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Handling Uncertainties in Linear Programming Models

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PREFACE

A frequently used approach to linear programming problems with only vaguely known coefficients of the objective function is to treat these coefficients as random variables; this means that the lack of knowledge is described by a distribution function. For the case in which such a procedure cannot be justified, S.Ya. Chernavsky and A.D. Virtzer of the Working Consultative Group for the President of the Academy of Sciences of the USSR developed a decision theoretical approach, some aspects of which are described here for pedagogical purposes.

In this paper first the problem of handling uncertainties in linear programming models is outlined, and the decision criteria to be used are explained. Thereafter, a method of finding optimal strategies under uncertain values of the objective function coefficients is described. Finally, the method is applied to a simple uncertainty case of the MESSAGE model.

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1. INTRODUCTION

Many authors have paid attention to the importance of considering the uncertainty problem in forecasting, e.g. in Ref.[1] the problem of building up an energy supply system is described as follows:

> "A variety of energy supply and conversion technologies can compete to meet demands. Here, it is assumed that technologies compete primarily on a cost basis, the cheapest technology available being used first. But there are constraints on the rates at which new resources can be exploited, or new facilities built, and on the total amount of any single activity (such as coal mining) that a society will tolerate. And deliberate planning to maintain flexibility--for example, to provide diversity of supply in order to cope better with unexpected changes in energy supply systems--can affect decisions which would otherwise be dominated by cost considerations alone."

Deliberate planning, however, needs data. The problem of getting these data is described in the same place:

> "These data, while arrived at by averaging many sources, are still highly judgmental. And while they will surely change over time, perhaps dramatically, just one cost estimate for each technology is used here for the entire planning horizon. Sensitivity analyses can test alternative cost estimates. Yet the possibility that the cost figures used here might be greatly understated should not be overlooked. It can be observed that the real costs of complex energy supply systems today invariably exceed expectations, and this may not change in the future. (The 1970 to 1977 costs of power plants in the U.S., for example, rose much faster than the domestic

> consumer price index.) This possibility could well heighten interest in the potential economic attractiveness of energy efficiency improvements (or energy productivity increases) as an alternative "supply" source. The cost estimates used here are, for better or for worse, no more than a composite of the presently best recognized estimates."

Because of these problems one has to try to take into account these uncertainties in forecasting. Frequently one describes them by considering all uncertain parameters as random variables the distribution of which is known. Thus one is led to stochastic programming problems. In fact, the real situation in forecasting is uncertain. For handling uncertainty in these cases, S.Ya. Chernavsky and A.D. Virtzer developed some methods [2, 3, 4, 5], one of which is described in this paper.

While being Research Scholar at IIASA during the months of September and October 1980, one of the authors (S.Ya. Chernavsky) presented the results of the methods developed. He implemented them by using one example of the MESSAGE model [1, 6] with the help of another author of this paper (L. Schrattenholzer). Because of the great interest of energy modellers in these methods two authors (R. Avenhaus and R. Beedgen), who were Research Scholars at IIASA during the first months of 1980, made an attempt to present one of these methods in a pedagogical way, because energy modellers are in general not specialists in decision theory. Later, S.Ya. Chernavsky, during his stay at IIASA in September and October 1980, continued his work [5], joined this effort, and together with R. Avenhaus put the paper into its present form. With the above-mentioned educational purposes in mind, it appeared possible to the authors to present the proofs in a form different from that originally given in [2-5].

2. FORMULATION OF THE PROBLEM

Let us consider the following problem:

$$\underset{\underline{x}\in X}{\text{minimize}}\ \underline{c}'\cdot\underline{x}\ , \tag{2-1}$$

where $X = \{\underline{x}'\} = \{(x_1 \ldots x_n)\}$ is given by

$$X: = \{\underline{x}:\underline{\underline{A}}\cdot\underline{x} = \underline{b},\ \underline{x} \geq 0\}\ , \tag{2-2}$$

and where $\underline{\underline{A}}$ is an m×n-matrix and $\underline{b}$ a vector with m elements.

In case the values of the elements of the matrix $\underline{\underline{A}}$ and the vectors $\underline{b}$ and $\underline{c}$ are precisely known, this is the well-known linear programming (LP) problem usually solved by way of the simplex algorithm.

Let us assume now that the values of the elements of the vector $\underline{c}$ are *not* precisely known, but that the vector is supposed to be an element of an n-dimensional polyeder C. We will also assume that the polyeder C is convex. In the simplest case, in which all components of $\underline{c}$ are independent of each other, we have

$$C = \{\underline{c}' = (c_1 \ldots c_n) \ : \ c_i^l \le c_i \le c_i^u \ , \ i = 1 \ldots n\} \quad . \qquad (2\text{-}3)$$

However, we also will consider more complicated sets.

The question arises as to the way in which to take into account the uncertainty of $\underline{c} \in C$ in the optimization procedure.

It should be stressed here that we cannot express our lack of knowledge by a distribution function on C and thus obtain a stochastic optimization problem.

For illustrative purposes, the following two examples will be used throughout the paper:

First Example (see Figure 1)

Minimize

$$K(x_1, x_2) := c_1 \cdot x_1 + c_2 \cdot x_2$$

with respect to (x_1, x_2) subject to

$$\begin{aligned} 3 \cdot x_1 + x_2 &\ge 3 \\ x_1 + 3 \cdot x_2 &\ge 3 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

where $\underline{c}' = (c_1, c_2)$ is element of C, which is a two-dimensional polyeder with corners (1,1); (1,4); (4,1), and (4,4).

Second Example (see Figure 2)

Minimize

$$K(x_1, x_2) = c_1 \cdot x_1 + c_2 \cdot x_2$$

with respect to (x_1, x_2) subject to

$$\begin{aligned} x_1 + 2 \cdot x_2 &> 2 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

where $\underline{c}$ is element of C, which is a two-dimensional simplex with corners (1,1); (4,1); and (1,9). ○

As one can see, there exists no uniquely defined solution to our problem for all $\underline{c} \in C$. For instance, for points $\underline{c}' = (1,1)$ and $(4,4)$ in the first example, the optimal solution would be point $(3/4, 3/4)$ of X (see Figure 1), for point $\underline{c}' = (1,4)$ the optimal solution would be point $(3,0)$ of X and for point $\underline{c}' = (4,1)$ the optimal solution would be point $(0,3)$ of X. In order to select optimal solutions, we have to use another decision criterion but not in the form (2-1).

3. MEANING OF DECISION CRITERIA [7,8]

In this chapter we will explain the meaning of four decision criteria to be used in the following. Only for this purpose we assume finite numbers of states of nature and strategies available to decision makers, but, as mentioned above, these criteria will be applied to a more general case.

The idea of these decision criteria is to minimize somehow the loss one suffers if a specific strategy is taken and one specific, a priori unknown state of nature is true. It is however important to realize that these criteria are constructed in order to determine the strategy to be used rather than the losses to be expected.

Let us assume that the set of the states of nature is discrete and that there are S states of nature $\underline{c} \in C$. When the decision maker has to select the optimal strategy from the domain X he does not yet know which of the states of nature will be true. The loss to the decision maker will be K_{ij} if he selects the i-th strategy and if the j-th state of nature is true. The decision matrix is called matrix $\underline{\underline{K}}$.

The following illustrates the four criteria by way of the numerical example given below:

$$\underline{\underline{K}} = \begin{pmatrix} 4 & 9 & 8 & 8 & 10 & 9 \\ 10 & 0 & 5 & 13 & 14 & 12 \\ 11 & 7 & 0 & 13 & 15 & 4 \\ 17 & 14 & 15 & 7 & 8 & 20 \\ 6 & 3 & 8 & 14 & 9 & 5 \end{pmatrix}$$

As one can see, strategy 1 would be the best strategy if state 1 of nature were true, strategy 2 if state 2 of nature were true, strategy 3 if states 3 or 6 were true, strategy 4 if states 4 or 5 were true, but strategy 5 has bigger losses in comparison to some other strategy for all the states of nature. Thus, in our numerical example, there is no such strategy which minimizes the loss for all states of nature.

Minimax (Wald) Criterion

According to this criterion the decision maker has to select that strategy which minimizes the maximum of the losses corresponding to each strategy with respect to all states of nature.

In our numerical example, the following maximum losses result:

10 for strategy 1
14 for strategy 2
15 for strategy 3
17 for strategy 4
14 for strategy 5

Therefore the selection of strategy 1 leads to the minimum of the maximum losses. Having decided in favor of strategy 1 corresponding to the Wald criterion, the decision maker need not be afraid that his maximum loss by any state of nature will be larger than that determined by the Wald criterion, i.e. 10 in our example.

Laplace Criterion

According to this criterion, the decision maker has to select that strategy which minimizes his arithmetic mean losses, calculated for each strategy, with respect to all states of nature.

In our example, the arithmetic mean losses are given by

8 for strategy 1
9.8 for strategy 2
8.3 for strategy 3
13.5 for strategy 4
7.5 for strategy 5

Therefore strategy 5 is optimal according to this criterion. It is interesting that strategy 5 does not minimize the loss for any state of nature.

Hurwitz Criterion

According to this criterion, the decision maker has to select that strategy which minimizes a linear combination of the maximum and the minimum of the losses, calculated for each strategy, with respect to all states of nature.

In our example, the linear combinations of the maximum and the minimum of the losses with pessimism parameter λ are given by

$\lambda \cdot 10+(1-\lambda) \cdot 4 = 4+6\lambda$ for strategy 1
$\lambda \cdot 14+(1-\lambda) \cdot 0 = 14\lambda$ for strategy 2
$\lambda \cdot 15+(1-\lambda) \cdot 0 = 15\lambda$ for strategy 3
$\lambda \cdot 20+(1-\lambda) \cdot 7 = 7+13\lambda$ for strategy 4
$\lambda \cdot 14+(1-\lambda) \cdot 3 = 3+11\lambda$ for strategy 5

Therefore strategy 2 is the optimal strategy for values of λ with $0 \leq \lambda \leq 0{,}5$, and strategy 1 is the optimal strategy for values of λ with $0{,}5 \leq \lambda \leq 1$. For $\lambda=1$ the Hurwitz criterion is equal to the Wald criterion.

A difficulty of this criterion is the appropriate selection of λ between 0 and 1.

Savage-Niehans (Minimax-Regret) Criterion

According to this criterion, the decision maker determines first for each state of nature the minimum loss and subtracts it from the losses related to this state and all the possible strategies. In other words, he determines for each strategy the difference of the actual loss and the minimum loss under the fixed state of nature. This difference is named regret. Each strategy is characterized by its own maximum regret. So the decision maker determines a strategy that has a minimum of maximum regret.

In our example the matrix of regrets $\underline{\underline{R}}$ looks as follows

$$\underline{\underline{R}} = \begin{pmatrix} 0 & 9 & 8 & 1 & 2 & 5 \\ 6 & 0 & 5 & 6 & 6 & 8 \\ 7 & 7 & 0 & 6 & 7 & 0 \\ 13 & 14 & 15 & 0 & 0 & 16 \\ 2 & 3 & 8 & 7 & 1 & 1 \end{pmatrix}$$

and each strategy has the following maximum regrets

9 for strategy 1
8 for strategy 2
7 for strategy 3
16 for strategy 4
8 for strategy 5

Therefore strategy 3 is the optimal strategy according to this criterion.

One should notice that in our numerical example the decision maker using the set of the fourth criteria does not have an opportunity to select one strategy which minimizes values of all of these criteria at the same time.

Nevertheless, even in such a difficult case of selecting the optimal strategy the decision maker has an opportunity to get important results if he uses the set of these criteria. Thus, in our example such a result is the condition of the optimality for strategy 5 obtained by using the Laplace criterion.

4. A METHOD OF FINDING OPTIMAL STRATEGIES UNDER UNCERTAINTY

Let us come back to the problem formulated in the second chapter: if the decision maker knew vector $\underline{c}'$ exactly, he would then determine the optimal strategy $\underline{x} \in X$, where X is given by (2-1), with the help of the criterion

$$\min_{\underline{x} \in X} \underline{c}' \cdot x \quad .$$

However, as he knows only that $\underline{c}' \in C$ he does not have to apply this criterion.

Wald (Minimax) Criterion

According to this criterion, the decision maker determines one of the optimal strategies under uncertainty by solving the following optimization problem:

$$\underset{\underline{x} \in X}{\text{minimize}} \max_{\underline{c}' \in C} \underline{c}' \cdot x \tag{4-1}$$

For the simple above-mentioned case of independent coordinates of vector $\underline{c}'$

$$C := \{\underline{c} : c_i^l \leq c_i \leq c_i^u,\ i=1,\dots,n\}$$

we have, because of the assumption $\underline{x} \geq 0$,

$$\max_{\underline{c} \in C} \underline{c}' \cdot x = \underline{c}^{u'} \cdot \underline{x} \quad ,$$

where $\underline{c}^{u'} := (c_1^u,\dots,c_n^u)$. Thus, in order to solve the problem (4-1) in this simple case it is enough to solve the normal LP problem

$$\underset{\underline{x} \in X}{\text{minimize}}\ \underline{c}^{u'} \cdot \underline{x} \quad . \tag{4-2}$$

Let us consider a more complicated case, in which the coordinates of vector $\underline{c}'$ depend on each other. In accordance with our assumption C is a convex polyeder. Then we have

Theorem 1: Let C_o be the set of extreme points of the set C. Then we have

$$\min_{\underline{x} \in X} \max_{\underline{c} \in C} \underline{c}' \cdot x = \min_{\underline{x} \in X} \max_{\underline{c} \in C_o} \underline{c}' \cdot x \quad . \tag{4-3}$$

Proof: C is a convex and compact subset of a local convex set, $G_{\underline{x}}(\underline{c}') := \underline{c}' \cdot \underline{x}$ for any $\underline{x}$ is a continuous convex function of $\underline{c}$. It is known that the maximum of the convex function lies at an extreme point (a convex function has the form as given in Figure 3); there exists a $\underline{c}'_o \in C_o$ with $G_{\underline{x}}(\underline{c}'_o) = \max_{\underline{c}' \in C} G_{\underline{x}}(\underline{c}')$. □

In accordance with this theorem it is sufficient to only consider the extreme points of C. It is not enough, however, to consider the extreme points of space X only, as the second example will show.

Second Example

According to the formulation of the second example in the second chapter we have

$$\max_{\underline{c}'} \underline{c}'\cdot\underline{x} = \begin{cases} 4\cdot x_1+x_2 & \text{for} \quad 4\cdot x_1+x_2\geq x_1+9\cdot x_2 \\ x_1+9\cdot x_2 & \phantom{\text{for}} \quad 4\cdot x_1+x_2\leq x_1+9\cdot x_2 \end{cases} .$$

The border line is given by

$$4\cdot x_1+x_2=x_1+9\cdot x_2$$

which is equivalent to

$$x_2=\frac{3}{8}\cdot x_1 \quad .$$

Therefore we get

$$\min_{\underline{x}} \max_{\underline{c}'} \underline{c}'\cdot\underline{x} = \begin{cases} \min\limits_{\underline{x}}(4\cdot x_1+x_2) & \text{for} \quad x_2\leq\frac{3}{8}\cdot x_1 \\ \min\limits_{\underline{x}}(x_1+9\cdot x_2) & \phantom{\text{for}} \quad x_2\geq\frac{3}{8}\cdot x_1 \end{cases} ,$$

which leads to the solution

$$\min_{\underline{x}} \max_{\underline{c}'} \underline{c}'\cdot\underline{x} = 5$$

at the point $(x_1,x_2) = (\frac{8}{7},\frac{3}{7})$ which is *not* an extreme point of X.

