CONSISTENCY BETWEEN THE MULTIREGIONAL AND SINGLE REGIONAL PROJECTION MODELS

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INTRODUCTION

When making projections for a population closed to migration and exposed to a fixed regime of growth, two phases may be observed. The first one is the stabilization of the age composition toward a stable proportional distribution. During the second phase, the stable population grows with a rate known as the intrinsic rate of natural increase.

The mathematical techniques used to analyze the stabilization process have been thoroughly investigated by Keyfitz (1968, chapter 3) and others. It has been shown that the two stages underlined above can be studied with the aid of the two largest (in absolute values) eigenvalues of the growth matrix: the unique positive root, defining the stable growth ratio, and the complex eigenvalue with the largest absolute value (together with its conjugate), defining the population wave--a phenomenon which is caused by the stabilization of the age structure. The unique positive eigenvector defines the stable (and the stable equivalent) population.

When a multiregional population open to migration is considered, one must add an additional feature to the model: space. A detailed mathematical description of this model can be found in Rogers (1975). It is an extension of the single-region model (i.e. for a population of the one region closed to migration), but also includes spatial distribution. In order to incorporate this new concept, the forms of analysis must be enlarged to produce the multiregional life table, the multiregional growth matrix and population projection, the multiregional stable population, the stable growth rate.

The two phases of the single-region population projection process also exist in the multiregional case. The first one consists of the achievement of age and regional stability. During the second phase the regional populations grow at a constant rate, retaining unchanged age compositions and regional shares.

Rogers (1975) has shown that the stability of the multiregional population can be studied with the aid of the dominant eigenvalue of the multiregional growth matrix and its corresponding eigenvector. He also has suggested (Rogers, 1976) that the stabilization process passes through two phases before stability is reached: first, when the regional age compositions are stabilized , and second, when regional shares are stabilized.

A more detailed discussion of these problems is the subject of this paper. In Section I, the eigenvalues and eigenvectors of the multiregional growth matrix will be studied with respect to the single-region cases in order to find a theoretical background for further discussions.

In Section II the phases of the multiregional projection process are discussed. Three phases will be considered: 1) the stabilization of the regional age compositions and the subsequent waves; 2) the stabilization of the regional shares and the modified age structures (effects of migration); 3) stable growth. Problems still remain for further theoretical investigators.

The discussions throughout the paper are illustrated by a three-region example: northwestern Bulgaria (referred to as N. West), northeastern Bulgaria (N. East), and Sofia. This example has been selected from a large number of analyses with two or three regions because it reveals the most common properties of the multiregional projection process.

I. CONSISTENCY BETWEEN THE MULTIREGIONAL AND SINGLE-REGIONAL MODELS

One of the main objectives of the population policy in some countries is to lower the number of migrants as much as possible. In order to study the effect of lowering migration rates, a simulation model can be made that diminishes them to a lower level in a population projection. If this level becomes so low that some of the migration flows are neglected, a problem will arise: does the multiregional model collapse to the single-region one or not? It will be shown in this paper that this is true of all of the multiregional demographic measures obtained from the eigenvalues and the eigenvectors of the multiregional growth matrix except for the stable equivalent population.

The multiregional growth matrix has the following arrangement of its elements, as suggested by Rogers (1966):[1)]

$$\underset{\sim}{G} = \begin{pmatrix} \underset{\sim}{G}_{11} & \underset{\sim}{G}_{21} \cdots \underset{\sim}{G}_{n1} \\ \underset{\sim}{G}_{12} & \underset{\sim}{G}_{22} \cdots \underset{\sim}{G}_{n2} \\ \cdots & \cdots \\ \underset{\sim}{G}_{1n} & \underset{\sim}{G}_{2n} \cdots \underset{\sim}{G}_{nn} \end{pmatrix} \quad (1)$$

where n = number of regions and $\underset{\sim}{G}_{ij}$ are matrices with the structure of a Leslie matrix:

$$\underset{\sim}{G}_{ij} = \begin{pmatrix} 0 & 0 & b_{ij}(\alpha - 5) \ldots b_{ij}(\beta - 5) \ldots & & 0 \\ s_{ij}(0) & & & & \\ & s_{ij}(5) & & & \\ & & \ddots & & \\ & & & s_{ij}(Z - 5) & 0 \end{pmatrix} \quad (2)$$

$i,j = 1,n$;

α and β are the beginning and the end of the child-bearing interval;

$b_{ij}(x)$, $\alpha - 5 \leq x \leq \beta - 5$, is the average number of babies born during the 5-year projection period to an individual aged x to x + 4 in region i, who survive 5 years later in region j;

$s_{ij}(x)$ is the proportion of individuals aged x to x + 4 in region i, who migrate to region j and survive there to age x + 5 to x + 9 (survivorship proportions);

Z defines the last age group.

