INVESTMENT PROGRAMMING FOR INTERDEPENDENT PRODUCTION PROCESSES

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ABSTRACT

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Interdependent Production Processes

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In this study a solution procedure is developed for a class of industrial investment planning models which incorporates the following features: economies-of-scale in production, intermediate input-output relationships among production activities, and joint production of different products having common processing requirements (capacity sharing). The model is formulated as a mixed integer programming problem. It is single-period and disregards spatial considerations. The choice is between domestic production and imports (make-buy) to satisfy exogenously stated demands for a given set of interrelated products.

A two-stage solution procedure was developed and specialized to various specifications of the planning model. At the first stage simple sufficiency conditions for import and for domestic production of a given product are systematically applied in an attempt to reduce

the size of the problem. At the second stage (solution stage) an LP-based branch-and-bound (B-B) algorithm is used.

Data from the mechanical engineering (metal working) sector of the Republic of Korea was used to implement the proposed two-stage solution procedure. The results from 25 test problems generated from the Korean data provide strong evidence of the efficiency of the approach. Moreover, computational experience with the B-B stage alone indicates that very large problems can be efficiently solved without dependence on the success of any form of problem reduction attempt.

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CHAPTER 1

INTRODUCTION

The objective of this study is to analyze an important class of investment planning models and to develop efficient solution procedures that are applicable under a broad range of model specifications. We are specifically concerned with problems of industrial investment analysis in the context of development planning, the aim of the analysis being the identification of those projects whose establishment is economically desirable.

The problems of industrial project selection are greatly complicated by the complex interdependencies that exist among production activities. The models of this study focus mainly on those interdependencies among activities within an industry, a subsector, or even a whole sector of an economy, that stem largely from the strong economies of scale in processing activities entailed by joint production using shared production facilities (capacity sharing), and from intermediate input-output technological relationships within the sector.

The optimization problem can be posed as one of choosing investments in industrial processes so as to meet given demands or output targets at minimum cost by taking account of intra-sectoral flows and potential gains from capacity sharing. The models are of the make-buy type, restricted to a single period and disregarding spatial considerations. A solution to any of these models may provide decisions as to which projects should be undertaken, or it may merely provide a comparative-advantage ranking of production activities, depending on the particular questions being addressed.

The present study was inspired by the work of Westphal and Rhee [1978] on the mechanical engineering sector (also called metal working industry) of the Republic of Korea.[1/] This type of investment analysis falls under the general heading of process analysis. Other prominent applications of this type of analysis include the work of Kendrick [1967] for planning investments in the Brazilian steel industry, Gately [1971] for the Indian electricity generating sector, de la Garza and Manne [1973] for the Mexican energy sector, and the study reported in Stoutjesdijk and Westphal [1978] for planning the East African fertilizer industry.[2/]

Manne and Markowitz [1963] distinguish three areas of activity involved in process analysis: model building, which begins with an investigation of technology to obtain a mathematical description or model; the development of computationally efficient solution procedures or algorithms through which these models may be used for purposes of analysis; and, finally, the actual use of the models to address practical issues of public policy. While Westphal and Rhee in the Korea study were largely concerned with the first and third

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1/ As this study is referred to numerous times throughout this dissertation, it is hereafter called simply the Korea study, and the model used in the study is referred to interchangeably as the Korea model or the Westphal and Rhee model.

2/ This study appears in the same volume as the Korea study.

areas in the context of project identification, this study concentrates on the second. The emphasis here is on efficient computational approaches to this class of problems.

Because interdependence among activities is the focus of concern of the various models dealt with in this study, a brief discussion of its implications for investment decisions is provided in Chapter 2. The explicit recognition of interdependencies among interrelated production activities is in fact what distinguishes process analysis, in the context of project identification, from the traditional methods of project appraisal. While process analysis can be viewed as a systems analysis approach, project appraisal merely deduces the consequences of undertaking a particular project by taking nearly everything as given.

Chapter 2 then proceeds with the mathematical formulation of the planning problem as a mixed integer programming (MIP) problem. The model differs from Westphal and Rhee's formulation in only one respect: while theirs allows for the simultaneous investment in more than one processing unit (production facility, plant, etc.) of a given kind, our formulation limits it to one.

MIP is increasingly becoming a favorite tool of analysis for planning problems because of the flexibility that it allows in modeling techno-economic relationships. This flexibility, however, is not without a price, for MIP problems are generally very difficult to solve. Although simple and efficient branch-and-bound algorithms have been devised for many different classes of problems, no efficient general purpose computer software exists for large MIP problems.

Westphal and Rhee, for example, did not attempt to obtain a proven globally optimal solution to the Korea model, as this would have been "prohibitively expensive given the available computational software for mixed integer programming."[1/]

If optimal solutions are to be obtained in any reasonable amount of computational time for realistic size problems, the structure of the problem must be exploited in order to reduce the computational effort required to obtain the solution. This is essentially what was done by Westphal and Rhee in their attempt to solve the Korea model, and by Crémer [1976] to solve a similar version of the problem. It is also the approach taken in this study. A common feature, in fact, of the three approaches is that each has two distinct stages or phases: the first exploits the structure of the model in an attempt to reduce the size of the problem, and the second obtains a solution (not necessarily globally optimal). Although the problem reduction stage of these three approaches is essentially the same, completely different directions are taken at the second stage.

Our approach, which yields the globally optimal solution, is presented in Chapter 3. It exploits the input-output structure of the problem in order to obtain bounds to be used in a branch-and-bound scheme. Computational experience is provided in Chapter 4 using data from the Korea study. Chapter 4 also contains a brief review of the approach taken by Westphal and Rhee in the estimation of the Korea model; it serves to illustrate some of the issues and diffi-

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1/ Westphal and Rhee [1978], Chapter 15.

culties that arise in model estimation.

In Chapter 5 we drop the capacity sharing feature of the model and proceed to generalize it by incorporating such features as alternative products, choice among alternative production techniques for each product, piecewise-linear concave investment cost functions, and finally, general concave cost functions. All of these versions of the model can equally be viewed as generalizations of Leontief substitution systems or generalized versions of break-even analysis.

The models of Chapter 5 apply in situations in which capacity sharing is not important. This may occur as the sector develops and demands rise to a point that justifies a higher degree of specialization, that is, production activities become more end-product oriented rather than process oriented. These models are also applicable in situations in which the planning problem is specified at such a level of aggregation -- say at the plant or even interindustry level -- that capacity sharing loses its meaningfulness.

The planning models of this study are discussed within the context of development planning since the most likely beneficiaries of this type of investment analysis are semi-industrial, developing countries. Because of economies-of-scale characteristic of industrial activities and the reduced size of their markets, investment in certain production activities can be justified economically in such countries only if full advantage is taken of capacity sharing and of the external economies[1/] generated by the intermediate input-output relationships within the sector. The models studied here, however, are not limited to be used in studies conducted by planning agencies of developing countries. Investment analysis of the types discussed here could also be performed by a group of major firms within a sector or subsector, or even a single firm which occupies a position of importance within the subsector it operates. The decision maker implied in these models could thus equally well be the board of directors of a large, multi-unit corporation, or the planning board of a developing country.

A summary of the study and suggestions for further research are provided in Chapter 6.

1/ We use Chenery's [1959] definition of external economies as applied to the effects of investment: "... industries A, B, C, ..., provide external economies to industries K, L, M, ..., if investment in industries A, B, C, ..., causes a decrease in the cost of supplying the demands for the products of K, L, M, ..."

CHAPTER 2

THE INVESTMENT PLANNING PROBLEM

2.1 Introduction

It is possible to categorize investment projects by the degree to which their evaluation requires the simultaneous evaluation of other investment projects. At one extreme are those projects that legitimately may be appraised in isolation since their impact on the profitability of other projects is negligible or non-existent, or, equivalently stated, no external economies are generated by these projects. At the other extreme lie those investment projects whose impact is sufficiently great that external effects reach across the entire economy and must clearly be evaluated in an economy-wide framework. Under this extreme would fall those projects that require a large fraction of total planned investment over a medium term planning horizon and could significantly alter the structure of supply and demand for major commodities and resources. Somewhere between these two extremes lie those cases for which it is necessary to evaluate simultaneously all investment projects falling within a given sector or subsector of the economy. These are the cases of interest in this study.

A brief discussion of the various types of interdependencies caused by the presence of economies of scale in production activities is given in section 2.2. We are particularly concerned with the effects of these interdependencies on production costs and the extent to which they may affect investment decisions. The discussion of section 2.2 disregards the effects of interdependencies between sectors, the most important of which being the competition (between the sectors) for a number of scarce resources.

The investment planning model that captures the effects of two important types of interdependencies is presented in section 2.3. Its formulation is adapted from Westphal and Rhee [1978], which should be consulted for a detailed discussion of the model.

2.2 Interdependencies in Investment Decisions

The presence of economies-of-scale in production activities gives rise to two important types of interdependencies which make it necessary to evaluate simultaneously the investment projects within a sector. The first one is due to intermediate product relationships, which is referred to as input-output or material interdependence, and stems from the use of intermediate inputs produced at decreasing unit costs. Due to economies-of-scale, the unit cost of each product depends on its output or demand level and on the unit cost of its inputs, which in turn depend on their demand levels and on the costs of their inputs, and so on. Clearly under this situation a group of products that are considered profitable when analyzed jointly, may separately appear unprofitable and would not be undertaken by an individual investor who does not take into account the increase in the profitability of investment in related projects. Chenery [1959] provides a very illuminating analysis of the input-output type of interdependence. He makes use of an interindustry model to study the extent and the circumstances under which coordinated investment decisions would lead to more efficient resource utilization than would individual decisions based on existing market information. He concludes that besides the fact that profitable projects may not be identified if input-output interdependence in production is not explicitly taken into account, the lack of coordination of production decisions may also lead to suboptimal timing of plant construction and suboptimal scale of plants constructed to supply intermediate inputs. Westphal[1/] makes use of a simpler version of Chenery's model to illustrate this.

The second type of interdependency, process interdependence, occurs when different products require processing in similar equipment or processing facilities. The possibility of joint production, which we refer to as capacity sharing, gives rise to an important interdependency among all production activities. Because of economies-of-scale there are large potential benefits to be derived by exploiting process interdependence. It is the joint effects of input-output interdependence and process interdependence that are captured in the planning model formulated in the next section. These effects are further discussed there.

Two other types of interdependencies that are also potentially very important in the presence of economies-of-scale but which are not incorporated in the models studied ~~in this dissertation~~ here are temporal and spatial interdependence. Failure in explicitly recognizing the time element in investment analysis may lead to suboptimal

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1/ In Stoutjesdijk and Westphal [1978], Chapter 5.

decisions, since, in the presence of economies-of-scale, it may be efficient to build capacity in anticipation of future growth in demand, or it may pay to delay the construction of new capacity until demand levels have increased sufficiently, with interim demands being met by imports.[1/] Temporal interdependence may thus alter the structure of the optimal investment pattern.

It is important to note, however, that in the context of the project identification model presented in the following section, the consequence of disregarding temporal interdependence does not appear to be a serious one. Since economies-of-scale make desirable the establishment of plants in advance of the growth in demand, the effect of an analysis that takes the time element into account would thus be to lower the demand level at which domestic production is justified, that is, to lower the break-even point between domestic production and imports. This has two important implications. First, the optimal timing for the projects identified by the model must be now (time zero of the planning period) and not at any later point in time; it could have possibly been earlier, but the projects were not identified earlier. Secondly, it is possible that some projects not identified by the model should optimally be implemented in the current planning period, in advance of demand growth. It is unlikely, however, that the benefits from undertaking any project that is not identified when the external economies from input-output and process interdependence are fully exploited would be very large. Moreover, under any form of

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1/ See Erlenkotter [1967].

binding budget constraint it is highly unlikely that these marginal projects would be undertaken in the current planning period.

It remains to discuss the effects of temporal interdependence on the scale (sizing) of the projects. If we accept the above argument that the projects not identified by our planning model, but that would be by a dynamic model, are not important, then the optimal timing for the identified projects is known. Consequently, the sizing of the projects can be determined independently of the investment decision; that is, they need not be determined simultaneously with the decision of which projects should be undertaken. Since the projects are interdependent, however, the sizing decision must be determined jointly for all the identified projects. With growing demands, the scales obtained from our (static) model provide lower bounds on the optimal scales of the projects.

Finally, spatial interdependence across all production decisions is introduced by the existence of transportation costs. The interdependence takes the form of a trade-off between the gains from economies-of-scale attained in building larger plants or processing facilities and the increased transportation costs of serving larger market areas and/or spatially dispersed user plants, depending on whether products are produced for final consumption or as inputs in the production of other products. In the models studied in this dissertation a pre-specified location for each production activity is assumed. For the types of industry characterized by capacity sharing, e.g., the mechanical engineering industry, transport costs of products and raw materials would seem to be of much less relative significance than in heavy process industries such as cement or fertilizers.

2.3 Formulation of the Investment Planning Model

The planning problem can be simply stated as follows: select investments in production activities so as to minimize the cost of satisfying the exogenously stated demands for a given "bill of goods." The choice is between domestic production and imports, and the model, formulated below, is thus of the make-buy type.

Four distinct sets of activities are specified in the model:

x_i = level of domestic production of product i;

y_i = level of imports of product i;

z_k = new capacity in the k^{th} type of process element;

Δ_k = zero-one variable associated with the k^{th} type of process element; $\Delta_k = 1$ if investment is undertaken in process element type k, 0 otherwise.

The term "process element" is used to designate the individual elements of production; it may represent a piece of equipment, a group of equipment that jointly perform a certain processing function, or an entire plant, depending on the specific questions being addressed by the model. The index k is consistently used to denote a process element, and i or j to denote products. We use the symbols I and K to denote the set of products and process elements respectively, as well as the cardinalities of these sets.

The objective function is to minimize the total cost of meeting the final demands for the I products and can be stated as follows:

$$\text{Min} \left\{ \sum_{k \in K} (F_k \Delta_k + V_k z_k) + \sum_{i \in I} G_i x_i + \sum_{i \in I} W_i y_i \right\} , \qquad (2.1)$$

where

F_k = fixed charge portion of annual cost of capacity for process element type k;

V_k = variable charge portion of annual cost of capacity and operation for process element type k;

G_i = exogenous cost of producing one unit of product i; it includes labor, raw materials, and any other intermediate inputs that are exogenous to the model;

W_i = unit import price for product i.

All these parameters are nonnegative and the V_k are strictly positive. Obviously $G_i < W_i$, for otherwise product i should clearly be imported.

A material balance constraint states that the sum of domestic production and imports must be equal to the final demand plus the endogenously generated demand for each product:

$$x_i - \sum_{j \in I} a_{ij} x_j + y_i = D_i , \qquad i \in I , \qquad (2.2)$$

where

a_{ij} = input-output requirement: one unit of product j requires a_{ij} (≥ 0) units of product i;

D_i = exogenous (final) demand for product i.