○

This example shows that in general, after the usual simplex algorithm has been applied by solving the LP-program (2-1), it is impossible to solve the problem (4-1). The following theorem shows that we can solve the entire problem by solving a single LP-problem of a higher dimension:

Theorem 2: The solution to the problem

$$\underset{\underline{x}\in X}{\text{minimize}}\ \underset{\underline{c}\in C}{\max}\ \underline{c}'\cdot\underline{x} \quad ,$$

where X is given by (2-1), and C_o is the set of extreme points of C, is equivalent to the solution of the problem

$$\underset{(\underline{x},y)\in X'}{\text{minimize}}\ y \quad , \tag{4-4}$$

where X' is given by

$$X' = \{(\underline{x},y):\underline{x}\in X,\ y\geq\underline{c}_i'\cdot\underline{x},\ y\geq 0,\ \underline{c}'_i\in C_o\} \ . \tag{4-5}$$

Proof: If y is greater or equal to all $\underline{c}_i'\cdot\underline{x}$, then it is also greater or equal to the maximum of $\underline{c}_i'\cdot\underline{x}$. Therefore, the minimization of y on the space X' just leads to the solution of the original problem.

□

First Example

According to the formulation of the first example in the second chapter and according to Theorem 2, we have to solve the problem

$$\underset{(\underline{x},y)\in X'}{\text{minimize}}\ y$$

where X' is given by

$3\cdot x_1+x_2\geq 3$	(1)	$y-x_1-x_1\geq 0$	(5)
$x_1+3\cdot x_2\geq 3$	(2)	$y-x_1-4\cdot x_2\geq 0$	(6)
$x_1 \geq 0$	(3)	$y-4\cdot x_1-x_2\geq 0$	(7)
$x_2\geq 0$	(4)	$y-4\cdot x_1-4\cdot x_2\geq 0$.	(8)

As conditions (5) to (7) are dominated by (8), we have to look for the corners of the simplex in the (x_1,x_2,y)-space determined by (1) to (4) and (8). There are only three corners, determined by

i) $3\cdot x_1+x_2=3$	ii) $x_1+3\cdot x_2=3$	iii) $3\cdot x_1+x_2=3$
$x_1 =0$	$x_2=0$	$x_1+3\cdot x_2=3$
$y-4\cdot x_1-4\cdot x_2=0$	$y-4\cdot x_1-4\cdot x_2=0$	$y-4\cdot x_1-4\cdot x_2=0$

which leads to

$$(x_1,x_2,y) = \begin{cases} (0,3,12) & \text{i)} \\ (3,0,12) \quad \text{for} & \text{ii)} \\ (\frac{3}{4},\frac{3}{4},6) & \text{iii)} \end{cases}$$

We get $\min_{(\underline{x},y)\in X'} y=6$, that is, the solution is given at a corner of X.

Second example

According to the formulation of the second example in the second chapter and according to Theorem 2, we have to solve the problem

$$\underset{(\underline{x},y)\in X'}{\text{minimize}}\ y\ ,$$

where X' is given by

$$\begin{array}{llll} x_1+2\cdot x_2\geq 2 & (1) & y-x_1-x_2-0 & (4) \\ x_1 \qquad \geq 0 & (2) & y-4\cdot x_1-x_2-0 & (5) \\ \qquad x_2\geq 0 & & y-x_1-9\cdot x_2-0\ . & (6) \end{array}$$

As condition (4) is dominated by (5) and (6), the corners are determined by

$$\begin{array}{lll} \text{i)}\ x_1+2\cdot x_2=2 & \text{ii)}\ x_1+2\cdot x_2=2 & \text{iii)}\ x_1+2\cdot x_2=2 \\ \quad x_1 \qquad =0 & \qquad\qquad x_2=0 & \quad y-4\cdot x_1-x_2=0 \\ \quad y-x_1-9\cdot x_2=0 & \quad y-4\cdot x_1-x_2=0 & \quad y-x_1-9\cdot x_2=0 \end{array}$$

which leads to

$$(x_1,x_2,y) = \begin{array}{ll} (0,1,9) & \text{i)} \\ (2,0,8) \quad \text{for} & \text{ii)} \\ (\frac{8}{7},\frac{3}{7},5) & \text{iii)}\ , \end{array}$$

and $\min_{(\underline{x},y)\in X'} y = 5$. ○

Laplace Criterion

According to this criterion the decision maker determines one of the optimal strategies under uncertainty by solving the following optimization problem

$$\underset{\underline{x}\in X}{\text{minimize}}\ \frac{1}{V(C)} \int_C \underline{c}'\cdot\underline{x}\cdot d\underline{c} \qquad (4\text{-}6a)$$

where V(C) is the volume of the n-dimensional convex polyeder C,

$$V(C) := \int_C d\underline{c} \quad .$$

It is obvious that

$$\min_{\underline{x}\in X} \frac{1}{V(C)}\cdot\int_C \underline{c}'\cdot\underline{x}\cdot d\underline{c} = \min_{x\in X} \underline{c}_{av}\cdot\underline{x} \qquad (4\text{-}6b)$$

where $\underline{c}_{av} := \frac{1}{V(C)} \cdot \int_C \underline{c}'\cdot d\underline{c}$ is the centre of the weight of domain C.

The different methods for the determination of the centre of the weight of convex polyeder are known. One of them is suggested by A. Hölzl who proved the following theorem which could be used for the general case.

Theorem 3: Let C_n be an n-dimensional simplex, defined by

$$C_n := \{\underline{c} : \underline{c} = \underline{c}_o + \sum_{i=1}^{n} t_i\cdot(\underline{c}_i - \underline{c}_o),\ 0 \le t_i \text{ for } i=1,\ldots,n;\ \sum_{i=1}^{n} t_i \le 1\} \quad ,$$

where $\underline{c}_o \ge 0$, $\underline{c}_i > 0$ and $\{\underline{c}_i - \underline{c}_o, i=1,\ldots,n\}$ are linearly independent. Then we have

$$\int_{C_n} \underline{c}'\cdot\underline{x}\cdot d\underline{c} = (\frac{1}{n+1} \cdot \sum_{i=o}^{n} \underline{c}_i'\cdot\underline{x})\cdot V(C_n) \quad ,$$

where $V(C_n)$ is the volume of the simplex C_n.

Proof: Given in Annex A to this paper. □

Second Example

We have

$$V(C) = \int_1^4 dc_1 \cdot \int_1^{-\frac{8}{3}\cdot c_1+\frac{35}{3}} dc_2 = 12$$

and

$$\int_C d\underline{c}\cdot\underline{c}'\cdot\underline{x} = \int_1^4 dc_1 \cdot \int_1^{-\frac{8}{3}\cdot c_1+\frac{35}{3}} dc_2\cdot(c_1\cdot x_1+c_2\cdot x_2) = 24\cdot x_1+44\cdot x_2 \quad .$$

Therefore the problem is to

$$\underset{\underline{x}\in X}{\text{minimize}} \; (2\cdot x_1+\frac{11}{3}\cdot x_2)$$

which leads to

$$(x_1,x_2)^{opt} = (0,2); \quad 2\cdot 0+\frac{11}{3}\cdot 2 = \frac{22}{3} \quad .$$

According to Theorem 3, with

$$\underline{c}_o'=(1,1),\ \underline{c}_1'=(4,1),\ \underline{c}_2'=(1,9)$$

this is equivalent to

$$\underset{\underline{x}\in X}{\text{minimize}} \; \frac{1}{3}\cdot(\underline{c}_o'+\underline{c}_1'+\underline{c}_2')\cdot\underline{x}$$

which, in fact, again leads to

$$\underset{\underline{x}\in X}{\text{minimize}} \; (2\cdot x_1+\frac{11}{3}\cdot x_2) \quad .$$

○

It should be noted that for practical calculations in which the coordinates of vector $\underline{c}$ are usually independent, the Laplace criterion is written in the form

$$\underset{\underline{x}\in X}{\text{minimize}} \ \frac{1}{S}\cdot\sum_{i=1}^{S} \underline{c}_i'\cdot\underline{x}$$

where S is the number of corners of polyeder C. For the more general case one should use the Laplace criterion in the form (4-6b).

Hurwitz Criterion

According to this criterion, the decision maker determines the optimal strategy by solving the following optimization problem:

$$\underset{\underline{x}\in X}{\text{minimize}} \ [\lambda\cdot\max_{\underline{c}\in C} \underline{c}'\cdot\underline{x}+(1-\lambda)\cdot\min_{\underline{c}\in C} \underline{c}'\cdot\underline{x}] \quad ,$$

where the value of the pessimism parameters $\lambda\in[0,1]$ has to be chosen appropriately.

As in the case of the minimax criterion, we can restrict our considerations to the extreme points of the set C:

Theorem 4: Let C_o be the set of extreme points of the set C, defined by (4-2). Then we have

$$\min_{\underline{x}\in X}[\lambda\cdot\max_{\underline{c}\in C} \underline{c}'\cdot\underline{x}+(1-\lambda)\cdot\min_{\underline{c}\in C} \underline{c}'\cdot\underline{x}]=\min_{\underline{x}\in X}[\lambda\cdot\max_{\underline{c}\in C_o} \underline{c}'\cdot\underline{x}+(1-\lambda)\cdot\min_{\underline{c}\in C_o} \underline{c}'\cdot\underline{x}] \quad . \tag{4-7}$$

Proof: For $\max_{\underline{c}\in C} \underline{c}'\cdot\underline{x}$ see Proof of Theorem 1 and for $\min_{\underline{c}\in C} \underline{c}'\cdot\underline{x}$ see the properties of the solution of an LP problem. □

In a way similar to that one given by Theorem 2 for the minimax-criterion, we can simplify the computational procedure:

Theorem 5: The solution to the problem

$$\underset{\underline{x}\in X}{\text{minimize}} \ [\lambda\cdot\max_{\underline{c}\in C_o} \underline{c}'\cdot\underline{x}+(1-\lambda)\cdot\min_{\underline{c}\in C_o} \underline{c}'\cdot\underline{x}] \quad ,$$

where X is given by (2-1), and C_o is the set of extreme points of C as given by (4-2), is equivalent to the solution of the problem

$$\underset{i=1,\dots,S}{\text{minimize}}\ K_i \quad , \tag{4-8a}$$

where

$$K_i := \min_{(\underline{x},y)\in X_i} [\lambda\cdot y+(1-\lambda)\cdot \underline{c}_i'\cdot\underline{x}] \tag{4-8b}$$

and where

$$X_i := \{(\underline{x},y): \underline{x}\in X, y\geq \underline{c}_k'\cdot\underline{x}, k=1\dots S,\ y\geq 0,\ \underline{c}_i'\cdot\underline{x}\leq\underline{c}_j'\cdot\underline{x},\ j=1,2,\dots i-1,i+1\dots S\}. \tag{4-8c}$$

Proof: Obvious. □

First Example

Let us consider first $\underline{c}_1=(1,1)$. We have to determine

$$\min_{(\underline{x},y)\in X_1} [\lambda\cdot y+(1-\lambda)\cdot(x_1+x_2)]$$

subject to $(\underline{x},y)\in X_1$, where X_1 is determined by the following set of inequalities:

$3\cdot x_1+x_2\geq 3$	(1)	$\lambda-x_1-x_2\geq 0$	(5)	$x_1+x_2\leq x_1+4\cdot x_2$	(9)
$x_1+3\cdot x_2\geq 3$	(2)	$\lambda-x_1-4\cdot x_2\geq 0$	(6)	$x_1+x_2\leq 4\cdot x_1+x_2$	(10)
$x_1 \geq 0$	(3)	$\lambda-4\cdot x_1-x_2\geq 0$	(7)	$x_1+x_2\leq 4\cdot x_1+4\cdot x_2$	(11) ,
$x_2\geq 0$	(4)	$\lambda-4\cdot x_1-4\cdot x_2\geq 0$	(8)		

which reduces to (1) through (4) and (8).

The corners of X_1 are

$$\begin{array}{lll} 3\cdot x_1+x_2=3 & 3\cdot x_1+x_2=3 & x_1+3\cdot x_2=3 \\ x_1+3\cdot x_2=3 & x_1 \quad =0 & x_2=0 \\ y-4\cdot x_1-4\cdot x_2=0 & y-4\cdot x_1-4\cdot x_2=0 & y-4\cdot x_1-4\cdot x_2=0 \end{array}$$

which leads to

$$(\tfrac{3}{4},\tfrac{3}{4},6) \qquad (0,3,12) \qquad (3,0,12)$$

Thus we get

$$K_1 = \min_{(\underline{x},y)\in X_1} [\lambda\cdot y+(1-\lambda)\cdot(x_1+x_2)]$$

$$= \min \quad [\lambda\cdot y+(1-\lambda)\cdot(x_1+x_2)] =$$

$$= \min \quad [4.5\cdot\lambda+1.5,\ 9\cdot\lambda+3]$$

$$= 4.5\cdot\lambda+1.5 \text{ for } 0\leq\lambda\leq 1 \quad ,$$

which is given at $(x_1,x_2) = (\frac{3}{4},\frac{3}{4})$.

As for $\underline{c}_i$, $i=2,3,4$, we get $X_i=\phi$, and therefore

$$\min_i K_i = K_1 \quad .$$

Second Example

With $\underline{c}_1=(1,1)$, $\underline{c}_2=(4,1)$, $\underline{c}_3=(1,9)$ we get the set x_1 as follows ($X_2=\phi$, $X_3=\phi$):

$$\begin{array}{llllll} X_1: & x_1+2\cdot x_2\geq 2 \quad (1) & y-4\cdot x_1-x_2\geq 0 & (4) & x_1+x_2\leq 4\cdot x_1+x_2 & (5) \\ & x_1 \quad \geq 0 \quad (2) & y-x_1-9\cdot x_2\geq 0 & (5) & x_1+x_2\leq x_1+9\cdot x_2 & (6) \\ & x_2\geq 0 \quad (3) & & & & \end{array}$$

As (5) and (6) are identical to (2) and (3), we get the following corners for X_1

$x_1+2\cdot x_2=2$	$x_1+2\cdot x_2=2$	$x_1+2\cdot x_2=2$	$x_1+2\cdot x_2=2$
$x_1 \quad =0$	$x_2=0$	$x_1 \quad =0$	$x_2=0$
$y-4\cdot x_1-x_2=0$	$y-4\cdot x_1-x_2=0$	$y-x_1-9\cdot x_2=0$	$y-x_1-9\cdot x_2=0$

which leads to

$(0,1,1) \qquad (2,0,8) \qquad (0,1,9) \qquad (2,0,2).$

Thus we get

$$\begin{aligned} K_1 &= \min_{(\underline{x},y)\in X_1} [\lambda\cdot\underline{c}_1'\cdot\underline{x}+(1-\lambda)\cdot y] = \\ &= \min\ [\lambda(x_1+x_2)+(1-\lambda)\cdot y] = \\ &= \min\ [\lambda\cdot 1+(1-\lambda)\cdot 1,\lambda\cdot 2+(1-\lambda)\cdot 8,\lambda\cdot 1+(1-\lambda)\cdot 9,\lambda\cdot 2+(1-\lambda)\cdot 2] = \\ &= \min\ [1,8-6\cdot\lambda,9-8\cdot\lambda,2] \\ &= 1 \text{ for } 0 \leq \lambda \leq 1; \end{aligned}$$

it is given at $(x_1,x_2) = (0,1)$ for $0 \leq \lambda \leq 1$. ○

Savage-Niehans (Minimax-Regret) Criterion

According to this criterion, the decision maker determines the optimal strategy by solving the following optimization problem:

$$\underset{\underline{x}\in X}{\text{minimize}} \max_{\underline{c}\in C} [\underline{c}'\cdot\underline{x}-\min_{\underline{x}\in X} \underline{c}'\cdot\underline{x}] \quad .$$

As in the case of the minimax criterion, we can restrict our considerations to the extreme points of set C.