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1) An alternative arrangement has been suggested by Feeney (1970). There is no difference between the two notations from theoretical viewpoint, but it is believed that Feeney's notation is more suitable for computations, while Rogers' is better for analytical studies.

From the definition of $\underset{\sim}{G}_{ij}$ we may write

$$\underset{\sim}{G}_{ij} < \underset{\sim}{G}_{ii}, \quad i,j = 1,n, \; i \neq j \quad , \tag{3}$$

when the migration movements and the regional differences in fertility and mortality are reasonable,(i.e. can be those of an existing population).

Suppose that the population of each region is closed to migration. Then the growth process of a region's population can be defined by a Leslie matrix as described in (2). The multi-regional growth matrix in such a case will be reduced to a block-diagonal matrix (i.e. $\underset{\sim}{G}_{ij} = 0$, when $i \neq j$), which will be denoted by $\underset{\sim}{G}_r$:

$$\underset{\sim}{G}_r = \begin{pmatrix} \underset{\sim}{G}_1 & \underset{\sim}{O} \ldots \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{G}_2 \ldots \underset{\sim}{O} \\ \ldots & \ldots \\ \underset{\sim}{O} & \underset{\sim}{O} \ldots \underset{\sim}{G}_n \end{pmatrix} \tag{4}$$

where $\underset{\sim}{G}_i$ $(i = 1,\ldots,n)$ is the conventional growth matrix of the population in region i. $\underset{\sim}{G}_i$ differs from $\underset{\sim}{G}_{ii}$ because the latter includes the impact of outmigrations.

When corresponding elements from the matrices $\underset{\sim}{G}_r$ and $\underset{\sim}{G}$ are compared for their magnitude, it can be seen that they are very close to one another. For the three-regional example N. West - N. East - Sofia, the largest difference was found between the elements $S_{11}(15) = 0.9216$ from $\underset{\sim}{G}_{11}$, and $S_1(15) = 0.9953$ from $\underset{\sim}{G}_1$. Table 1 shows the elements of the matrix $\underset{\sim}{G}_1$ (single region N. West) and of the matrices $\underset{\sim}{G}_{11}$, $\underset{\sim}{G}_{12}$ and $\underset{\sim}{G}_{13}$, which give the spatial distribution in the projection process of N. West's population. Note that $G_1(x) \neq \sum_{j=1}^{3} G_{1j}(x)$ and $S_1(x) \neq \sum_{j=1}^{3} S_{1j}(x)$, because of the regional differences in fertility and mortality levels.

Table 1. Non-zero elements of the matrices $\underset{\sim}{G}_1$, $\underset{\sim}{G}_{11}$, $\underset{\sim}{G}_{12}$ and $\underset{\sim}{G}_{13}$.

AGE	$\underset{\sim}{G}1$	$\underset{\sim}{G}11$	$\underset{\sim}{G}12$	$\underset{\sim}{G}13$
		FIRST ROW		
0	0.000000	0.000000	0.000000	0.000000
5	0.000437	0.000427	0.000006	0.000010
10	0.097059	0.091326	0.001012	0.003419
15	0.350508	0.328785	0.002636	0.014555
20	0.384078	0.374780	0.001147	0.008435
25	0.172536	0.170106	0.000368	0.002465
30	0.055346	0.054705	0.000094	0.000646
35	0.015522	0.015362	0.000023	0.000143
40	0.003126	0.003096	0.000004	0.000025
45	0.000247	0.000244	0.000000	0.000002
50	0.000000	0.000000	0.000000	0.000000
55	0.000000	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000	0.000000
		SURVIVORSHIP PROPORTIONS		
	$\underset{\sim}{G}1$	$\underset{\sim}{G}11$	$\underset{\sim}{G}12$	$\underset{\sim}{G}13$
0	0.985527	0.970793	0.002519	0.012217
5	0.997767	0.983066	0.004361	0.010339
10	0.997146	0.946454	0.008857	0.041855
15	0.995262	0.921550	0.008332	0.065455
20	0.994103	0.948800	0.004034	0.041305
25	0.993757	0.973317	0.002366	0.018073
30	0.992076	0.981018	0.001232	0.009836
35	0.988773	0.982275	0.000628	0.005871
40	0.981470	0.977266	0.000382	0.003831
45	0.971743	0.968964	0.000238	0.002541
50	0.959684	0.957353	0.000180	0.002147
55	0.934410	0.931849	0.000172	0.002389
60	0.883338	0.880770	0.000115	0.002523
65	0.831057	0.828706	0.000074	0.002129
70	0.762239	0.760366	0.000036	0.001654
75	0.639006	0.637360	0.000000	0.001414
80	0.444137	0.442596	0.000000	0.000878