The D_i specify demand requirements from activities exogenous to the model, and may be for final consumption or investment use, or even for intermediate inputs to be used in production activities that are exogenous to the model.

For each type of process element a capacity balance constraint requires that capacity be at least as large as the total volume of processing:

$$z_k - \sum_{i \in I} b_{ki} x_i \geq 0 , \qquad k \in K , \tag{2.3}$$

where

b_{ki} = amount of capacity of process element type k required in the production of one unit of product i.

Fixed cost constraints require that the fixed cost associated with a given process element be incurred if capacity in that process element is required:

$$C_k \Delta_k - z_k \geq 0 , \quad k \in K , \tag{2.4}$$

where C_k is an upper bound on z_k.

Integrality constraints:

$$\Delta_k = 0 \text{ or } 1 , \qquad k \in K \tag{2.5}$$

Finally, nonnegativity constraints complete the mathematical formulation of the planning model:

$$\begin{aligned} x_i &\geq 0 , \quad i \in I \\ y_i &\geq 0 , \quad i \in I \\ z_k &\geq 0 , \quad k \in K \end{aligned} \tag{2.6}$$

It should be obvious that the input-output and process interdependencies discussed in the previous section enter the model through the material and capacity balance constraints. Production decisions for activities i and j are interdependent whenever at least one of the following two occurrences takes place:

(i) $a_{\ell i} > 0$ and $a_{\ell j} > 0$ for one or more $\ell \in I$;

(ii) $b_{ki} > 0$ and $b_{kj} > 0$ for one or more $k \in K$.

In the first case activities i and j have at least one endogenous intermediate input in common. The external economies created by this type of interdependence take various forms. Assume, for example, that the domestic production of product i is economically justified but intermediate input ℓ is imported. The domestic production of product j would have the effect of increasing the demand for intermediate product ℓ. Due to economies-of-scale this increase in demand might warrant the domestic production of product ℓ. Obviously these effects may occur in either direction or jointly, that is, product j may be produced only if the added demand that its domestic production would generate for product ℓ justifies the domestic production of ℓ. Assume next that both product i and the intermediate product ℓ are domestically produced. Because of economies-of-scale the unit cost of producing the additional amount of input ℓ (required by product j) decreases. This decrease might be sufficient to justify the domestic production of product j. In this case we can say that products i and j <u>indirectly</u> share capacity through common inputs.

In case (ii) there is direct capacity sharing between products i and j in at least one process element. Again, under the economies-of-scale that characterize these production acitivites, the domestic production of either product might justify the production of the other, or domestic production may only take place if both products contribute toward covering the fixed costs of capacity in common process elements.

It is the joint or combined effect of these interdependencies across all production activities that are of interest for investment decisions. The programming model formulated above provides the tool to explore the effects of these interdependencies.

An implicit assumption of the model is that the demands for the I products are such that the upper bound on the capacity of any process element that may be built will not be exceeded. That is,

$$\sum_{i \in I} b_{ik} \left[\sum_{j \in I} a_{ij} \bar{D}_j + D_i \right] \leq C_k \,, \qquad k \in K \,, \tag{2.7}$$

where $\bar{D}_j$ is the production level of product j if all I products are domestically produced, that is, a bound on xj. This restriction is essentially what distinguishes the above model from the one used by Westphal and Rhee [1978] to study the mechanical engineering sector of Korea. As it was discussed in Chapter 1, restriction (2.7) was not imposed on the Korea model. As a consequence, the Δ_k were not binary variables in that case, but were only required to be nonnegative integers. Upper bounds on the Δ_k could be easily obtained, however, from (2.7) for those k for which it was not satisfied. It is easy to see that this type of formulation implies a capacity cost function with jump-type discontinuities C_k, $2C_k, \ldots$, the jumps being equal to the fixed costs F_k. This type of cost function not only undermines the effects of economies-of-scale,[1/] but it also assumes that the capacity of a process element is a "hard" number that can be precisely and unambiguously determined. While there are situations under which this type of cost function may be appropriate, in this study we are interested in those cases where the simple fixed charge cost function is the appropriate one to specify. It is assumed that it is valid over the relevant range of expected output if domestic production is to take place. The upper limit of this range is obviously given by the left-hand-side of inequality (2.7). Although alternative production techniques are not explicitly incorporated in the model, the cost function specified may reflect the fact that different techniques that perform the same processing function may be employed at different output levels.

Next, the planning model given by (2.1) - (2.6) is put in a slightly different form. The new parameters that will appear in the objective function contain more meaningful economic information than in the form given previously. It is in this form that the model will be studied in the next chapter.

Since there is a positive cost associated with each unit of capacity built, in any optimal solution to the problem constraints (2.3) will be satisfied as strict equality. We can thus replace

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1/ As it implies that process elements of larger capacity do not cost proportionately less over a wide range of output levels.

(2.3) by

$$z_k - \sum_{i\in I} b_{ki}x_i = 0 , \qquad k \in K . \tag{2.3a}$$

If we use (2.3a) to eliminate the variables z_k, the following equivalent programming problem is obtained:

$$\mathrm{Min}\left\{\sum_{k\in K} F_k\Delta_k + \sum_{i\in I}\left(\sum_{k\in K} V_k b_{ki} + G_i\right)x_i + \sum_{i\in I} W_i y_i\right\} \tag{2.8}$$

$$x_i - \sum_{j\in I} a_{ij}x_j + y_i = D_i , \qquad i \in I \tag{2.9}$$

$$C_k\Delta_k - \sum_{i\in I} b_{ki}x_i \geq 0 , \qquad k \in K \tag{2.10}$$

$$\Delta_k = 0 \text{ or } 1 , \qquad k \in K \tag{2.11}$$

$$x_i \geq 0 , \qquad i \in I . \tag{2.12}$$

The model in this form corresponds to a fixed charge version of the one studied by Crémer [1976], who specified a general concave investment cost function. The two previously treated versions of this problem and the one studied here thus differ only in the form of the cost function for capacity that is specified.

If we now use constraints (2.9) to eliminate the import variables y_i from (2.8), the objective function takes the following form:

$$\mathrm{Min}\left\{\sum_{k\in K} F_k\Delta_k + \sum_{i\in I}\left(\sum_{k\in K} V_k b_{ki} + G_i + \sum_{j\in I} a_{ji}W_j - W_i\right)x_i + \sum_{i\in I} W_i D_i\right\} . \tag{2.8a}$$

Constraints (2.9) become:

$$x_i - \sum_{j\in I} a_{ij}x_j \leq D_i , \qquad i \in I . \tag{2.9a}$$

If we drop the constant term $\sum_{i\in I} W_i D_i$ from (2.8a), change the objective to maximization and define

$$H_i = W_i - \sum_{j\in I} a_{ji}W_j - \sum_{k\in K} V_k b_{ki} - G_i ,$$

we obtain the following problem:

$$\text{Max} \left\{ \sum_{i\in I} H_i x_i - \sum_{k\in K} F_k \Delta_k \right\}$$

subject to (2.9a), (2.10), (2.11) and (2.12) .

The H_i can be interpreted as the unit savings of domestic production over import cost for product i when only variable production and capacity costs are considered <u>and</u> all endogenous intermediate inputs are imported.

As we shall be referring to the planning model in this form throughout this study, we call it problem (P). We refer to the problem in the form given by (2.8) - (2.12) as (P'). Obviously (P) and (P') have the same optimal solution and $v(P') = \sum_{i\in I} W_i D_i - v(P)$, where $v(\cdot)$ is the optimal value of problem $(\cdot)$.

CHAPTER 3

A SOLUTION APPROACH TO THE INVESTMENT PLANNING MODEL

3.1 Introduction

In this chapter we study the planning model (P) formulated in Chapter 2. The aim is the development of an efficient solution approach to this class of problems. For ease of reference (P) is rewritten in full below:

$$\text{(P)} \qquad \text{Max} \left\{ \sum_{i \varepsilon I} H_i x_i - \sum_{k \varepsilon K} F_k \Delta_k \right\}$$

$$x_i - \sum_{j \varepsilon I} a_{ij} x_j \leq D_i \, , \qquad i \;\varepsilon\; I \qquad (3.1)$$

$$C_k \Delta_k - \sum_{i \varepsilon I} b_{ki} x_i \geq 0 \, , \qquad k \;\varepsilon\; K \qquad (3.2)$$

$$\Delta_k = 0 \text{ or } 1 \, , \qquad k \;\varepsilon\; K \qquad (3.3)$$

$$x_i \geq 0 \, , \qquad i \;\varepsilon\; I \, . \qquad (3.4)$$

There are two distinct stages to the approach taken here to solve (P). At the first stage an attempt is made to reduce the size of the problem by exploiting some properties that its optimal solution must satisfy. These properties, which were developed by Westphal and Rhee for the Korea model and extended by Crémer [1976] to the general concave cost version, take the form of very simple sufficiency conditions for import and for domestic production of a given product. They allow for the identification of activities that are not competitive with imports when maximum advantage is taken of interdependence, on the one hand, and of activities that should clearly be undertaken even if no advantage is taken of interdependence, on the other.

At the second stage a branch-and-bound (B-B) scheme based on the linear programming (LP) relaxation of the planning problem is used. We show that the problem obtained when the integrality constraints of (P) are relaxed is a simple maximization of a linear objective function over a Leontief substitution structure, for which a very efficient solution approach exists.

Our approach at the second stage is in sharp contrast to the approach taken by Westphal and Rhee to solve the Korea model, and by Crémer to solve his version of the problem. Westphal and Rhee developed an approximate solution method based on the solution to single-product models, while Crémer developed a condition similar to the sufficient condition for domestic production used in the first stage that applies to combinations of two activities at a time, then three, four, etc. Once a combination of products satisfying the condition is identified, the first stage condition for domestic production is re-applied and the process is repeated. Crémer's approach at the second stage is basically an extension of the first stage.

While the computational effort required by both Crémer's and Westphal and Rhee's approach seems to depend heavily on the degree to which the first stage succeeds in reducing the size of the problem, the computational experience provided in Chapter 4 indicates that the approach developed here does not depend in any significant way on the outcome of the first stage.

We assume initially that the input-output matrix $A = \{a_{ij}\}$ is upper triangular, with zeros in the main diagonal. In other words, we are assuming that the products can be numbered in such a way that $a_{ij} = 0$ whenever $i \geq j$. Although this assumption may appear somewhat limiting, for models specified at the product level it is not very restrictive. In the Korea study, for example, the input-output matrix is in fact upper triangular. We do nevertheless generalize our analysis to the case where A is not triangular.

The organization of this chapter is as follows. Section 3.2 deals with the relaxation of (P); the solution to the relaxed problem is discussed and a two-step algorithm is developed. In section 3.3 the structure of (P) is thoroughly analyzed. Based on the theoretical results of sections 3.2 and 3.3, the solution approach to the planning model is formalized in section 3.4. Finally, in section 3.5 it is shown that a very efficient solution also exists for the relaxed problem when the input-output matrix A is not triangular.

3.2 Relaxation of the Planning Model

In this section we give the linear programming relaxation of (P), on which the branch-and-bound approach is based, and describe a very simple and efficient solution technique for the relaxed problem.

If the integrality constraints (3.3) in (P) are relaxed, the following LP problem is obtained:

$$\text{Max} \quad \sum_{i \varepsilon I} H_i x_i - \sum_{k \varepsilon K} F_k \Delta_k$$

$$x_i - \sum_{j \varepsilon I} a_{ij} x_j \leq D_i , \qquad i \ \varepsilon \ I$$

$$C_k \Delta_k - \sum_{i \varepsilon I} b_{ki} x_i = 0 , \qquad k \ \varepsilon \ K$$

$$0 \leq \Delta_k \leq 1 , \qquad k \ \varepsilon \ K$$

$$x_i \geq 0 , \qquad i \ \varepsilon \ I .$$

With the Δ_k being continuous variables and having objective function coefficients $F_k \geq 0$, clearly an optimal solution to the relaxed problem exists with the capacity constraints (3.2) satisfied as strict equalities. Also, the constraints $0 \leq \Delta_k \leq 1$, $k \ \varepsilon \ K$ may be dropped from the relaxed model since the Δ_k can never be greater than one (by the implicit assumption (2.7) of Chapter 2), nor can they be less than zero, as can be seen by examining the capacity constraints (because $b_{ki} \geq 0$, $x_i \geq 0$ and $C_k > 0$). We can therefore use the capacity constraints to substitute for the Δ_k in the objective function to obtain the following equivalent formulation for the LP relaxation, which we call $(\overline{P})$:

$$(\overline{P}) \qquad \text{Max} \quad \sum_{i \varepsilon I} \left(H_i - \sum_{k \varepsilon K} \frac{F_k}{C_k} b_{ki} \right) x_i$$

$$x_i - \sum_{j \varepsilon I} a_{ij} x_j \leq D_i , \qquad i \ \varepsilon \ I$$

$$x_i \geq 0 , \qquad\qquad i \in I .$$

It is easy to see that $(\overline{P})$ has the structure of a <u>general</u> input-output model, distinguished from a <u>simple</u> input-output model by the presence of alternative production techniques for each product. The slack variables of $(\overline{P})$ are in effect the alternative "techniques" for supplying each product; they are the import variables y_i which in our formulation of the model have zero coefficients in the objective function. Since $(\overline{P})$ is also "productive" (as $x_i = 0$, and thus $y_i = D_i$ for each $i \in I$ is a feasible solution that satisfies all the final demands), all the conditions of the substitution theorem of general input-output theory are satisfied. Such systems are said to possess Leontief substitution structure. See Gale [1960] for a statement and proof of the substitution theorem, or, for a more extensive treatment of this subject, the chapters by Samuelson, Koopmans and Arrow in Koopmans [1951]. The substitution theorem states the conditions under which the technique used to satisfy each product's demand is unique, irrespective of the exogenous demand levels D_i $(D_i \geq 0)$. Applied to $(\overline{P})$ this says that each product will be entirely imported or domestically produced to satisfy all its exogenous plus endogenous demands, since its optimal basis is independent of the right-hand-side quantities D_i. This is discussed in more detail in section 3.5 where the analysis of this section is extended to the case where A is not triangular.

For LP problems with upper triangular input-output matrix, Dantzig [1955] has shown that the optimization can be carried out sequentially, as the solution to the i^{th} activity depends only on the solution to the first $i - 1$ activities.[1/] This result can also be very easily established by the use of sequential projection following Geoffrion [1970]. To develop the computational approach to $(\overline{P})$ we make use of this result, as well as of the substitution theorem, in obvious ways. There are two steps to the approach: the first step simply determines which products should be produced, and, at the second step it is determined at what level they should be produced.

