$$\int_0^\infty e^{-rt} \left(k \cdot p_C(t) - [p_B(t) + p_C(t)]^\xi s(t)\right) dt \tag{3.11}$$

is maximized, while the system is developing according to (3.4), (3.5) and (3.8). For $\xi = 0$ the costs for the state intervention do not depend on the numbers of speakers of language $B$. Here one can think of adding language $B$ to (street-)signs. For $\xi = 1$ the costs linearly increase with the number of speakers—one could think of bilingual education in schools.

## 4. Optimal control and optimal steady states

Substituting $p_B + p_C$ by $1 - p_A$ in the objective function, the Hamiltonian can be expressed as

$$H(p_A, p_B, S, s) = k \cdot p_C - (1 - p_A)^\xi s + \lambda_A \dot{p}_A + \lambda_B \dot{p}_B + \lambda_S \left(f(S, s) - \mu S\right), \tag{4.12}$$

where $\lambda_A$, $\lambda_B$ and $\lambda_S$ are the costate variables measuring the marginal value of the corresponding state variables $p_A$, $p_B$ and $S$, respectively.

We assumed that the control variable is bounded, i.e. that the budget for language policies fostering bilingualism is limited. This budget constraint is formalized by the inequality $s \leq 1$. To include the constraint in the formal model we define the Lagrangian $L := H + \omega(1 - s)$, where $\omega$ is the Lagrange multiplier. For the identification of the optimal intervention at a given state we consider the derivative of $L$ with respect to the control variable $s$:

$$L_s = -(1-p_A)^\xi + \lambda_S \frac{\partial f(S,s)}{\partial s} - \omega. \tag{4.13}$$

To identify optimal intervention, we are looking for $s$ and $\omega$ such that $L_s = 0$ and $\omega(1-s) = 0$. We have

$$L_s = 0 \Leftrightarrow (1-p_A)^\xi + \omega = \lambda_S \cdot \underbrace{\frac{\partial f(S,s)}{\partial s}}_{\geq 0} \Rightarrow \lambda_S \geq 0.$$

Note, if $p_A < 1$ then we even have $\lambda_S > 0$. For the explicit form of the function $f$ defined in (3.8) we get

$$L_s = 0 \Leftrightarrow (1-p_A)^\xi + \omega = \lambda_S \cdot \frac{\nu}{2} \frac{1-2S}{\sqrt{s^*}}$$

$$\Leftrightarrow s^* = \left(\lambda_S \frac{\nu}{2} \frac{1-2S}{(1-p_A)^\xi + \omega}\right)^2. \tag{4.14}$$

The second derivative of $L$ with respect to the control variable $s$ is non-positive if $\lambda_S > 0$ in which case the Legendre Clebsch condition is satisfied. Whenever $p_A = 1$, in which case $\lambda_S = 0$ could be possible, $s = 0$ is obviously optimal. Applying the optimal control we have

$$\dot{S} = f(S, s^*) - \mu S = \lambda_S \frac{\nu^2}{2} \frac{(1-2S)^2}{(1-p_A)^\xi + \omega} - \mu S. \tag{4.15}$$

If the constraint is inactive, i.e. $s < 1$, then $\omega = 0$. If, in contrast, the constraint is active ($s = 1$), then

$$\omega = \lambda_S \frac{\nu}{2}(1-2S) - (1-p_A)^\xi \geq 0. \tag{4.16}$$

### 4.1. Stationary points

To state the co-state equations we first introduce some functions. For $L = A, B$ set

$g_L(p_A, p_B, S) := 2p_L\alpha_L(LC; p_A, p_B, S) + p_C\alpha_L(CC; p_A, p_B, S) - p_L,$

which equals $\dot{p}_L/(\theta p_C)$ whenever $p_C > 0$. Then,

$H = p_C(k + \theta\lambda_A g_A + \theta\lambda_B g_B) - (1-p_A)^\xi s + \lambda_S(f(S,s) - \mu S).$

Using this notation we have

$$H_{p_A} = -(k + \theta\lambda_A g_A + \theta\lambda_B g_B) + \theta\lambda_A p_C \frac{\partial g_A}{\partial p_A} + \theta\lambda_B p_C \frac{\partial g_B}{\partial p_A} + \frac{\xi}{(1-p_A)^{1-\xi}} s, \tag{4.17}$$

$$H_{p_B} = -(k + \theta\lambda_A g_A + \theta\lambda_B g_B) + \theta\lambda_A p_C \frac{\partial g_A}{\partial p_B} + \theta\lambda_B p_C \frac{\partial g_B}{\partial p_B}, \tag{4.18}$$

$$H_S = \theta p_C \left(\lambda_A \frac{\partial g_A}{\partial S} + \lambda_B \frac{\partial g_B}{\partial S}\right) + \lambda_S \left(\frac{\partial f(S,s)}{\partial S} - \mu\right). \tag{4.19}$$

The co-state equations are then given by

$\dot{\lambda}_A = r\lambda_A - H_{p_A},$

$\dot{\lambda}_B = r\lambda_B - H_{p_B},$

$\dot{\lambda}_S = r\lambda_S - H_S.$

To find inner stationary points we try to identify solutions $(\hat{p}_A, \hat{p}_B, \hat{S}, \hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_S)$ to

$0 = \dot{p}_A = \dot{p}_B = \dot{S} = \dot{\lambda}_A = \dot{\lambda}_B = \dot{\lambda}_S.$

For $\hat{p}_A$ and $\hat{p}_B$ to be stationary we need either $\hat{p}_C = 0$ or $g_A(\hat{p}_A, \hat{p}_B, \hat{S}) = g_B(\hat{p}_A, \hat{p}_B, \hat{S}) = 0.$

Note, any steady state status $0 < \hat{S} < \bar{S}$ corresponds to a steady state control variable $0 < \hat{s}^* < 1$ and hence to some $\hat{\omega} = 0$. In this case, the stationarity of the status ($\dot{S}(\hat{S}, \hat{\lambda}_S) = 0$) yields an explicit relation between $\hat{S}$ and $\hat{\lambda}_S$, cf. (4.15):

$$\hat{\lambda}_S = \frac{2\mu}{\nu^2} \frac{\hat{S}}{(1-2\hat{S})^2}(1-\hat{p}_A)^\xi. \tag{4.20}$$

Plugging this into (4.14) we get for the stationary optimal intervention

$$\hat{s}^* = \left(\frac{\mu}{\nu} \frac{\hat{S}}{1-2\hat{S}}\right)^2 < 1. \tag{4.21}$$

If $\hat{S} = \bar{S}$, then $\hat{s}$ has to be equal to one and thus $\hat{\lambda}_S \geq 2\frac{2\nu+\mu}{\nu\mu}(1 - p_A^*(\bar{S}))^\xi$ has to hold true, cf. (4.16).

