NESTED OPTIMIZATION IN DOA
ESTIMATION FOR NONLINEAR DYNAMICAL
SYSTEMS: SPACECRAFT LARGE ANGLE
MANOEUVRES

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ABSTRACT

This paper discusses the formulation and numerical develop-
ment of an algorithm for the estimation of the domain of
attraction of a general nonlinear autonomous dynamical system.
The method is based on stability analysis using Lyapunov's
direct method with quadratic Lyapunov functions. It requires
the nesting of an unconstrained and a constrained optimization
problem--both highly nonlinear. The Powell '64 conjugate direc-
tion algorithm and the BFGS quasi-Newton algorithm may be used
as alternatives at the outer loop, while the recent Powell-Han
projected Lagrangean algorithm is used for the inner loop non-
linear programme. Difficulties intrinsic to the Powell-Han
algorithm, in obtaining global constrained minima and in pro-
viding sensitivity analysis of the inner loop problem in order
to use BFGS at the outer loop are discussed in the context of
stable control of large angle manoeuvres for astronomical
satellites.

Keywords: Dynamical systems, nested optimization, sensitivity
           analysis of nonlinear programming, Lyapunov's direct
           method, satellite large angle manoeuvres, stability
           analysis, unconstrained optimization.

CONTENTS

1. INTRODUCTION

This paper investigates the feasibility of the Lyapunov function approach to stability analysis of a (possibly controlled) dynamical system through the development of suitable software to estimate the *domain of attraction (DOA)* of the (target) equilibrium point of the system. It describes the theoretical and computational development of a technique for assessing the stability behavior of two satellite attitude control schemes, based on Lyapunov's *second* or *direct* method. This approach is a generalization of "energy" sink ideas involving "energy-like" functions and their time rate of change along motions of the system under investigation. In qualitative terms, system stability is assured if the total "energy" decreases as the system motion evolves in time (*cf.* Lasalle & Lefschetz, 1961; Rouche *et al*, 1977).

In order to make this paper self-contained the concepts necessary for an understanding of the stability analysis technique developed are outlined briefly in the next two sections of the paper.

The practical implementation of the technique for the stability analysis of engineering,economic and public policy control systems requires the solution of two sets of optimization problems: an inner-loop constrained problem and an outer-loop unconstrained problem. The complex nonlinear nature of the problems involved requires that the algorithms for finding the optima be carefully selected. The performance of the three methods chosen were first carefully tested on an experimental design of appropriate test problems (see Dempster *et al*, 1979). A similar design of dynamical system test problems was used for preliminary evaluation of the overall Lyapunov stability technique developed. It will be discussed in Section 4.

A computer system--DOMATT--which involves the selected optimization procedures and implements the stability evaluation technique was developed and applied to the test dynamical systems. It will be described briefly in Section 4 along with the numerical experiments performed with it. Section 5 describes the preliminary application of the stability analysis technique developed to two spacecraft large angle attitude control systems designed in the study on which this paper partially reports. In the study digital simulations of various large angle slew manoeuvres were first performed to check the specification and stability behaviour of the attitude control systems analyzed (*cf*, Dempster *et al*, 1979; Dempster, 1980).

Utilization of the latest optimization techniques makes the Lyapunov function approach to stability analysis of general realistic models of satellite large angle manoeuvre--and other complex--control systems potentially feasible for the first time. Sources of difficulties experienced with the Powell-Han inner loop optimization technique--recently independently encountered by other researchers--are precisely identified in Section 4 and 5. Directions to overcome these and other difficulties in order to establish definitely the usefulness of the method in applications are indicated in Section 6.

In this regard, it should be noted that numerical DOA estimation, as investigated in this paper, currently provides the only hope of *proof* of global (asymptotic) stability of satellite large angle attitude control systems. Indeed, cascaded nonlinearities in several dimensions--as incorporated in the reaction wheel attitude control system -- are currently beyond the reach of frequency domain stability analysis techniques. Moreover, simulation studies can *never* establish stability properties beyond doubt -- even at immense computational cost.

2. STABILITY OF DYNAMICAL SYSTEMS

Consider the autonomous nonlinear *dynamical system* given by the vector differential equation

$$(1) \qquad \dot{x} = f(x)$$

where x is an n-vector in the (Euclidean) *state space* $\mathbb{R}^n$ of the system, f is a continuously differentiable n-vector valued function of an n-vector argument ($f:\mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$) and dot denotes time derivative.

A state vector x_e is an *equilibrium point* of the system (1) if, and only if,

$$(2) \qquad f(x_e) = 0 \quad .$$

Without loss of generality, we may translate the origin of the state space to x_e and take $x_e := 0$. (In large angle attitude control system analysis the equilibrium point of interest will be a point of zero body rates and zero attitude errors relative to a prescribed inertial target attitude for the spacecraft.)

The origin 0 of state space is an *asymptotically stable* equilibrium point of the system (1) with respect to a domain Ω in state space if, and only if, for all intitial points $x(0) := x_0$ in Ω the corresponding solution trajectories of (1) tend to 0 as t tends to infinity. That is, the system eventually returns to equilibrium from any initital point in the domain Ω, which is termed a *region of asymptotic stability* (RAS) of 0 for the system.

The maximal RAS of 0 is called the *domain of attraction* (DOA) of 0 for the system.

In order to apply *Lyapunov's second* or *direct method* to the identification of an RAS for the system (1) an appropriate definition of an "energy-like" function is needed. A real valued function V defined on a domain Ω of state space ($V : \Omega \rightarrow \mathbb{R}$) is *positive definite* on Ω if, and only if, $V(x) > 0$ for all non-zero state vectors x in Ω . Such a function V is *negative definite* if, and only if, -V is positive definite.

The following theorem, due to Lasalle, gives a method for using more closely specified "energy-like" functions to identify regions of asymptotic stability, see Lasalle & Lefschetz (1961), Chapter 2, §9.

THEOREM: (Lasalle). *Let Ω be a domain in $\mathbb{R}^n$ and let V be a real valued function of an n-vector argument which is continuously differentiable and positive definite. Consider the open region*

$$(3) \qquad \Omega^* = \{x \in \Omega : V(x) < v\}$$

inside the contour of V at level $v > 0$. If Ω^* is bounded -- i.e. V has closed contours--and the time derivative $\dot{V}$, given by

$$(4) \qquad \dot{V}(x) := \nabla V(x) f(x)$$

*of V is negative definite on Ω^* except at the origin 0, then 0 is an asymptotically stable equilibrium point of the dynamical system (1) and Ω^* is an RAS of 0.*

In expression (4) $\nabla V(x)$ denotes the *gradient* of the function V, i.e., the n-row vector of partial derivatives of V given by

$$(5) \qquad \nabla V(x) := (\partial V(x)/\partial x_1 \ldots \partial V(x)/\partial x_n) \ .$$

Figure 1:

Illustration of the Use of an "Energy-like"
Lyapunov Function to Determine an RAS
of a Dynamical System Using Lasalle's Theorem