We first define $S_i = H_i - \sum_{k\varepsilon K} \frac{F_k}{C_k} b_{ki}$. S_i is thus the coefficient of x_i in the objective function of $(\overline{P})$.

Step I: Identification of activities at positive level in the optimal solution

1. Initialize $i = 1$
2. Compute $\overline{S}_i = S_i + \sum_{j=1}^{i-1} \overline{S}_j a_{ji} \delta_j$. If $\overline{S}_i \geq 0$ set $\delta_i = 1$, otherwise set $\delta_i = 0$.
3. If $i = I$ go to Step II. Otherwise $i \leftarrow i+1$ and go to 2.

1/ Actually Dantzig showed this result for a block-triangular input-output matrix, of which A is a special case.

Step II: Determination of the optimal production levels

1. If $\delta_i = 0$ set $\bar{x}_i = 0$ and go to 3.

2. Compute $\bar{x}_i = D_i + \sum_{j=i+1}^{I} a_{ij}\bar{x}_j$

3. If $i = 1$, STOP. Otherwise $i \leftarrow i-1$ and go to 1.

The $\bar{S}_i$ computed at Step I are the coefficients of x_i in the sequential optimization process. $\bar{S}_i$ gives the savings of domestic production over imports for product i given optimal decisions with respect to products 1, 2, ..., $i-1$, that is, with minimum-cost input source.[1/] Therefore, if the savings are positive product i should be produced in the solution to $(\bar{P})$ and we set $\delta_i = 1$. Knowing then from Step I which products should be domestically produced, at Step II we obtain recursively the production levels. A bar over the x_i is used to designate the optimal solution to $(\bar{P})$.

We can see from the results of this section that the solution to the LP relaxation of (P) is an extreme point of the material balance constraints (3.1), i.e., of the Leontief substitution system. Veinott [1969] has shown that also for the case of minimization of

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1/ If we let π_i be the dual variable associated with the i^{th} constraint of $(\bar{P})$, and $\bar{\pi}_i$ its optimal value, then it is easy to see that

$$\bar{\pi}_i = \max \{0, \bar{S}_i\} = \bar{S}_i \, \delta_i \, .$$

Step I thus consists of sequentially obtaining the dual solution of $(\bar{P})$. $\bar{\pi}_i > 0$ implies that product i should be domestically produced.

a concave objective function (equivalently, the maximization of a convex function) an optimal solution exists which is an extreme point of the Leontief system. Because any extreme point of constraints (3.1) satisfies the capacity balance constraints (3.2)[1/] of (P) with the appropriate Δ_k set to one, an optimal solution to (P) also exists which is an extreme point of the Leontief system. This is in fact also true for the version of the planning problem studied by Crémer, but not for the Westphal and Rhee model, as the investment cost function they specified in the Korea study is not concave.[2/]

3.3 Further Theoretical Development

In this section we derive conditions under which domestic production is optimal and under which imports are optimal, based on the relaxed problem $(\overline{P})$. Some of these results are useful for problem size reduction, or fathoming in the B-B approach to be described in the next section, and still others merely provide some insight on the problem.

If we set all fixed costs F_k to zero in $(\overline{P})$, then $S_i = H_i$ for each $i \in I$. We define for this case $\overline{H}_i$, similarly to $\overline{S}_i$, so that a distinction can be made between a problem that only considers

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1/ Since the capacity bounds C_k satisfy condition (2.7).

2/ See section 2.3, Chapter 2.

variable costs from one in which fixed costs allocation is also included. Obviously if all $F_k = 0$ then $\overline{S}_i = \overline{H}_i$. The $\overline{H}_i$ can thus be interpreted as the savings per unit of domestic production over imports for product i when minimum-cost input sources are used and only variable costs of domestic production are considered.

We use consistently throughout this paper a bar over a variable to designate its optimal value in $(\overline{P})$, and an asterisk to designate its optimal value in (P), that is, its global optimum. We also make use later of the following additional notation:

$$K_i = \{k \in K \mid b_{ki} > 0\} \quad ,$$

and

$$I_k = \{i \in I \mid b_{ki} > 0\} \quad .$$

<u>Theorem 3.1</u> If $S_i < 0$ for each $i \in I$, then all I products should be imported in the optimal solution of (P), that is, for each $i \in I$ $x_i^* = 0$ and $y_i^* = D_i$.

<u>Proof</u>. Assume we are solving $(\overline{P})$ by the simplex method. We add slack variables y_i to the constraints of $(\overline{P})$ and use an all-import solution ($y_i = D_i$, $i \in I$) as the initial basic feasible solution. The simplex multipliers associated with this basis are all zero and thus the relative cost factors for the activities not in the basis are identically equal to the S_i. Since all $S_i < 0$, by the theory of the simplex method the current basis must be optimal. This establishes that $\overline{x}_i=0$ for all $i \in I$. Since this solution implies that $\overline{\Delta}_k=0$ for all $k \in K$ (since $\overline{\Delta}_k = \sum_{i \in I_k} b_{ki}\overline{x}_i / C_k$), and it is feasible in (P), it must then be optimal in (P), and thus $x_i^* = 0$ for all $i \in I$.

Although this result is very simple and straightforward, it is somewhat counterintuitive as it implies that if we analyze each product individually (a pricing-out operation) with all endogenous inputs to its production imported and fixed costs charged proportionately to its capacity requirements, and it is cheaper to import than to produce each product, then all products should optimally be imported. That is, no benefits from coordination of investments can possibly justify any domestic production. Thus, at least one S_i must be positive if any domestic production is to take place. Obviously Theorem 3.1 is valid even when A is not triangular, as no triangularity assumption is needed for its proof.

Theorem 3.2 $x_i^* = 0$ for all $i \in I$ such that $\overline{H}_i \leq 0$.

This result, due to Westphal and Rhee [1978], although obvious, is a very important one. It says that any product that is imported in the solution of $(\overline{P})$ with all $F_k = 0$, should optimally be imported in the solution of (P). $\overline{H}_i \leq 0$ implies that the minimal variable cost of domestic production for product i is at least as large as the import price W_i. Thus it should be obvious that $\overline{H}_i \leq 0$ is a sufficient condition for it not to be optimal to produce product i.

Theorem 3.2 can thus be used to eliminate from our model all products for which $\overline{H}_i \leq 0$, thereby reducing the size of the problem. Another important implication of Theorem 3.2 is that the solution to $(\overline{P})$ with all $F_k = 0$ gives upper bounds on the optimal levels of domestic production. These upper bounds can be used to compute upper bounds on the effective capacity requirements for each process element, which in turn can be used in place of the C_k, thereby rendering $(\overline{P})$ a tighter LP relaxation of (P).

Fact 3.1 If $H_i D_i > \sum_{k \varepsilon K_i} F_k$ then product i alone pays for the fixed costs associated with all the process elements k in K_i.

Fact 3.2 If $\sum_{i \varepsilon I'} H_i D_i > \sum_{k \varepsilon K'} F_k$ then it is better to produce all I' products than to import them all, where $I' \subseteq \{i \varepsilon I \mid H_i > 0\}$ and

$$K' = \{k \varepsilon K \mid b_{ki} > 0 \text{ for some } i \varepsilon I'\} .$$

Fact 3.1 provides a sufficient condition for domestic production of a given product, whereas Fact 3.2 gives a sufficient condition for domestic production of at least some of the products in the set I'. The condition of Fact 3.2 can, of course, be strengthened by defining the set I' as follows:

$$I' \subseteq \{i \varepsilon I \mid H_i > 0 \text{ \underline{and} the condition of Fact 3.1 is not satisfied}\} ,$$

where the set K' in this case excludes the process elements identified by Fact 3.1.

It is also important to observe that the condition of Fact 3.1 can only be satisfied for those products i with $S_i > 0$. Products with $S_i < 0$ but $\overline{H}_i > 0$ will be domestically produced only if some of the fixed costs are "paid for" by other products which share capacity with product i in one or more process elements. That is, without coordination of investments their domestic production could not possibly be economically justified. Finally, if the condition of Fact 3.1 is satisfied for at least one product i, then we can set $F_k = 0$ for all $k \in K_i$ (i.e., fix open the process elements included in K_i) and re-apply the condition once again to all those products that are process interdependent with product i. This can be repeated until no new product satisfies the condition.

Similarly to Fact 3.2, the following fact provides an alternative sufficient condition for some domestic production.

<u>Fact 3.3</u> If $\sum_{i \in I''} \bar{H}_i D_i > \sum_{k \in K''} F_k$ then it is better to produce all I'' products than to import them all, where $I'' = \{i \in I \mid \bar{H}_i > 0\}$ and K'' is the set of all process elements needed in the production of all I'' products.

The condition of Fact 3.3 can be similarly strengthened as that of Fact 3.2. We note that while here the set I'' is uniquely defined, the set I' in Fact 3.2 could be any subset of $\{i \in I \mid \bar{H}_i > 0\}$. Of course if I' is a singleton Fact 3.2 degenerates into Fact 3.1.

3.4 The Solution Approach to the Planning Model

We now turn to the formulation of the algorithm for the solution of (P). There are two stages to our approach, and both make use of the theoretical developments of the two preceding sections: in the first stage for problem size reduction, and in the second for the development of a branch-and-bound procedure.

3.4.1 Problem Reduction Stage

Assume that initial screening of products has been done using Theorem 3.2, and that the set I now contains only products i such that $\overline{H}_i > 0$.

As pointed out in the preceding section, Fact 3.1 can be used to further reduce the size of the problem; if the condition $H_i D_i - \sum_{k \in K_i} F_k > 0$ is satisfied for some i, this allows us to fix at the level 1 all Δ_k for $k \in K_i$. We use, however, for this case a strengthened version of Fact 3.1. It is strengthened by continuous updating of the H_i to reflect the savings of production over imports given that certain products are known to be optimally produced and also by considering the endogenous demands generated by these production activities. For ease of exposition we present the procedure within the context of the simplex method. However, no LP problem will be solved by the simplex method since the solution approach developed for $(\overline{P})$ allows for updating the costs by means of very simple computations.

Consider $(\overline{P})$ and let the initial basic feasible solution be formed by all the import activities (i.e., all the slack variables $y_i = D_i$). Rather than introducing into the basis at a given iteration the production activities that price out negatively[1/] (which would

1/ In section 3.5 we show that block pivoting can be used when solving $(\overline{P})$ and that once an activity has been introduced into the basis it will not be removed in succeeding iterations.

be optimal in $(\overline{P})$), our approach consists of introducing into the basis only activities that satisfy a sufficient condition for optimality in (P).

For the first iteration the sufficient condition is clearly that of Fact 3.1, that is,

$$H_i D_i \;-\; \sum_{k \in K_i} F_k \;>\; 0 \;, \tag{3.5}$$

where an iteration consists of the pricing-out of each nonbasic production activity and the introduction into the basis of each one that satisfies the sufficient condition. Let the set I^* contain the production activities that satisfy (3.5), i.e., become basic in the first iteration, and let $K^* = \left\{ k \in K \mid k \in \bigcup_{i \in I^*} K_i \right\}$.

At the second iteration, the condition for a given production activity i, $i \notin I^*$, to enter the basis permanently becomes

$$\hat{H}_i \hat{D}_i \;-\; \sum_{\substack{k \in K_i \\ k \notin K^*}} F_k \;>\; 0 \;, \tag{3.6}$$

where $\hat{H}_i$ is the savings of variable domestic production cost over import price for product i when inputs j are domestically produced if $j \in I^*$ and imported if $j \notin I^*$, and $\hat{D}_i$ is the total demand for product i in that solution. The $\hat{H}_i$, for each $i \notin I^*$, are obtained sequentially by

$$\hat{H}_i \;=\; H_i \;+\; \sum_{j \in I^*} a_{ji}\, \hat{H}_j \;,$$

and the $\hat{D}_i$ recursively by

$$\hat{D}_i = D_i + \sum_{j \epsilon I^*} a_{ij} \hat{D}_j .$$

This is similar to the way the $\bar{S}_i$ are obtained in Step I and the $\bar{x}_i$ in Step II of the solution procedure to $(\bar{P})$. Obviously $\hat{H}_i \geq H_i$ and $\hat{D}_i \geq D_i$. Condition (3.6) is thus strengthened each time a new activity is identified that should optimally be undertaken, (i.e., enter the basis). It is also strengthened whenever the set K^* is augmented as a result of a process interdependent activity being identified.

At each succeeding iteration the set I^* is augmented by the activities that are introduced into the basis in the immediately preceding iteration, the $\hat{H}_i$ updated and condition (3.6) reapplied for each activity $i \notin I^*$. The process is repeated until no new activity is added to I^*.

This ends the problem reduction stage, which yields a set of products known to be optimally produced (I^*), as well as a set of process elements (K^*) known to be open (built) in the optimal solution to (P).

3.4.2 Branch-and-Bound Stage

The method of branch-and-bound (B-B) is based on an enumeration technique that implicitly considers all possible solutions of an integer or mixed integer programming problem. In the case of our planning model, in which the variables required to assume integer values are binary, explicit enumeration of all possible solutions would require the solution of 2^K linear programming problems, each with a given combination of fixed values for the K binary variables. The B-B procedure allows us to eliminate from explicit enumeration those solutions which, based on bounding considerations, are known to be nonoptimal. An enumeration tree is built for the zero-one variables Δ_k, with each node of the tree characterized by a set of Δ_k variables having fixed values. That is, each node contains a partial solution of (P) and thus represents a problem with the same form of (P) but with the appropriate set of Δ_k variables fixed. We refer to these problems in the B-B algorithm as candidate problems and to all such problems at any point in time as the candidate list. The LP relaxation of the problem at each node is solved to obtain an upper bound on the optimal solution of the problem at that node. This upper bound represents the best value attainable by any completion of the partial solution at that node. The node may be fathomed if the bound is less than the value of the incumbent, that is, the best currently known feasible solution of (P). The bounds obtained by the LP relaxation can also be used to direct the search along the branches of the tree.[1/]

We refer to (P) with some of the Δ_k variables fixed as $(P_{\overline{K}})$ and defined $\overline{K}_0 = \{k \in K \mid \Delta_k$ is assigned the value 0$\}$, $\overline{K}_1 = \{k \in K \mid \Delta_k$ is assigned the value 1$\}$, and $\overline{K} = \overline{K}_0 \cup \overline{K}_1$. $\overline{K}$ is thus the index set of all variables Δ_k in a partial solution to (P). The

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1/ For a very comprehensive exposition of the branch-and-bound method see Geoffrion and Marsten [1972].