Using the explicit expression for the function $f$ introduced in Section 3, the equation $\dot{\lambda}_S = 0$ yields

$$0 = -\theta\hat{p}_C \left(\hat{\lambda}_A \frac{\partial g_A}{\partial S} + \hat{\lambda}_B \frac{\partial g_B}{\partial S}\right) + \hat{\lambda}_S \left(r + \mu + 2\nu \left(\left[\hat{\lambda}_S \cdot \frac{\nu}{2} \frac{1-2\hat{S}}{(1-\hat{p}_A)^\xi}\right] \wedge 1\right)\right). \tag{4.22}$$

#### 4.1.1. Monolingual stationary points

First we want to consider stationary points with $\hat{p}_C = 0$. Obviously, if $p_C = 0$, then the linguistic composition wont change anymore, since families of type *AB* are impossible, while no bilinguals, which function as a kind of language transmitters, are part of the population. In the steady state all families are of types *AA* and *BB* and children of such families are raised monolingual in the respective language. Hence, both monolingual language groups reproduce themselves independent of the statuses of both languages. Thus, the state does not invest any money to support the status minority language, which would produce costs without having any positive effect, i.e. $\hat{S} = \hat{s}^* = 0$.

#### 4.1.2. Bilingual stationary points

Now we want to consider stationary points with a bilingual sub-population, i.e. $p_C > 0$. Using the notation introduced above this yields that whenever $\hat{p}_L > 0$ the stationarity implies $g_L(\hat{p}, \hat{S}) = 0$.

#### 4.1.3. Bilingual stationary points with $p_B = 0$

The most interesting case is when monolingualism in the minority language $B$ vanishes and only monolinguals in $A$ and bilinguals remain. Such a state is desirable, since all society members are able to communicate with each other, while speakers of $B$ can still preserve their cultural identity. If $p_B = 0$ we need $\dot{p}_B \leq 0$. This is equivalent to $\hat{p}_A \geq \hat{p}_A^\Delta(S)$.

Let

$\underline{S} < S \leq \min\{\bar{S}, \tilde{S}\}$

and $p_A = p_A^*(S)$. The co-state equation $\dot{\lambda}_A = r\lambda_A - H_{p_A} = 0$ is independent of $\lambda_B$, since $g_B = 0$ and $\partial g_B/\partial p_A = 0$, see A.2. Hence, we can derive $\lambda_A(S) = \lambda_A(p_A, S)$. Given this $\lambda_A$ we can choose some $\lambda_B$ such that $\dot{\lambda}_B = 0$. In A.2 it is also shown that $\partial g_B/\partial S = 0$.

To identify optimal steady states we have to distinguish two possibilities. First we can check if there is a steady state at $\bar{S}$. To do so, it has to be investigated if there exists a $\hat{\lambda}_S > 2\frac{2\nu+\mu}{\nu\mu}(1-p_A^*(\bar{S}))^\xi$ which solves

$0 = \dot{\lambda}_S(\hat{\lambda}_S) = \dot{\lambda}_S(\bar{S}, p_A(\bar{S}), \lambda_A(\bar{S}), \hat{\lambda}_S).$

The second case covers $\underline{S} < S < \bar{S}$. Here, let $\lambda_S(S)$ be defined by (4.20). In this case steady states can be found by identifying statuses $S$ which solve

$0 = \dot{\lambda}_S(S) = \dot{\lambda}_S(S, p_A(S), \lambda_A(S), \lambda_S(S)).$

Depending on the parameter constellation and especially depending on $k$, $\nu$ and $\mu$ such a solution exists. If $k$ is too small, then no such solution exists, that means it is not profitable to maintain the minority language $B$.

**Lemma 4.1.** *For $k$ sufficiently large there exists at least one solution $\hat{S}^* \in (\underline{S}, \min\{\overline{S}, \tilde{S}\}]$ such that*

$$0 = \dot{\lambda}_S(\hat{S}^*) = \dot{\lambda}_S(\hat{S}^*, p_A(\hat{S}^*), \lambda_A(\hat{S}^*), \lambda_S),$$

*where $\lambda_S = \lambda_S(\hat{S}^*)$ if $\hat{S}^* < \overline{S}$, and $\lambda_S > 2\frac{2\nu+\mu}{\nu\mu}(1 - p_A^*(\overline{S}))^{\xi}$ if $\hat{S}^* = \overline{S}$.*

For a proof see Appendix.

*4.1.4. Bilingual stationary points with $p_B > 0$*

For an optimal steady state with $p_A, p_B, p_C > 0$ we need

$$\underline{S} \vee \tilde{S} < \hat{S} \leq (1 - \tilde{S}) \wedge \overline{S}.$$

This is only possible if $\tilde{S} < \overline{S} < 1/2$, which does not hold true for all parameter constellations, cf. Example 5.1.

For fix $S$ we need the following for any steady state: $\alpha_A(AC), \alpha_B(BC), \alpha_B(CC) > 0$. The last inequality is due to $S < 1/2$ and $\zeta < 1$. If $\alpha_A(CC) = 0$, then $\zeta(1 - S) > 1/2$ has to hold true, else $\alpha_A(CC) > 0$.