System Trajectories
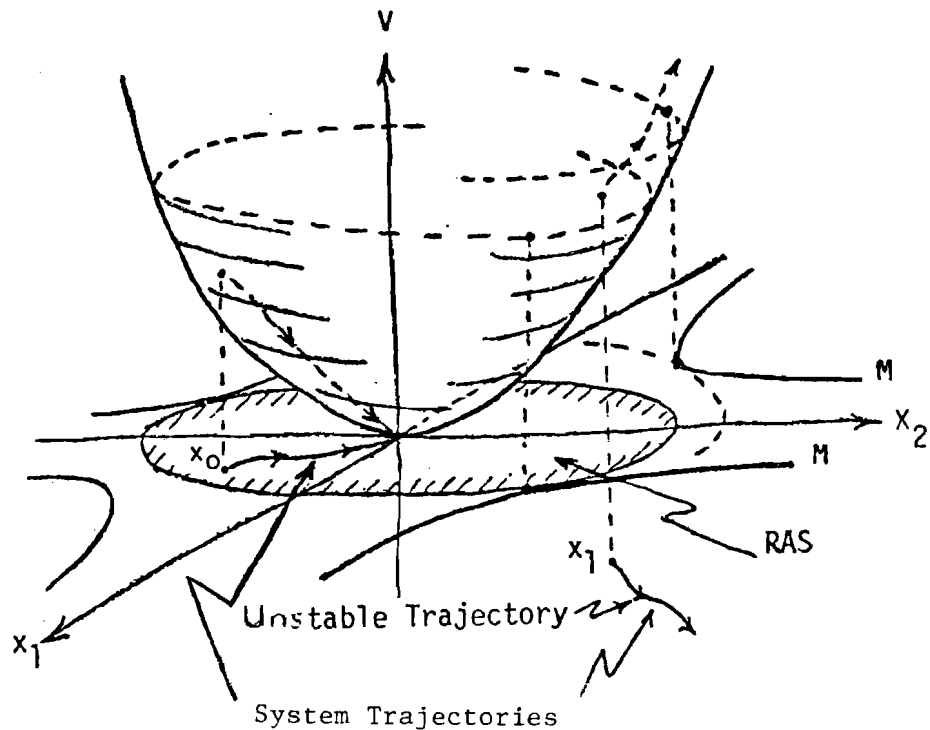
The function V of Lasalle's theorem is the required "energy-like" function; it is called a *Lyapunov function* for the system (1) in honour of its creator. The situation is illustrated in Figure 1. Note that a trajectory which begins at a point x_0 *inside* the RAS decreases the value of V--"energy"--as it converges to the origin, while a trajectory beginning at a point x_1 *outside* the RAS need not converge.

3. DOMAIN OF ATTRACTION ESTIMATION

If the dynamical system (1) is not *globally (asymptotically) stable*, i.e. stable from an initital position anywhere in the state space, or "stable-in-the-large", Lasalle's theorem implies that there exists a state vector x at which $\dot{V}(x) = \nabla V(x) f(x) > 0$ -- that is, the time rate of change of "energy" is *increasing*. Since

the Lyapunov function V is assumed continuously differentiable,
it follows that there must exist a nonlinear *manifold*

(6) $M: = \{x \in \mathbb{R}^n : \dot{V}(x) = 0, \; x \neq 0\}$

on which $\dot{V}$ vanishes. This manifold is in general *multibranched*
(see Figure 1) and it is a practically important open mathe-
matical problem to completely characterize the manifold M for
suitable classes of Lyapunov functions and dynamical systems
relevant to aerospace and other applications, *cf*. Shields (1973).

Now it follows from the above that a *maximal* RAS Ω^* corres-
ponding to a *specific* Lyapunov function $\dot{V}$ may be generated by
isolating the appropriate *local solution(s)* x* of the nonlinear
programming problem

(7) $\min_x V(x)$ subject to $\dot{V}(x) = 0$.

Note that since V and $\dot{V}$ vanish at 0, an equilibrium point of the
system, it follows that 0 is a *trivial solution* of (7). Thus we
actually seek a solution x* of the nonlinear programming problem

(8) $\min_x V(x)$ subject to $\dot{V}(x) = 0, \; x \neq 0$

or, using (6),

(8') $\min_{x \in M} V(x)$.

Figure 2 depicts the state space projection of an *actual* situa-
tion similar to that shown in Figure 1. The system is the second
order van der Pol equation (represented in canonical form in $\mathbb{R}^2$)
and a specific quadratic Lyapunov function is used, *cf*. Davison
& Kurak (1971). The actual DOA is bounded by the dotted
"prismoidal" curve and the maximal ellipsoidal RAS is shown
shaded. Note that the maximal RAS is determined by the oscu-
lation of two radially symmetric branches of the $\dot{V} = 0$ manifold
M at x* and -x*. In general the discrete solution set of the
programme (8) contains one or more radially symmetric pairs,

Shields & Storey (1975). The other pair of radially symmetric branches osculates another elliptical level set of the given Lyapunov function at x^0 and $-x^0$, but this lies in part outside the DOA and hence is not a valid RAS. This illustrates the general fact that the optimization problem (8) has local (up to first order) solutions which are *not* global. In this paper, methods are devised both for eliminating the trivial solution of problem (7) at the origin and for locating a global, rather than a local, solution of the programme (8).

In order to obtain a *maximal* estimate of the DOA of the dynamical system (1), we may consider a parametric class of Lyapunov functions, solve the problem (8) for each of them, and choose the "largest" with respect to a suitable measure of the size of the candidate regions of asymptotic stability corresponding to the Lyapunov functions chosen at each step. More formally, consider a parametric class of Lyapunov functions

Figure 2:

Geometric Illustration of the Mathematical Programming Problem Determining a Maximal RAS for the van der Pol Equation Using a Specific Quadratic Lyapunov Function
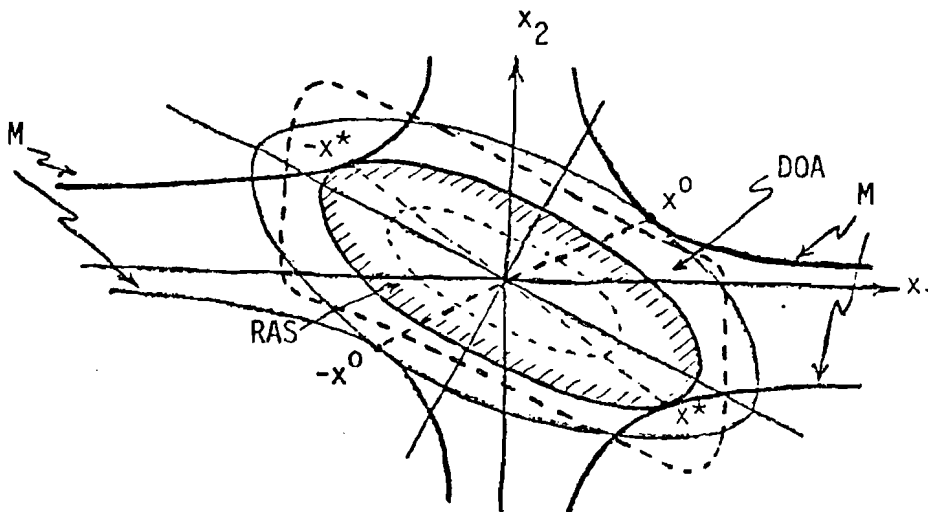
(9) $\mathcal{Y} := \{V_z\}_{z \in Z}$

involving a parameter vector z in a set Z. Let the optimal value
of the programme (8) for the Lyapunov function parametrized by z
be denoted by

(10) $V_z^* := V_z(x^*)$.

Thus to obtain the best estimate of the DOA using Lyapunov
functions from the class $\mathcal{Y}$ we must solve the nonlinear
(unconstrained) programming problem

(11) $\max_{z \in Z} F(V_z^*, z)$,

where F is a suitable measure of the size of the maximal RAS
corresponding to a specific Lyapunov function given by the solu-
tion to (8).

Thus, combining the problems (8) and (11), we see that in
order to obtain a maximal estimate for the DOA of the dynamical
system (1) using a given class of Lyapunov functions, we must
solve the difficult *nested* nonlinear optimization problem

(12) $\max_{z \in Z} F(\max_{x \in M} V_z(x), z)$,

being careful to isolate the *global* solution of the *inner loop*
(RAS) optimization problem (8).

The major difficulty in applying Lyapunov's direct method
for stability analysis of dynamical systems to practical problems
is that in general there is no systematic method for finding
appropriate Lyapunov functions and solving related problems for
various classes of differential equations and systems. A general
review of the properties of dynamical systems relevant to aero-
space applications and of computable Lyapunov function classes
for them was made in Dempster *et al* (1979). Suffice it to say
here that of the three types treated in the literature:-

1. Quadratic (Rodden, 1965; Weissenberger, 1969;
   Geiss *et al*, 1971; Davison & Kurak, 1971;
   Shields & Storey, 1975)

2. Polynomial (Zubov, 1955)

3. Piecewise Linear (Rosenbrock, 1962)

quadratic Lyapunov functions have proved the most reliable to date.