Theorem 6: Let C_o be the set of extreme points of set C, defined by (4-2). Then we have

$$\min_{\underline{x}\in X} \max_{\underline{c}\in C} [\underline{c}'\cdot\underline{x}-\min_{\underline{x}\in X} \underline{c}'\cdot\underline{x}] = \min_{\underline{x}\in X} \max_{\underline{c}\in C_o} [\underline{c}'\cdot\underline{x}-\min_{\underline{x}\in X} \underline{c}'\cdot\underline{x}]$$

Proof: It is well known that the function

$$z(\underline{c}') := \min_{\underline{x}\in X} \underline{c}'\cdot\underline{x}$$

is a concave function in $\underline{c}$, i.e., that the relation

$$\min_{x \in X} (\lambda \cdot \underline{c}_1' + (1-\lambda) \cdot \underline{c}_2') \cdot \underline{x} \geq \min_{x \in X} \lambda \cdot \underline{c}_1' \cdot \underline{x} + \min_{x \in X} (1-\lambda) \cdot \underline{c}_2' \cdot \underline{x}$$

holds. An illustration is given in Figure 3a.

Now, as $z(\underline{c}')$ is a concave function,

$$\underline{c}' \cdot \underline{x} - z(\underline{c}')$$

is a convex function. It is known that the maximum of the convex function lies at one of the extreme points of C. An illustration of that is given in Figure 3b.

□

In a way similar to that one given by Theorem 2 for the minimax criterion, we can simplify the computational procedure:

Theorem 7: The solution to the problem

$$\underset{\underline{x} \in X}{\text{minimize}} \max_{\underline{c} \in C_o} [\underline{c}' \cdot \underline{x} - \min_{\underline{x} \in X} \underline{c}' \cdot \underline{x}] \quad ,$$

where X is given by (2-1), and C_o is the set of extreme points of C as given by (4-2), is equivalent to the solution of the problem

$$\underset{(\underline{x}, y) \in X'}{\text{minimize}} \; y \quad ,$$

where X' is given by

$$X' = \{ (\underline{x}, y) : \underline{x} \in X, y \geq \underline{c}_i' \cdot \underline{x} - z_i, \; i=1 \ldots S, y \geq 0 \}$$

and where z_i is defined by

$$z_i := \min_{\underline{x} \in X} c_i' \cdot \underline{x}, \quad i=1 \ldots S \quad .$$

Proof: Obvious. □

First Example

Let us determine first the z_i, $i=1...4$:

$$z_1 = \min_{\underline{x}} (x_1+x_2) = \frac{3}{2}$$

$$z_2 = \min_{\underline{x}} (4\cdot x_1+x_2) = 3$$

$$z_3 = \min_{\underline{x}} (x_1+4\cdot x_2) = 3$$

$$z_4 = \min_{\underline{x}} (4\cdot x_1+4\cdot x_2) = 6$$

According to Theorem 7 we have to solve the problem

$$\underset{(\underline{x},y)\in X'}{\text{minimize}}\ y$$

where X' is given by

$$\begin{array}{llll} 3\cdot x_1+x_2\geq 3 & (1) & y-x_1-x_2+\frac{3}{2}\geq 0 & (5) \\ x_1+3\cdot x_2\geq 3 & (2) & y-x_1-4\cdot x_2+3\geq 0 & (6) \\ x_1 \geq 0 & (3) & y-4\cdot x_1-x_2+3\geq 0 & (7) \\ x_2\geq 0 & (4) & y-4\cdot x_1-4\cdot x_2+6\geq 0 & (8) \end{array}$$

There are 3 corners, given by

i) $3\cdot x_1+x_2=3$, $x_1=0$, $y-x_1-x_2=-\frac{3}{2}$

ii) $x_1+3\cdot x_2=3$, $x_2=0$, $y-4\cdot x_1-x_2=-3$

iii) $3\cdot x_1+x_2=3$, $x_1+3\cdot x_2=3$, $y-x_1-4\cdot x_2=-3$

iv) $3\cdot x_1+x_2=3$, $x_1+3\cdot x_2=3$, $y-4\cdot x_1-x_2=-3$

which lead to

i) $(o,3,\frac{3}{2})$ ii) $(3,0,9)$ iii) $(\frac{3}{4},\frac{3}{4},\frac{3}{4})$ iv) $(\frac{3}{4},\frac{3}{4},\frac{3}{4})$.

Thus, we get min $y=\frac{3}{4}$, the solution is given at $(x_1,x_2)=(\frac{3}{4},\frac{3}{4})$.

Second Example

Let us determine first the z_1, i=1,2,3:

$$z_1 = \min_{\underline{x}} (x_1+x_2) = 1$$
$$z_2 = \min_{\underline{x}} (4\cdot x_1+x_2) = 1$$
$$z_3 = \min_{\underline{x}} (x_1+9\cdot x_2) = 2 \quad .$$

According to Theorem 7 we have to solve the problem

$$\underset{(\underline{x},y)\in X'}{\text{minimize}}\ y$$

where X'is given by

$$\begin{array}{llll} x_1+2\cdot x_2\geq 2 & (1) & y-x_1-x_2+1\geq 0 & (4) \\ x_1 \geq 0 & (2) & y-4\cdot x_1-x_2+1\geq 0 & (5) \\ x_2\geq 0 & (3) & y-x_1-9\cdot x_2+2\geq 0\ . & (6) \end{array}$$

There are 3 corners, given by

i) $x_1+2\cdot x_2=2$, $x_2=0$, $y-4\cdot x_1-x_2+1=0$

ii) $x_1+2\cdot x_2=2$, $x_2=0$, $y-x_1-9\cdot x_2+2=0$

iii) $x_1+2\cdot x_2=2$, $y-4\cdot x_1-x_2+1=0$, $y-x_1-9\cdot x_2+2=0$

which leads to

i) (2,0,7) ii) (0,1,7) iii) $(1,\frac{1}{2},\frac{7}{2})$.

Thus, we get min $y=\frac{7}{2}$, the solution is given at $(x_1,x_2)=(1,\frac{1}{2})$.

○

5. APPLICATION TO MESSAGE [1],[6]

A number of primary energy sources and their associated conversion technologies are considered. These include resources and technologies that could permit an essentially unlimited supply of energy--the fundamental point of the exercise being to explore possible transitions to energy systems states based on more or less unlimited resources such as 232Th, 238U, and solar energy.

Each primary energy source (except solar and hydroelectric power) is subdivided into an optional number of classes in MESSAGE, taking account of the price of extraction, quality of resources, and location of deposit. These primary sources are then converted directly (e.g., by crude oil refining) or indirectly (e.g., electrolytic hydrogen) into secondary energy. Secondary energy is exogenous to MESSAGE and is provided by the MEDEE-2 model as time series data for electricity, soft solar, solid, liquid, and gaseous fuels.

The variables of the model are expressed in period-averages of annual quantities.

The objective function is the sum of discounted costs for fuels (primary energy)--operation/maintenance and capital costs for providing the energy demand over the planning horizon (1980-2030).

In the equations of the models--given roughly below--indices are sometimes omitted if it seems to facilitate understanding.

Objective Function

The objective function of the MESSAGE model is the sum of discounted costs of capital, operating-maintenance, and fuels (primary energy):

$$\sum_{t=1}^{n} \beta(t)\cdot 5\cdot\{\underline{b}'\cdot\underline{r}(t) + \underline{c}'\cdot\underline{x}(t) + \underline{d}'\cdot\underline{y}(t)\} \quad ,$$

where

- t is current index of time period
- n is number of time periods
- $\beta(t)$ is discount factor
- 5 is number of years per period
- $\underline{b}$ is vector of energy resources costs
- $\underline{r}$ is vector of resource activities (LP variables)
- $\underline{c}$ is vector of operation/maintenance costs
- $\underline{x}$ is vector of energy conversion activities (LP variables)
- $\underline{d}$ is vector of capital (investment) costs
- $\underline{y}$ is vector of capacity increments (LP variables)

The discount factor is calculated from an annual discount rate of 6%, applied to a constant dollar investment stream. As MESSAGE is intended to minimize societal costs this discount rate is to be understood as a pre-tax one.*

The cost of increments to capacity still operating at the end of the planning horizon is corrected by a "terminal valuation factor", tv:

$$tv(t) = (1-\beta^{5\cdot(n+1-t)}) \quad ;$$

for example, the terminal valuation factor for the last time period is

$$tv(n) = 1 - \beta^5 \quad .$$

Constraints

The following resource constraint is defined for each resource and for each category:

$$\sum_{t=1}^{n} 5\cdot r(t) \leq Av$$

where

r(t) is annual extraction in period t
Av is availability of resource

The following resource requirement is specified for each time period for each resource:

$$\sum_{j=1}^{J} r_j(t) \geq \sum_{1} (v_1\cdot x_1(t) + 5\cdot w_1\cdot y_1(t) - 5\cdot w_1\cdot y_1(t-6))$$

where

j is index of resource category
J is number of resource categories
v_1 is specific consumption by production activity x_1
w_1 is inventory requirement for capacity increment y_1

*In these analyses, taxes are taken as part of the difference between prices and costs and so are not included in these cost-minimization calculations. Because of this fact, the discount factor here may be thought of as a "social" discount factor, applied equally to all world regions.

The following capacity constraint is specified for each technology and for each load region supplied by this technology:

$$x_j \le Cap \cdot h_j \cdot pf \quad . \qquad Cap(t) = \sum_{\tau=t-5}^{t} 5 \cdot y(\tau)$$

where

j is index of load region
Cap is capacity
h_j is load duration of load region j
pf is plant factor

The following demand constraint is specified for each time period, for each demand sector, and for each load region:

$$\sum_i \eta_{ij} \cdot x_i \ge DM_j$$

where

j is index of demand sector
η_{ij} is conversion efficiency (or equal to 0 if x_i does not supply demand sector j)
DM_j is annual secondary energy demand

The following build-up constraint is specified for some (primarily new) technologies and for each time period:

$$y(t) \le \gamma \cdot y(t-1) + g$$

where

γ is growth parameter
g is constant, allowing for start-up.

Numerical Illustration

In the following we illustrate the methodology discussed so far with the help of this MESSAGE model. As a reference case we consider the low scenario for World Region I (North America) as described in [1].* We assume only two parameters of the objective function to be uncertain, namely capital costs for Fast Breeder Reactors (FBR) and for Solar Thermal Electric Conversion (STEC).

*In [1] only those data are given which are necessary for the understanding of the procedure and of the results. A documentation of all input data of MESSAGE is being prepared by one of the authors (Leo Schrattenholzer).

In Table 1a reference values as well as ranged of uncertainties for those two parameters are given. This leads to the set of extreme points C_o, defined by (4-2), the elements of which are listed in Table 1b.

In Table 2 the results of MESSAGE runs with the data given in Table 1 are represented. Overall costs, and electricity production by FBR's and STEC in 2030. First the results for the reference data of Table 1a are given, thereafter the results for the four extreme points, i.e., the elements of C_o, and finally the results of the application of the decision criteria as discussed in Chapter 4. As we have chosen independent intervals for the two cost parameters, the minimax criterion simply means to take $\underline{c}_4$, the Laplace criterion means to take the mid-values of the intervals, and the Hurwitz criterion means to take the weighted mean of $\underline{c}_1$ and $\underline{c}_4$.

In Figures 4 and 5 the electricity production by means of the various technologies are given as functions of time: In Figure 4a to 4d the electricity production is given for the four extreme points according to Table 1b. In Figures 5a and 5b the electricity production according to the Minimax (Wald) criterion and according to the Laplace criterion are given. In Figures 5c and 5e the electricity production according to the intermediate steps of the Hurwitz criterion as formulated by Theorem 4 are given (the fourth case is dominated as can be concluded from Table 2). In Figure 5f finally the electricity production according to the Savage-Niehaus criterion is given.

It should be emphasized that it was *only for illustrative purposes* that we considered the FBR and STEC capital costs to be uncertain and all other parameters as precisely known. Nevertheless, one may draw some general conclusions from these results.

One realizes that the application of different decision criteria leads to extremely different strategies, even though the resulting overall costs vary by less than 1 percent. This can be explained by the fact that both alternatives will play a role only after the year 2000 and therefore the discounting factor decreases the influence on the overall costs.

Thus, if we use relative estimates for decision making we will come to the conclusion that the relative difference does not matter for us in this example. But if we consider absolute differences between total costs for different strategies we will notice that in comparison with today's costs for development these differences are rather large and so we can use these absolute estimates for decision making.

This conclusion does not mean that in estimating strategies one should not consider other important criteria, for example environmental burdens and qualitative criteria, such as public opinion and so on.

In such cases one should come back to the reformulation of the original problem on the basis of the multiobjective optimization approach. An outline of such an approach with different but definite objective functions is given in Annex B.

Of course, it is not certain that the differences between the values of the total costs of the different strategies appear only in the fourth decimal as in the numerical example given. In [4] another numerical example is given in which the differences appear already in the second decimal.

6. CONCLUDING REMARKS

There are many theoretical and practical aspects of the methods for handling uncertainties in LP-problems discussed here which were not described in the paper. Some of them will be mentioned in the following, additional ones see in [5].

Use of Decision Criteria

First of all one should apply not only one criterion for the determination of optimal strategies under uncertainty but all four criteria mentioned above together. However, what should the decision maker do if the different criteria lead to different optimal strategies?

In [5] is is shown that the further analysis can be continued with the help of the multi-objective optimization approach. In general having in mind only pedagogical aspect it is not reasonable to give some further recommendations without considering the specific features of the concrete problem. An example for such a procedure in the case of nuclear energy systems is given in [4].

Uncertainties in Further Coefficients

In this paper the description was limited to the treatment of uncertainties of coefficients of the objective function. It is clear that in a real situation in forecasting $\underline{\underline{A}}$ and $\underline{b}$ can be uncertain as well. In [2,5] the case with $\underline{\underline{A}}$ and $\underline{b}$ being certain for the near future in a forecasting problem, but with $\underline{\underline{A}}$ and $\underline{b}$ being uncertain for the distant future, has been treated. It should be remarked that the solution of such problems leads to higher-dimensional LP-problems.

Computational Effort

For practical applications it is highly interesting to estimate the number of additional constraints in the new LP-problems arising by the use of the procedures described. If we have in our original model (without taking into account uncertainties) m constraints, then using

- the Wald criterion, we have to solve one LP-problem with (m+S) constraints where S is the number of extreme points of the convex polyeder C;

- the Laplace criterion,we have to solve as well one LP-problem with m constraints;
- the Hurwitz criterion, we have to solve S LP-problems each of which has (m+2S-1) constraints;
- the Savage-Niehans criterion, we have to solve S LP-problems each of which has m constraints plus one LP problem with (m+S) constraints.