As before, for suitable $S$ (here $\max\{\underline{S}, \tilde{S}\} < S < \overline{S}$), we can find $p_A(S)$ and $p_B(S)$ such that $\dot{p}_A = \dot{p}_B = \dot{S} = 0$. For some parameter constellations there can be more than one stable solution $p_A(S)$ and $p_B(S)$ such that $\dot{p}_A = \dot{p}_B = 0$. Furthermore we get a unique $\lambda_S(S)$. The co-state equations yield a linear system in $\lambda_A, \lambda_B$ with 3 equations and coefficients depending on $S$. To identify the optimal status, one has to check if this linear system has a solution for some suitable $S$. This also holds true at the left boundary. At the right boundary one has to check if the linear system in $\lambda_A, \lambda_B$ and $\lambda_S$ has a solution with a sufficiently large $\lambda_S$, see above.

## 5. Numerical calculations

In this section we numerically investigate the linguistic behavior of the population under the optimal policy. We show the existence of different stable and optimal steady states. Moreover, we illustrate the dependence of the selected steady state on the initial distribution of speakers as well as on how much bilingualism is valued with respect to expenditures by the decision maker (parameter $k$). To analyze the evolution towards the steady states we plot exemplary trajectories.

Two examples are considered. For both of them we set $\eta = 0$. In Example 5.1 we choose $\mu$, the rate of decline of the minority language status $S$, to be 0.2, which is relatively high. In contrast, Example 5.2 depicts a case where the status of the minority language declines rather slowly over time ($\mu = 0.01$). Furthermore, the parameter $\zeta$, which measures the aggregated weight that is put on the status in the decision of $LC$ families, $L = A, B$, to socialize their children as monolinguals in $L$, is slightly higher in Example 5.1. In both example we chose the discount rate $r$ to be 0.5.

**Example 5.1.** $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.8$; $\nu = 0.5$, $\mu = 0.2$, $\theta = 1$ and $\xi = 0$.

**Example 5.2.** $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.7$; $\nu = 0.5$, $\mu = 0.01$, $\theta = 1$ and $\xi = 0$.

First we calculate the $S$- thresholds, cf. Section 3.2. In Example 5.1 we have $\underline{S} = 0.375$, $\overline{S} = 0.417$ and $\tilde{S} = 0.492$, while in Example 5.2, $\underline{S} = 0.286$, $\overline{S} = 0.495$ and $\tilde{S} = 0.463$. According to these numbers and the statements made in Section 3.2, stable equilibria with $p_A, p_C > 0$ and $p_B = 0$ are possible for both examples. In Example 5.2 furthermore equilibria with $p_A, p_C > 0$ and $p_B > 0$ are possible, since $\tilde{S} < \overline{S}$. This is not the case for Example 5.1, since there $\overline{S} < \tilde{S}$. The actual stable bilingual equilibria are displayed in Table 3. For Example 5.1 we investigate the influence of different values of $k$, namely $k = 60$, $k = 75$ and $k = 90$. For Example 5.2 we concentrate on the case of $k = 20$. For any parameter constellation there also is a manifold of steady states at $(\hat{p}_A, \hat{p}_B, \hat{S}) = (\hat{p}_A, 1 - \hat{p}_A, 0)$, where $\hat{p}_A$ can take any value between zero and one. In these steady states it is optimal to have $\hat{s} = 0$. Note, however, that not every point on this manifold is a candidate for the optimal long run solution due to its stability properties, cf. Lemma 3.2. Next, we analyze the two examples in greater detail.

**Table 3**
This table contains stable bilingual steady states – if such exist – for Examples 5.1 and 5.2 for different values of $k$. The steady state values of the status $\hat{S}$, the optimal control $\hat{s}^*$, the fraction of speakers $\hat{p}_A$ and $\hat{p}_B$ as well as the steady state objective $k\hat{p}_C - \hat{s}^*$ are listed. Here $r = 0.5$ and $\xi = 0$.

| | $k$ | $\xi$ | $\hat{S}$ | $\hat{s}^*$ | $\hat{p}_A$ | $\hat{p}_B$ | $k\hat{p}_C - \hat{s}^*$ |
|---|---|---|---|---|---|---|---|
| Example 5.1 | 60 | 0 | – | – | – | – | – |
| | 75 | 0 | 0.41 | 0.74 | 0.85 | 0 | 10.3 |
| | | 1 | $\overline{S} \approx 4.2$ | 1 | 0.81 | 0 | 13.4 |
| | 90 | 0 | $\overline{S} \approx 4.2$ | 1 | 0.81 | 0 | 16.3 |
| Example 5.2 | 20 | 0 | 0.47 | 0.03 | 0.44 | 0.03 | 10.6 |

Example 5.1, $k = 60$

If $k$ is small the decision maker does not have a particularly high incentive to support the status of the minority language $B$ in the long run. As can be seen in the first row of Table 3 there is no bilingual steady. The following happens. Let us consider a situation where the fraction of $A$ speakers, $p_A$, is relatively high, while $p_B$ and $p_C$ and the status variable $S$ are small. Because of the dominance of $A$ speakers, most families are of type $AA$. Thus, $p_A$ increases. Initially $p_C$ decreases due to the low status of $B$ and the low chances of $A$ speakers of meeting a bilingual partner. This development is challenged by the decision maker who invests much into raising the status of $B$. Under such a policy the incentive to raise their children bilingual increases for $AC$ and $CC$ couples. This yields an increase in the number of bilinguals. An other effect of is that $BC$ couples have a stronger incentive to raise their children as $B$-monoglots. However, since the fraction of $B$ and $C$ speakers is small, the policy does not have a strong effect on the overall development of the language and over all $p_B$ decreases even further. As a result, it soon does not pay off anymore to invest into the status of the language as these measures affect less and less people. Thus, the status of $B$ decreases again. Consequently, the incentive to raise children bilingual and therefore the fraction of bilinguals decreases as well. In the long-run the majority of the population only speaks $A$ and bilingual speakers disappear completely in the long run. This behavior is illustrated in Figs. 2 and 3.

Example 5.1, $k = 75$, $\xi = 0$

Table 3 shows that for $k = 75$ there exists a steady state with 15% bilinguals and no monolingual speakers of the minority language $B$. To obtain this fraction of bilingual speakers in the long run, 75% of the budget has to be used. If this bilingual steady is reached or not depends on the initial state values. For the initial states considered in Figs. 4 and 5 the system converges to that steady states. If the initial $p_B$, $p_C$ and $S$ would be even smaller than in Fig. 4, the system is likely to converge to a steady state with almost only $A$-monolingual speakers, few $B$-monoglots and no bilinguals.