The use of quadratic Lyapunov functions in turn requires the generation of arbitrary sign definite matrices and the solution of a matrix equation (Lyapunov's equation). Computationally efficient procedures (and codes) for performing these operations were carefully selected (and obtained) for the present work (see Appendix I of Dempster *et al*, 1979).

The practical advantages of employing quadratic Lyapunov functions in stability studies are twofold. First--and of greatest importance--is that at constant value the Lyapunov fuction $V_P(x)$ given by the *quadratic form* x'Px represents a hyper-ellipse in state space (see Figure 2). Hence it is easy to visualize and its "size", i.e. a monotone function of its *hypervolume* given at a *(global)* optimum of the inner loop program (8) by

(13) $h(P) := n \log V_P^* - \log \det P$

where V_P^* denotes the corresponding optimal value, is easy to compute.

Secondly, quadratic forms are easily generated. The method employed in this paper is to select an arbitrary negative definite matrix -Q and then solve the *Lyapunov matrix equation*

(14) $A'P + PA = -Q$

for the kernel P of the quadratic form. To see how this equation arises, consider the dynamical system (1) written in first order Taylor series expansion as

(15) $\dot{x} = Ax + g(x)$

where the n×n matrix $A := \nabla f(0)$ and g contains second and higher order terms. Neglecting g in (15) and computing $\dot{V}$ directly yields

$$\dot{V}(x) = \frac{d}{dt}(x'Px) = \dot{x}'Px + x'P\dot{x}$$

(16) $= x'(A'P+PA)x := -x'Qx$.

Hence it follows from Lasalle's theorem (§2) that the linearization about the origin of the nonlinear system (1) is globally asymptotically stable if, and only if, Q in (16) is positive definite. To generate by solving (14) a positive definite matrix P(Q), given a positive definite matrix Q, an $0(n^3)$ iterative algorithm due to Smith (1971) is available.

The best technique for generating positive definite matrices Q was utilized in an earlier (small angle attitude control) study by Geiss *et al* (1971). It is well known that all real symmetric matrices are orthogonally similar to a diagonal matrix, whose entries are its *eigenvalues*, and thus that all positive definite matrices are orthogonally similar to a diagonal matrix with positive diagonal elements. Hence the parametrization of all n×n positive definite matrices may be effected in the required n(n+1)/2 parameters by combining a parametrization of the group of orthogonal matrices with the n diagonal elements of a diagonal matrix in the form

(17) $Q(z) := G'(\theta,\emptyset) \Lambda G(\theta,\emptyset)$

where the row vector $z' := (\lambda',\theta',\emptyset')$, $\Lambda := \mathrm{diag}(\lambda_1,...,\lambda_n)$ and G is an orthogonal matrix defined by an (n-1)(n-2)/2 - vector θ and an (n-1)- vector $\emptyset$ (see Dempster *et al*, 1979, for details). The advantage of this parametrization over other possible, but badly behaved, parametrizations in the same number of parameters is that with it separate adjustment of the lengths and orientations of the principal axes of the hyperellipsoidal Lyanpunov function contours is possible.

We are now in a position to define precisely the objective function $F: \mathbb{R}_m \to \mathbb{R}, z \mapsto F(z)$ of the unconstrained problem (11) as

(18) $F(z) := hoPoQ(z)$.

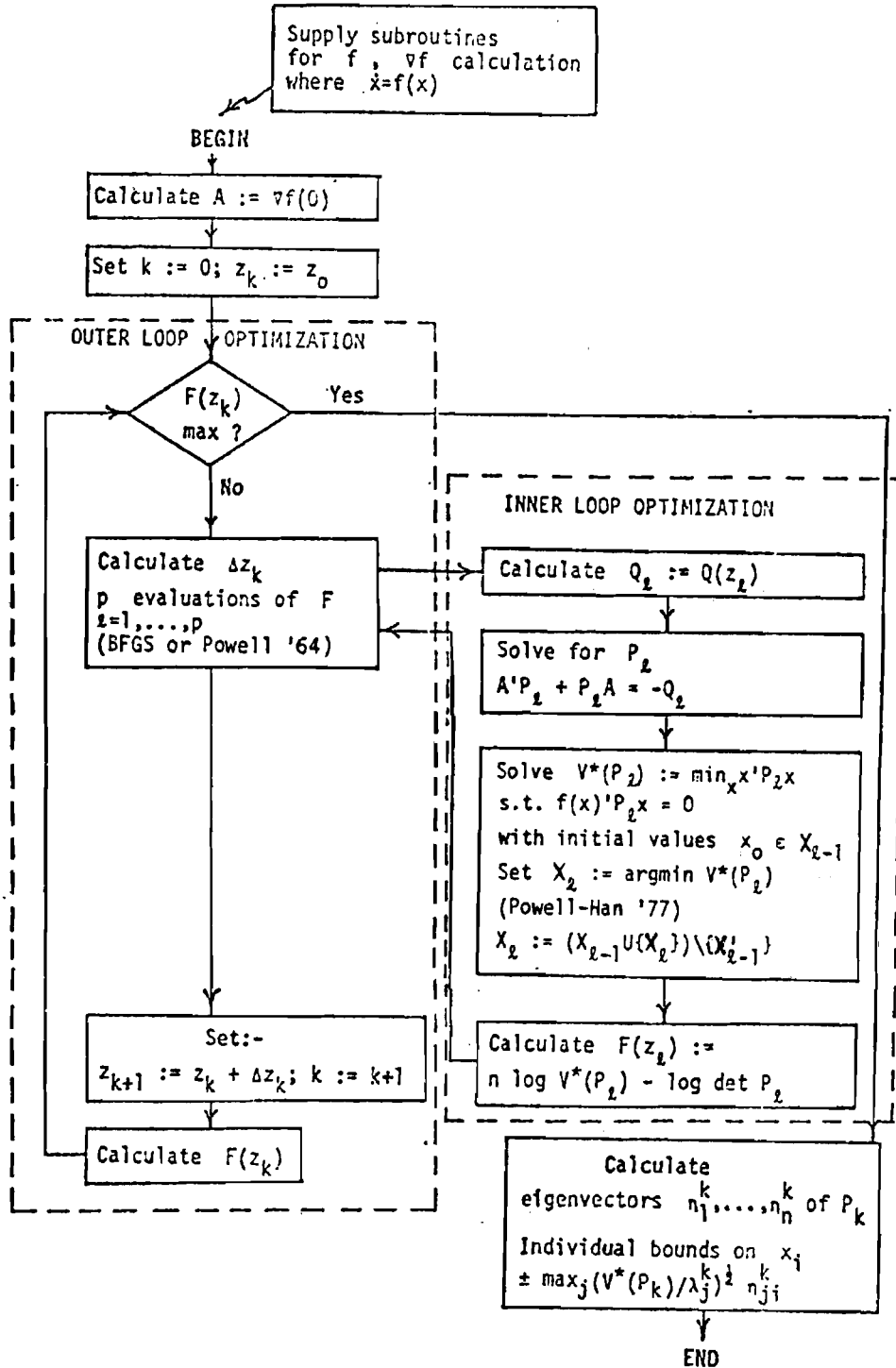## 4.   THE DOMATT SYSTEM AND NUMERICAL EXPERIMENTS

This section discusses the implementation of the technique for assessing the stability behavior of a (controlled) dynamical system based on Lyapunov's direct method discussed in the previous two sections.

Having presented the necessary background material in Section 3, Figure 3 outlines the structure of an algorithm to estimate the domain of attraction of an equilibrium point of a dynamical system. There are two parts to this algorithm. For any given (quadratic) Lyapunov function we wish to find the maximum *region of asymptotic stability (RAS)*. Measures of the size of this region estimate the size of the *domain of attraction* of the equilibrium point relative to the given Lyapunov function. By interpreting the relevant theorem (Lasalle's theorem) geometrically the RAS problem can be formulated as a nonlinear programming problem. This forms the *inner loop* of the algorithm.

The second part, or *outer loop*, of the algorithm seeks the quadratic Lyapunov function which yields the largest RAS, and hence the optimal estimate of the domain of attraction. Again this is formulated as a nonlinear programming problem.