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ANNEX A

Proof of Theorem 3

by

Alois Hölzl

Theorem 3: Let C_n be an n-dimensional simplex, defined by

$$C_n:=\{\underline{c}:\underline{c}=\underline{c}_o+\sum_{i=1}^{n} t_i\cdot(\underline{c}_i-\underline{c}_o);\ 0\leq t_i \text{ for } i=1,\ldots n;\ \sum_{i=1}^{n} t_i\leq 1\} \quad ,$$

where $\underline{c}_o\geq 0$, $c_i>0$ and $\{\underline{c}_i-\underline{c}_o, i=1,\ldots n\}$ are linearly independent. Then we have for $x>0$

$$\int_{C_n}\underline{c}'\cdot\underline{x}\cdot d\underline{c} = (\frac{1}{n+1}\cdot\sum_{i=o}^{n}\underline{c}_i'\cdot\underline{x})\cdot V(C_n) \quad ,$$

where $V(C_n)$ is the volume of the simplex C_n.

Proof: Because of the conditions $\underline{c}_o\geq 0$, $\underline{c}_i>0$ for $i=1,\ldots,n$ and $\underline{x}>0$, the expression $\underline{c}'\cdot\underline{x}$ is non-negative for every $\underline{c}\in C_n$. The integral can therefore be considered as the volume of the domain

$$D_{n+1}:=\{(\frac{\underline{c}}{d}):\underline{c}\in C_n;\ 0\leq d\leq\underline{c}'\cdot\underline{x}\} \quad .$$

It will be shown that the domain D_{n+1} can be split into n+1 disjoint simplices C_{n+1}^i with volume $V(C_{n+1}^i)=(\frac{1}{n+1}\cdot\underline{c}_i'\cdot\underline{x})\cdot V(C_n)$ for $i=0,1,\ldots,n$, so that

$$\int_{C_n}\underline{c}'\cdot\underline{x}\cdot d\underline{c}=V(D_{n+1})=\sum_{i=0}^{n} V(C_{n+1}^i)=(\frac{1}{n+1}\cdot\sum_{i=0}^{n}\underline{c}_i'\cdot\underline{x})\cdot V(C_n) \quad .$$

For notational convenience, a simplex will be defined by listing its corners, i.e.,

$$C_n:=\{\underline{c}:\underline{c}=\underline{c}_o+\sum_{i=1}^{n} t_i\cdot(\underline{c}_i-\underline{c}_o);\ 0\leq t_i \text{ for } i=1,\ldots,n;\ \sum_{i=1}^{n} t_i=1\}$$

will be written in abbreviated form as $C_n:=<\underline{c}_o, \underline{c}_1,\ldots,\underline{c}_n>$.

Step A: The (n+1)-dimensional simplices C^i_{n+1}, i=0,1,...,n, which are defined as

$$C^1_{n+1} := < \begin{pmatrix} \underline{c}_1 \\ \underline{c}_1' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \underline{c}_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_n \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix} >$$

$$C^2_{n+1} := < \begin{pmatrix} \underline{c}_1 \\ \underline{c}_1' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ \underline{c}_2' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \underline{c}_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_n \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix} >$$

$$C^3_{n+1} := < \begin{pmatrix} \underline{c}_1 \\ \underline{c}_1' \cdot x \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ \underline{c}'_2 \cdot x \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ \underline{c}_3' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \underline{c}_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_n \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix} >$$

$$\vdots$$

$$C^n_{n+1} := < \begin{pmatrix} \underline{c}_1 \\ \underline{c}'_1 \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ \underline{c}_2' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ \underline{c}_3' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_4 \\ \underline{c}_4' \cdot \underline{x} \end{pmatrix}, \ldots, \begin{pmatrix} \underline{c}_n \\ \underline{c}_n' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_n \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix} >$$

$$C^o_{n+1} := < \begin{pmatrix} \underline{c}_1 \\ \underline{c}_1' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_2 \\ \underline{c}_2' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_3 \\ \underline{c}_3' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_4 \\ \underline{c}_4' \cdot \underline{x} \end{pmatrix}, \ldots, \begin{pmatrix} \underline{c}_n \\ \underline{c}_n' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ \underline{c}_o' \cdot \underline{x} \end{pmatrix}, \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix} >$$

form a partitioning of the domain D_{n+1}.

Proof of Step A: The simplices $C^1_{n+1}, \ldots, C^n_{n+1}$ are well-defined due to the conditions that (1) $<\underline{c}_o, \underline{c}_1, \ldots, \underline{c}_n>$ is an n-dimensional simplex, so that the vectors $\underline{c}_1-\underline{c}_o, \ldots, \underline{c}_n-\underline{c}_o$ are linearly independent, and (2) $\underline{c}_i>0$ for i=1,...,n and $\underline{x}>0$, so that $\underline{c}_i' \cdot x>0$ for i=1,...,n. If $\underline{c}_o>0$, then C^o_{n+1} is also well-defined; if $\underline{c}_o=0$, then $\begin{pmatrix} \underline{c}_o \\ \underline{c}_o' \cdot \underline{x} \end{pmatrix} = \begin{pmatrix} \underline{c}_o \\ 0 \end{pmatrix}$, and the simplex C^o_{n+1} vanishes. In order to prove that the simplices C^i_{n+1}, i=0,1,...,n defined above form a partitioning of the domain D_{n+1}, one must show that

(i) $\quad (\frac{\underline{c}}{d}) \in C^j_{n+1} \Rightarrow (\frac{\underline{c}}{d}) \in D_{n+1}$;

(ii) $\quad (\frac{\underline{c}}{d}) \in D_{n+1} \Rightarrow \exists j \in \{0,1,\ldots,n\}: (\frac{\underline{c}}{d}) \in C^j_{n+1}$

(iii) $\quad (\frac{\underline{c}}{d}) \in C^k_{n+1} \cap C^l_{n+1}$ with $k \neq l \Rightarrow (\frac{\underline{c}}{d})$ is a point on the surface of both C^k_{n+1} and C^l_{n+1}.

ad (i):

$\binom{\underline{c}}{d} \in D_{n+1}$ has the representation

$$(1)\quad \begin{cases} \underline{c}=\underline{c}_o+\sum_{i=1}^{n} t_i\cdot(\underline{c}_i-\underline{c}_o); & 0\le t_i \text{ for } i=1,\dots,n; \quad \sum_{i=1}^{n} t_i\le 1 \\ d=\alpha\cdot\underline{c}'\cdot\underline{x}; & 0\le\alpha\le 1. \end{cases}$$

$\binom{\underline{c}}{d} \in C^{j}_{n+1}$, $1\le j\le n$ has the representation

$$(2a)\quad \begin{cases} \underline{c}=\underline{c}_o+\sum_{i=1}^{j-1} u_i\cdot(\underline{c}_i-\underline{c}_o)+(u_j+v_j)\cdot(\underline{c}_j-\underline{c}_o)+\underbrace{\sum_{i=j+1}^{n} v_i\cdot(\underline{c}_i-\underline{c}_o)}_{=\,0 \text{ for } j=n} \\ d=\sum_{i=1}^{j} u_i\cdot\underline{c}_i'\cdot\underline{x} \\ \text{where} \\ 0\le u_i \text{ for } i=1,\dots,j; \quad 0\le v_i \text{ for } i=j,\dots,n; \quad \sum_{i=1}^{j} u_i+\sum_{i=j}^{n} v_i\le 1. \end{cases}$$

$\binom{\underline{c}}{d} \in C^{o}_{n+1}$ has the representation

$$(2b)\quad \begin{cases} \underline{c}=\underline{c}_o+\sum_{i=1}^{n} u_i\cdot(\underline{c}_i-\underline{c}_o) \\ d=\sum_{i=1}^{n} u_i\cdot\underline{c}_i'\cdot\underline{x}+u_o\cdot\underline{c}_o'\cdot\underline{x} \\ \text{where} \\ 0\le u_i \text{ for } i=0,1,\dots,n; \quad \sum_{i=0}^{n} u_i\le 1. \end{cases}$$

(a) Let $\binom{\underline{c}}{d} \in C^{j}_{n+1}$, $1\le j\le n$, i.e., $\binom{\underline{c}}{d}$ has the representation (2a). If one defines the coefficients

$$\begin{cases} t_i:=u_i \text{ for } i=1,\dots,j-1 \\ t_j:=u_j+v_j \\ t_i:=v_i \text{ for } i=j+1,\dots,n \end{cases}$$

the representations (1) and (2a) are idential with respect to $\underline{c}$. Furthermore,

$$d=\sum_{i=1}^{j} u_i\cdot\underline{c}_i'\cdot\underline{x}$$

$$\leq \sum_{i=1}^{j} u_i \cdot \underline{c}_i' \cdot \underline{x} + \sum_{i=j}^{n} v_i \cdot \underline{c}_i' \cdot \underline{x} + \underbrace{(1 - \sum_{i=1}^{j} u_i - \sum_{i=j}^{n} v_i)}_{\geq 0} \cdot \underline{c}_o' \cdot \underline{x}$$

$$= [\underline{c}_o + \sum_{i=1}^{j} u_i \cdot (\underline{c}_i - \underline{c}_o) + \sum_{i=j}^{n} v_i \cdot (\underline{c}_i - \underline{c}_o)]' \cdot \underline{x}$$

$$= \underline{c}' \cdot \underline{x},$$

i.e., $\binom{\underline{c}}{d} \in D_{n+1}$.

(b) Let $\binom{\underline{c}}{d} \in C^o_{n+1}$, i.e., $\binom{\underline{c}}{d}$ has the representation (2b). If one defines $t_i := u_i$ for $i=1,\ldots,n$, the representations (1) and (2b) are idential with respect to $\underline{c}$. Furthermore,

$$d = \sum_{i=1}^{n} u_i \cdot \underline{c}_i' \cdot \underline{x} + u_o \cdot \underline{c}_o' \cdot \underline{x}$$

$$\leq \sum_{i=1}^{n} u_i \cdot c_i' \cdot \underline{x} + (1 - \sum_{i=1}^{n} u_i) \cdot \underline{c}_o' \cdot \underline{x}$$

$$= [\underline{c}_o + \sum_{i=1}^{n} u_i \cdot (\underline{c}_i - \underline{c}_o)]' \cdot \underline{x}$$

$$= \underline{c}' \cdot \underline{x} \quad ,$$

i.e., $\binom{\underline{c}}{d} \in D_{n+1}$.

ad (ii):
Let $\binom{\underline{c}}{d} \in D_{n+1}$, i.e., $\binom{\underline{c}}{d}$ has the representation (1). If one defines for $1 \leq j \leq n$ in the representation (2a)

$u_i := t_i$ for $i=1,\ldots,j-1$

$v_i := t_i$ for $i=j+1,\ldots,n$

$u_j^1 := 0$, $v_j^1 := t_j$ and $u_j^2 := t_j$, $v_j^2 := 0$

and for $j=0$ in the representation (2b)

$u_i := t_i$ for $i=1,\ldots,n$

$u_o^1 := 0, u_o^2 := (1 - \sum_{i=1}^{n} t_i)$, the representation of $\underline{c}$ is identical in (1) and (2a) for $1 \leq j \leq n$ and in (1) and (2b) for $j=0$. Furthermore,

$$d_j^1 = \underbrace{\sum_{i=1}^{j-1} t_i \cdot \underline{c}_i' \cdot \underline{x}}_{=0 \text{ for } j=1}, \quad d_j^2 = \sum_{i=1}^{j} t_i \cdot \underline{c}_i' \cdot \underline{x} \quad \text{for } 1 \leq j \leq n$$

and

$$d_o^1 = \sum_{i=1}^{n} t_i \cdot \underline{c}_i' \cdot \underline{x}, \quad d_o^2 = \sum_{i=1}^{n} t_i \cdot \underline{c}_i' \cdot \underline{x} + (1 - \sum_{i=1}^{n} t_i) \cdot \underbrace{\underline{c}_o' \cdot x}_{=0 \text{ for } \underline{c}_o=0}$$

$$= [\underline{c}_o + \sum_{i=1}^{n} t_i \cdot (\underline{c}_i - \underline{c}_o)]' \cdot \underline{x}$$

$$= \underline{c}' \cdot \underline{x}.$$

Thus, the intervals $I_j := [d_j^1, d_j^2]$, $j=1,\ldots,n$ and $I_o := [d_o^1, d_o^2]$ form a partitioning of the interval $[0, \underline{c}' \cdot \underline{x}]$. (If $\underline{c}_o = 0$ and the simplex C_{n+1}^o vanishes, the interval I_o vanishes.) Therefore, $d = \alpha \cdot \underline{c}' \cdot \underline{x}$ with $0 \leq \alpha \leq 1$ must be an element of some interval I_j, $0 \leq j \leq n$. If one defines

$$u_j := \frac{d - d_j^1}{d_j^2 - d_j^1} \cdot t_j, \quad v_j := t_j - u_j \quad \text{for } 1 \leq j \leq n$$

$$u_o := \frac{d - d_o^1}{d_o^2 - d_o^1} \cdot (1 - \sum_{i=1}^{n} t_i) \qquad \text{for } j=0$$

the representations (1) and (2a) or (2b) are also identical with respect to d, i.e., $\binom{\underline{c}}{d} \in C_{n+1}^j$ for some j, $0 \leq j \leq n$.

ad (iii):
As shown in (ii) above, the intervals $I_j(\underline{c}) := \{d : \binom{\underline{c}}{d} \in C_{n+1}^j\}$ form a partitioning of the interval $[0, \underline{c}' \cdot \underline{x}]$. Therefore, any element $\binom{\underline{c}}{d} \in C_{n+1}^k \cap C_{n+1}^l$ for $k \neq l$ must be a point on the surface of both C_{n+1}^l and C_{n+1}^l.