It can be seen that the largest differences are to be found in some of the survivorship proportions, but they are not greater than 10%. Thus we can speak of the elements of $\underset{\sim}{G}$ as the <u>perturbated</u> elements of $\underset{\sim}{G}_r$.

Perturbation theory studies the effects of small changes in some matrix elements on the eigenvalues and their corresponding eigenvectors. It is presented in detail, for instance, in Wilkinson (1965). A basic theorem which can be applied in the case of population growth matrices is the following (Wilkinson, 1964, p.67):

<u>Theorem</u>: If the absolute value of each element of the two matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is smaller than 1, and λ_i is a simple eigenvalue of $\underset{\sim}{A}$, there exists a simple eigenvalue $\lambda_i(\varepsilon)$ of the matrix $\underset{\sim}{A} + \varepsilon\underset{\sim}{B}$ which is given by a convergent power series, when ε is small ($\varepsilon > 0$):

$$\lambda_i(\varepsilon) = \lambda_i + k_1\varepsilon + k_2\varepsilon^2 + \cdots \quad , \tag{5}$$

or $\lambda_i(\varepsilon) \to \lambda_i$, when $\varepsilon \to 0$.

If $\{x_i\}$ is the eigenvector corresponding to λ_i, then

$$\{x_i(\varepsilon)\} = \{x_i\} + \{\ell_1\}\varepsilon + \{\ell_2\}\varepsilon^2 + \cdots \quad , \tag{5'}$$

or $\{x_i(\varepsilon)\} \to \{x_i\}$, when $\varepsilon \to 0$.

Bellman (1960) has shown that the coefficient k_1 from (5) can be estimated as

$$k_1 = (\{x_i\}, \underset{\sim}{B}\{x_i\}) \quad , \tag{6}$$

where $(\cdot,\cdot)$ denotes the inner (scalar) product of two vectors. The vector-coefficient $\{\ell_1\}$ from (5') can be presented as

$$\{\ell_1\} = \sum_{j \neq i} \frac{d_j\{x_j\}}{R_j - R_i} + d_i\{x_i\} \quad . \tag{6'}$$

An appropriate combination of d_j when $d_i = 0$ will give the vector $\{\ell_1\}$ as normalized, so that its length equals unity. Note that when $\{x_i\}$ is a real vector $\{\ell_1\}$ is also real, because the complex eigenvectors appear in conjugate pairs.

Following Bellman's techniques of estimation of k_1, k_2 may be estimated as

$$k_2 = (\{x_i\}, \underset{\sim}{B}\{\ell_1\}) \quad , \tag{7}$$

and $\{\ell_2\}$ is a linear combination similar to (6').

Note that Bellman's results as stated here are true for distinct eigenvalues (and distinct eigenvectors) only. This is not a restriction, however, because in practice the eigenvalues of matrices constructed for existing populations are always different.

This theorem can be applied in the case of the matrices $\underset{\sim}{G}_r$ and $\underset{\sim}{G}$. We shall define ε as the largest , in absolute value, difference between corresponding elements of $\underset{\sim}{G}_r$ and $\underset{\sim}{G}$. It was pointed out earlier that for the example considered $\varepsilon = S_1(15) - S_{11}(15) = 0.0757$. Surely, this value of ε can be treated as small, because powers of ε higher than 2 give values which should be practically negligible. It ought to be noted that this value of ε is quite large when compared with other multiregional systems, but even if it is twice as large, powers higher than 2 should also be negligible.

We can further define the matrix $\underset{\sim}{A}$ from the theorem to be the block-diagonal matrix $\underset{\sim}{G}_r$ from (2), and each element of $\underset{\sim}{B}$ can be presented as

$$b_{ij} = (g_{ij} - g_{ij}(r))/\varepsilon \quad , \tag{8}$$

Table 2. Nonzero elements of the three submatrices $\tilde{B}_{11}$, $\tilde{B}_{12}$ and $\tilde{B}_{13}$ of $\tilde{B}$.