The key to a successful procedure for estimating the domain of attraction lies in the choice of algorithms for performing the two nonlinear optimizations described above. The selection, specification, implementation and testing of the optimization routines used in the method was carefully addressed. After an extensive general review of current optimization techniques relevant to DOA estimation for the controlled dynamics of satellite large angle manoeuvres (conducted early in the study), it was decided that the latest proven constrained and unconstrained techniques of classical type afforded the best change of efficient optimization calculations. In particular, for the relatively few

Figure 3

Flowchart of the Basic DOA Estimation Algorithm

Supply subroutines
for f , ∇f calculation
where $\dot{x} = f(x)$

BEGIN

Calculate $A := \nabla f(0)$

Set $k := 0$; $z_k := z_0$

OUTER LOOP OPTIMIZATION

$F(z_k)$ max ? — Yes

No

Calculate Δz_k
p evaluations of F
$\ell = 1, \ldots, p$
(BFGS or Powell '64)

INNER LOOP OPTIMIZATION

Calculate $Q_\ell := Q(z_\ell)$

Solve for P_ℓ
$A'P_\ell + P_\ell A = -Q_\ell$

Solve $V^*(P_\ell) := \min_x x'P_\ell x$
s.t. $f(x)'P_\ell x = 0$
with initial values $x_0 \in X_{\ell-1}$
Set $X_\ell := \arg\min V^*(P_\ell)$
(Powell-Han '77)
$X_\ell := (X_{\ell-1} \cup \{X_\ell\}) \setminus \{X_{\ell-1}'\}$

Calculate $F(z_\ell) :=$
$n \log V^*(P_\ell) - \log \det P_\ell$

Set:-
$z_{k+1} := z_k + \Delta z_k$; $k := k+1$

Calculate $F(z_k)$

Calculate
eigenvectors $n_1^k, \ldots, n_n^k$ of P_k
Individual bounds on x_i
$\pm \max_j (V^*(P_k)/\lambda_j^k)^{\frac{1}{2}} n_{ji}^k$

END

nondifferentiable points in the baseline system dynamics (see §5) due to sign (gas jet) and to saturation, window and Heaviside (reaction wheel) function modelling of control actuators, infant (and largely inefficient first order) nondifferentiable optimization techniques were not deemed worth the computational overheads. (In this regard see Lemarechal (1979b).) Instead smooth and other suitable approximations to the baseline system control actuator nonlinearities have been utilized.

Further, random line and grid search methods have not proven the most effective of search techniques for locating global optima, let alone for locating optima of more well-behaved functions. In any event, research performed in the present study has shown that the bulk of the difficulties in the face of multiple optima reported in the recent literature (cited in the previous section) with the older optimization techniques used previously for DOA estimation have largely been due to a failure to parametrize the problem properly and to limit step sizes sensibly. Undoubtedly this has come about through a general lack of problem understanding which was advanced, but not completed, in the present study.

The above considerations were fully developed in Dempster *et al* (1979) where basic requirements for numerical optimization algorithms in this context were set out. Two alternative algorithms were proposed for the outer loop--the Powell (1964) *conjugate direction method*, and the BFGS *quasi-Newton* (or variable metric) *method* (see, e.g. Adby & Dempster, 1974) and one for the inner loop--the new Powell-Han *projected Lagrangian algorithm*, Powell (1977,1978). The performance of these algorithms was evaluated on a number of carefully specified test problems (details of which are to be found in Appendix II to Dempster *et al* , 1979).

For computation using the Powell-Han algorithm, the inner loop programme (8) has been replaced by the analytically equivalent problem

(19) $$V^*(P) := \min_x x'Px$$

$$s.t. \quad f(x)'P\,x\,p(x) \geqq 0$$

where p is a nonnegative radially decreasing function with a single pole at the origin of at least the same order as the zero at the origin of f(x)'Px. Multiplication of the equation constraint of (8) by p removes the trivial solution of (7) at the origin, while use of the inequality appears to result in improved performance of the Powell-Han algorithm.

Of course the Powell-Han routine is only capable of finding *local* optima of (19) corresponding to solutions on various branches of the manifold $M = \{x \in \mathbb{R}^n : f(x)'Px = 0\}$. In general several local minima exist (in radially symmetric pairs) and the global solution must be selected from these as yielding the smallest value of V. As the outer loop algorithm varies the positive definite matrix Q, the Lyapunov function kernel P is deformed via the Lyapunov matrix equation and the branches of the manifold M move in a manner which makes it impossible to predict on which branch the global solution lies. While a small change in Q may move the manifold M by only a small amount, the global solution may, as a result, jump to a different branch of M. This difficulty may be overcome by keeping track of all candidate local optima as Q varies, and by choosing at each outer loop function evaluation the global inner loop optimum from this set.

In the computer implementation of the DOA estimation algorithm an initial set X_o of state space starting points for the Powell-Han algorithm is supplied by the user. At each outer loop function evaluation ℓ, the result of every successful inner loop optimization is added to the current set X_ℓ, while all other elements of X_ℓ in a neighbourhood of this local optimum are deleted in order to prevent unlimited growth of the set X. In numerical experiments, it has been observed that when the manifold M moves slowly with changes in Q, each local solution of (19) replaces the starting point which led the Powell-Han

algorithm to it.(In the code, the number of starting points may be limited by the user.)

Since at the outer loop optimum there are usually multiple *global* solutions to the inner loop problem (8), the outer loop optimum is often located when two or more inner loop solutions of (19) (other than radially symmetric pairs) yield the same value of V.

Figure 3 suggests schematically how the algorithm outlined in Section 3 and incorporating the optimization routines discussed above are implemented as the structured FORTRAN code DOMATT. The general design principles and skeletal outline of the DOMATT system as a piece of applications software are indicated in Figure 4, while the module structure of DOMATT's FORTRAN code-- including the Atomic Energy Research Establishment, Harwell, U.K. optimization routines VA04A, VA13AD, VFO2AD and POWHAN -- is depicted in Figure 5.

The DOMATT code was tested on 8 specially chosen dynamical system test problems culled from the open literature. The criteria used in selecting these test problems were as follows:

1. To include at least one system to which the solutions of the RAS problem and the DOA are known analytically.

2. To include a low order system for one of the control actuator categories represented in the aerospace systems, i.e. gas jet (see §5) as a simple test of the performance of the DOMATT system in the presence of "hard" nonlinearities.

3. To grade the orders of the test problems (2 through 5) in such a way as to give some indication of the increase in computation time with increase in problem complexity.

(The specification of the test dynamical systems can be found in Appendix III of Dempster *et al*, 1979.) The computations were performed to an accuracy of 10^{-3} by an ICL2980 computer running under the VME/B operating system.