LP relaxation $(\overline{P}_{\overline{K}})$ of $(P_{\overline{K}})$ can be written as

$$(\overline{P}_{\overline{K}}) \qquad \text{Max} \left\{ \sum_{i\varepsilon I} \left(H_i - \sum_{k\varepsilon K_i} \frac{\overline{F}_k}{C_k} b_{ki} \right) x_i \right\} - \sum_{k\varepsilon \overline{K}_1} F_k$$

$$x_i - \sum_{j\varepsilon I} a_{ij} x_j \leq D_i , \qquad i \varepsilon I$$

$$x_i \geq 0 , \qquad i \varepsilon I ,$$

where

$$\overline{F}_k = \begin{cases} F_k & \text{if } k \notin \overline{K} \\ 0 & \text{if } k \varepsilon \overline{K}_1 \\ \infty & \text{if } k \varepsilon \overline{K}_0 \end{cases} .$$

Since the Δ_k variables disappear in the relaxed problem, the above definition of $\overline{F}_k$ allows us to fix process elements open $(\overline{F}_k = 0)$ or closed $(\overline{F}_k = \infty)$.

For the presentation of the B-B procedure to solve (P) we appeal to the general framework of Geoffrion and Marsten [1972]. The algorithm is:

1. Initialize the candidate list to consist of $(P_{\overline{K}})$ alone, with $\overline{K} = K^*$, and set $Z^* = \sum_{i\varepsilon I^*} \hat{H}_i D_i - \sum_{k\varepsilon K^*} F_k$, $\overline{K}_0 = \emptyset$, and $\overline{K}_1 = K^*$.
2. Stop if the candidate list is empty: the incumbent solution is optimal in (P).
3. Select one of the problems from the candidate list to become the current candidate problem (CP), using a LIFO

rule. Let $(P_{\overline{K}})$ be the chosen (CP).

4. Solve $(\overline{P}_{\overline{K}})$, the LP relaxation of $(P_{\overline{K}})$. Obtain the δ_i and $\overline{x}_i$, and compute the $\overline{\Delta}_k$ using

$$\overline{\Delta}_k = \sum_{i \varepsilon I_k} b_{ki}\overline{x}_i \,/\, C_k \,.$$

5. If $v(\overline{P}_{\overline{K}}) \leq Z^*$ go to 2.

6. If the optimal solution of $(\overline{P}_{\overline{K}})$ is feasible in $(P_{\overline{K}})$ go to 9.

7. Pick a k such that $\overline{\Delta}_k$ is fractional and compute

$$E_k = \sum_{i \varepsilon I_k} \hat{H}_i D_i \delta_i - \sum_{\substack{k \varepsilon K_i \\ i \varepsilon I_k}} \overline{F}_k \,.$$

If $E_k \geq 0$ set $\overline{F}_k = 0$ for each $k \,\varepsilon\, K_i$, $i \,\varepsilon\, I_k$ and go to 4; otherwise repeat step until there are no more k such that $\overline{\Delta}_k$ is fractional.

8. Separate (CP). From among the Δ_k fractional select (by some rule) $\Delta_{\overline{k}}$ as the separation variable and add its descendants to the candidate list in the order $(CP \mid \Delta_{\overline{k}} = 0)$ then $(CP \mid \Delta_{\overline{k}} = 1)$. Tag $(CP \mid \Delta_{\overline{k}} = 0)$ as follows:

$$UB(\overline{K} \mid \Delta_{\overline{k}} = 0) = v(\overline{P}_{\overline{K}}) - \sum_{i \varepsilon I_{\overline{k}}} \overline{S}_i D_i \delta_i + F_{\overline{k}}\overline{\Delta}_{\overline{k}} \,.$$

Update $\overline{K}$ for each problem and recompute the effective capacity bounds for all process elements $k \,\varepsilon\, K_i$, $i \,\varepsilon\, I_{\overline{k}}$ (i.e., tighten the C_k) in $(CP \mid \Delta_{\overline{k}} = 0)$. Go to 3.

9. If $v(CP) > Z^*$ record this solution as the new incumbent and set $Z^* = v(CP)$. Purge the candidate list of those problems with upper bounds $UB(\cdot) \leq Z^*$. Go to 2.

The root node of the B-B tree is represented by problem (P_{K^*}), or, equivalently $(P_{\overline{K}})$ with the $\overline{K} = K^*$. K^* is the index set of the Δ_k variables determined in the first stage to be optimally set at level one. Thus at the root node $\overline{K}_1 = K^*$ and $\overline{K}_0 = \emptyset$, and consequently $\overline{K} = K^*$. If $K^* = \emptyset$ then the initial candidate list contains precisely (P). Z^* is initialized at the value of savings over imports obtained from the domestic production of the I^* products identified in the first stage. If $I^* = \emptyset$ then $Z^* = 0$, which is the value of savings over imports if all I products continue to be imported.

In step 7 an effort is made to peg some Δ_k at the level one after the fathoming criteria of steps 5 and 6 fail and before resorting to separation of (CP). The $\hat{H}_i$ are known from the first stage.

In step 8 separation occurs. Two separation rules are computationally tested in Chapter 4. The first one makes use of the values of E_k computed in step 7 for each k such that $\overline{\Delta}_k$ is fractional in (CP). It consists in selecting $\Delta_{\overline{k}}$ such that $E_{\overline{k}} = \max_k \{E_k\}$ as the separation variable. A second promising rule consists in selecting a $\overline{\Delta}_k$ such that $I_{\overline{k}}$ is the largest among those I_k for k such that $\overline{\Delta}_k$ is fractional. This has the potential effect of making more $\overline{S}_i > 0$, i.e., increasing the set of products $i \in I_{\overline{k}}$

with $\delta_i = 1$, thus strengthening the test $E_k \geq 0$ in step 7 by picking a larger set or process interdependent products to share the fixed costs.

Note that whenever separation is necessary (step 8), the order in which the problems are added to the candidate list ensures that $(CP \mid \Delta_{\bar{k}} = 1)$ is always selected first and thus the candidate list contains only problems in which the separation variable was fixed at the level 0. Consequently, (still by step 8) all problems in the candidate list are tagged with an upper bound on their optimal values. These à priori or conditional upper bounds are used for fathoming purpose whenever a new incumbent solution is obtained (step 9).

Steps 7 and 8 require some further elaboration. The δ_i in the expression for the E_k in step 7 ensure that the sum will be only over those i such that $\bar{S}_i > 0$, as from the solution approach to the LP relaxation we know that $\delta_i = 1$ iff $\bar{S}_i > 0$. Also, the test $E_k \geq 0$ cannot possibly pass for any k such that $\sum_{i \varepsilon I_k} \delta_i = 1$ since in this case the test is equivalent to the condition of Fact 3.1, which was already used in the problem reduction phase. The expression for the conditional upper bound $UB(\bar{K} \mid \Delta_{\bar{k}} = 0)$ on $(CP \mid \Delta_{\bar{k}} = 0)$ in step 8 is very straightforward. We know that $v(\bar{P}_{\bar{K}})$ is an upper bound on $v(P_{\bar{K}})$, and if we fix $\bar{\Delta}_{\bar{k}}$, fractional in the optimal solution to $(\bar{P}_{\bar{K}})$, to zero, then we also know that the objective function of $(\bar{P}_{\bar{K}})$ must decrease by at least $\sum_{i \varepsilon I_{\bar{k}}} \bar{S}_i D_i \delta_i - F_{\bar{k}} \bar{\Delta}_{\bar{k}}$, where the first term is the variable savings foregone when products

$i \in I_{\overline{k}}$ with $\delta_i = 1$ are no longer domestically produced, and the second term is the fraction of fixed cost for process element $\overline{k}$ that will no longer be incurred. The reason why $\sum_{i \in I_{\overline{k}}} \overline{S}_i D_i \delta_i - F_{\overline{k}} \overline{\Delta}_{\overline{k}}$ is a lower bound on the decrease in $v(\overline{P}_{\overline{K}})$ when we fix $\overline{\Delta}_{\overline{k}} = 0$ is that input source switching will necessarily occur for products $j \in I$ for which one or more products $i \in I_{\overline{k}}$ are inputs to their production processes. Input source will switch from the domestically produced source at $W_i - \overline{S}_i$ per unit to the import source at W_i per unit, with $W_i - \overline{S}_i < W_i$ as $\delta_i = 1$ implies that $\overline{S}_i > 0$. Finally, the tightening of the capacity bounds C_k (still in step 8) are easily accomplished by recomputing the bounds on the x_i, $i \notin I_{\overline{k}}$, knowing that $\overline{x}_i = 0$ if $i \in I_{\overline{k}}$.

3.5 Solution to the Relaxed Problem when A is not Triangular

Although $(\overline{P})$ is a general LP problem when A is not triangular, we still can do better than a direct application of the simplex method. It will be shown in fact that a bound can be placed on the number of iterations required to solve $(\overline{P})$. For the development that follows we find it convenient to add slack variables y_i to $(\overline{P})$:

$$\text{Max} \sum_{i \in I} S_i x_i$$

$$x_i - \sum_{j \in I} a_{ij} x_j + y_i = D_i \,, \qquad i \in I$$

$$x_i \geq 0 \,, \quad y_i \geq 0 \,, \qquad i \in I \,.$$

There are I equations in the constraints and thus any optimal solution to $(\overline{P})$ need not contain more than I nonzero x_i and y_i variables. If each $D_i > 0$, then any feasible basis must contain at least one "technique" that produces each product (i.e., for each i we cannot have both $x_i = 0$ and $y_i = 0$). Since there are only I activities in the basis, any feasible solution, and thus the optimal solution, to $(\overline{P})$ must be such that each product will be produced by one and only one technique.

The above argument actually constitutes a proof of the substitution theorem as it applies to $(\overline{P})$. As discussed below, this result enables a short-cut in the simplex method to this problem.

Assume we have a basic feasible solution[1/] to $(\overline{P})$ that is non-optimal. If the i^{th} activity vector is to enter the basis at the next iteration, then, by implication of the above development, it must replace in the basis that activity that produces product i. That is, no computation is needed to determine which activity should leave the basis at any iteration, which in turn implies that the optimal basis is independent of the exogenous demand levels $D_i \geq 0$, since in the simplex method the D_i have no influence on the activity to enter the basis, only on the activity to be removed. This implication of the substitution theorem was explored in section 3.2 to develop the two-step computational approach to $(\overline{P})$, and is used

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1/ Any combination of I activities, with precisely one "technique" for each, yields a basic feasible solution.

again here to simplify the simplex method.

For the problem at hand, an initial basic feasible solution readily available is an all-import solution, i.e., $y_i = D_i$ for all $i \in I$. We then proceed to price out each activity not currently in the basis in order to determine if any should enter. If none should enter, then clearly the all-import solution is optimal.[1/] If an activity is found that should become basic, then the corresponding activity that "produces" the same product is removed from the basis and the pricing-out operation is repeated. Below we give a result that has the following implication: starting with an all-import basic solution, any activity that enters the basis at a given iteration of the simplex method will never be removed at succeeding iterations. Equivalently stated, any activity that is removed from the basis at a given iteration can never be part of the optimal basis. This equivalence follows from the substitution theorem.

Let $\bar{\pi}_i^{(n)}$ be the value of the dual variable associated with the i^{th} constraint of $(\bar{P})$ at the n^{th} iteration of the simplex method, and $\beta^{(n)}$ the corresponding basis matrix. The following theorem establishes the desired result.

Theorem 3.3 Starting with an all-import basic solution, the value of the dual variables $\pi_i^{(n)}$ are nondecreasing from iteration to

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1/ Given the interpretation of the coefficients of the x_i in the objective function, this has a counterintuitive implication identical to that of Theorem 3.1 of section 3.3.

iteration, for each $i \varepsilon I$.

Proof. Assume that $S_i > 0$ for at least one $i \varepsilon I$, since otherwise, by Theorem 3.1, $x_i^* = 0$ for all $i \varepsilon I$. We show by induction that $\bar{\pi}_i^{(n+1)} \geq \bar{\pi}_i^{(n)}$, $n = 1, 2, \ldots$. Clearly $\bar{\pi}_i^{(1)} = 0$ for all $i \varepsilon I$.

Let $S_j > 0$. Then production activity j is introduced into the basis and the corresponding import activity is removed. If $S^{(n)}$ is the vector containing the objective function coefficients corresponding to the basic variables at the n^{th} iteration of the simplex method, and $[\cdot]_i$ denotes the i^{th} component of the vector $[\cdot]$ and $[\cdot]_{ii}$ the element ii of the matrix $[\cdot]$, then for the second iteration we have:

$$\bar{\pi}_j^{(2)} = \left[S^{(2)} \left(\beta^{(2)}\right)^{-1}\right]_j$$

$$= S_j \left[\left(\beta^{(2)}\right)^{-1}\right]_{jj}$$

$$> 0 \text{ since } S_j > 0 \text{ and } \left[\left(\beta^{(2)}\right)^{-I}\right]_{jj} > 0 \ ,$$

and $\bar{\pi}_i^{(2)} = 0$ for all $i \varepsilon I$, $i \neq j$. Thus $\bar{\pi}_i^{(2)} \geq \bar{\pi}_i^{(1)} = 0$ for all $i \varepsilon I$. Assume now that $\bar{\pi}_j^{(n)} > 0$ and that a given nonbasic production activity is to enter the basis at iteration n+1. We have then that

$$\bar{\pi}_j^{(n+1)} - \bar{\pi}_j^{(n)} = \left[S^{(n+1)} \left(\beta^{(n+1)}\right)^{-1} - S^{(n)} \left(\beta^{(n)}\right)^{-1}\right]_j \qquad (3.7)$$

It must also be true at iteration n+1 that $Z^{(n+1)} \geq Z^{(n)}$, where $Z^{(n)}$ is the objective function value at the n^{th} iteration. And

$$Z^{(n+1)} \geq Z^{(n)} \dashrightarrow S^{(n+1)} \left(\beta^{(n+1)}\right)^{-1} D \geq S^{(n)} \left(\beta^{(n)}\right)^{-1} D ,$$

or

$$\sum_{i \varepsilon I} \left[S^{(n+1)} \left(\beta^{(n+1)}\right)^{-1} D\right]_i \geq \sum_{i \varepsilon I} \left[S^{(n)} \left(\beta^{(n)}\right)^{-1} D\right]_i . \qquad (3.8)$$

But since the activity to be removed from the basis at any given iteration does not depend on the right-hand-side vector D (by the substitution theorem), (3.8) must be satisfied for any vector $D \geq 0$, which in turn implies that

$$\left[S^{(n+1)} \left(\beta^{(n+1)}\right)^{-1}\right]_i \geq \left[S^{(n)} \left(\beta^{(n)}\right)^{-1}\right]_i \qquad (3.9)$$

for each $i \varepsilon I$.

Combining (3.9) with (3.7) we obtain $\bar{\pi}_j^{(n+1)} \geq \bar{\pi}_j^{(n)}$ for all $j \varepsilon I$, which completes the proof.