For the base case ($p_A = 0.85$, $p_B = 0.05$, $S = 0.1$), see Fig. 4 and the left panel of Fig. 6, the fraction of the bilingual population first

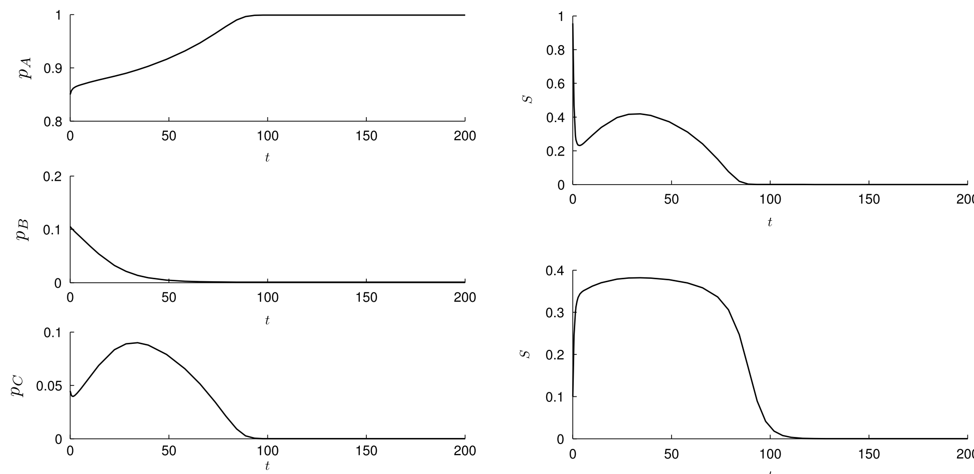



**Fig. 2.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 60$).

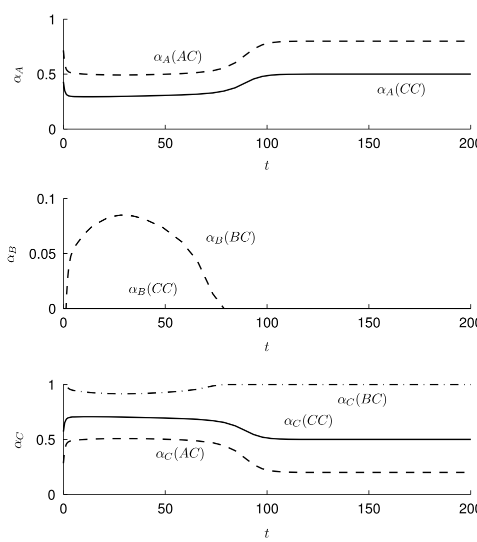



**Fig. 3.** $\alpha$-functions for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 60$).

decreases, since the status of language $B$ is low, as are the fractions of $B$ and $C$ speakers, so the majority of couples consist of $A$ speakers. Due to the dominance of $AA$ couples and the high likelihood that $AC$ and $CC$ couples raise their children as $A$-monoglots, $p_A$ first increases. Initially one would invest as much as possible into the status to increase it. As a first result of this policy $BC$ couples get a stronger incentive to raise their children as $B$-monoglots. Furthermore, $AC$ and $CC$ couples become less likely to raise their children just as speakers of language $A$ and instead are more likely to raise the children bilingual than before. Consequently, $p_A$ now decreases while $p_C$ increases see Fig. 4. Hence, the negative term in $\alpha_B(BC)$ decreases and even more $BC$ families raise their children as $B$'s. This is a problem as long as $p_B$, which is continuously decreasing, is above some threshold. To avoid this effect, the increase of $S$ is slowed down for a while, until $p_B$ is small enough and then increased again to obtain the steady state status.

If, in contrast to the base case, the initial status is high, see Fig. 5 and the right panel of Fig. 6, then initially the state does not have to invest as much into increasing the status of the minority language. Due to the high status of $B$, many $AC$ couples will raise their kids bilingual. As a result, at the beginning $p_A$ decreases while $p_C$ increases. Furthermore, the fraction of language $B$ speakers is so low that $BB$ and $BC$ couples are rather unlikely and $p_B$ decreases. To further support the growth of $p_C$ it is optimal to increase $s$ for some time. Due to the smaller fraction of $B$ speakers, $AA$ and $AC$ couples are more likely than $BC$ or $CC$ couples, thus, $p_A$ recovers after some time and even grows. At some point of time the status $S$ and the fraction of bilingual speakers $p_C$ are high enough while $p_B$ is very low, such that $s$ can be lowered again until it reaches its steady state.

Example 5.1, $k = 75$, $\xi = 1$

For $\xi = 1$ the costs for state intervention increase with the number of speakers of $B$, i.e. $B$-monoglots as well as bilinguals. Thus, the higher $p_A$, the lower are the costs for state intervention. In Fig. 7 we can see that for the base case, the system behaves quite similar to the case of $\xi = 0$. The major difference is that state intervention is not just maximal in the beginning, but the entire budget is used over the entire time horizon. Due to the large amount of $A$-monolinguals the intervention is much cheaper compared to the case where $\xi = 0$ (more than 80% cheaper). Therefore, in the long run the status and $p_C$ are higher while the $p_A$ is smaller, cf. Table 3.

Example 5.1, $k = 90$

If $k$ is large, then it is optimal to approach a steady state where the state invests the entire budget to reach the maximal possible status for minority language $B$, see Table 3. This yields a maximal amount of bilingual speakers while no $B$-monolinguals remain within the population. For the base case, see Fig. 8, initially the state spends as much as possible for improving the status of $B$. For similar reasons as before, $p_A$ first increases while $p_B$ and $p_C$ first decrease. This changes after some time. Once $p_B$ has become small enough, the state can afford to decrease efforts. However, to ensure a growth in the number of bilingual speakers, it is necessary to increase expenditures after some time again. This is the main difference to the case with a low $k$; where one would first decrease, then increase, and then decrease the expenditures $s$. i.e. the later increase is apparently necessary to reach a steady state with a proper bilingual population.