The DOMATT code solved all 2-dimensional problems to the
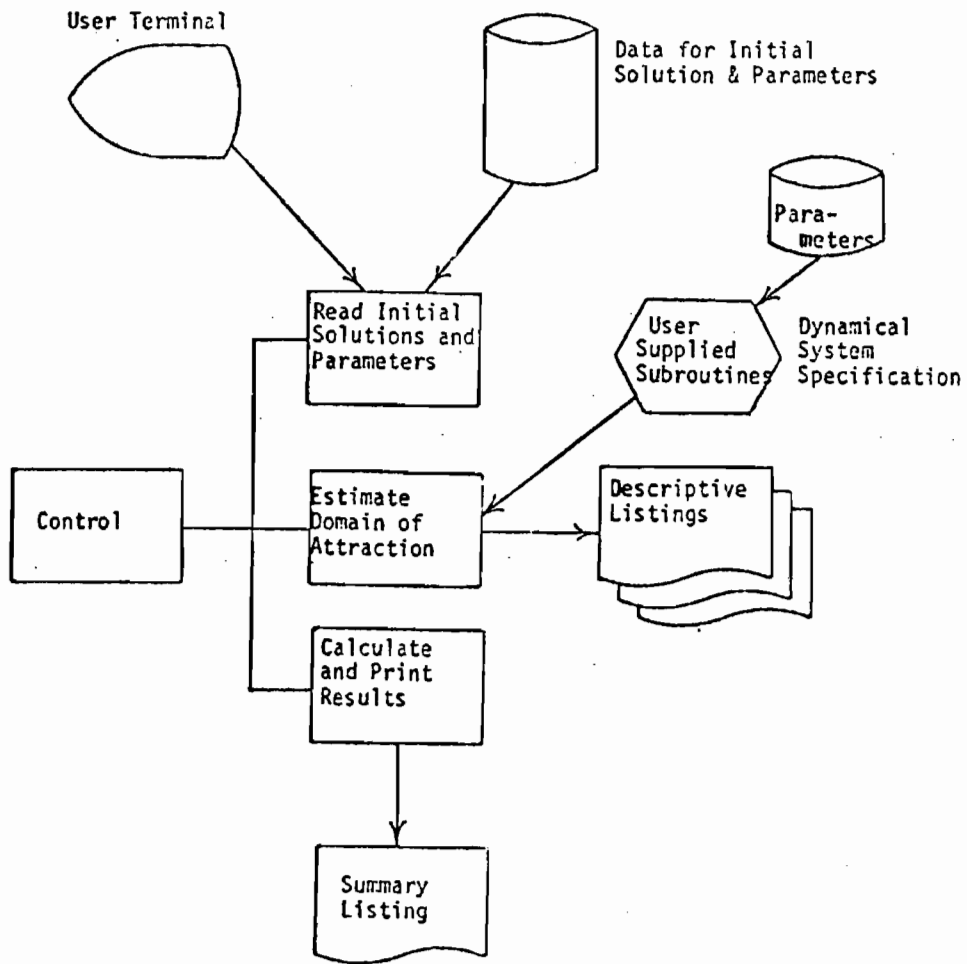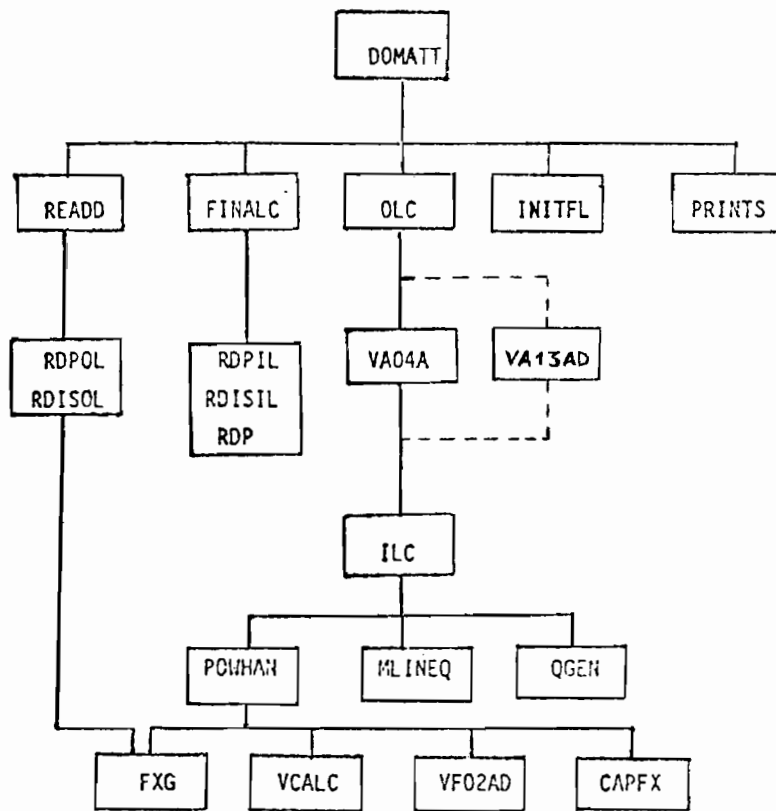
Figure 4:

DOMATT System Diagram

Figure 5:
DOMATT Module Chart

required accuracy except the jet-type problem involving a "hard" sign function nonlinearity. The outside bound error termination difficulties encountered with the Powell-Han algorithm on this problem were exacerbated on the higher dimensional "high curvature problem" -- a 3-dimensional cubic model of a servo-mechanism. DOMATT failed to make significant progress on the remaining 3-dimensional problems and the 4-dimensional problem over the course of several runs. It is not clear to what extent further algorithmic tuning could reduce computation times, although there was some indication on the largest test problem that the inner loop Powell-Han algorithm was taking prematurely small steps. More importantly, relatively slow progress was experienced at the outer loop by the Powell '64 conjugate direction method. If (approximate) gradient information from perturbation analysis of the inner loop problem (8) were available, faster outer loop progress could be expected with the BFGS quasi-Newton method on the basis of comparative tests of the two methods performed in the study and general experience, see. e.g. Adby & Dempster (1974). However, it is not clear what difficulties would be caused by nondifferentiability of the outer loop RAS measure due to multiple global solutions of the inner loop problem. This remains a topic for future research to be discussed further in Section 6.

An analysis of the average time per outer loop -- i.e. candidate quadratic Lyapunov -- function evaluation, showed this average inner loop run time to a global optimum to increase very reasonably with state space dimension. Due to limited computer budget remaining after two sets of numerical experiments and extensive simulation analysis, it was therefore decided at this point in the study to proceed to the application of the DOMATT code to the baseline aerospace systems. These will be set out in the next section, along with the results of the relatively unsuccessful numerical experiments so far performed.

5. AEROSPACE APPLICATION TO SPACESCRAFT LARGE ANGLE MANOEUVRES

One of the fundamental limitations on the amount of scientific data returned from orbiting astronomical observatories is

is due to the time spent in performing slew manoeuvres between stellar targets of interest. Current state-of-the-art attitude control laws implemented on such missions are simple but reliable and are designed to admit the required large angle manoeuvres by slowly rotating the spacecraft about one axis at a time in an Euler sequence. Many different control laws have been proposed in the literature to reduce manoeuvring time, the most attractive of which is to perform the required attitude change by rotating the satellite simultaneously about all three control axes. In this case however the control laws to realize the solution are almost certainly nonlinear, as are the equations describing the large angle dynamics and kinematics of the satellite itself. The overall control problem is immediately more complicated; in particular, it is difficult to assure the desired system stability behavior from all possible initial conditions.

In applications requiring both accurate fine pointing and a large angle slew capacity, satellites are generally controlled actively about three orthogonal axes. Control torques are provided either internally, by momentum exchange devices such as reaction wheels or control moment gyros, or externally, by mass expulsion devices such as cold gas or hydrazine thrusters. Coarse pointing control systems also make use of environmental torques, such as are due to solar radiation, gravity gradient and the Earth's magnetic field.

Two control actuator types -- mass expulsion and momentum exchange devices -- were examined in detail in Dempster *et al* (1979) and models suitable for control system analysis and design were developed for particular devices. In the modelling task emphasis was placed on the requirement that the models be simple but representative. Significant nonlinearities have been retained.

In designing a large angle attitude control system two control law synthesis techniques may be considered; the first is to use *optimal control* theory on the grounds that designs that minimize time, fuel and energy are of obvious interest in this application, the second is to rely on intuition and technically sound *heuristics*. The *ad hoc* designs resulting from the

second approach are more attractive from a practical point of view because they can be structured in feedback form. By comparison, optimal control solutions are difficult to implement. The associated computational requirements are generally large, certainly beyond the capability of the envisaged on-board computers. In particular, optimal controls are usually of the open loop type and as a consequence are sensitive to parameter changes and disturbances unless made closed loop through the model reference approach -- at significant computational cost.

We turn now to the mathematical definition of the two baseline large angle satellite attitude control systems developed in the study -- one *reaction jet* controlled and the other *reaction wheel* controlled. Realistic actuator models are incorporated in each. In particular, it will be assumed that reaction jets operate like ideal relays (without dead zones or rise times) and reaction wheels are subject to torque saturation and wheel speed limitations.