<u>Step A</u>

Step B: An n-dimensional simplex

$$C_n := \langle \underline{c}_o, \underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n \rangle$$

has the volume

$$V(C_n) = \frac{1}{n!} \cdot |\det(\underline{c}_1 - \underline{c}_o, \underline{c}_2 - \underline{c}_o, \ldots, \underline{c}_n - \underline{c}_o)| .$$

Proof of Step B: The simplex C_n is obtained by applying to the canonical simplex

$$C_n^* := \langle \underline{o}, \underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n \rangle$$

the transformation

$$g: \underline{y} \to \underline{c} := \underline{c}_o + \underbrace{(\underline{c}_1 - \underline{c}_o, \ldots, \underline{c}_n - \underline{c}_o)}_{C} \cdot \underline{y}.$$

The canonical simplex C_n^* has the volume $V(C_n^*) = \frac{1}{n!}$, as can be shown by induction using the principle of Cavalieri. According to the substitution theorem for integrals, the integral of a function $f: \mathbb{R}^n \to \mathbb{R}$ over the domain $g(C_n^*)$ can be expressed as an integral over the domain C_n^* as follows:

$$\int_{g(C_n^*)} f(\underline{c}) \cdot d\underline{c} = \int_{C_n^*} f(g(\underline{y})) \cdot \left|\det \frac{dg(\underline{y})}{d\underline{y}}\right| \cdot d\underline{y}.$$

In this special case with $f \equiv 1$ and $\frac{dg(\underline{y})}{d\underline{y}} \equiv C$ one obtains

$$V(C_n) = V(g(C_n^*)) = \int_{g(C_n^*)} d\underline{c} = \int_{C_n^*} |\det C| \cdot d\underline{y} = |\det C| \cdot \int_{C_n^*} d\underline{y} = |\det C| \cdot V(C_n^*)$$

i.e.,

$$V(C_n) = |\det C| \cdot V(C_n^*) = \frac{1}{n!} \cdot |\det C|.$$

<u>Step B</u>

Step C: The simplices C_{n+1}^j, $0 \leq j \leq n$ as defined in Step A have the volume

$$V(C_{n+1}^j) = \left(\frac{1}{n+1} \cdot \underline{c}_j' \cdot \underline{x}\right) \cdot V(C_n) \quad \text{for } 1 \leq j \leq n$$

$$V(C_{n+1}^o) = \left(\frac{1}{n+1} \cdot \underline{c}_o' \cdot \underline{x}\right) \cdot V(C_n) \quad \text{for } j=0.$$

Proof of Step C: If one chooses the corner $\binom{\underline{c}_o}{0}$ as the basic corner in each simplex C^j_{n+1}, then the n+1 vectors, the convex closure of which represents the simplex C^j_{n+1}, are given by

$$C^j := \left\{ \begin{pmatrix} \underline{c}_i - \underline{c}_o \\ \underline{c}_i' \cdot \underline{x} \end{pmatrix} \text{ for } i=1,\ldots,j; \begin{pmatrix} \underline{c}_i - \underline{c}_o \\ 0 \end{pmatrix} \text{ for } i=j,\ldots,n \right\} \text{ for } 1 \leq j \leq n$$

$$C^o := \left\{ \begin{pmatrix} \underline{c}_i \cdot \underline{c}_o \\ \underline{c}_j' \cdot \underline{x} \end{pmatrix} \text{ for } i=1,\ldots,n; \begin{pmatrix} \underline{o} \\ \underline{c}_o' \cdot \underline{x} \end{pmatrix} \right\} \text{ for } j=0.$$

If one expands the determinant of the matrix C^j for $1 \leq j \leq n$ by the minors of the elements of the last row, the minors of the elements $c^j_{n+1,i}$ for $i \neq j$ vanish since for $i<j$ the vector $(\underline{c}_j - \underline{c}_o)$ appears twice and for $i>j$ the last column in the corresponding minor is the null vector. Therefore, one obtains for $1 \leq j \leq n$

$$|\det C^j| = \underline{c}_j' \cdot \underline{x} \cdot |\det(\underline{c}_1 - \underline{c}_o, \ldots, \underline{c}_n - \underline{c}_o)|$$
$$= \underline{c}_j' \cdot \underline{x} \cdot n! \cdot V(C_n)$$

so that

$$V(C^j) = \frac{1}{(n+1)!} \cdot |\det C^j| = \left(\frac{1}{n+1} \cdot \underline{c}_j' \cdot \underline{x}\right) \cdot V(C_n).$$

If one expands the determinant of the matrix C^o by the minors of the elements of the last row, the minors of the elements $c^o_{n+1,i}$ for $1 \leq i \leq n$ vanish since the last column is the null vector. Therefore one obtains

$$|\det C^o| = \underline{c}_o' \cdot \underline{x} \cdot |\det(\underline{c}_1 - \underline{c}_o, \ldots, \underline{c}_n - \underline{c}_o)|$$
$$= \underline{c}_o' \cdot \underline{x} \cdot n! \cdot V(C_n)$$

so that

$$V(C^o) = \frac{1}{(n+1)!} \cdot |\det C^o| = \left(\frac{1}{n+1} \cdot \underline{c}_o' \cdot \underline{x}\right) \cdot V(C_n).$$

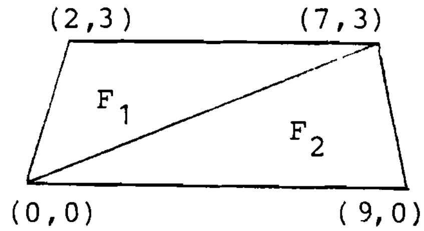
<u>Step C</u>

Since the simplices C^j_{n+1}, $0 \leq 1 \leq j$ form a partitioning of the domain D_{n+1}, one obtains

$$\int_{C_n} \underline{c}' \cdot \underline{x} \cdot d\underline{c} = V(D_{n+1}) = \sum_{i=o}^{n} V(C^i_{n+1}) = \left(\frac{1}{n+1} \cdot \sum_{i=o}^{n} \underline{c}_i' \cdot \underline{x}\right) \cdot V(C_n).$$

□

Concluding Remark: We will demonstrate with an example for n=2 that the Theorem does not hold for an arbitrary polyeder. Let us consider a polyeder given by the following figure



We have

$$F_1=\frac{1}{2}\cdot|\det(\binom{2}{3}\binom{7}{3})|=\frac{15}{2};F_2=\frac{1}{2}|\det\ (\binom{7}{3}\binom{9}{0})|=\frac{27}{2}$$

and therefore,

$$F=F_1+F_2=21.$$

Now, because of the Theorem we have

$$I_1:=\int_{F_1}(c_1\cdot x_1+c_2\cdot x_2)\cdot d\underline{c}=\frac{1}{3}((0+7+2)\cdot x_1+(0+3+3)\cdot x_2)\cdot F_1=(3\cdot x_1+2\cdot x_2)\cdot\frac{15}{2}$$

$$I_2:=\int_{F_2}(c_1\cdot x_1+c_2\cdot x_2)\cdot d\underline{c}=\frac{1}{3}((0+9+7)\cdot x_1+(0+0+3)\cdot x_2)\cdot F_2=(\frac{16}{3}\cdot x_1+x_2)\cdot\frac{27}{2}$$

and therefore

$$I:=\int_{F}(c_1\cdot x_1+c_2\cdot x_2)\cdot d\underline{c}=I_1+I_2=\frac{189}{2}\cdot x_1+\frac{57}{2}\cdot x_2.$$

This, however, is different from

$$\frac{1}{4}((0+9+7+2)\cdot x_1+(0+0+3+3)\cdot x_2)\cdot F=\frac{189}{2}\cdot x_1+\frac{63}{2}\cdot x_2\ .$$ ○

ANNEX B

MULTIOBJECTIVE OPTIMIZATION APPROACH

It is possible to use the procedures which have been described in the main part of this paper for finding optimal strategies in those cases where principally different objectives (e.g., costs, environmental pollution, etc.) exist. This problem was considered and solved in [9] and the results are given below. For the sake of simplicity we assume in this Annex that the values of all exogenous parameters of our model are certain.

Let us assume that one set of strategies X is described by the set

$$X := \{\underline{x} : \underline{\underline{A}} \cdot \underline{x} = \underline{b}, \underline{x} \geq 0\}$$

of linear constraints, and that we have s different objective functions

$$\underline{c}_i' \cdot \underline{x}, \ i=1,\ldots,S.$$

The problem is to determine that strategy which somehow minimizes all these objective functions. Let us suppose that there is no strategy which minimizes all S objective functions simultaneously, then any optimal strategy reflects one compromise or another between different objectives. The set of such optimal strategies is defined as the Pareto set: Strategy $\underline{x}$ does not belong to the Pareto set if there exists one strategy $\tilde{\underline{x}} \in X$ for which there exists at least one value $\underline{c}_j' \cdot \tilde{\underline{x}} < \underline{c}_j' \cdot \underline{x}$, $j=1\ldots S$. The question remains how one should select a reasonable compromise among different objective functions considering Pareto set strategies.

Again we can use the decision criteria described in Chapter 3. In the following, we will only give the formal representation of this criteria as applied to our problem.

Wald (Minimax) Criterion

$$\underset{\underline{x} \in X}{\text{minimize}} \ \max_{i=1,\ldots S} \underline{c}_i' \cdot \underline{x} \quad .$$

In order to solve this problem one has to solve the LP-problem

$$\underset{(\underline{x},y)}{\text{minimize}} \ y \quad ,$$

where $(\underline{x},y) \in \{\underline{\underline{A}}\underline{x}=b, \ \underline{x} \geq 0, \ y \geq \underline{c}_i' \underline{x}, \ i=1,\ldots S\}$.

Laplace Criterion

$$\underset{\underline{x} \in X}{\text{minimize}} \ \frac{1}{S} \cdot \sum_{i=1}^{S} \underline{c}_i' \cdot \underline{x} \quad .$$

This is a conventional form of an LP-problem.

Hurwitz Criterion

$$\underset{\underline{x}\in X}{\text{minimize}} \ (\lambda\cdot\max_{i=1,\ldots S} \underline{c}_i'\cdot\underline{x}+(1-\lambda)\cdot\min_{i=1,\ldots S} \underline{c}_i'\cdot\underline{x}) \ .$$

In order to solve this problem one has to solve S LP-problems, each of which has the form

$$\underset{(\underline{x},y)}{\text{minimize}} \ (\lambda\cdot y+(1-\lambda)\cdot\underline{c}_i'\cdot\underline{x}), \quad i=1,\ldots,S$$

where $(\underline{x},y)\in\{\underline{\underline{A}}\cdot\underline{x}=b,\underline{x}\geq 0,\ y\geq\underline{c}_j'\cdot\underline{x};\ \underline{c}_i'\underline{x}\geq c_k'\underline{x}\ ,$
$j=1,\ldots,S,\ k=1,\ldots,i-1,i+1,\ldots,S\}$

and furthermore,

$$\underset{i=1,\ldots S}{\text{minimize}}\ \min_{(\underline{x}\cdot y)} (\lambda\cdot y+1-\lambda)\cdot\underline{c}_i'\cdot\underline{x}) \ .$$

Savage-Niehans Criterion

$$\underset{\underline{x}\in X}{\text{minimize}}\ \max_{i=1,\ldots S} (\underline{c}_i'\cdot\underline{x}-\min_{\underline{x}\in X} \underline{c}_i'\cdot\underline{x})$$

In order to solve this problem one has to first solve S LP-problems

$$\underset{\underline{x}\in X}{\text{minimize}}\ \underline{c}_i'\cdot\underline{x},\ i=1,\ldots,S,$$

and further one LP-problem

$$\underset{(\underline{x},y)}{\text{minimize}}\ y$$

where $(\underline{x},y)\in\{\underline{\underline{A}}\cdot\underline{x}=\underline{b},\ \underline{x}\geq 0,\ y\geq(\underline{c}_i'\underline{x}-\min_{\underline{x}\in X} c_i'\underline{x}),\ i=1,\ldots,S\}$

It is reasonable to use in this case such a criterion which provides a finding of such a strategy belonging to the Pareto set for which relative differences between the values of the objective functions and their minima are equal. This criterion is described by the following constraints

$$\frac{\underline{c}_1'\cdot\underline{x}-\min_{\underline{x}\in X}\underline{c}_1'\cdot\underline{x}}{\min_{\underline{x}\in X} c_1'\cdot\underline{x}} = \frac{\underline{c}_2'\cdot\underline{x}-\min_{\underline{x}\in X}\underline{c}_2'\cdot\underline{x}}{\min_{\underline{x}\in X}\underline{c}_2'\cdot\underline{x}} = \ldots\ldots = \frac{\underline{c}_S'\cdot\underline{x}-\min_{\underline{x}\in X}\underline{c}_S'\cdot\underline{x}}{\min_{\underline{x}\in X}\underline{c}_S'\cdot\underline{x}}$$

This criterion represents a special form of the so-called bliss point criterion [10].

Table 1a. Reference Values and Ranges for Capital Costs of Fast Breeder Reactors (FBR) and of Solar Thermal Electric Conversion (STEC).

	Reference Value [$/kWe]	Ranges [$/kWe]
FBR	920	850-1275
STEC	1900	1000-2500

Table 1b. Elements of the Set C_o of Extreme Points According to (4-2).

	FBR	STEC
$\underline{c}_1$	850	1000
$\underline{c}_2$	1275	1000
$\underline{c}_3$	850	2500
$\underline{c}_4$	1275	2500

Table 2. Results of MESSAGE-runs for the Low Scenario and World Region I With Data Given by Table 1.

Decision Criterion	Value of Criterion [G$]	FBR Electricity Production in 2030 [GWyr]	STEC Electricity production in 2030 [GWyr]
Reference Case	2988.05	330	0
$\min_{\underline{x}} \underline{c}_1' \cdot \underline{x}$	2985.92	283	62.6
$\min_{\underline{x}} \underline{c}_2' \cdot \underline{x}$	2988.97	0	150
$\min_{\underline{x}} \underline{c}_3' \cdot \underline{x}$	2986.38	367	0
$\min_{\underline{x}} \underline{c}_4' \cdot \underline{x}$	2989.26	0	0
Minimax (Wald) (Theorems 1 and 2)	2989.26	0	0
Laplace (as given by (4-6))	2989.24	28	0
Hurwitz 1	2989.26	0	0
(λ=0.8) 2	2989.26	0	0
3	2988.30	255	0
4*	2989.26	0	0
(Theorems 4 and 5)			
Savage-Niehans (Theorems 6 and 7)	2.23	148	3

*As according to Table 1b already $\underline{c}_2$ leads to a zero installed STEC capacity, $\underline{c}_4$ does so, too.

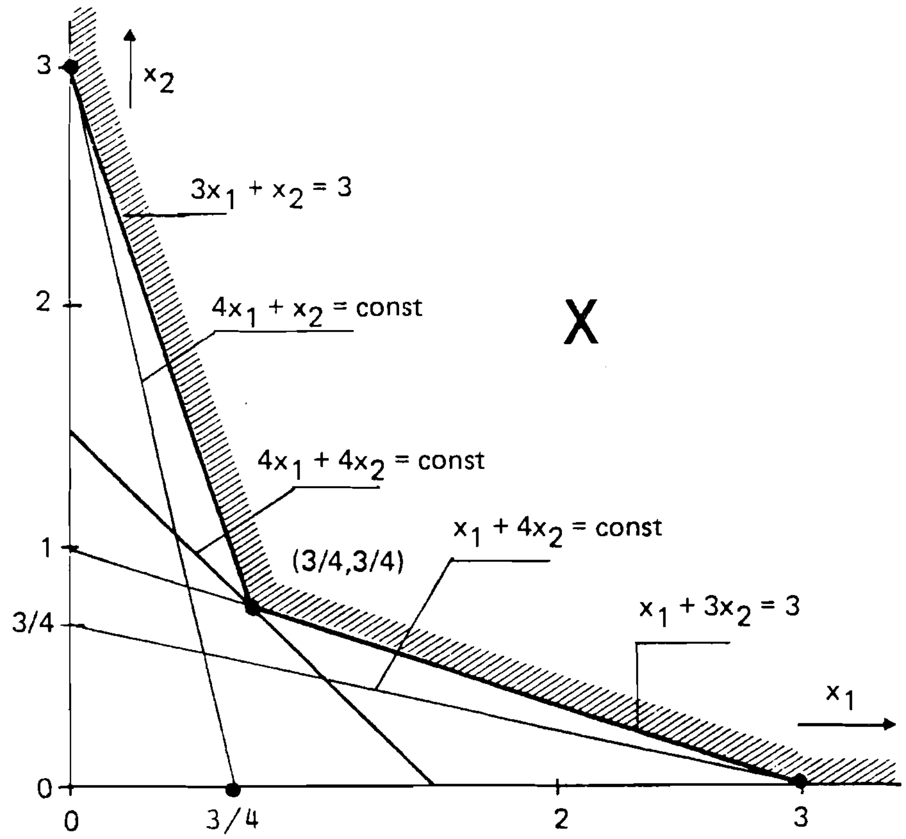


Figure 1a. Variable space and lines of levels of the objective function of the first example.

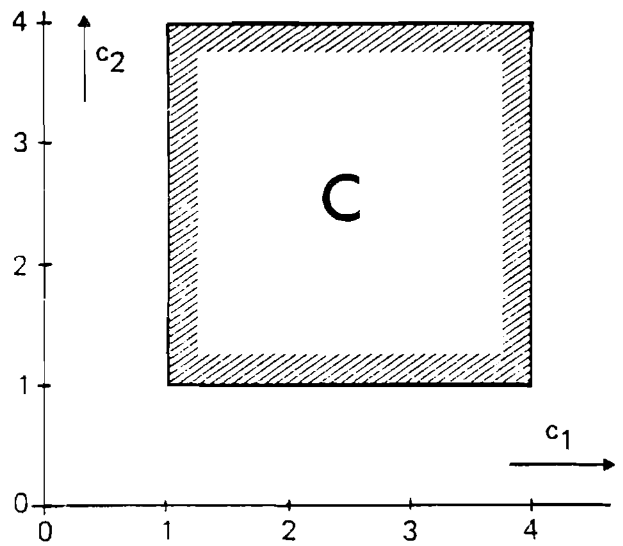


Figure 1b. Space of the uncertain parameters of the first example.