Example 5.2, $k = 20$

Table 3 shows that in the bilingual steady state for the parameter constellation considered in Example 5.2 all three linguistic repertoires remain intact in the long run. This is the major difference to Example 5.1 and is mainly due to the much lower value of $\mu$ ($\mu = 0.2$ in Example 5.1 and $\mu = 0.01$ in Example 5.2). Here with the low $\mu$ it is much less costly to keep the status at a high level. The development of the population groups is similar to before, however, $p_B$ only decreases for a certain time, then the status of language $B$ is so high that even $CC$ couples have a small incentive to teach their children only language $B$. Due to the small

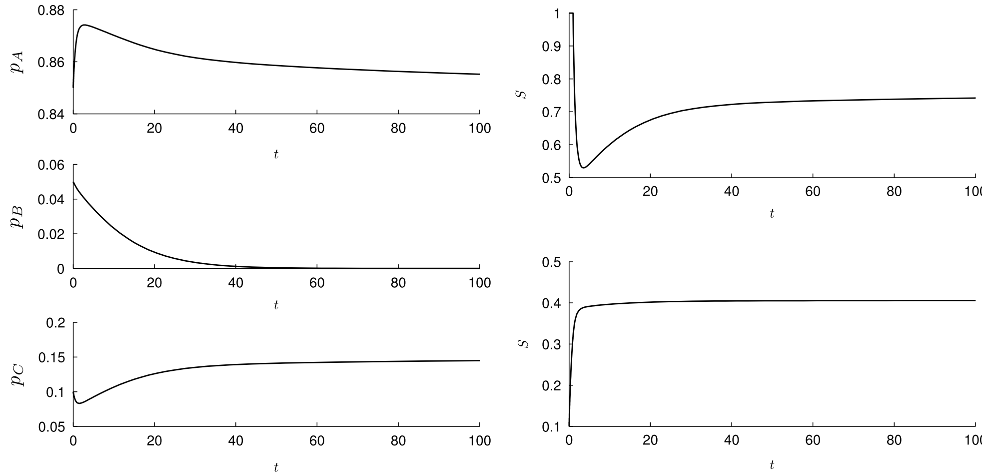



**Fig. 4.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 75$, $\xi = 0$).

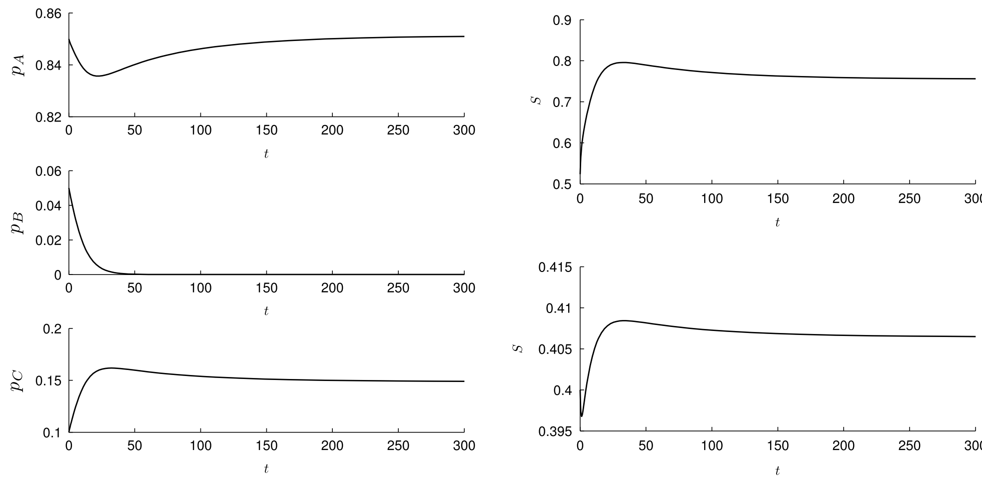



**Fig. 5.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.4$ (Example 5.1, $k = 75$, $\xi = 0$).

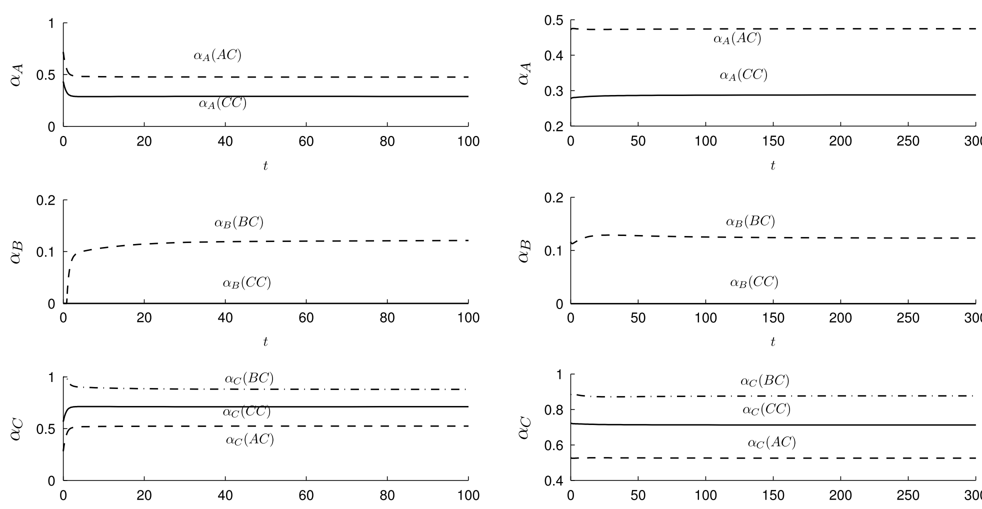



**Fig. 6.** $\alpha$-functions. In the both panels $p_A(0) = 0.85$ and $p_B(0) = 0.05$. In the left panel $S(0) = 0.1$, while in the right one $S(0) = 0.4$ (Example 5.1, $k = 75$, $\xi = 0$).

depreciation of $S$ it is not necessary to spend much for keeping the status high, so one would only invest much into the status in the beginning to get it to a high level and then decrease control efforts over time. Example 5.2 with $k = 20$ is visualized in Fig. 9. Note, in the long run only 3% of the budget is used to guaranty that more than half of the population is bilingual.