Consider first the *body rate dynamics* of a rigid vehicle measured relative to a body-fixed, body-centered reference frame b. These are given by

$$(20) \qquad \dot{\omega} = J^{-1}u + J^{-1}S(\omega)h_b,$$

where ω denotes the 3-vector of vehicle angular velocities -- the *body rates*, J is the 3 × 3 matrix of vehicle *inertias* (resolved in the body frame), u is the 3-vector of *control actuator outputs*, h_b is the 3-vector of *vehicle angular momenta* resolved in the body frame, and $S(\omega)h_b$ denotes the matrix-vector representation of the vector cross product $\omega \times h_b$, i.e. S is given by

$$(21) \qquad S(x) = \begin{bmatrix} 0 & x_3 & -x_2 \\ 0 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}.$$

The corresponding *kinematic equations* are given in terms of *Gibbs vector* p attitude representation as

(22) $\dot{p} = \frac{1}{2} [I_3 - S(p) + pp']\omega$,

where I_3 denotes the 3-dimensional identity matrix.

Thus the state space of the (controlled) dynamical system given by (20) and (22) is Euclidean 6-dimensional space $\mathbb{R}^6$ and the desired equilibrium point is

(23) $x := \binom{\omega}{p} = 0$,

where $p = 0 (\epsilon\ \mathbb{R}^3)$ is the *target* inertial vehicle attitude and $\omega = 0 (\epsilon\ \mathbb{R}^3)$, i.e. *rest*, is the target body rate vector. It follows that the vector p represents vehicle attitude *error* relative to an inertially fixed body-centered target reference frame t taken to coincide with a *fixed* inertial reference frame s. On actual astronomical missions, definition of current attitude error relative to an *arbitrary* inertial target frame is easily computed on-board by simple formulae (set out in Dempster *et al*, 1979).

After investigation of various slew control laws proposed in the literature, the *Mortensen '63* control law was chosen to define the *control actuator input* signal as

(24) $s := K\omega + k(1 + p'p)p$,

where $K = diag\{k_1, k_2, k_3\}$ is a diagonal matrix of strictly positive diagonal entries and k is a positive constant. The control law (24) defines a *feedback* control with proportional rate, and nonlinear position, gain. It can be shown -- by a new simple proof, see Dempster *et al* (1979) -- to be globally asymptotically stable *in the absence of actuator nonlinearities*, i.e. when control reaction torque output u equals minus control input signal s . A principal aim of the research described in this paper and currently in progress is to determine the stability properties of control systems incorporating (24) with realistic nonlinear actuator output modelling.

To specify the controlled dynamics of the baseline systems, it remains only to specify the control actuator output u , as a function of the control input signal s given by (24), and the related vehicle angular momentum vector h_b , in the body rate dynamical equation (20).

Gas Jet (External Torque) Baseline Attitude Control System

In this case, the control actuator output is given by

(25) $u := -sgn(s)$

as a 3-vector of simple sign functions each of which operates on the appropriate coordinate of the input signal s_i to produce the idealized gas jet torque $\pm T_{max}$ (see Figure 6.1) for each of the three thruster pairs. The angular momentum vector corresponding to such external torque devices is given by

(26) $h_b = J\omega$,

to result in *closed loop dynamics* of the form

(27) $\dot{\omega} = -J^{-1}sgn(s) + J^{-1}S(\omega)J\omega$.

Reaction Wheel (Angular Momentum Exchange) Baseline Attitude Control System

This system is considerably more complicated that the gas jet system and involves two aspects newly developed during the course of the study. The complicated actuator output modelling -- which ignores relatively insignificant electric motor nonlinearities -- was developed with the benefit of preliminary simulation studies.

Consider first the angular momentum vector corresponding to momentum exchange torques. By the conservation law the *system* angular momentum resolved in inertial coordinates is a *constant* vector h and the transportation b = As between the inertial coordinates s and the current coordinates b of the body reference frame may be expressed in terms of current attitude, see

Figure 6:

Graphical Representations of "Hard" Nonlinearities
Used in Control Actuator Modelling

6.1

$\text{sgn}(s_i)$

6.2

$\text{sat}(s_i)$

6.3

6.4

$1-H(s_i\omega_i^W)$
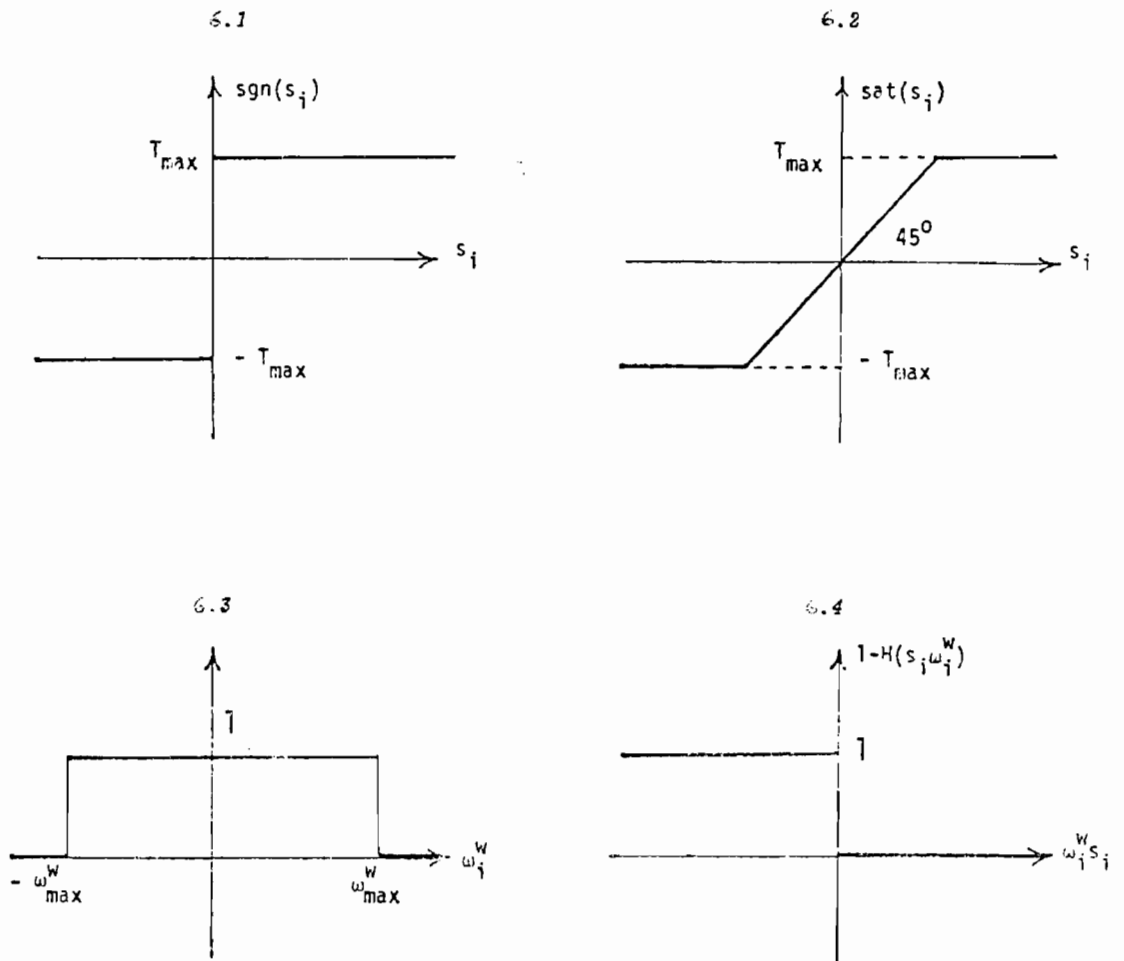
Dempster (1980), Table 1. Hence it follows that the appropriate angular momentum vector is given by

(28) $\qquad h_b = (1+p'p)^{-1} [(1+p'p)I_3 + 2pp' + 2S(p)]h$.

If a slew manoeuvre is begun from *rest*, the vehicle angular momentum is initially zero and all system angular momentum must be stored in the spinning reaction wheels. (In practice, auxiliary thrusters are typically used to dump system momentum from time to time in the course of a mission.) Expression (28) obviates the need to *explicitly* consider differential equations for wheel rates of momenta -- thereby enlarging the state space of the system to $\mathbb{R}^9$ -- as has been done previously in the literature.