An important implication of the above is that we may use block pivoting, that is, we may introduce into the basis in a single iteration of the simplex method every production activity that prices out negatively (with the corresponding import activities removed), and that only production activities are candidates for entering the basis. This allows for a very simple solution to $(\bar{P})$. If $\bar{I}$ is the set of products in an optimal solution to $(\bar{P})$, then it will take at most $\bar{I}$ simplified iterations of the simplex method to obtain the optimal solution (where by simplified we mean that no computations are needed to decide which activities are to be removed from the basis). Note that $\bar{I}$ is a bound on the number of iterations even if no advantage is taken of block pivoting.

The above result is highly significant in the context of our solution approach to (P). Since bounds can also be easily obtained with very little computational effort when A is not triangular, a solution approach to the more general planning model is readily available. Simply disregard the problem reduction stage, and make the following changes in the B-B algorithm: In step 1, since $K^* = \emptyset$, initialize the candidate list to consist of (P) alone, and set $Z^* = 0$, and $\overline{K}_0 = \overline{K}_1 = \overline{K} = \emptyset$; and, either replace $\hat{H}_i$ by H_i in step 7, or drop the step altogether from the algorithm.

The sufficiency conditions used in the problem reduction stage for the triangular case are no longer valid here due to the circular nature of the interdependence in this case. An initial elimination of activities not competitive with imports at marginal cost can still be carried out, however, by solving $(\overline{P})$ with all F_k set to zero (i.e., set $S_i = H_i$ for each $i \in I$). As in the triangular case, the import activities in the optimal basis of such a problem correspond to products that are optimally imported in the solution of (P).

A problem reduction stage could conceivably be developed for the nontriangular case, although not in the simple form allowed by the triangularity of A, and probably not in as strong a form. One such weaker form of a problem reduction stage, for example, consists of disregarding all the elements of A below the main diagonal[1/] and use the same conditions as for the upper triangular case. Because all the elements of A are nonnegative, if the

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1/ This can be viewed as a form of "data relaxation."

sufficient condition for domestic production of a certain product is satisfied when the lower triangular elements are ignored, it must also be satisfied when they are taken into account. The effect of ignoring the lower elements corresponds to taking less than full advantage of the potential benefits from input-output interdependence. Numbering the endogenous activities in such a way that maximum upper triangularity is obtained for A would strengthen these conditions and render such a problem reduction stage potentially more effective.

CHAPTER 4

COMPUTATIONAL RESULTS

4.1 Introduction

This chapter reports on the computational results obtained from the implementation of the solution approach described in Chapter 3. We make use of the same data that was estimated for the implementation of the Korea model and used by Westphal and Rhee to analyze experimentally possible investments in the Republic of Korea's mechanical engineering sector during its Third Five-Year Plan. Here, however, the data is merely used to test computationally the performance of our proposed solution approach to the investment planning problem.

The products produced by the mechanical engineering or metal working sector include: fabricated metal products, non-electrical machinery, electrical products and machinery, and transport equipment.[1/] The sector is characterized by the great variety of its output; mechanical engineering products are highly differentiated and number in the millions. Another characteristic of this sector that greatly complicates planning at the sectoral or even subsectoral level is that many products can and often are produced by multi-purpose equipment that is not tailored to the production of a specific product. Finding an appropriate model specification that adequately treats this type of "capacity sharing" was considered by Westpahl and Rhee as the major stumbling block to constructing

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1/ See Westphal and Rhee [1978], Chapter 14.

sector-wide planning models to the mechanical engineering sector.

Products share capacity in "process elements," a term we used generically in the presentation of the planning model in Chapter 2 to designate an individual element of production capacity. In section 4.2 it is given specific meaning in the context of the mechanical engineering industry as it was applied in the Korea case study. For a discussion of the methodology of decomposing production facilities into process elements within the format of models such as the one studied here, see Nam, Rhee and Westphal [1973]. Section 4.2 also describes very briefly the estimation of the model; it serves to illustrate some of the important issues that arise in model estimation. Particularly important is the issue of aggregation that must necessarily be confronted when modeling sectors such as the mechanical engineering. For a detailed discussion of the model estimation Westphal and Rhee [1978] should be consulted.

In section 4.3 some features of the Korea data are examined within the context of computational complexity. Finally, computational experience from the implementation of our solution approach to the planning problem is provided in section 4.4.

4.2 Model Estimation

The model used in the Korea case study was intended merely to serve as a screening device to obtain initial comparative-advantage ranking of production activities within the mechanical engineering sector prior to the design of specific projects. The methodology used by Westphal and Rhee can be regarded as a refinement of that of

Vietorisz [1972], which in turn is a refinement of the methodology used to study the Soviet Union's mechanical engineering sector by a team of researchers at the Institute for Research in Social Science, University of North Carolina (1958-9). It will be apparent from the following description of Westphal and Rhee's model estimation methodology that the issue of aggregation is a very important one in sectors characterized by a great multitude of products, such as the mechanical engineering sector. Aggregation is in fact what essentially distinguishes the Korea study methodology from its two predecessors.

Two features characterize Westphal and Rhee's methodology. First is the aggregation of equipment into shops. A "shop," or process element, is defined to be "a collection of complementary equipment (and associated labor) that perform closely related processes; for example, a machine shop or a foundry."[1/] Process elements are thus specified at the subplant level and correspond to "shops," which are the building blocks that make up individual plants within the sector. In order to place a limit on the extent to which the model can exploit capacity sharing, each shop type is further divided into "shop classes." Two elements describe a shop class: the type of processing activity, and the collection of products that can be processed together within a single shop. Westphal and Rhee make use of the following example, which appears in the Korea study, in order to illustrate this. A given type of shop conducts stamping operations using a press of less than 50 tons maximum force. The production of any of the

1/ Westphal and Rhee [1978], Chapter 15.

following products involves processing in shops of this type: household blenders, household ovens, bicycles, motorcycles, passenger cars, 3-wheel trucks, and 4-wheel trucks. The stamping operation could, in principle at least, be carried out in a single shop. If this pattern of production organization is considered feasible, this could be expressed in the model by specifying a single constraint for this type of shop; it could process any combination or all of the above seven products. In this case there would be a single class of shops with respect to the corresponding stamping operation involved in the production of these products. In the Korea model three different classes of shops are specified for the stamping operation to reflect the judgment that capacity sharing with respect to this operation is unlikely across all seven products. A shop class was specified for each of the following sets of products: household blenders and ovens; bicycles and motorcycles; and passenger cars and 3-wheel and 4-wheel trucks. Capacity sharing was thus allowed, for example, between household blenders and ovens, but not between either of these and bicycles and/or motorcycles. We can see from this illustration that the number of shop classes specified for each type of shop implies a restriction on the feasible pattern of production organization.[1/] All shop classes of a given type were assumed to have identical fixed-charge cost functions which considerably reduced the

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1/ There are many reasons why one pattern of production organization would be preferable over another, and they may be as diverse as: engineering judgement, institutional factors, and marketing considerations, to name a few. See Westphal and Rhee [1978], Chapter 15.

amount of parameter estimation required to implement the model. The Korea model contains 37 different types of shops and 272 shop classes.

The second feature that characterizes Westphal and Rhee's approach is the use of "representative products" to aggregate over heterogeneous products. Each of the 116 individual products appearing in the Korea model represents in fact a whole class of products.1/ As an example, one of the "products" that appears in the study is "fractional horsepower electric motor" and it represents the class of all electric motors of less than one horsepower. According to the "representative product" concept, the technical coefficients used for describing the production of the product class are the ones corresponding to the single, most "representative" product within the class. The most representative products are in general taken to be those which have the largest share of demand or expected growth in demand within their respective classes, with the boundaries between product classes drawn on the basis of similarities among individual products with respect to intermediate input and processing requirements. Demands for the products are specified in units of physical weight, with the nonrepresentative products in each class converted in terms of units (metric tons) of the representative product. The products selected to be included in the study were those for which it was judged that Korea could most likely achieve production costs competitive with imports over the medium term horizon.

1/ A list of the 116 products as well as the 37 shop types can be found in Westphal and Rhee [1978], annex of Chapter 17, and Chapter 16, respectively.

To obtain the processing coefficients b_{ki}, a "reference shop" was designed for each shop class. Engineering estimates for the coefficients were then obtained in relation to these reference shops, in total machine hours, by summing over the machines used within the reference shop the time required in actual processing. Total machine time requirement is then converted to shop time requirement by dividing it by the number of machines in the reference shop. The b_{ki} thus gives the number of hours of processing in shop class k required in the production of one metric ton of product i. The capacity of a shop is measured by the number of hours within a year that the shop may be operated, multiplied by the number of machines it contains.

Finally, the following procedure was used by Westphal and Rhee to estimate the parameters of the fixed-charge cost function for each type of shop. A "double-reference-capacity" shop was designed with twice the physical output capacity of the reference shop. These double-reference shops, however, are not merely proportional blow-ups of the corresponding reference shops. Individual pieces of equipment may be found in a given reference shop that are not found in the associated double-reference shop. The capacity in effective shop hours of the double-reference shop is taken by definition, regardless of its actual machine hours capacity, to be twice that of its associated reference shop. Shop time is thus a relative concept used to measure the volume of processing activity.

Having obtained estimates of the costs of building and operating reference and double-reference-capacity shops, the fixed-charge cost function for each type of shop was obtained by fitting a straight line through the point estimates of annual cost for the reference and double-reference shops. These point estimates were obtained by adding annualized capital costs to annual operating costs, which include expenditures on labor, fuel and other consumed inputs which do not appear as an identifiable part of the products endogenous to the model.

4.3 Structure of the Data

Some features of the Korea data are examined in this section, as they provide a rough indication of the degree of interdependence among the production activities included in the study.

The Korea model specifies 272 different process elements (shop classes) which are required in the production of the 116 endogenous products. The model thus contains 272 zero-one variables (one for each shop class) and 232 continuous variables (a production and an import variable for each product).

Both the matrix of input-output coefficients, A, and the processing requirements coefficient matrix B are fairly sparse. The A matrix, which is upper triangular, has a density of 4.3%. Fourteen activities require no endogenous intermediate inputs, and the average number of endogenous intermediate inputs over the remaining 102 production activities is 5.8. This somewhat low number is in part due to the level of aggregation of the products included in the study. A more narrowly defined "product" would necessarily increase the number of endogenous intermediate inputs required in its production. The sparsity is also partly due to the fact that it was assumed that many intermediate inputs would continue to be imported, and are thus exogenous to the model.

The density of the B matrix clearly depends on how many shop classes are specified for each type of shop.[1/] The lowest possible density would occur if no capacity sharing is allowed in any process element. In this case, however, the problem could be cast into the form of problem (P1) of Chapter 5, for which it is shown there that a very simple solution technique exists. This specification implies a pattern of production organization that is totally end-product oriented. The density of B, on the other hand, would be the maximum possible whenever only one shop class is specified for each type of shop; that is, a shop type would be identically equal to a shop class. This is a situation under which maximum advantage can be taken of capacity sharing, implying a pattern of production organization that is totally process oriented within the sector. This higher density of the B matrix would, on the one hand, most likely increase the difficulty in obtaining globally optimal solutions to this class of problems. On the other hand, however, it would result in a smaller number of constraints (fewer rows for the B matrix) and consequently fewer 0 - 1 variables. Under this specification the Korea model would contain only 37 0 - 1 variables. It is probably safe to infer from this that intermediate specifications with regard to the allowable pattern of production organization would constitute problems computationally more complex than either of the

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1/ See the discussion on production organization in the previous section.

extreme situations. The actual density of B, based on 272 shops classes, is 2.4%. Of the 272 shops, 148 may be shared.

4.4 Computational Experience

To implement the solution approach described in Chapter 3 a FORTRAN H code was developed and run on an IBM 3033 computer.

Test problems were generated based on the data from the Korea study using different parameter specifications. These parametric changes are similar to the ones made by Westphal and Rhee for the purpose of sensitivity analysis of the solution to the Korea model, and they correspond to different specifications of the foreign exchange rate, hourly wage rate, interest rate, and demand levels. A change in the foreign exchange rate affects not only the import price of the endogenous products, but also domestic production costs through exogenous imported inputs and capacity costs through imported production equipment. Hourly wage rates directly affect domestic production costs, and changes in interest rates affect the investment cost for production capacity. Because of economies-of-scale, changes in the demand levels affect the average domestic production costs attainable for each product.

Fifteen alternative problems were generated from the Korean data, and the computational experience with these problems is summarized in Table 4.1. Problem 1 corresponds to the basic specification of the Korea model. The remaining problems correspond to different combinations of parametric changes applied to the basic data; different demand levels, exchange rates, and labor wage rates were

TABLE 4.1

COMPUTATIONAL RESULTS WITH THE KOREAN DATA

Problem number	(1) Number of products with $\bar{H}_i \leq 0$	(2) Number of 0-1 variables eliminated due to condition of column (1)	(3) Number of 0-1 variables eliminated in Stage 1	(4) Number of free variables at end of Stage 1	(5) Number of nodes evaluated	(6) CPU time (in seconds)
1	57	102	154	16	1	0.69
2	57	102	152	18	1	0.67
3	57	102	163	7	1	0.68
4	73	147	101	24	1	0.68
5	53	91	172	9	1	0.72
6	60	108	137	27	17	1.02
7	68	130	123	19	1	0.68
8	40	71	197	4	1	0.67
9	40	71	192	9	1	0.66
10	40	71	200	1	1	0.75
11	46	77	190	5	1	0.66
12	51	90	177	5	1	0.71
13	36	63	199	10	1	0.66
14	46	83	183	6	1	0.67
15	49	90	173	9	1	0.66

assumed, either individually, two at a time, or all three simultaneously. As the purpose of this study is not to analyze the sensitivity of investment decisions in the mechanical engineering sector of Korea to changes in some important economic variables, but merely to test the computational efficiency of our solution approach, the effects of interest rate changes on investment cost for production capacity were not considered. Nor were the effects of exchange rate changes on investment cost through the imported components of new production capacity.[1/] Instead, we simply considered a decrease of 25%, and increases of 25% and 50% respectivley in the total cost of investment (i.e., applied to the fixed-charge function) as three additional specifications.

Table 4.1 should be read as follows. For Problem 1, for example, column (1) shows that 57 of the 116 production activities specified in the model were identified as not competitive with imports at marginal production costs using minimum-cost input sources. As a result of the elimination from the model of these 57 activities, it is shown in column (2) that 102 zero-one variables (one for each process element) were also eliminated. Of the 170 (= 272 - 102) remaining binary variables, the problem reduction stage (Stage 1) succeeded in identifying 154 which should be at level one in the optimal solution (column (3)). Thus, only 16 zero-one variables remained free at the end of Stage 1; this is shown in column (4). In column (5) we can

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1/ The consideration of these effects would require manipulating the individual cost components of the investment cost function.

see that only one node of the B-B tree needed to be evaluated in order to obtain the completion of the optimal solution of Problem 1. That is, the LP relaxation solved at the root node of the B-B tree was naturally integer. This in fact was true for all but one of the 15 problems; problem 6 required 17 node evaluations. The computational times, which include input-output time, are shown in column (6).