## 6. Conclusions

The state aims at ensuring wide communication possibilities, while recognizing and supporting – if this is not *too costly* – minority language rights. This trade-off between a commonly spoken language and the preservation of a minority language is

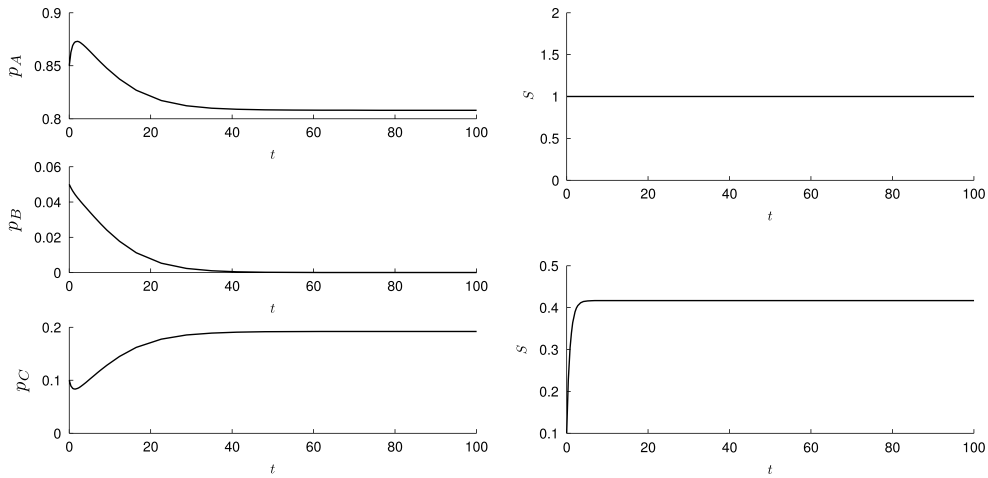



**Fig. 7.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.4$ (Example 5.1, $k = 75$, $\xi = 1$).

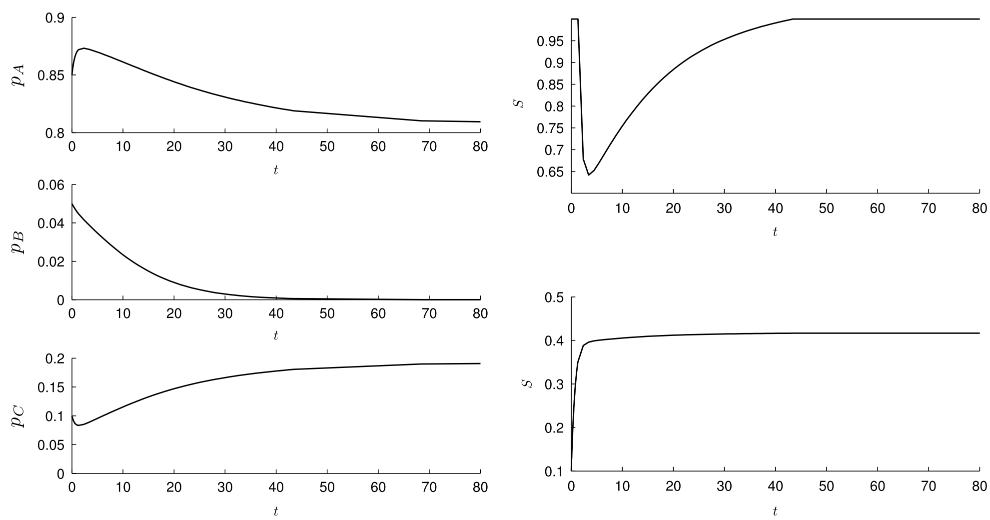



**Fig. 8.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 90$).

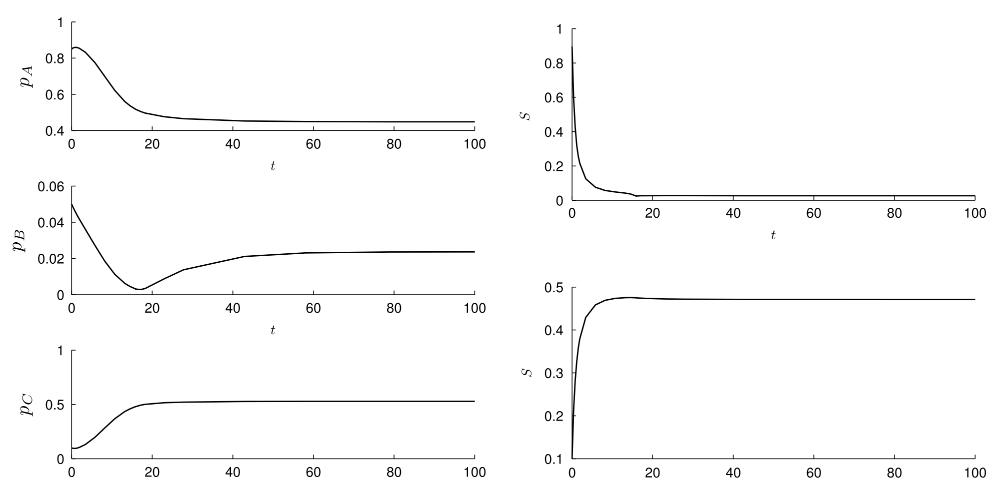



**Fig. 9.** Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.2, $k = 20$).

approached through bilingualism. To investigate how language policies can be used to preserve a minority language in a bilingual subpopulation we developed an abstract language dynamics model. The point of departure is individual utility maximization, while here only intergenerational language transmission is considered. Families decide to bring up their children either as monolinguals in the majority or the minority language, or as bilinguals. This decision is based on how they value the communicational value of each language and their emotional attachment to the languages at hand. Through a continuous investment into language policies the state can increase the status of the minority language and thereby foster bilingual parenting in families with one or two bilingual parents. It is assumed that the state wants to maximize the number of bilingual speakers at minimal costs.