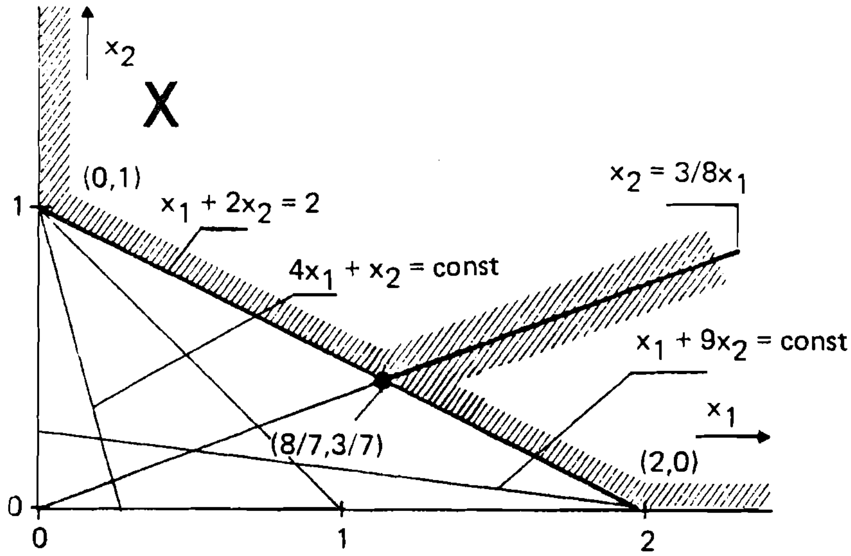


Figure 2a. Variable space and lines of levels of the objective function of the second example.

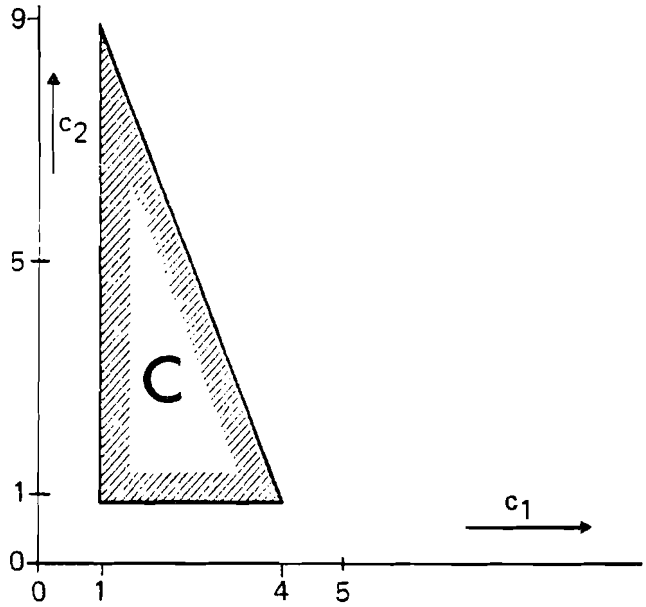
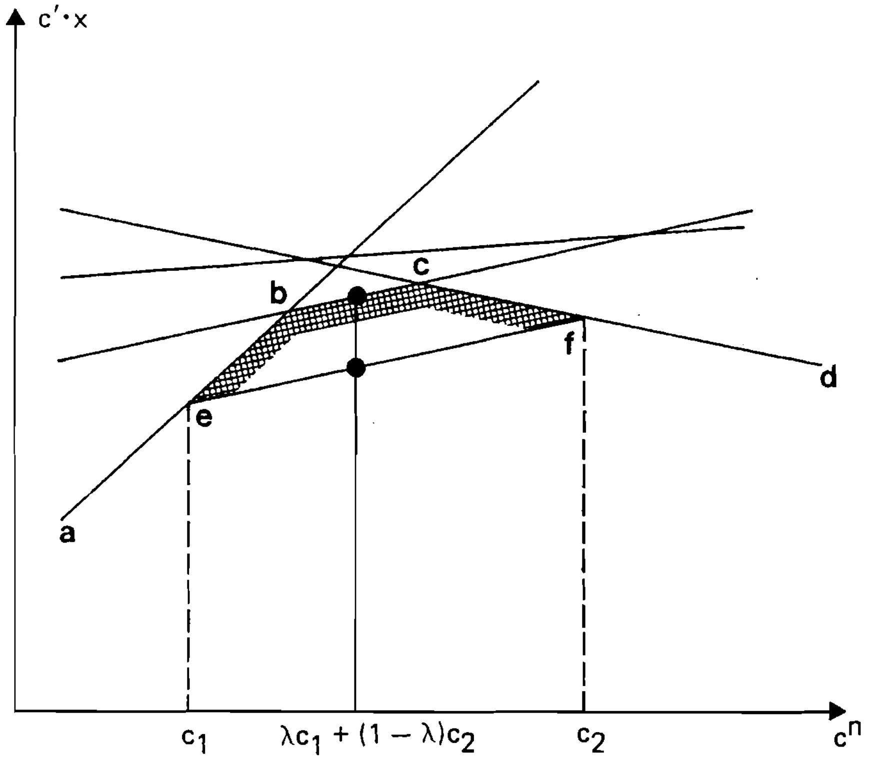


Figure 2b. Space of the uncertain parameters of the second example.



$\min_{x \in X} c' \cdot x$ lies on the line abcd.

$(\lambda \cdot \min_{x \in X} c_1 \cdot x + (1 - \lambda) \min_{x \in X} c_2 \cdot x)$ is the line ef

Any point of ef is below or equal to any point of ebcf.

Figure 3a. Illustration of the concavity of the function $z(\underline{c}') := \min_{\underline{x} \in X} \underline{c}' \cdot \underline{x}$

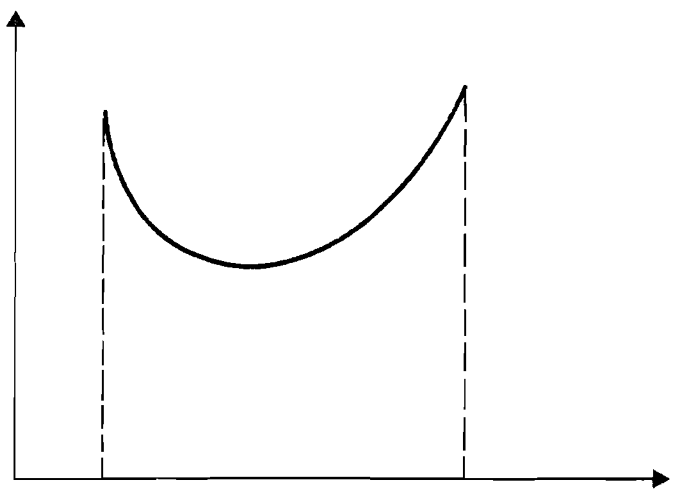


Figure 3b. Illustrating the position of the maximum of the convex function as lying at one of the extreme points.

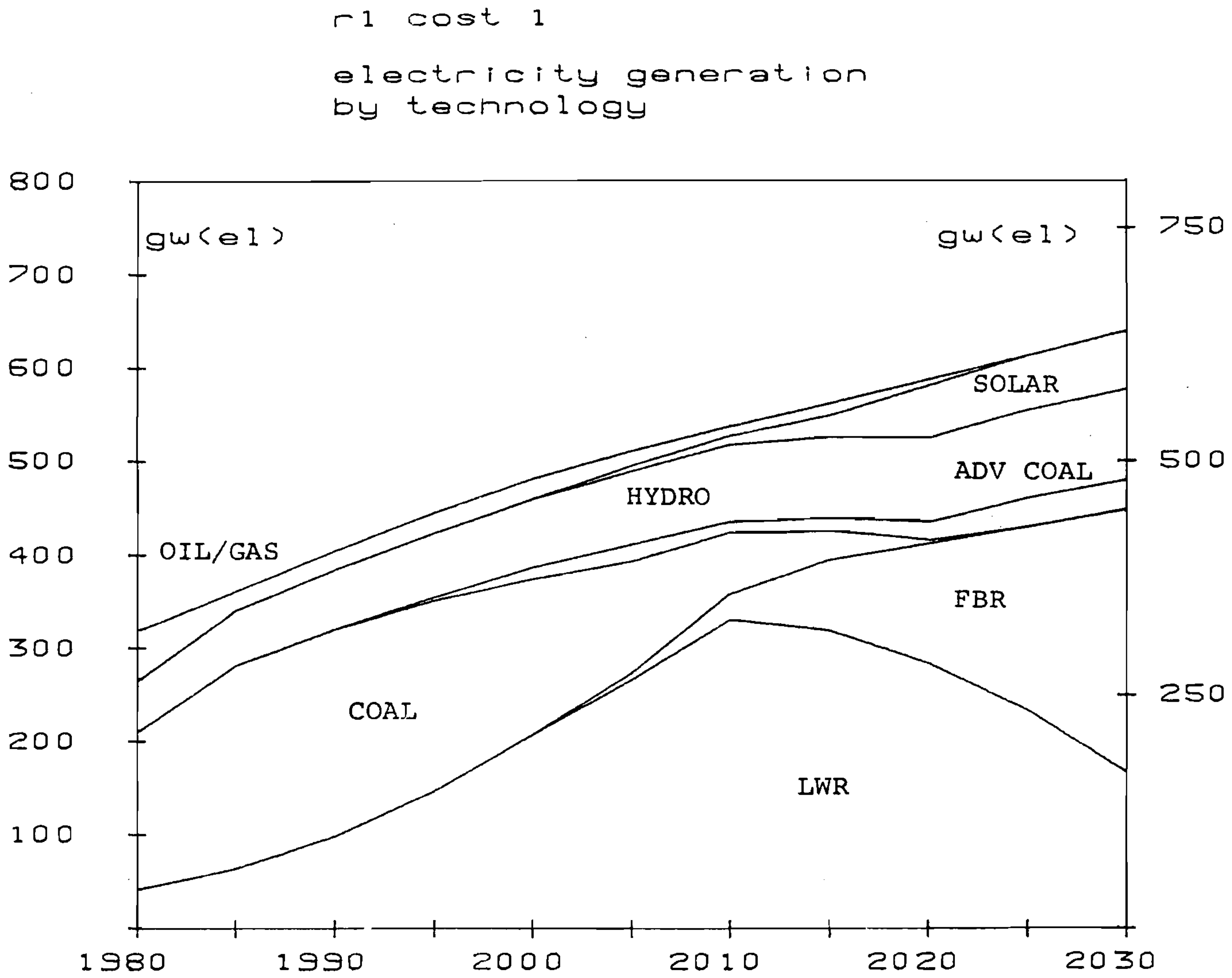


Figure 4a. Results of MESSAGE-runs for world region 1: electricity production for extreme point $\underline{c}_1$ (see Table 1b) as a function of time.

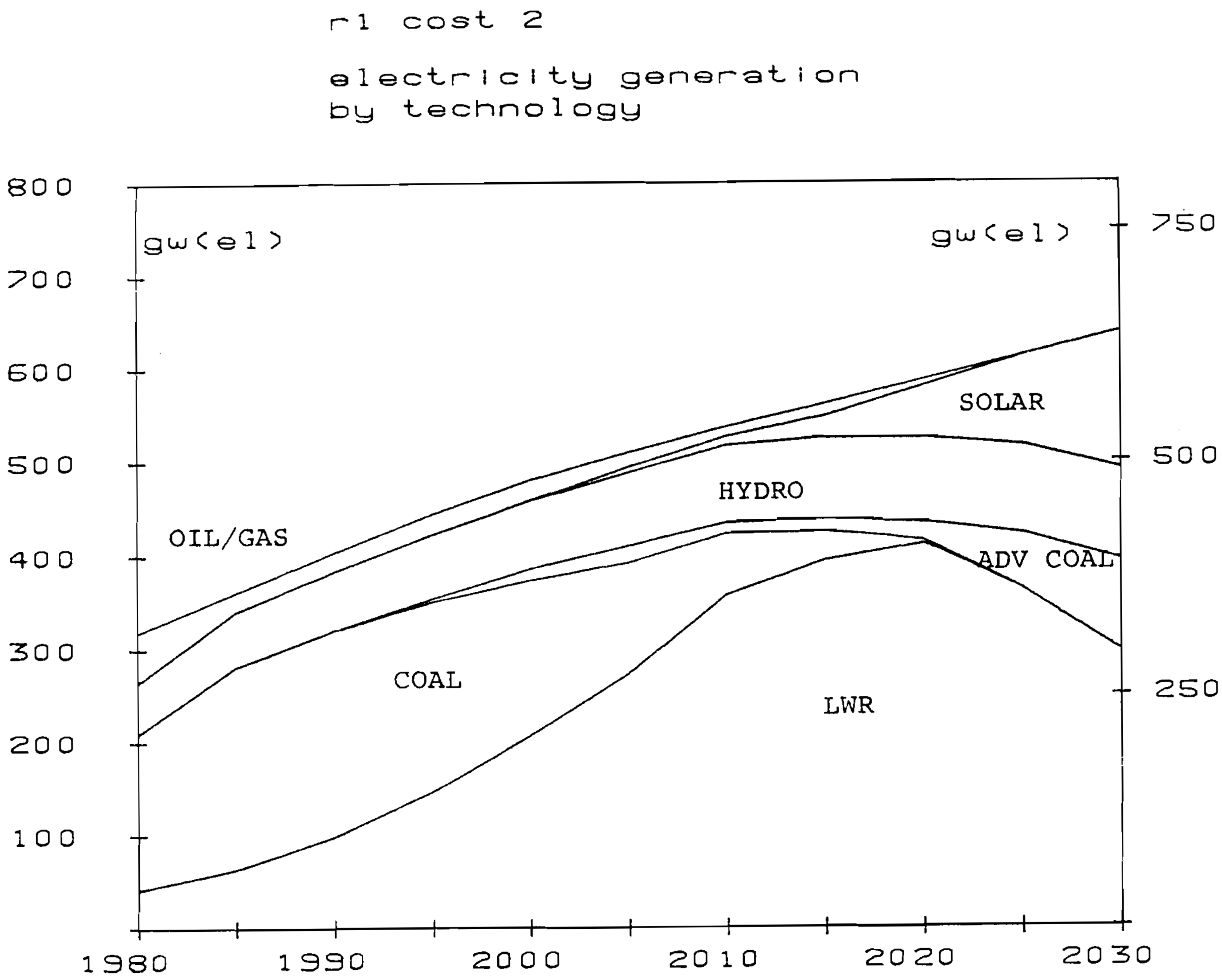


Figure 4b. Results of MESSAGE-runs for world region 1: electricity production for extreme point $\underline{c}_2$ (see Table 1b) as a function of time.