In control actuator output modelling however one needs an expression for the *wheel speed* vector ω^W which may be expressed in terms of the *reaction wheel inertia* matrix J^W and the *vehicle inertia* matrix (with locked wheels minus the wheel inertias) J as

(29) $\qquad \omega^W = (J^W)^{-1} [h_b - (J+J^W)\omega]$.

We are now is a position to set out reaction wheel control actuator output, in terms of controller input signal and current wheel speeds, as the following cascade of "hard" nonlinear functions, *viz.*

(30) $\qquad u := -sat(s)h(\omega^W, s)$.

The saturation function is a 3×3 diagonal matrix of saturation functions representing *torque limited responses*, each of which operates on the appropriate coordinate of the input signal s_i to produce the idealized reaction wheel torque (see Figure 6.2). The second factor h in (30) is a 3-vector of 0-1 valued functions h_i representing wheel motor shutdown and restart for the purposes of *wheel speed limitation* to prevent wheel break up. Each coordinate function is of the form

(31) $\qquad h_i(\omega_i^W, s_i) := \max\{\text{win}(\omega_i^W),\ 1 - H(s_i \omega_i^W)\} \qquad i = 1, 2, 3$.

The window function serves to *limit* wheel speed to within $\pm\omega_{max}^W$ (see Figure 6.3), while one minus the Heaviside function H calls for *restart* of a free spinning wheel in the opposite direction of rotation (see Figure 6.4). The effect of the maximum in (31) is to allow wheel torque output when *either* (or both) of these conditions hold.

Numerical Experiments

Note that the (closed loop) dynamical systems generated by both baseline attitude controllers, respectively, (22),(27) and (20),(22),(29-31) are of the form

(32) $\qquad \dot{x} = \tilde{f}(x) + \tilde{g}(x)$,

where $\tilde{f}$ is an analytical function of the state vector x and $\tilde{g}$ is a discontinuous, piecewise continously differentiable function. The theory of §§2 and 3 on the other hand applies only to dynamical systems whose right hand sides are continuously differentiable. Unfortunately, an analytical theory of piecewise quadratic Lyapunov functions for discontinuous systems is well developed only for the case of a unidimensional discontinuity, see Weissenberger (1965,1969) and Dempster *et al* (1979). As can be seen from inspection of Figure 6 however, for both baseline systems the discontinuous or nondifferentiable points of $\tilde{g}$ are explicitly known and few in number. It is therefore possible to provide continuously differentiable arctangent approximations to the discontinuous/nondifferentiable functions, although these are extremely complicated, see Dempster *et al* (1979), Appendix IV, for details. In the numerical experiments with the DOMATT code therefore a gradient evaluation in a specified neighbourhood of a nondifferentiability was replaced by a median sub- or supergradient value, while in such a neighbourhood of a discontinuity a gradient evaluation was replaced by a specified large number of appropriate sign.

For detailed numerical parameter values used in the mathematical models of the baseline large angle attitude control systems, the reader is referred to Dempster *et al* (§1.5). It is sufficient here to note that all numerical experiments were conducted with system parameters corresponding to a large vehicle of the NASA Orbiting Astronomical Observatory (OAO) type.

Early in the study it was decided that simulation of the closed loop dynamics of the baseline systems *prior* to DOA estimation of their dynamics was the most scientifically (and cost) effective sequence. The main advantage seen was that if realistic highly *nonlinear* actuator modelling resulted in serious instabilities in the controlled dynamics of the baseline systems -- known to be globally asymptotically stable with *linear* actuators -- this would be uncovered before applying the DOMATT code developed for DOA estimation. In any event, the stability behaviour uncovered by the simulations would be useful in guiding the tuning of the DOA estimation procedure for use on realistic aerospace systems. An added advantage -- actually realized -- was correction in both modelling and parameter setting of the baseline systems previously agreed, before DOA estimation computer runs were attempted. It should be borne in mind that the thrust of the present research is nevertheless ultimately to replace computationally expensive and scientifically *inconclusive* stability analysis of dynamical systems in engineering, economics and policy analysis by *simulation* with a computationally efficient and *decisive* procedure.

The objective of the simulations performed in the present application was to investigate the stability boundaries of the proposed controllers of a hypothetical spacecraft during 3-axis slew manoeuvres. The controllers under consideration were:-

1. the gas jet baseline system,
2. the reaction wheel baseline system with *zero* initial spacecraft momenta,
3. the reaction wheel baseline system with *large* initial spacecraft momenta.

It was assumed in all cases that the initial angular velocities of the vehicle were zero, and that the desired rotation angle was as close to 180° (*reverse-point* manoeuvre) as possible. A unit eigenaxis and an angle of 179° were arbitrarily chosen to define the initial Gibb's vector representation p(0) of the inertial attitude error of the body relative to the target intertial body attitude. Simulations of all the 8 combinations of sign of the components of p(0) were made to verify the stability of the proposed controllers. All simulations were run on a PDP 11/45 computer with floating-point hardware running under the IAS operating system. The simulation program used was Oxford Systems Associates' Extended System Modelling Program (ESMP) which is a general-purpose block-oriented simulator allowing for extensive user interaction. In summary, the simulation experiments *indicated* that -- with suitable parametrization -- *both* baseline large angle attitude control systems are globally stable. A more detailed discussion may be found in Dempster (1980) and Dempster *et al* (1979).

Given the difficulties uncovered in the application of the DOMATT code to the eight test continuously differentiable problems described in §4, it was not expected that the application to the nondifferentiable baseline systems would prove very successful without a deeper understanding of the problems experienced by the inner loop optimization routine. Thus, while both systems were coded in the study, the DOMATT code has been applied so far only to the gas jet system. The results of this application will be described briefly in the sequel, but it was felt that it would be an unwise research strategy to devote too many resources to the application of the DOMATT code to the baseline systems before a more robust code had been tested on the sample test problems.

The gas jet baseline system was therefore presented to DOMATT with the intention of finding at least a crude RAS estimate. The first such run processed 66 inner loop optimizations -- i.e. 66 candidate Lyapunov functions -- in 30s. This is a suspiciously rapid average rate when compared to such rates for the lower dimensional test problems. In fact, in every case

the inner loop global solution was found to lie in state space
within the specified tolerance of the origin. When similar
near zero inner loop solutions were found with the test problems,
a clear contrast emerged between inner loop optimizations find-
ing a genuine solution and those finding the trivial solution
at the origin. Various constraint addition and rescaling
techniques were attempted, but none proved successful. In view
of this, there was little to be gained from making extensive
runs on the coded reaction wheel baseline system without further
research.

Since the completion of these experiemnts, the theoretically
effective nonlinear constraint rescaling technique -- which
removes the trivial solution to the inner loop constraint equa-
tion at the origin -- described above in connection with the
numerical inner loop problem (19) has been developed. The
application of this technique to the gas jet system is a
first priority of research currently in progress.

6.   CONCLUSIONS AND DIRECTIONS FOR ALGORITHM DEVELOPMENT

A wide study of Lyapunov's *direct method* was made with
reference to dynamical systems with both continuous and dis-
continuous right hand sides. The concept of Lyapunov stability
was examined in some detail and methods of generating Lyapunov
functions were reviewed. Quadratic Lyapunov functions were
selected as being most suitable for examining the stability of
general dynamical systems which might arise in engineering or
policy analysis applications.

A careful study was also made of available optimization
routines and as a result three routines were selected for more
extenstive testing -- the Powell-Han constrained optimization
routine and the Powell '64 and BFGS unconstrained optimization
routines. As little was known about the practical capabilities
of these routines in the present application, a set of test
problems was designed with a view to evaluating the strengths
and weaknesses of each. In the event, the routines performed
in a predictable fashion and the Powell '64 and Powell-Han
routines were chosen for the optimization problems envisaged
in DOA estimation.