In Wickström (2005) it was already proven that for a constant status and proper parameter constellations stable bilingual steady states are possible. Here we could furthermore show that such bilingual steady states can even be optimal when costs for language policies are taken into account. It was illustrated that for some cases there are steady states only with monolingual speakers of the majority language and bilinguals but without any monolingual speakers of the minority language. In such a state all individuals within the population can – in principle – communicate with each other while the minority can preserve its language. For other cases we could see that small subpopulation with monolingual speakers of the minority language survives in the long run optimal state. As one would expect, bilingual steady states are only optimal, if bilingualism is valued high enough with respect to expenditures.

Whether or not a bilingual steady state is not only possible but really targeted by the decision maker, depends on the initial distribution of speakers as well as the initial status of the minority language. If both the status and number of speakers of the minority language are too low, then it is not worthwhile to invest in language maintenance in the long run, which results in a purely monolingual population. In most of the examples considered in the numerical analysis, the initial values were high enough and it was illustrated how expenditures change over time to achieve an optimal bilingual steady state in the long run.

For future research the current model will be extended. To get closer to the real-world complexity of language acquisition and transmission within a large population, we will add to the model language learning in formal education as well as adult language learning. Furthermore, language policies will be investigated in greater detail. We also intend to adjust the model to cases of *new* minorities, that means minorities which are based on temporary or permanent migration.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Part of the research of Bengt-Arne Wickström and Torsten Templin leading to this paper has received funding from the European Community's Seventh Framework Program under grant agreement No. 613344 (Project MIME). Andrea Seidl and Gustav Feichtinger were supported by the Austrian Science Fund (FWF) under Grant P25979-N25.


## Appendix

### A.1. Family type distribution

Consider a population of size $2N$, where $N$ is large. Suppose that the population consists of $N$ female and $N$ male individuals. Let $N_A$ denote the number of female $A$-monolinguals, $N_B$ the number of female $B$-monolinguals and $N_C$ the number of female bilinguals. We assume that these numbers are the same for the male population. The distribution of speakers is given by $p_A = N_A/N$, $p_B = N_B/N$ and $p_C = N_C/N$.

Family formation can be conceptualized as a repeated random procedure of choosing pairs. Let us start with one randomly chosen pair consisting of a female $X$ and a male $Y$. The probability that $X = L_1$ and that $Y = L_2$, $L_1, L_2 \in \{A, B, C\}$, is given by

$$\mathbb{P}[X = L_1, Y = L_2] = \mathbb{P}[X = L_1]\mathbb{P}[Y = L_2] = p_{L_1}p_{L_2}.$$

Note that a family type $L_1L_2$, $L_1 \neq L_2$, is obtained either by $X = L_1, Y = L_2$ or by $X = L_2, Y = L_1$ (for the family type we do not take the gender of the parents into account). Hence, the probability of obtaining a pair of type $L_1L_2$ is given by

$$\mathbb{P}[L_1L_2] = \begin{cases} p_{L_1}^2 & : \quad L_1 = L_2 \\ 2p_{L_1}p_{L_2} & : \quad L_1 \neq L_2. \end{cases}$$

Now, all $N$ pairs are chosen randomly after one another. The total expected number of $L_1L_2$-type pairs equals $N \cdot \mathbb{P}[L_1L_2]$ and hence the expected fraction of $L_1L_2$-type pairs is $\mathbb{P}[L_1L_2]$. After this first step we have $N$ pairs with $2Np_Ap_B$ of them being of type $AB$. Recall, we assume that parents shall be able to properly communicate with each other, and therefore we exclude $AB$ families. Splitting these $AB$ pairs again and repeating the random selection we obtain new pairs of types $AA$, $BB$ and $AB$. This procedure is repeated until only $AA$ and $BB$ pairs are left. This way, half of the $2Np_Ap_B$ pairs of type $AB$ will be transformed into $AA$ pairs, while the other half will form $BB$ pairs. As a result, we obtain the numbers presented in Table 1. Note, due to the law of large numbers ($N$ is assumed to be large), the realized number of $L_1L_2$-type of pairs can be approximated by the expected number.

### A.2. Partial derivatives of $g_A$ and $g_B$ when $p_B = 0$

Given the definition of $g_A$ and $g_B$, their partial derivatives are given by

$$\frac{\partial g_A}{\partial p_A} = (1 - S)\left[2\zeta - (\varepsilon + \gamma p_A) + (1 - p_A)\gamma\right] - 1$$
$$\frac{\partial g_B}{\partial p_A} = -\alpha_B(CC) - \delta(1 - S)(1 - p_A)1_{\{\alpha_B(CC)>0\}}$$
$$\frac{\partial g_A}{\partial p_B} = -\left[S(2\beta p_A + \delta p_C) + (1 - S)(\varepsilon + \gamma p_A)\right]$$
$$\frac{\partial g_B}{\partial p_B} = 2\alpha_B(BC) - \alpha_B(CC) + \gamma S(1 - p_A)1_{\{\alpha_B(CC)>0\}} - 1$$
$$\frac{\partial g_A}{\partial S} = -2\zeta p_A - (\varepsilon + \gamma p_A)(1 - p_A)$$
$$\frac{\partial g_B}{\partial S} = (1 - p_A)(\varepsilon + \delta p_A)1_{\{\alpha_B(CC>0)\}}.$$

Note, if $p_A \geq p_A^{\Delta}$, then $\alpha_B(CC) = 0$ and hence $\partial g_B/\partial p_A = \partial g_B/\partial S = 0$.