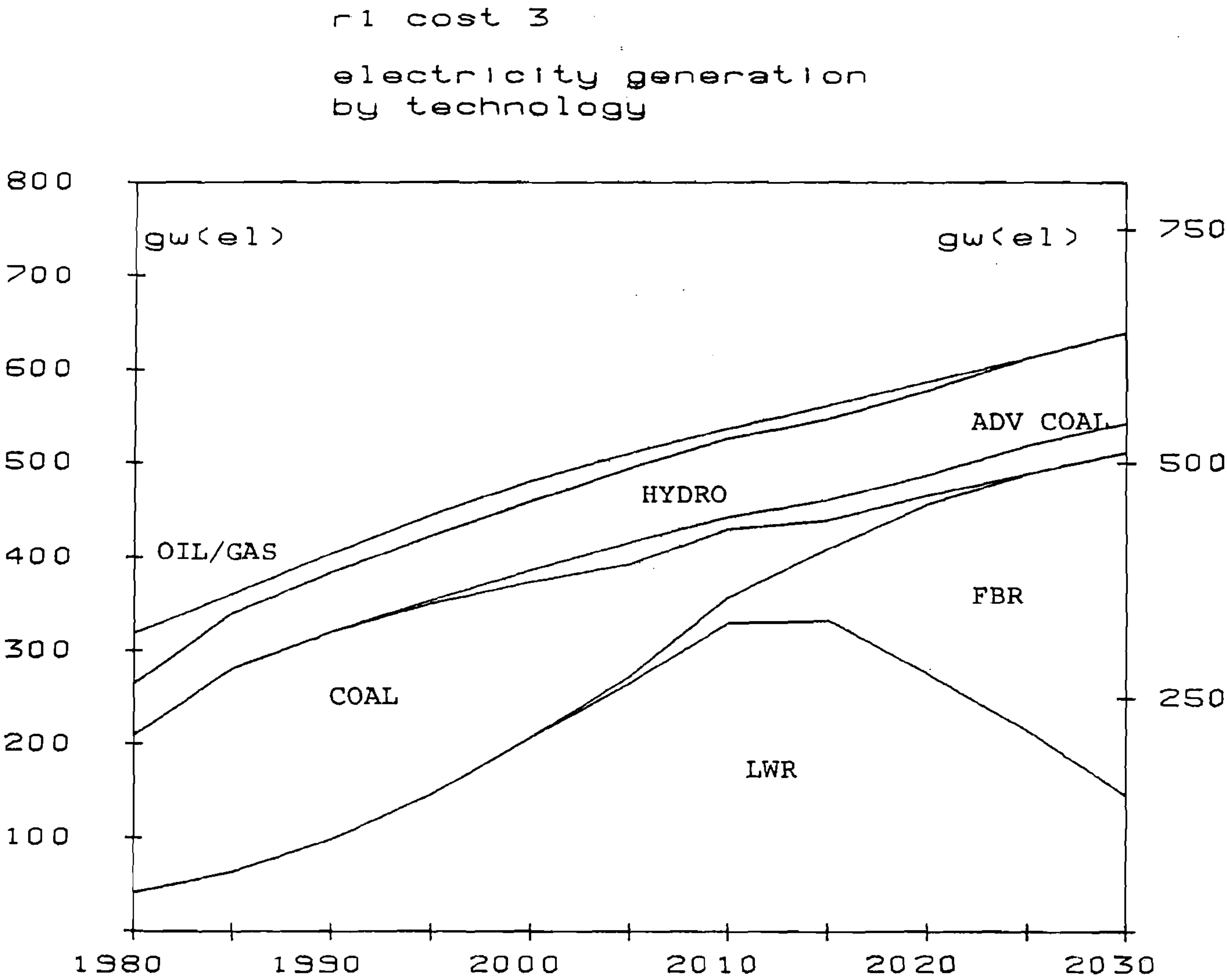


Figure 4c. Results of MESSAGE-runs for world region 1: electricity production for extreme point $\underline{c}_3$ (see Table 1c) as a function of time.

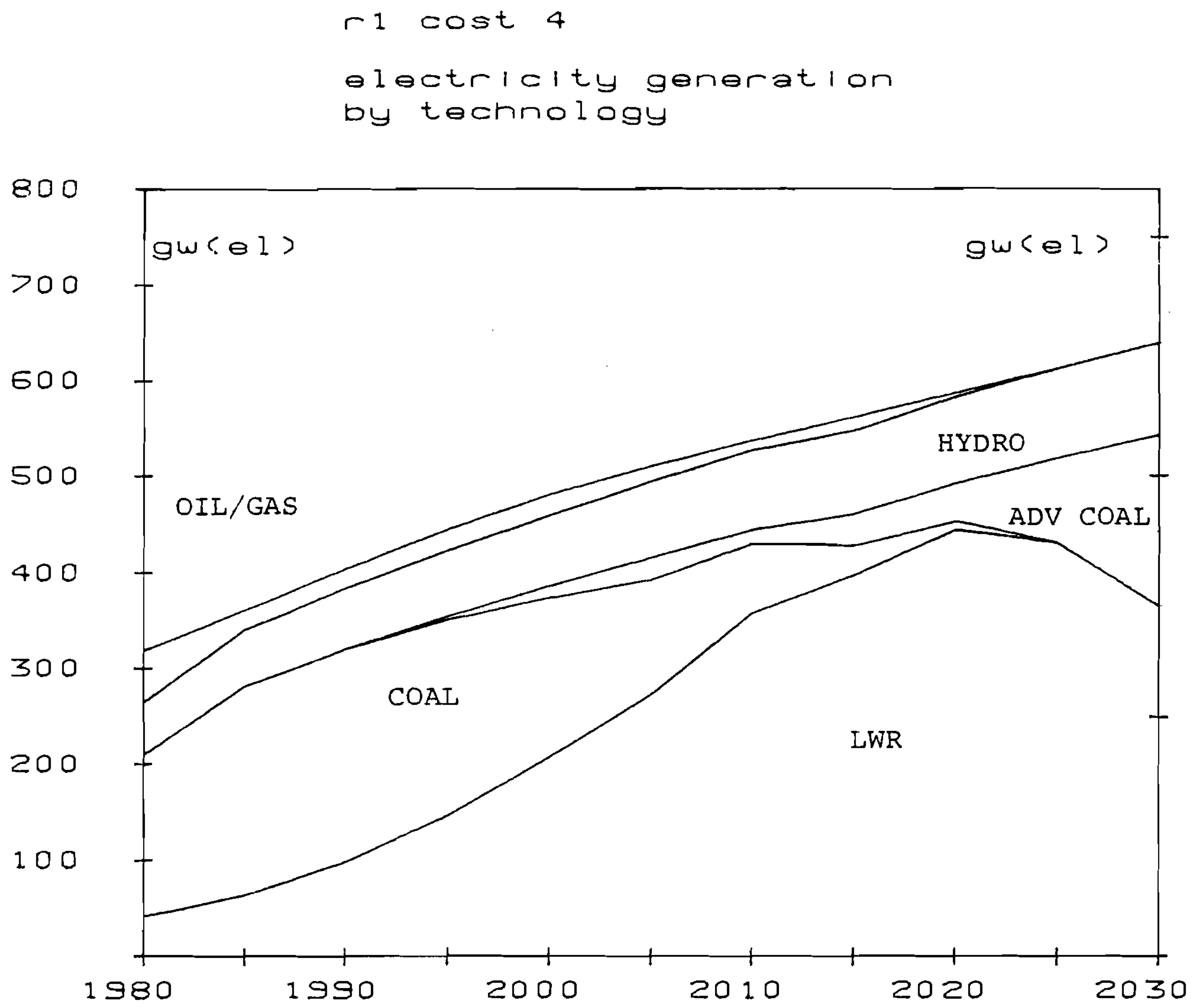


Figure 4d. Results of MESSAGE-runs for world region 1: electricity production for extreme point $\underline{c}_4$ (see Table 1d) as a function of time.

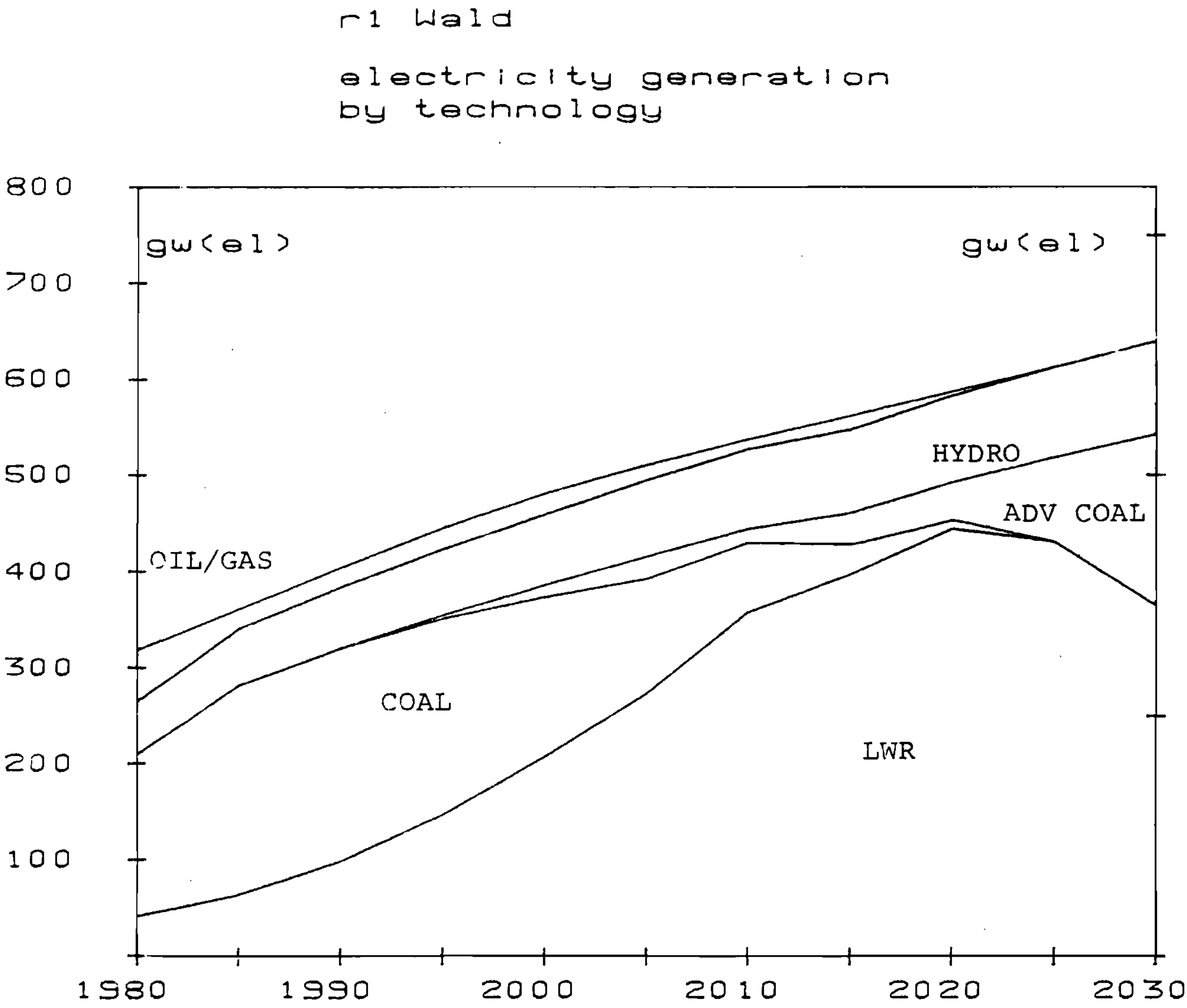


Figure 5a. Results of MESSAGE-runs for world region 1: electricity production according to the Minimax-(Wald) criterion as a function of time.

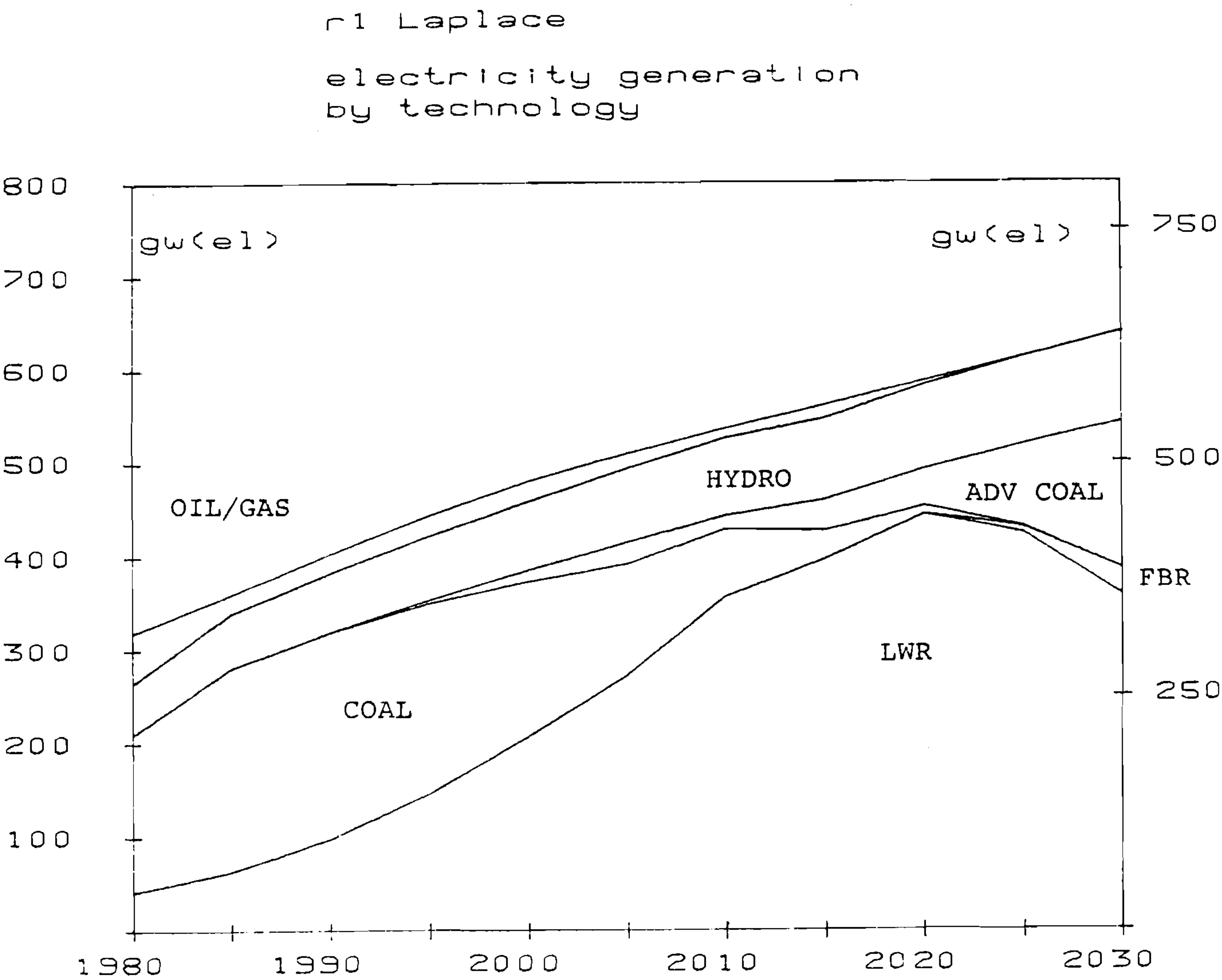


Figure 5b. Results of MESSAGE-runs for world region 1: electricity production according to the Laplace criterion as a function of time.