We can see from Table 4.1 that Stage 1 was extremely successful in reducing the size of the problem. An important fact that does not appear in the table, however, is that for 11 of the 15 problems Stage 1 succeeded in identifying <u>all</u> profitable production activities. Of course this could not be known at the end of Stage 1, as at that stage of the solution process nothing can be said of the activities not identified. The solution of one LP relaxation, however, established the optimality of the solution. For these cases exactly the condition of Theorem 3.1 of Chapter 3 occurred, with the cost coefficients S_i obviously adjusted to reflect the fact that the products identified by Stage 1 are optimally produced.

It is evident from Table 4.1 that a true test has not been given to the B-B stage of the solution process. In order to test its performance all 15 problems were re-solved without using the problem reduction stage. In other words, it was left to the B-B stage alone to find the optimal solution to these problems. The computational results obtained are shown in Table 4.2. Column (1) gives the number of free zero-one variables remaining after the marginal condition for importing $\bar{H}_i \leq 0$ was applied. It is worth noting, however, that exactly the same number of nodes would have

TABLE 4.2

COMPUTATIONAL RESULTS WITH THE KOREAN DATA: B - B STAGE ALONE

Problem number	(1) Number of free 0-1 variables	(2) Number of nodes evaluated	(3) CPU time (in seconds)
1	170	60	1.95
2	170	72	2.30
3	170	25	1.20
4	125	18	0.99
5	181	24	1.19
6	164	61	2.05
7	142	38	1.33
8	201	95	3.21
9	201	110	3.76
10	201	82	2.82
11	195	20	1.10
12	182	12	0.90
13	209	11	0.96
14	189	23	1.16
15	182	19	1.13

been evaluated if the import condition had not been applied. This is true because all the Δ_k that were set to zero because of the said condition, would be zero in every relaxation solved along the B-B tree. Also, all computational times reported in this chapter include this pre-Stage 1 identification and elimination of noncompetitive activities.

As column (2) indicates, in no case did the B-B tree grow very large; and the computational times, although larger than those of Table 4.1, are nevertheless quite small for the size of the problems being solved.

Encouraged by these results, some further testing was conducted. Additional problems were generated from the Korea data by making parametric changes as before, but with one important difference. Rather than making a percentage change in one or more parameters across all activities, changes were made by individually examining each H_i value. Some parameters were adjusted upward and some downward for each product in an attempt to increase the effect of the interdependencies among the activities. Some changes in the fixed-charge cost functions were also made in some of the runs. The computational results obtained with these "fabricated" problems are reported in Tables 4.3 and 4.4, with and without Stage 1, respectively. These tables have the same format as Tables 4.1 and 4.2.

The computational experience with this latter set of problems lends further empirical support to the efficiency of our approach to the planning problem. The efficiency of the B-B stage is particularly significant for the following reason. As it was discussed in section

TABLE 4.3

COMPUTATIONAL RESULTS WITH "FABRICATED" DATA

Problem number	(1) Number of products with $\bar{H}_1 \leq 0$	(2) Number of 0-1 variables eliminated due to condition of column (1)	(3) Number of 0-1 variables eliminated in Stage 1	(4) Number of free variables at end of Stage 1	(5) Number of nodes evaluated	(6) CPU time (in seconds)
16	30	39	205	28	7	0.91
17	30	39	208	25	5	0.80
18	29	39	222	11	1	0.70
19	32	43	203	25	7	0.91
20	30	39	205	28	1	0.77
21	30	39	208	25	7	0.94
22	32	43	156	73	35	1.87
23	36	53	191	28	11	1.15
24	60	108	137	27	17	1.07
25	47	83	156	33	15	1.03

TABLE 4.4

COMPUTATIONAL RESULTS WITH "FABRICATED" DATA: B - B STAGE ALONE

	(1)	(2)	(3)
Problem number	Number of free 0-1 variables	Number of nodes evaluated	CPU time (in seconds)
16	233	157	5.96
17	233	73	3.02
18	233	71	2.97
19	229	211	7.54
20	233	105	4.23
21	233	116	4.74
22	229	127	5.01
23	219	99	3.65
24	164	59	1.84
25	189	87	2.89

3.5 of Chapter 3, if we want to solve problems with nontriangular input-output structure, we cannot count on an efficient problem reduction stage to decrease the size of the problem to a more manageable size. Because of the circular nature of the interdependence in such situations, any problem reduction stage that may be devised will necessarily be either more cumbersome, or weaker (as the data relaxation suggested in section 3.5), and in either case may prove not to be worth the computational effort required. Since it was shown in Chapter 3 that the solution to the LP relaxation when A is not triangular is essentially no more difficult than for the case in which A is upper triangular, the computational experience provided in this chapter with the B-B stage alone (Tables 4.2 and 4.4) indicates that our LP-based B-B approach would be rather efficient in solving problems with nontriangular input-output structures without any problem-reduction attempt being made. Although improvements could possibly be made by such attempts, our results indicate in the least that it is feasible to solve very large problems without dependence on the success of any form of problem reduction techniques. Table 4.5 gives the summary statistics from Tables 4.2 and 4.4 on the performance of the B-B stage alone for the two sets of problems.

Thus far nothing has been said concerning the particular implementation of the B-B algorithm that yielded the results just discussed. We next analyze in turn: (1) the effectiveness of step 7 of the B-B algorithm; (2) the alternative separation strategies proposed in Chapter 3; (3) the quality of the bounds obtained from the LP relaxation; and (4) the tightness of the conditional upper bounds UB(·)

TABLE 4.5

SUMMARY STATISTICS FOR THE B - B STAGE

Problems	Average number of 0-1 variables*	Average number of nodes evaluated*	CPU time (in seconds)
1 - 15	179	45	1.74
16 - 25	219	110	4.18

Note: * Rounded to nearest integer.

computed at each separation.

At step 7 of the B-B algorithm an attempt is made to peg at the level 1 some of the Δ_k fractional in the solution to the relaxed problem at a given node. For this purpose E_k is computed for each k corresponding to a fractional Δ_k. Computational tests performed with a sample of the problems, both with and without Stage 1, showed that although it succeeded in many cases in pegging some variables, it did not prove to be worth the computational effort expended in computing the E_k. Even in those cases in which the size of the tree decreased significantly as a result of using step 7, computational times were generally higher than those in which the step was bypassed. This was particularly true for problems with a large number of free variables, regardless of the separation strategy used. As a result of these experimental runs step 7 was discarded, and along with it the separation rule of selecting the Δ_k corresponding to the largest E_k. The simplest implementation of the B-B algorithm thus proved the most effective, and all the computational results given in this chapter pertain to this simpler form of the algorithm.

The last two items ((3) and (4)) concerning the B-B algorithm are best discussed in the context of the ineffectiveness of step 7, as it serves to illustrate the success of the approach. The ineffectiveness of step 7 may be explained by several factors. First, node evaluations are carried out extremely fast. Independently of any other factor this implies that an all-integer solution (i.e., the first feasible solution of (P)) can be obtained at a very small computational cost.[1/] Secondly, if the first factor is combined with an efficient separation strategy one can conclude that the first all-integer solution obtained should be reasonably good. In our approach this is evidenced by the fact that in nearly 50% of the test problems the first feasible solution obtained was in fact optimal. Thirdly, if the two previous factors are combined with a relaxation that yields tight bounds, one can confidently expect that the B-B tree should not grow very large. The computational experience provided here shows that the number of node evaluations required in any of the problems was never very large in relation to the number of 0-1 variables. In fact, in no case did the number of node evaluations exceed the number of variables! Moreover, for most of the cases in which the first all-integer solution obtained was optimal, every active node of the tree subsequently examined was fathomed by <u>bound</u> without any further branching taking place. Finally, the conditional upper bounds UB(·)

[1/] Even if a large number of nodes must be evaluated.

proved relatively effective in fathoming at all levels of the B-B tree, causing in most cases a significant reduction in the number of node evaluations required.

It is fairly safe to conclude from the above discussion that all the ingredients for a successful B-B are present in our solution approach to such an extent that attempts as those of step 7 are rendered ineffective or "not worth the effort." It is important to note, in conclusion, that the simpler B-B implementation seems to provide the best of two worlds; not only it appears to be more efficient for the cases in which the input-output matrix A is upper triangular, but, most importantly, it is also the version that applies when A is not triangular.[1/]

[1/] A weaker form of step 7 could actually be used, as indicated in section 3.5 of Chapter 3. In view of the results of this chapter, however, it is extremely unlikely that it would be effective.

CHAPTER 5

RELATED PROBLEMS AND EXTENSIONS

5.1 Introduction

In this chapter we study several versions of the planning problem (P) in which the capacity sharing feature is eliminated. The most basic version studied here is identical to (P) with $B = I$, where I is the identity matrix. We show that for this version of the planning problem stronger results than those of Chapter 3 can be obtained for both stages of the solution approach. The results obtained are then extended to the cases in which the following features are added to this basic version of (P): alternative products, choice among alternative production techniques for each product, and piecewise and general concave investment cost functions. These models will be discussed within the general context of the solution approach to (P); that is, we assume that they will be solved by the two-stage approach and discuss how the results derived here can improve each stage of the solution for these special problems.

We assume throughout this chapter that the input-output matrix is upper triangular.

5.2 Models of Input-Output Interdependence

Consider the problem, which we label (P1):

$$\text{(P1)} \qquad \text{Max} \sum_{i \in I} (H_i x_i - F_i \Delta_i)$$

$$x_i - \sum_{j \in I} a_{ij} x_j \leq D_i , \qquad i \in I$$

$$C_i \Delta_i - x_i \geq 0 , \qquad i \in I$$

$$\Delta_i = 0 \text{ or } 1 , \qquad i \in I$$

$$x_i \geq 0 , \qquad i \in I ,$$

where $H_i = W_i - \sum_{j \in I} a_{ji} W_j - V_i - G_i$, and $C_i \geq x_i$ for all possible x_i.

(P1) is a no-capacity-sharing version of (P). It can be viewed alternatively as a Leontief substitution problem with economies-of-scale, or as a generalization of break-even analysis[1/] for interdependent products.

Since (P1) is (P) with $B = I$, it is obvious that the solution approach to (P) can be directly applied to (P1). For this simpler structure, however, we show that a sufficient condition for imports similar to the condition for domestic production used in the problem reduction stage (Stage 1), can be obtained. Stronger results can also be derived for the B-B stage based on $(\overline{P1})$, the LP relaxation of (P1), given below:

$(\overline{P1})$ $$\text{Max} \sum_{i \in I} \left(H_i - \frac{F_i}{C_i} \right) x_i$$

[1/] If all $a_{ij} = 0$ (P1) decomposes into I simple make-buy problems.

$$x_i - \sum_{j \in I} a_{ij} x_j \leq D_i\ , \qquad i \in I$$

$$x_i \geq 0\ , \qquad i \in I\ .$$

$(\overline{P1})$ has precisely the structure of $(\bar{P})$, and the two-step solution approach applies, with the $\bar{S}_i$ computed at Step I (with $S_i = H_i - \frac{F_i}{C_i}$) and the $\bar{x}_i$ obtained at Step II. Theorem 3.1 clearly applies to $(\overline{P1})$ also, and thus at least one S_i must be strictly positive if any domestic production is to take place. Theorem 3.2, as before, can be used to eliminate activities noncompetitive at marginal cost; as this theorem applies to all the versions of (P1) studied in this chapter, we assume hereafter that the set I contains only the competitive activities (i.e., $\bar{H}_i > 0$ for each $i \in I$).

The sufficient conditions for domestic production ((3.5) and (3.6)), of Stage 1, become respectively

$$H_i D_i > F_i \tag{5.1}$$

and

$$\hat{H}_i \hat{D}_i > F_i\ , \tag{5.2}$$

with the $\hat{H}_i$ and $\hat{D}_i$ having the same interpretation as in Chapter 3. If I^* is the set of activities that satisfy (5.1), then any activity $i \notin I^*$ that satisfies (5.2) is added to I^* in an iterative fashion as described in Chapter 3. At the end of this stage a set of activities $I^* \subseteq I$ is identified which is known to be optimally undertaken.

The problem reduction stage for (P1) does not end here, however. We give next a sufficient condition that can be used iteratively to identify activities that are optimally imported in (P1).

In the absence of capacity sharing the total variable savings attained by undertaking a given activity must cover the fixed cost incurred in undertaking it.[1/] In other words, total benefits must cover total costs; this is true whether production is for final consumption, for intermediate input in the production of other products, or both. If x_i^* is the optimal level for activity i and H_i^* its associated variable savings (i.e., $W_i - H_i^*$ is the variable cost of producing product i), then for each i such that $x_i^* > 0$ the condition $H_i^* x_i^* > F_i$ must be satisfied. It is easy to see that if this condition is not satisfied then the minimum average production cost attainable is larger than the import cost[2/] and thus no savings can be obtained by undertaking domestic production.

1/ Under capacity sharing an activity with positive marginal savings over import cost could optimally be undertaken which did not cover its associated fixed costs. It was only required that all process interdependent activities jointly covered fixed costs.

2/ Let MAC_i = minimum average domestic production cost attainable for product i.

Then $$MAC_i = (W_i - H_i^*) + \frac{F_i}{x_i^*}$$

$$= W_i - \left(\frac{H_i^* x_i^* - F_i}{x_i^*} \right)$$

$$\geq W_i \quad \text{if} \quad H_i^* x_i^* - F_i \leq 0 .$$

Based on the condition given above that must be satisfied by the activities at a positive level in the optimal solution to (P1), we show how a necessary condition for domestic production (equivalently, a sufficient condition for import) can be obtained which can be applied iteratively to identify activities that are optimally imported. The condition is presented within the context of the simplex method.

Let $C_i = \infty$ (equivalently, $F_i = 0$) for each $i \in I$ in $(\overline{P1})$, and assume we have a basic feasible solution which contains all the production activities. With the $F_i = 0$ this basic solution is in fact optimal in $(\overline{P1})$ since all activities with $\bar{H}_i \leq 0$ have been eliminated and are thus exogenous to the model. With the $\bar{H}_i$ and $\bar{x}_i$ corresponding to this basis at hand, the following theorem provides a sufficient condition for an activity to be optimally imported.

Theorem 5.1 If $\bar{H}_i\bar{x}_i \leq F_i$ then product i is optimally imported in the solution of (P1).

Proof.
$$F_i \geq \bar{H}_i\bar{x}_i$$
$$\geq \bar{H}_i x_i^* \quad \text{since} \quad \bar{x}_i \geq x_i^* \geq 0 ,$$
$$\geq H_i^* x_i^* \quad \text{since} \quad \bar{H}_i \geq H_i^* \geq 0 .$$

Hence $\bar{H}_i\bar{x}_i \leq F_i \Rightarrow H_i^*x_i^* \leq F_i \Rightarrow x_i^* = 0$.