### A.3. Proof of Lemma 3.2

Since $\eta = 0$, every constellation with $p_A + p_B = 1$, which implies $p_C = 0$, is a steady state ($\dot{p}_A = \dot{p}_B = 0$). In the following we investigate their stability. Let $f_{LL}$ denote the matrix

$$f_{LL} = \begin{pmatrix} \dfrac{\partial \dot{p}_A}{\partial p_A} & \dfrac{\partial \dot{p}_A}{\partial p_B} \\ \dfrac{\partial \dot{p}_B}{\partial p_A} & \dfrac{\partial \dot{p}_B}{\partial p_B} \end{pmatrix}$$

and define $a := p_A(1 - 2\alpha_A(AC))$ and $b := p_B(1 - 2\alpha_B(BC))$. For $p_C = 0$ the matrix $f_{LL}$ equals $\binom{a \quad a}{b \quad b}$ and has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a + b$. If $p_A = 1$ and hence $p_B = 0$ the non positivity of $a + b = a$ is equivalent to $S \leq 1 - 1/2\zeta$. If in contrast $p_A = 0$ and $p_B = 1$ we need for stability that $a + b = b \leq 0$. This is equivalent to $S \geq 1/2\zeta$ and cannot be true since $S < 1/2$ and $\zeta \leq 1$. If $p_A, p_B > 0$ we have

$$a + b = 1 - 2(p_A\alpha_A(AC) + p_B\alpha_B(BC)).$$

Consider the function

$$h(p) = p\alpha_A(AC; p, 1 - p, S) + (1 - p)\alpha_B(BC; p, 1 - p, S).$$

Then stability, i.e. $a+b \leq 0$, is equivalent to $h(p_A) \geq 1/2$. We will investigate the four possible cases separately. If $\alpha_A(AC) = \alpha_B(BC) = 0$, then $h = 0$. So we can except this first case. As a second case let $\alpha_A(AC) = 0$ and $\alpha_B(BC) > 0$. Then,

$$f(p) = (1-p)(\zeta S - \beta(1-S)p) \leq (1-p)\zeta S < 1/2, \tag{A.23}$$

since $S < 1/2$ and $(1-p), \zeta < 1$. Thus, we can exclude this case as well. As a third case let $\alpha_A(AC) > 0$ and $\alpha_B(BC) = 0$. Here,

$$\begin{aligned} f(p) &= (1-p)(\zeta(1-S) - \beta S(1-p)) \\ &= p\zeta - S(p\zeta + \beta p(1-p)). \end{aligned}$$

To get $f(p_A) \geq 1/2$ we need $p_A \geq 1/2$. Then, $f(p_A) \geq 1/2$ yields

$$S \leq \frac{p_A\zeta - 1/2}{p_A\zeta + \beta P - A(1-p_A)}.$$

The right hand side of the last inequality is increasing in $p_A$ for $p_A \geq 1/2$. Hence, to achieve $f(p_A) \geq 1/2$ we need at least

$$S \leq \frac{\zeta - 1/2}{\zeta} = 1 - \frac{1}{2\zeta}.$$

In case 4 we have $\alpha_A(AC), \alpha_B(BC) > 0$. Here, $f$ is a convex function in $p$:

$$f(p) = \zeta S + (\zeta - 2\zeta S - \beta)p + \beta p^2.$$

Hence, for all $0 < p < 1$, $f(p) \leq \max\{f(0), f(1)\}$. We have $f(0) = \zeta S < 1/2$ and $f(1) = \zeta(1-S)$. For $S > 1 - 1/2\zeta$, $f(1) < 1/2$. Summarizing we can see that in the first two cases no stable steady state exists, while in the last two cases a necessary condition for stability is given by $S \leq 1 - 1/2\zeta$. □

*A.4. Proof of Lemma 4.1*

For $S \in (\underline{S}, \min\{\bar{S}, \tilde{S}\}]$ let $p_A = p_A^*(S)$, while $p_B = 0$.

**Case 1**: $\bar{S} \leq \tilde{S}$

Set $S = \bar{S}$. The stationarity of $\lambda_A$ yields

$$0 = \left(r - \theta p_C \frac{\partial g_A}{\partial p_A}\right)\lambda_A + k - \frac{\xi}{(1-p_A)^{1-\xi}}.$$

Note that for $p_A = p_A^*(S)$ it is easy to check that $\partial g_A/\partial p_A \leq 0$. To achieve stationarity of $\lambda_S$, we have to find a $\lambda_S \geq 2\frac{2\nu+\mu}{\nu\mu}(1-p_A)^{\xi}$ such that

$$0 = \dot{\lambda}_S = -\theta p_C \lambda_A \frac{\partial g_A}{\partial S} + \lambda_S(r + \mu + 2\nu).$$

Since $\lambda_A < 0$ increases in $k$ and $\frac{\partial g_A}{\partial S} < 0$, the solution to the above linear equation is sufficiently large, if $k$ is sufficiently large.

**Case 2**: $\tilde{S} < \bar{S}$

Here the stationarity of $\lambda_A$ yields

$$0 = \left(r - \theta p_C \frac{\partial g_A}{\partial p_A}\right)\lambda_A + k - \frac{\xi}{(1-p_A)^{1-\xi}} s^*(S),$$

and $\lambda_S = \lambda_S(S)$ is given by (4.20). We seek for a proper $S$ such that $\dot{\lambda}_S = 0$ holds, cf. (4.22), where $\partial g_B/\partial S = 0$. If the first summand of (4.22) is denoted by $f_1(S)$ and the second one by $f_2(S)$, then we aim to solve $-f_1(S) = f_2(S)$. It is easy to check that at $\underline{S}$ (note, $p_A^*(\underline{S}) = 1$) we have $f_1(\underline{S}) = 0$. Depending on $\xi$ it holds $f_2(\underline{S}) > 0$ (for $\xi = 0$) or $f_2(\underline{S}) = 0$ (for $\xi > 0$). Furthermore, $f_2(S) \to \infty$ for $S \to 1/2$, while $-f_1$ is bounded. Since $f_2$ is independent of the parameter $k$ while $-f_1$ is growing linearly in $k$, we get for sufficiently large $k$ that $-f_1(S) > f_2(S)$ for some relevant $S$. Summarizing we have for sufficiently large $k$: $-f_1(\underline{S}) \leq f_2(\underline{S})$, $-f_1(S) > f_2(S)$ for some $S \in (\underline{S}, 1/2)$, $f_2(1/2) = \infty$, $-f_1(1/2) < \infty$ and $f_1, f_2$ are continuous functions on $(\underline{S}, 1/2)$. Hence, there exists at least one intersection between the two functions in the interval $(\underline{S}, 1/2)$. □

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