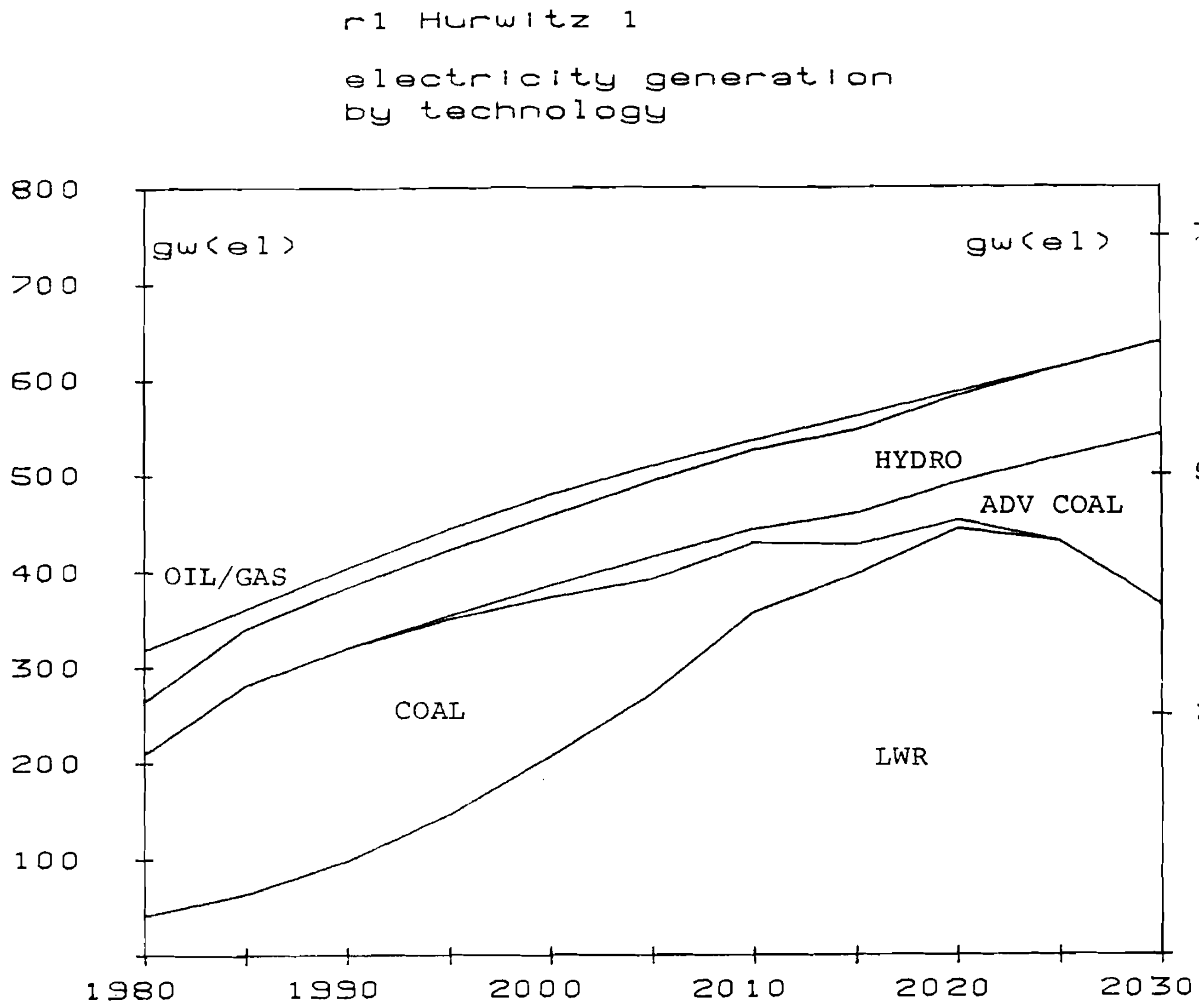


Figure 5c. Results of MESSAGE-runs for world region 1: electricity production according to the Hurwitz criterion, intermediate step 1 of Theorem 4, as a function of time.

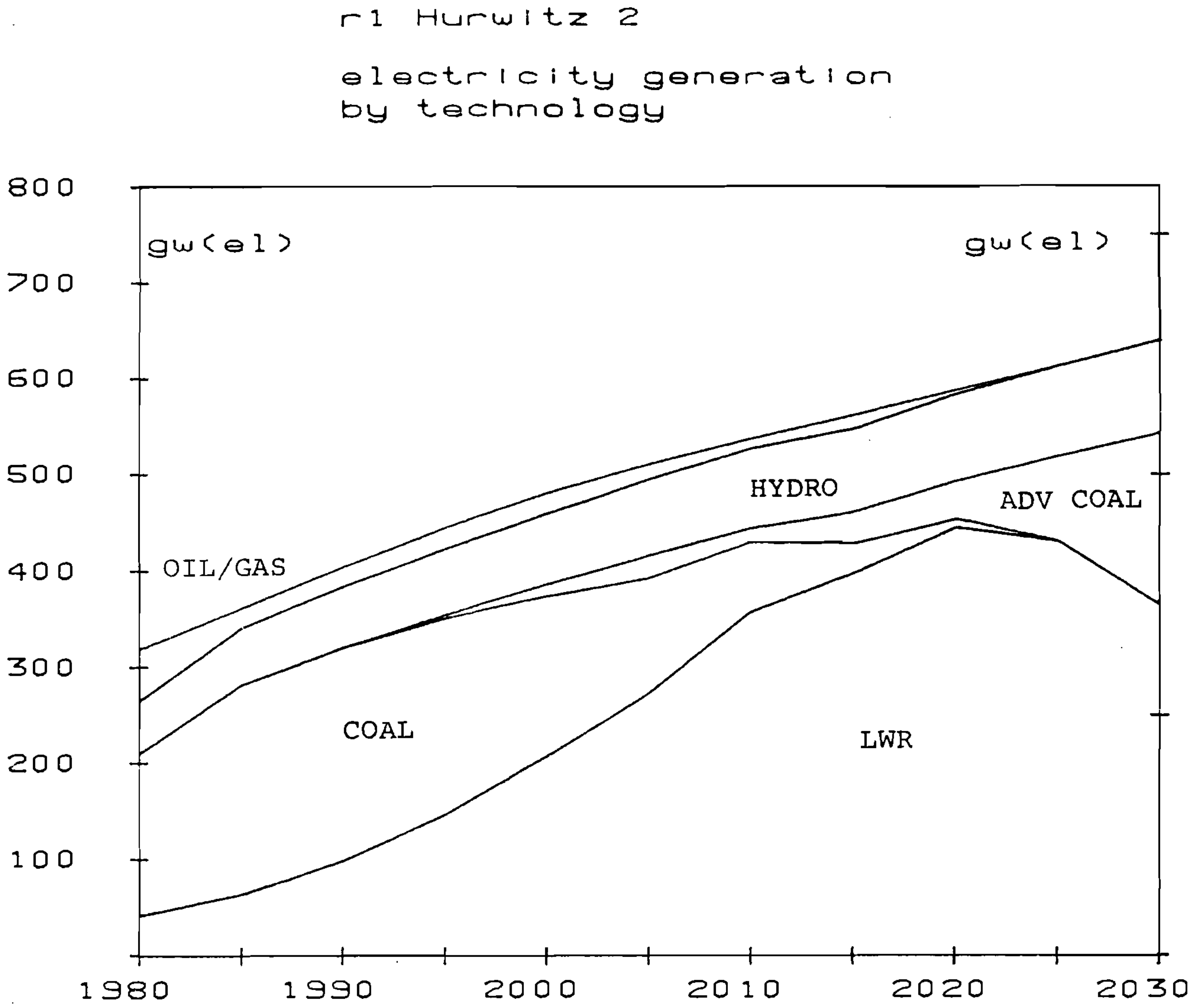


Figure 5d. Results of MESSAGE-runs for world region 1: electricity production according to the Hurwitz criterion, intermediate step 2 of Theorem 4, as a function of time.

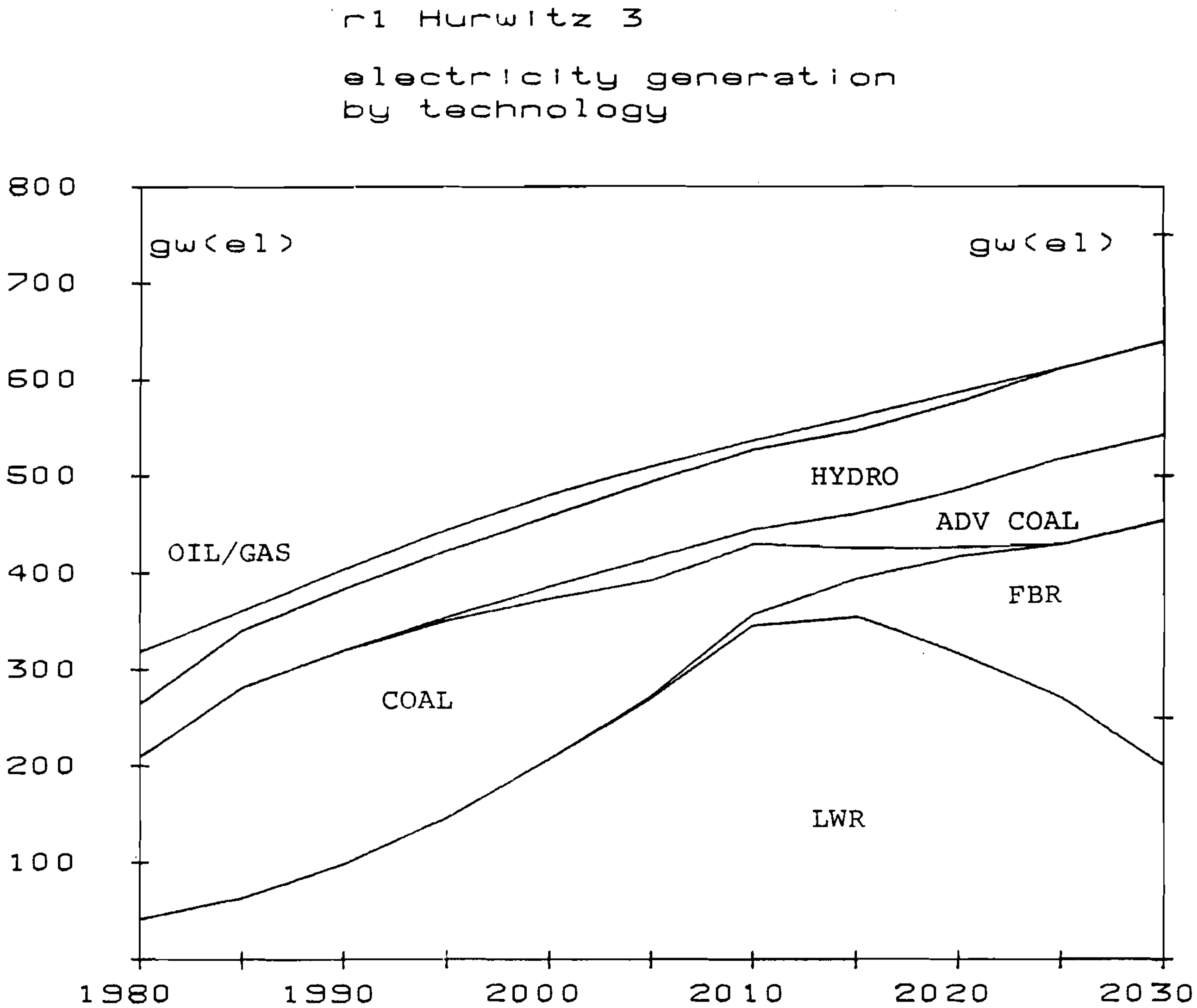


Figure 5e. Results of MESSAGE-runs for world region 1: electricity production according to the Hurwitz criterion, intermediate step 3 of Theorem 4, as a function of time.

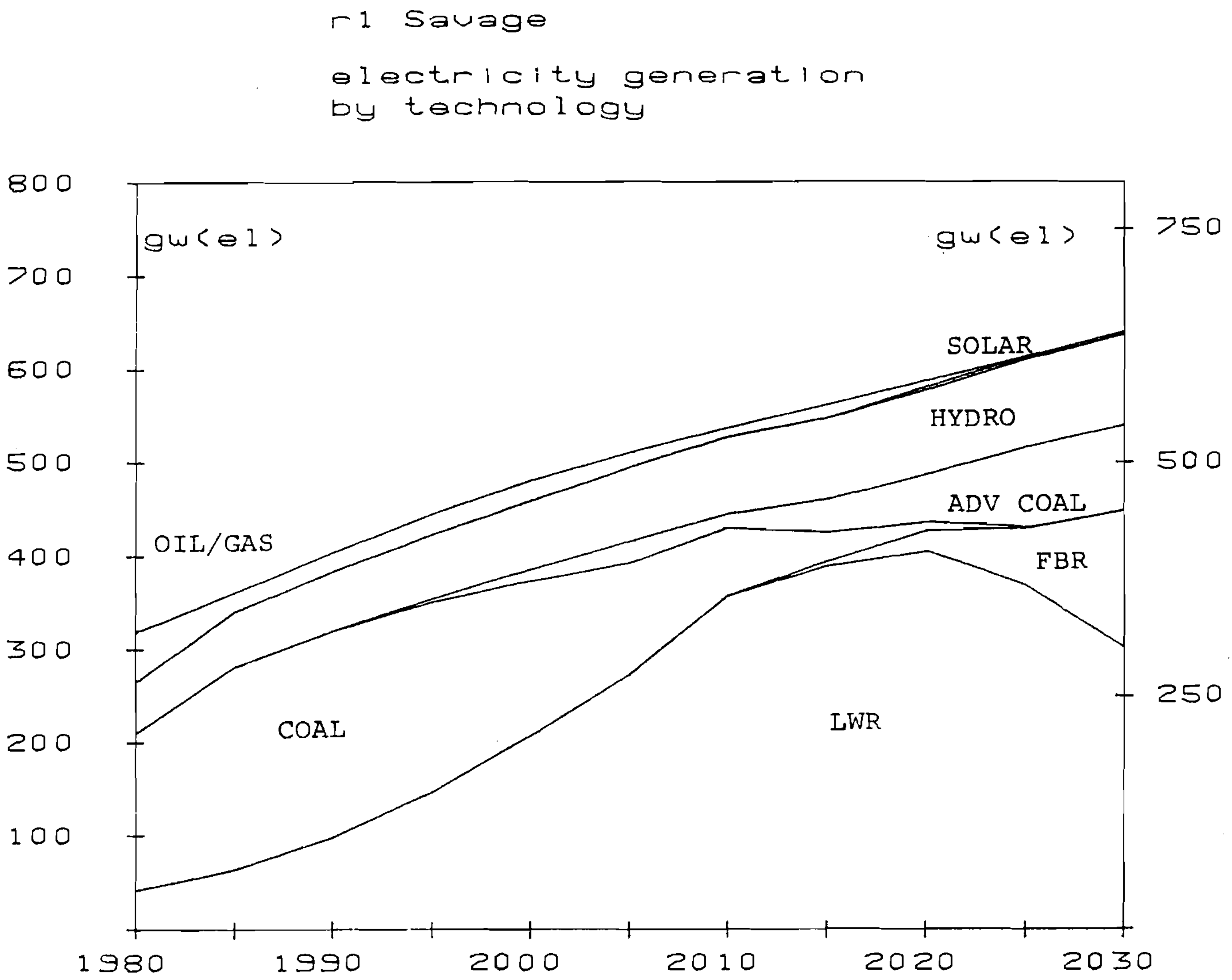


Figure 5f. Results of MESSAGE-runs for world region 1: electricity production according to the Savage criterion, as a function of time.