Assume that the condition of Theorem 5.1 is satisfied for one or more products and the corresponding activities are eliminated from the basis in $(\overline{P1})$. Let I^+ be the set of production activities remaining in the basis. If $\tilde{H}_i$ and $\tilde{x}_i$ correspond to the new values of $\bar{H}_i$ and $\bar{x}_i$ associated with the new basis, then, by the interpretation of the $\bar{H}_i$ and their updated values $\tilde{H}_i$, it should be clear that $\bar{H}_i \geq \tilde{H}_i \geq H_i^*$ for each $i \in I^+$. It is also true that $\bar{x}_i \geq \tilde{x}_i \geq x_i^*$, since the set I^+ still contains the set of activities optimal in (P1). $\tilde{H}_i\tilde{x}_i$ is thus still a valid upper bound on $H_i^*x_i^*$; hence any activity in the new basis that satisfies

$$\tilde{H}_i\tilde{x}_i \leq F_i \tag{5.3}$$

can be removed, as it cannot possibly be optimal in (P1). Since only activities that are optimally imported in (P1) are eliminated from the basis of $(\overline{P1})$, the $\tilde{H}_i$ and $\tilde{x}_i$ obtained at each iteration constitute upper bounds on H_i^* and x_i^* respectively, and the process can thus be repeated until (5.3) is no longer satisfied for any of the I^+ production activities remaining in the basis.

The set of products I^+ obtained at the end of the elimination procedure has the following property: If any one product is dropped in favor of imports, total savings over imports obtained from the domestic production of the remaining $I^+ - 1$ products is less than if all I^+ products are produced. In other words, the value of the objective function of (P1) decreases if any one of the I^+ activities is not undertaken. It is possible, however, that it increases if two or more interdependent activities are dropped in favor of imports. In principle at least, one could apply at this point the equivalent of condition (5.3) to combinations of two or more interdependent products from the set $I^+ - I^*$. This, however, would be rather tedious. The optimality or non-optimality of the current set of activities can be easily established with a very simple B-B stage.

At the root node of the B-B tree $(\overline{P1})$ with I^+ endogenous activities is solved. We set, for each $i \in I^+$, $C_i = \begin{cases} \infty & \text{if } i \in I^* \\ \tilde{x}_i & \text{otherwise} \end{cases}$. This corresponds respectively to setting $F_i = 0$ for each activity known to be optimal in (P1) and to tightening the capacity bounds for the remaining production activities by setting them equal to the best upper bounds currently known, namely the $\tilde{x}_i$. If the solution to this problem is naturally integer then it is obviously optimal in (P1). If it is not integer, however, it should be easy to see that in the absence of capacity sharing, all Δ_i variables $(\Delta_i = \frac{x_i}{C_i})$ that are at the level one may be fixed to one and the corresponding activities may be added to the set I^*. Moreover, any Δ_i integer in the solution to the relaxation at any other node of the B-B tree will be integer in any completion of the partial solution at that node.

5.2.1 Alternative Products

Assume now that each production process (or industry, whichever may be the interpretation) in (P1) may produce more than one product, but each product can only be produced by one production process. That is, the only substitution among activities is between domestic production and imports -- no alternative production techniques.

There are K production processes specified for the production of the I products, $K \leq I$. Industry k can produce any combination of products $i \in I_k$, and for any $k \neq \ell$ $I_k \cap I_\ell = \emptyset$. This problem, (P2), has the following formulation:

$$\text{(P2)} \qquad \text{Max} \left\{ \sum_{k \in K} \sum_{i \in I_k} H_i x_i - \sum_{k \in K} F_k \Delta_k \right\}$$

$$x_i - \sum_{j \in I} a_{ij} x_j \leq D_i , \qquad i \in I$$

$$C_k \Delta_k - \sum_{i \in I_k} b_{ki} x_i \geq 0 , \qquad k \in K$$

$$\Delta_k = 0 \text{ or } 1 , \qquad k \in K$$

$$x_i \geq 0 , \qquad i \in I ,$$

where $H_i = W_i - \sum_{j \in I} a_{ji} W_j - V_k b_{ki} - G_i$, and b_{ki} is defined as in (P). C_k is an upper bound on the total activity of industry k for any possible value of x_i, $i \in I_k$.

(P2) obviously includes (P1) as a special case, and the basic distinction between (P2) and (P) is that in (P) $I_k \cap I_\ell \neq \emptyset$ for at least one $k \neq \ell$. (P2) has the following LP relaxation:

$$(\overline{\text{P2}}) \qquad \text{Max} \sum_{k \in K} \left[\sum_{i \in I_k} \left(H_i - \frac{F_k}{C_k} b_{ki} \right) x_i \right]$$

$$x_i - \sum_{j \in I} a_{ij} x_j \leq D_i , \qquad i \in I$$

$$x_i \geq 0 , \qquad i \in I .$$

Theorem 3.1 obviously applies to $(\overline{\text{P2}})$; and conditions (5.1) and (5.2) become respectively

$$\sum_{i \in I_k} H_i D_i > F_k \tag{5.1a}$$

and

$$\sum_{i \in I_k} \hat{H}_i \hat{D}_i > F_k . \tag{5.2a}$$

If the condition of the Theorem 5.1 is replaced by $\sum_{i \in I_k} \bar{H}_i \bar{x}_i \leq F_k$, the same results obtained for (P1) follow for (P2). A characteristic of the solution of (P2) can be seen to be that each industry k will either produce all products $i \in I_k$, or none, where I_k contains only products i such that $\bar{H}_i > 0$, as we assume that all products with $\bar{H}_i \leq 0$ have been eliminated from the set I.

5.2.2 Alternative Production Techniques

In this subsection we extend the basic model (P1) to allow for choice among alternative production techniques for each product, and show how the results obtained for (P1) apply to this version of the problem.

Let T_i be the set of alternative techniques for product i, as well as the cardinality of the set. T_i fixed-charge cost functions are thus specified for each $i \in I$. This version of the problem, labeled (P3), has the following formulation:

(P3) $$\text{Max} \sum_{i \in I} \sum_{t \in T_i} (H_i^t x_i^t - F_i^t \Delta_i^t)$$

$$\sum_{t \in T_i} x_i^t - \sum_{j \in I} \sum_{t \in T_j} a_{ij} x_j^t \leq D_i , \qquad i \in I$$

$$C_i\Delta_i^t - x_i^t \geq 0\ , \qquad i \in I\ , \qquad t \in T_i$$

$$\Delta_i^t = 0 \text{ or } 1\ , \qquad i \in I\ , \qquad t \in T_i$$

$$x_i^t \geq 0\ , \qquad i \in I\ , \qquad t \in T_i\ ,$$

where $H_i^t = W_i - \sum_{j\in I} a_{ji}W_j - G_i^t - V_i^t$ and C_i is an upper bound on x_i^t, for any $t \in T_i$. Superscripts t have been added to the appropriate parameters to denote their dependence on the production techniques. A set of constraints of the form

$$\sum_{t\in T_i} \Delta_i^t \leq 1\ , \qquad i \in I$$

is not required in (P3), as the substitution theorem guarantees that only one technique t from each T_i will be used if product i is domestically produced.[1/] In other words, at most one x_i^t, $t \in T_i$, or y_i (the slack variable) will be at a positive level for each $i \in I$.

It is assumed in the formulation of (P3) that the input-output structure does not vary with the production technique. This is a reasonable assumption for choice of technique problems with input-output relationships specified at the product level. It merely says that each technique $t \in T_i$ under consideration for product i requires the same <u>endogenous</u> inputs (parts, components, modules, subassemblies, etc.) in its production process, but may have completely

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1/ See Chapter 3.

different exogenous input requirements, labor skills, etc. The cost components G_i^t account for these differences.

We also assume, without loss of generality, that the inequalities

$$V_i^1 + G_i^1 > V_i^2 + G_i^2 > \cdots > V_i^{T_i} + G_i^{T_i}$$

$$\underline{\text{and}} \quad F_i^1 < F_i^2 < \cdots < F_i^{T_i} \tag{5.4}$$

are satisfied for each $i \in I$. If (5.4) is not satisfied for one or more $i \in I$, then we can show that at least one production technique t can be eliminated from T_i. Let $V_i^{t'} + G_i^{t'} = \min_{t \in T_i} \left\{V_i^t + G_i^t\right\}$.

We can then eliminate from further consideration all techniques $t \in T_i$ such that $F_i^t \geq F_i^{t'}$, as they are completely dominated by technique t' (i.e., costs under these techniques are at least as high as under t' at all production levels). This simple dominance rule can be repeated for $V_i^{t''} + G_i^{t''} = \min_{t \in T_i - \{t'\}} \left\{V_i^t + G_i^t\right\}$, and so on for each $t \in T_i$ and $i \in I$. If we then renumber the "competitive" techniques for each product in decreasing order of $(V_i^t + G_i^t)$, $t \in T_i$, inequalities (5.4) will be satisfied and we will have obtained the concave envelope of the capacity cost curves for each product.

We note at this point that if we relax the assumption of constant marginal expansion cost in (P1) and specify instead a piecewise linear concave cost function for each product, the formulation of the problem would take exactly the form of (P3), with inequalities (5.4) automatically satisfied. The superscripts in the variables and parameters in this problem would be associated with the <u>segments</u> of the capacity cost functions. Thus, we can conclude that after the elimination of noncompetitive techniques, the "alternative production techniques" version of (P1) is identical to the "multi-segment cost function" version.

The LP relaxation of (P3) is

$$(\overline{P3}) \qquad \text{Max} \sum_{i\in I} \sum_{t\in T_i} \left(H_i^t - \frac{F_i^t}{C_i} \right) x_i^t$$

$$\sum_{t\in T_i} x_i^t - \sum_{j\in I} \sum_{t\in T_j} a_{ij} x_j^t \leq D_i , \qquad i \in I$$

$$x_i^t \geq 0 , \qquad i \in I, \ t \in T_i .$$

With A upper triangular, the constraint matrix of $(\overline{P3})$ is block triangular; row i now contains $T_i + 1$ positive elements, including the coefficient of the slack variable y_i.

If we solve $(\overline{P3})$ with all $C_i = \infty$ (equivalently, all $F_i^t = 0$) then obviously any product i that is domestically produced in the solution to such problems will be produced by the technique with the lowest $(V_i^t + G_i^t)$, namely technique T_i. Thus, rather than working with the full problem, the approach taken here consists of working with a sequence of problems such that each has a single production technique specified for each product. Initially we consider the problem with $t = T_i$ only, for each $i \in I$. To this problem Theorem 3.2 applies and all activities with $\bar{H}_i^{T_i} \leq 0$ can be eliminated, as they are optimally imported in (P3). Note that not only technique T_i is eliminated in this case, but all T_i techniques are, since $\bar{H}_i^{T_i} \leq 0 \Rightarrow \bar{H}_i^t \leq 0$ for any $t \in T_i$.

Rather than discussing both the sufficient conditions for domestic production and the sufficient conditions for imports as they apply to (P3), we choose to discuss in detail only the latter; it should become apparent from the discussion how the former can be applied.

For the development that follows, some additional notation is needed. Let a_i^t denote the activity vector associated with the production variable x_i^t and e_i the activity vector corresponding to the import variable y_i. In addition, let C_i^t denote the production level for product i at which the capacity cost function changes slope, that is, the break point between techniques $t - 1$ and t.

With a solution at hand for $(\overline{P3})$ with $t = T_i$ and all $C_i = \infty$, the sufficient condition for a given production activity to be permanently removed from the basis (given by Theorem 5.1) can be applied. In this case, however, if production activity a_i^t is removed from the basis, it may be replaced either by the import activity e_i, or by a different production activity $a_i^{t'}$, $t' \in T_i$. If a similar iterative procedure as that for (P1) and (P2) is to be applied to (P3), the new activity to become basic must be selected in a way that the validity of Theorem 5.1 is maintained at each iteration of the procedure. The following theorem replaces Theorem 5.1 for this problem.

Theorem 5.1a Let $t = T_i$. If $\bar{H}_i^t \bar{x}_i^t \leq F_i^t$ and $x_i^t \geq C_i^t$ then product i is optimally imported in the solution of (P3).

Proof. By condition (5.4) we know that $H_i^t > H_i^{t'}$ for all $i \varepsilon I$ and any $t' = 1, 2, \ldots, t-1$. It follows that $\bar{H}_i^t > \bar{H}_i^{t'}$ and that $\bar{x}_i^t \geq \bar{x}_i^{t'}$ for all $i \varepsilon I$ and any $t' = 1, 2, \ldots t-1$. Then, by Theorem 5.1, $\bar{H}_i^t \bar{x}_i^t \leq F_i^t$ is a sufficient condition for technique t to be discarded, that is, for production activity a_i^t to be permanently removed from the basis. Under alternative production techniques, two cases must be considered in order to determine which of the activities that produces product i should replace activity a_i^t in the basis:

(i) $\bar{x}_i^t < C_i^t$, and

(ii) $\bar{x}_i^t \geq C_i^t$.

To complete the proof of the theorem we need to show that in order for a_i^t to be replaced in the basis by the import activity e_i, condition (ii) must be satisfied. Assume first that $\bar{x}_i^t < C_i^t$. Since $\bar{x}_i^t$ is a valid upper bound on the production level for product i, then technique t cannot be used in the optimal solution of (P3) because by condition (5.4) there exists a production technique $t' < t$ $(t = T_i > 1)$ for which lower domestic production costs are attained at production level $\bar{x}_i^t$. Now, if $\bar{x}_i^t \geq C_i^t$, then by concavity of the investment cost function and the fact that $\bar{x}_i^{t'} \leq \bar{x}_i^t$ under any technique $t' = 1, 2, \ldots, t-1$, no technique can yield a lower domestic production cost for product i than technique t. Hence, the import activity e_i should replace a_i^t in the basis, that is, product i is optimally imported in the solution of (P3).

Under alternative production techniques, thus, the sufficiency condition for product i to be imported must include the additional requirement that $\bar{x}_i^t \geq C_i^t$, since if $\bar{x}_i^t < C_i^t$ the expansion cost used in the condition $\bar{H}_i^t \bar{x}_i^t \leq F_i^t$ overestimates the true cost at the production level $\bar{x}_i^t$, and this invalidates the condition for activity a_i^t to be dropped in favor of the import activity e_i. However, $\bar{x}_i^t < C_i^t$ is a sufficient condition for a_i^t to be replaced in the basis by a different <u>production</u> activity, that is, for switching production techniques. If $C_i^{t'-1} \leq \bar{x}_i^t \leq C_i^{t'}$, $t' \leq t$, then activity $a_i^{t'-1}$ should replace a_i^t in the basis regardless of whether condition $\bar{H}_i^t \bar{x}_i^t \leq F_i^t$ is satisfied or not, since $\bar{x}_i^t$ is still a valid upper bound on the level of domestic production for product i. If $\bar{H}_i^t \bar{x}_i^t > F_i^t$ and $\bar{x}_i^t \geq C_i^t$ then obviously activity a_i^t remains basic.