Lasalle's theorem has been interpreted for quadratic Lyapunov functions as a basis for an inner loop optimization procedure for estimating the *region of asymptotic stability* (RAS) of dynamical systems of the form

$$\dot{x} = f(x) \quad .$$

A well behaved parametrization of quadratic Lyapunov functions has been taken so that the optimal Lyapunov function (that producing the largest RAS) can be found. Thus a procedure for estimating the DOA of an equilibrium point of a (controlled) dynamical system has been devised which implements the latest optimization techniques in the quadratic Lyapunov function approach to DOA estimation. The procedure has been incorporated in a modular FORTRAN computer code DOMATT written to advanced modern software standards.

Before the applications study reported in this paper commenced the analysis of rigid spacecraft dynamics for simultaneous three-axis manoeuvres had largely ignored the nonlinearities known to be present in the operation of control actuators and attitude sensors. While for stability analysis of such manoeuvres it is valid to ignore the relatively negligible sensor nonlinearities, no study aimed at eventual practical implementation of a satellite large angle attitude control system can afford to ignore actuator nonlinearities. In the present study a *new* nonlinear mathematical model of reaction wheel control actuators has been developed which explicitly considers torque saturation and wheel motor speed limitation shutdown and restart.

Having regard for the computational overheads imposed on both DOA estimation and simulation techniques by state space dimension, the minimum dimension vehicle attitude representation -- in terms of Gibb's vectors -- has been utilized in this study. This representation is easily computed from the other common attitude representations, and conversely, see Dempster (1980), Table 1. Moreover, simulation studies performed have demonstrated that in spite of their mathematical singularity at 180° Gibb's vectors can be used to study 180° reverse points to within the

accuracy of a fine pointing control system. A *new* minimal
dimension representation of satellite body rate dynamics for
reaction wheel control systems has also been developed. The
new representation obviates the necessity to consider the wheel
rates *explicitly* as state variables in analysis or simulation.

Two large angle attitude controllers, a *gas jet* mass
expulsion system (external torque) and a *reaction wheel* momentum
exchange system (internal torque), were modelled with respect
to the new representation of body rate dynamics, using Gibb's
vector kinematics and the Mortensen '63 control law. As noted
above, the models account for the most important nonlinearities
in the control actuators. These two systems were adopted as the
baseline systems for the simulations and DOA estimations per-
formed in the latter part of the study.

A *new* simple proof has been provided of the global (asymptotic)
stability of the gas jet and reaction wheel actuated Mortensen '63
large angle attitude control system in the *absence* of actuator
nonlinearities, see Dempster *et al* (1979). The full baseline
systems were extensively simulated using Oxford Systems Associates'
Extended System Modelling Program (ESMP). These simulations
indicate that both the baseline systems are globally stable, in
spite of the inclusion of realistic nonlinearities in the control
actuator modelling. This is in itself a significant result.

In order to test the DOMATT code, tune algorithm parameters
and gain computational experience with the method, a carefully
designed set of dynamical system test problems were run. Although
the procedure both reproduced results of previous researchers on
low order test problems and produced plausible results where the
DOA is not known *a priori*, difficulties with the performance of
the inner loop optimization procedure were encountered on test
problems, and the higher order gas jet baseline system, which
incorporates "hard" nonlinearities. As a result it was decided
to postpone computational experience with the reaction wheel
baseline system -- which incorporate several "hard" nonlineari-
ties -- until a deeper understanding of these problems has been
obtained.

The outstanding research thus remains the refinement of the DOA estimation procedure -- in particular the inner loop Powell-Han constrained optimization method. Suggestions are made below for research directed at improvement of the inner loop optimization techniques. The study has uncovered additional work which might usefully be done to clarify various aspects of the work completed and to extend the principles established to more complex systems. These are discussed in the remainder of the paper.

Directions for Further Research

This study has identified a number of respects in which the DOMATT program should be refined. They are the following:-

(i)   Since the end of the study information has been obtained which shows that difficulties with the Powell-Han algorithm similar to those encountered at the inner optimization loop of DOMATT have been experienced by other researchers, Lemarechal (1979a), Madsen (1979). Two identified sources of deficiency with the best current implementation of the algorithm (Harwell VAO2AD as used in DOMATT) are relevant in the context of this study. The first deficiency is a tendency to singularity in the Lagrangian inverse Hessian update when steep-walled functions such as "hard" nonlinearities are encountered. The practical result is an attempt at a *large* algorithm step outside prescribed bounds for the state variables. The second shortcoming involves prematurely *small* algorithm steps on problems -- such as those posed by the baseline systems -- having highly curved constraint surfaces. The difficulties result from using Han's theoretical results on Lagrangian augmentation and quadratic approximation line search. In this regard it should be borne in mind that Powell-Han-type algorithms are currently universally regarded as the most promising approach to the numerical solution of difficult nonlinear programming problems such as those encountered at the inner loop of DOMATT. Hence a full investigation of international computational experience

and proposals to overcome the above shortcomings is in progress and the results will be incorporated in the code.

(ii) The DOMATT code includes inner loop starting point selection procedures designed to overcome the problem of selecting a local rather than the global optimum by the inner loop algorithm, and also for preventing the slow convergence at the outer loop caused by multiple global optima -- a problem identified by Shields and Storey (1975). These procedures can be refined by further application of combinatorial techniques. An important part of future study is an investigation of how tangency points (local optima) found in the inner loop move as the P matrix (quadratic Lyapunov function kernel) is altered at the outer loop. This is a form of parametric information about the solution of nonlinear programming problems under perturbations -- an area of considerable research activity at the moment. Unfortunately, current perturbation analyses of nonlinear programmes, see e.g. Fiacco & Hutzler (1979), Robinson (1980), Zlobec (1980), are not immediately applicable in that they assume too much smoothness in the problem functions and perturbation parameter dependencies. It would be interesting to extend such general analyses to the case of programmes with only *directionally differentiable* problem functions, *cf.* Dempster & Wets (1976), which arise frequently, for example, in aerospace engineering applications such as the present.

(iii) Another problem revealed in the present study to date is that of the trivial solution to the inner loop optimization problem presented by the origin. Techniques to overcome this using objective function constraints of the form

$$x'Px \geq V_{min}$$

have been tried and could be investigated further. More important however is the testing of the nonlinear constraint rescaling technique discussed in §4 which should completely eliminate the difficulty.

(iv) An investigation must also be made of steplength tuning
and outer loop gradient provision for the alternative BFGS
quasi-Newton algorithm utilized in the present study. These
provisions are designed to enhance the efficiency of the
outer loop algorithm. Regarding the second point, due
to the complex nonlinear, but analytical, formulae
represented by (17), see Dempster *et al* (1979), Appendix
I, the development of gradient information should be
investigated through the use of automatic differentiation
software which has recently become available, Robinson
(1979). Gradient information would also require of course
the results of the perturbation analysis of the inner loop
problem called for in (ii) above.

Given that these refinements of the DOMATT code can be developed
using test problems, the code will then be initially applied to
the gas jet and reaction wheel baseline aerospace systems. Exten-
sive tests should be carried out to allow the detailed mathematical
properties of the problems to be assessed, so that the DOMATT code
can be further tuned. For example, the precise nature of the
starting point procedures selected depends on how closely the
candidate inner loop solutions need to be tracked in successive
iterations of the outer loop. This is likely to be fairly problem
specific, so that detailed mathematical investigation of the $\dot{V}=0$
manifold could improve the efficiency of the DOMATT code.

Regarding aerospace applications in general, it would be
fruitful to study the extension of Weissenberger's piecewise
quadratic Lyapunov function techniques for single actuators to
the 3-dimensional actuators used on spacecraft. This would
allow study of the gas jet chattering phenomenon. The use of
piecewise linear Lyapunov functions (generated by linear pro-
gramming techniques) should also be investigated , *cf*. Rosenbrock
(1962). Such a method would give more careful mapping of DOA
boundaries than is possible with quadratic techniques and would
allow the extension of the DOMATT code to more general non-
differentiable systems.

Finally, assuming that the problems discussed above can be solved, experience with the quadratic Lyapunov function approach to dynamical system stability analysis -- as embodied in the DOMATT code -- should be accumulated with other systems arising in engineering, economics and policy analysis. The potential applications in systems studies -- for example, in energy policy research -- are widespread.

7.   ACKNOWLEDGEMENTS

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