Let I^+ be the set of production activities remaining in the basis after the above conditions are systematically applied for each product in the set I, and let $a_i^{t(i)}$ be the activity in the current basis, for each $i \in I^+$. Since the approach ensures that $\bar{H}_i^{t(i)}$ and $\bar{x}_i^{t(i)}$ are still valid upper bounds on variable savings over import costs and production levels respectively, (i.e., Theorem 5.3a is valid at each iteration with $t = t(i)$ for each $i \in I$) the elimination procedure can be repeated as long as technique switching occurs and/or production activities are eliminated in favor of imports, with the set I^+, and $\bar{H}_i^{t(i)}$ and $\bar{x}_i^{t(i)}$ updated at each iteration.

The approach for identifying production activities that are optimally undertaken in the solution of (P3) is basically the reverse of the process just described and yields a set of products I^* known to be optimally produced, although not necessarily the techniques by which they should be produced. The process starts with an all-import basic solution. Once a production activity a_i^t enters the basis it can only be replaced (in the basis) by another production activity $a_i^{t'}$; that is, only production technique switching can occur for such activities. Moreover, $t' > t$, since at each iteration we must have lower bounds both on variable savings over imports and on production levels; the first activity to enter the basis must therefore use technique $t = 1$. At each iteration the $\hat{D}_i^t$ give lower bounds on the levels of domestic production[1/] and can be used to substitute one production technique for another in the basis. At the first iteration after a_i^1 is introduced into the basis, for example, if $\hat{D}_i^1 \geq C_i^1$ then technique t such that $C_i^t \leq \hat{D}_i^1 \leq C_i^{t+1}$ will replace technique 1 for product i. Technique switching may occur for one or more activities at each iteration. As new production activities are introduced into the basis, apart from its effects on production costs of other products, endogenous demands are generated for the interdependent activities already in the basis; these added demands may justify a switch from production technique t to t', $t' > t$. With activity $a_i^{t'}$ replacing a_i^t in the basis, the marginal cost of producing product i decreases,[2/] which in turn decreases the production costs for all products $j \in I$ such that $a_{ij} > 0$. This in turn may cause either new activities to satisfy the sufficient condition for being

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1/ The $\hat{D}_i^t$ for this problem correspond to the $\hat{D}_i$ in (P1) and (P2) and are obtained the same way.

2/ By the concavity of the cost function.

introduced into the basis, or technique switching for the activities already in the basis (i.e., a_i^t such that $i \varepsilon I^*$), or both. This process is repeated, maintaining at each iteration $\hat{H}_i^t$ and $\hat{D}_i^t$ valid lower bounds on variable savings over imports and production levels respectively.

At the end of this problem reduction stage, as was also the case of problems (P1) and (P2), we have a set I^* of products that are known to be domestically produced in the optimal solution of (P3), and a set I^+ ($I^+ \supseteq I^*$) of products that have the property that if any one is dropped in favor of imports, total savings over imports from the remaining $I^+ - 1$ products decreases. Under alternative production techniques, however, the set T_i of competitive techniques for each product may have been reduced. Let $\overline{t}(i)$ be the technique corresponding to product i in the final basis formed by the I^+ activities, and define $\underline{t}(i)$ similarly for the I^* activities. If $t^*(i)$ is the production technique for product i in the optimal solution to (P3), then

$$\underline{t}(i) \leq t^*(i) \leq \overline{t}(i) \qquad \text{for each } i \varepsilon I^* ,$$

and

$$t^*(i) \leq \overline{t}(i) \qquad \text{for each } i \varepsilon I^+ - I^* .$$

The optimality of the solution given by activities $a_i^{\overline{t}(i)}$, $i \varepsilon I^+$ can be easily verified or disproved by a simple B-B stage. At each node a problem with only one technique specified for each product is solved, and the sufficient conditions used in the problem reduction stage can be used here for guiding separation.

5.2.3 General Concave Cost Function

Consider now the case in which the investment cost function for capacity is given by a general concave cost function $F_i(x_i)$, with $\frac{dF_i(x_i)}{dx_i} \geq 0$ and $\frac{d^2F_i(x_i)}{dx_i^2} \leq 0$ for all possible values of x_i, for each $i \in I$. This problem is formulated as

$$\text{(P4)} \qquad \text{Max} \sum_{i \in I} [H_i' x_i - F_i(x_i)]$$

$$x_i - \sum_{j \in I} a_{ij} x_j \leq D_i , \qquad i \in I$$

$$x_i \geq 0 , \qquad i \in I .$$

The H_i' in this case are distinguished from the H_i of the previous problems by the fact that variable investment costs are not included as components of variable domestic production costs.

In the previous subsection was shown the equivalence between (P3) and a version of (P1) with multi-segment concave cost functions. We show here that (P4) can be solved by solving an "equivalent" problem with a piecewise linear cost function.

The "equivalent" problem with piecewise linear approximation of the cost curves $F_i(x_i)$ is obtained sequentially. The equivalence is in the sense that at the optimal solution the piecewise linear approximation coincides with the true cost function. Let C_i be an upper bound on all possible values of x_i.[1/] Obtain then for each $F_i(x_i)$ a fixed-charge approximation that is tangent to it at $x_i = C_i$. Let F_i be the intercept, and the slope $V_i = \left.\frac{dF_i(x_i)}{dx_i}\right|_{x_i=C_i}$. The formulation of this fixed-charge version of (P4) is precisely (P1), with $H_i = H_i' - V_i$, and Theorem 3.2 applies to this problem since the minimum attainable V_i are used in H_i and consequently the $\overline{H}_i$ are not underestimated. We can thus eliminate from the problem all products i such that $\overline{H}_i \leq 0$ as they are optimally imported in (P4). If any product with $\overline{H}_i \leq 0$ is found, we can set $C_i = \overline{x}_i$, the solution of the relaxation of the fixed-charge problem with all $F_i = 0$, and obtain a new fixed-charge approximation at the new upper bounds $\overline{x}_i$. This can be repeated as long as products are eliminated.

The procedure developed for (P1) and used for (P2) and (P3) can now be applied to the fixed-charge approximation of (P4). Starting with all production activities in the basis, the condition of Theorem 5.1 can now be systematically applied here. At the end of the first iteration we set $C_i = \overline{x}_i$, where the $\overline{x}_i$, the new basic solution, are obtained like in (P1), and new fixed-charge approximations to $F_i(x_i)$, $i \in I^+$, are obtained. At this point of the iterative procedure

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1/ One such upper bound can be easily computed as follows:

$$C_I = D_I$$

$$C_i = D_i + \sum_{j>i} a_{ij} C_j \,, \qquad i = I-1, I-2, \ldots, 1 \,.$$

we have an opportunity to reapply Theorem 3.2 if one or more C_i changed from their previous values, since in this case some H_i may decrease as a result of possible increase in the variable investment cost V_i, as $\left.\frac{dF_i(x_i)}{dx_i}\right|_{x_i=a} \geq \left.\frac{dF_i(x_i)}{dx_i}\right|_{x_i=b}$ if $a < b$ by concavity of $F_i(x_i)$.

The above process is repeated until no production activity is dropped either in favor of a production activity with a different "technique" (i.e., one with higher V_i, which would occur if one or more $\bar{x}_i$ decrease from one iteration to the next, but remain at a positive level). We have at the end of this stage a set of products I^+ with the same property as in the previous problems treated, and fixed-charge functions which coincide with the $F_i(x_i)$ at the points $\bar{x}_i$, for each $i \in I^+$.

Again, the optimality or not of the solution from the first stage can be determined by a B-B stage, in which at the root node we have the last fixed-charge approximation obtained. The procedure of the first stage can be used to obtain fixed-charge approximations whenever separation occurs.

We note finally that the procedure for obtaining a set of products I^* known to be optimally produced in (P4) cannot be applied to this problem, as $\left.\frac{dF_i(x_i)}{dx_i}\right|_{x_i=0}$ may not be finite. It could be applied, however, to those activities, if any, which satisfy

$$\left.\frac{dF_i(x_i)}{dx_i}\right|_{x_i=0} < W_i - G_i .$$

CHAPTER 6

CONCLUSION

6.1 Summary

In this study a solution procedure was developed for a class of investment planning models which incorporates the following features: economies-of-scale in production, intermediate input-output relationships among production activities, and capacity sharing. The choice is between domestic production and imports to satisfy exogenously stated demands for a given bill of goods. The model was presented in Chapter 2, which also provides a brief discussion of the complex interdependencies that exist among production activities and their potential effects on investment decisions.

The theoretical analysis of the planning model was done in Chapter 3. Simple sufficiency conditions for import and for domestic production of a given product were discussed and a problem reduction stage was developed which applies these conditions in an iterative fashion. Activities which are not competitive with imports when maximum benefits from interdependence are assumed are identified at this stage, as well as activities that are profitable even when no advantage is taken of interdependence. For the solution stage an LP-based branch-and-bound (B-B) algorithm was developed. It was shown that the LP relaxation of the planning problem is a simple maximization over a Leontief substitution structure for which a very efficient solution approach exists. The development of this chapter assumed an upper-triangular input-output structure for the planning model. It was shown then how "data relaxation" allows for the same problem reduction stage to be used for nontriangular input-output structures, and, most importantly, it was shown that the LP relaxation can also be very efficiently solved in this case, and the B-B stage is thus readily applicable to this more general problem.

Computational results obtained from the implementation of the solution approach developed in Chapter 3 using the Korea study data were given in Chapter 4. The results from 25 test problems generated by making parametric changes in the Korean data provided strong evidence of the efficiency of our two-stage approach. The problem reduction stage was very successful in reducing the size of the problem. For several problems, in fact, all profitable production activities were identified at this stage, although this could only be known after the B-B stage. More important than the first stage, however, was the efficiency of the B-B stage. Computational experience obtained for the B-B stage alone showed that its efficiency does not depend in any significant way on the success of a problem reduction stage. This is important for two reasons: First, the effectiveness of the problem reduction stage decreases with the degree of interdependence among the activities; and secondly, for nontriangular input-output structures it is uncertain whether it "pays" to use a problem reduction stage and the solution to such problems must therefore rely more heavily or perhaps entirely on the B-B stage, or some other solution method.

The important conclusion from the computational experience of

Chapter 4 is that very large problems can be efficiently solved by our two-stage approach. Computational feasibility should thus no longer be an issue in deciding the level of aggregation at which such models should be estimated.

In Chapter 5 various models of input-output interdependence (no capacity sharing allowed) were studied, each incorporating different features. The two-stage approach was specialized to each. It was shown for these models that besides the possible identification of activities known to be optimally undertaken, the problem reduction stage yields a set of activities which contains all the activities in the optimal solution to the model and has the following property: if any one activity is dropped from the set, total savings from undertaking all the remaining activities decreases.

While it is unlikely that the simple models of Chapter 5 fit any real situation, they constitute nevertheless basic substructures of many important planning problems, including the process analysis models cited in Chapter 1. These substructures can be exploited in many ways in decomposition schemes to obtain efficient solution approaches to more complex real problems, and this is perhaps the main value of the analysis of Chapter 5.

6.2 Suggestions for Further Research

An obvious area for further research is the introduction of spatial and dynamic elements in the planning model (P). Another important extension of (P) would be the incorporation of alternative production techniques for each product. It seems that this particular extension could be incorporated in our two-stage approach with only minor changes along the lines of the alternative techniques extension of (P1).

It was pointed out in the previous section that the problems studied in Chapter 5 constitute important substructures of many planning problems. As most of the successful approaches to mixed integer programming problems rely heavily on exploiting structures, the results of Chapter 5 can be useful in solving more complex problems. A serious difficulty, however, occurs when constraints are added to these simple models, or even to (P), which explicitly place a limit on some resource. In these cases the dichotomy between domestic production and imports will not be in general a property of the optimal solution. This is the case, for example, if a budget constraint is added to (P) or to any of the models of Chapter 5. It is an important extension that would allow sectoral models like (P) to be imbedded in economy-wide models. This constitutes a very difficult problem, as no price can be found in general for the limited resource which would lead to the optimal production decisions.[1/]

An example is given next of an extension of one of the simple models of Chapter 5 which constitutes a very important class of problems and for which the difficulty described above does not occur. The general approach of this dissertation seems to be promising for this class of problems. Consider the following multi-period extension of (P1), the most basic structure studied in Chapter 5. We use a

1/ This immediately rules out an otherwise promising approach to this constrained version of (P), which is Lagrangian relaxation (dual decomposition) with respect to the added constraint(s).

cost minimization formulation of (P1) without substituting for the import variables so that all the parameters appear explicitly.

(P5) $$\text{Min} \sum_{i \in I} \sum_{t \in T} (V_i^t x_i^t + W_i^t y_i^t + F_i^t \Delta_i^t)$$

$$x_i^t - \sum_{j \in I} a_{ij} x_j^t + y_i^{t-1} - y_i^t = D_i^t , \quad i \in I , t \in T$$

$$C_i^t \Delta_i^t - x_i^t \geq 0 \quad i \in I , t \in T$$

$$\Delta_i^t = 0 \text{ or } 1 , \quad i \in I , t \in T$$

$$x_i^t \geq 0 , \quad i \in I , t \in T$$

$$y_i^t \geq 0 , \quad i \in I , t \in T$$

We assume that V_i^t includes the variable production cost component G_i^t, and, as before, that the input-output matrix A is upper triangular. If we give the following interpretation to the variables and parameters in (P5):

x_i^t = quantity of product i produced or ordered at the beginning of period t,

y_i^t = ending inventory of product i in period t,

V_i^t = variable cost of production for product i in period t

W_i^t = inventory holding cost per unit of product i in period t,

F_i^t = fixed cost component of production cost, set-up cost, or ordering cost, for product i in period t,

then (P5) represents an important class of problems occurring in material requirements planning (MRP) systems. (P5) is similar to Veinott's [1969] multi-facility economic-lot-size model, and it includes as special cases the formulation of Crowston, Wagner and Williams [1973], and Crowston and Wagner [1973].

An optimal solution to (P5) exists which is an extreme point of the material balance constraints, and the LP relaxation of (P5), as in all the models studied here, yields a Leontief substitution problem, It can be seen immediately, by the substitution theorem, that at most one of x_i^t and y_i^{t-1} will be at a positive level (i.e., $x_i^t y_i^{t-1} = 0$) in the optimal solution of (P5), and exactly one if $D_i^t > 0$.

The results of this study suggest that the specialization of our two-stage approach to this important class of problems might be a worthwhile research effort.

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