	FIRST ROW		
	$\tilde{B}11$	$\tilde{B}12$	$\tilde{B}13$
0	0.000000	0.000000	0.000000
5	-0.000136	0.000081	0.000136
10	-0.077776	0.013729	0.046383
15	-0.294701	0.035761	0.197458
20	-0.126140	0.015561	0.114432
25	-0.032966	0.004992	0.033441
30	-0.008696	0.001275	0.008764
35	-0.002171	0.000312	0.001940
40	-0.000407	0.000054	0.000339
45	-0.000041	0.000000	0.000027
50	0.000000	0.000000	0.000000
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

	MAIN SUBDIAGONAL		
	$\tilde{B}11$	$\tilde{B}12$	$\tilde{B}13$
0	-0.199886	0.034174	0.165740
5	-0.199438	0.059163	0.140262
10	-0.687704	0.120157	0.567818
15	-1.000000	0.113035	0.887983
20	-0.614594	0.054727	0.560357
25	-0.277295	0.032098	0.245184
30	-0.150016	0.016714	0.133438
35	-0.088154	0.008520	0.079648
40	-0.057032	0.005182	0.051973
45	-0.037701	0.003229	0.034472
50	-0.031623	0.002442	0.029127
55	-0.034743	0.002333	0.032410
60	-0.034838	0.001560	0.034228
65	-0.031894	0.001004	0.028883
70	-0.025409	0.000488	0.022439
75	-0.022331	0.000000	0.019183
80	-0.020909	0.000000	0.011911

where b_{ij} is an element of $\underset{\sim}{B}$, g_{ij} of $\underset{\sim}{G}$, and $g_{ij}(r)$ of $\underset{\sim}{G}_r$. Obviously, $\underset{\sim}{G}_r + \varepsilon \underset{\sim}{B} = \underset{\sim}{G}$.

Theoretically, the maximum absolute value of the elements of B is equal to unity. In practice, only one element (this is $b_{11}(15)$ in the example of Table 2) reaches the maximum because the migration schedules have a high peak situated somewhere in the 15-30 year age interval (Rogers, Raquillet, Castro, 1977), and because there usually exists a dominating outmigration flow among the regions. In the example considered, the strongest outmigration rates are exhibited for the N. West region, and the peak of the migration schedule is in the 15-20 year age interval.

Table 2 gives the elements of $\underset{\sim}{B}$ that correspond to $\underset{\sim}{G}_{11}$, $\underset{\sim}{G}_{12}$ and $\underset{\sim}{G}_{13}$; these are denoted by $\underset{\sim}{B}_{11}$, $\underset{\sim}{B}_{12}$ and $\underset{\sim}{B}_{13}$. Two elements only, situated next to the one with maximum absolute value, are close by magnitude to the maximum one, namely $b_{11}(10)$ and $b_{11}(20)$. All the remaining elements are much smaller.

Following these remarks, an estimation of (5) and (5') will be made by taking advantage of the demographic properties of the notions considered.

An upper limit of the absolute value $|(\{x_i\},\underset{\sim}{B}\{x_i\})|$, where $\{x_i\}$ is an eigenvector of $\underset{\sim}{G}_r$, can be found in the following way. First, we evaluate $\underset{\sim}{B}\{x_i\}$. Recalling the construction of $\underset{\sim}{G}_r$, the coordinates of $\{x_i\}$ can be divided into a number of subsets, equal to the number of regions, so that the j-th subset is the only one with non-zero elements. Denoting the j-th subset of the i-th eigenvector by a bar, for instance $\{\bar{x}_i\}$, we may write:

$$\overline{\underset{\sim}{B}\{x_i\}} = \underset{\sim}{B}_{jj}\{\bar{x}_i\} \quad .$$

The elements of $\{x_i\}$ are all nonnegative and not greater than unity, since the vector is given to be normalized. The element of highest absolute value of $\underset{\sim}{B}_{jj}$ is equal to unity, say, and the elements from the first row are all very small (Table 2). Hence,

$$|\underset{\sim}{B}_{jj}\{\bar{x}_i\}| \leq |\{\bar{x}_i\}| \quad ,$$

taking Euclidean norms of vectors. Because $(\{x_i\},\{x_i\}) = 1$, we derive

$$|(\{x_i\},\underset{\sim}{B}\{x_i\})| = |(\{\bar{x}_i\},\overline{\underset{\sim}{B}\{x_i\}})| = |(\{\bar{x}_i\},\underset{\sim}{B}_{jj}\{\bar{x}_i\})|$$

$$\leq |(\{\bar{x}_i\},\{\bar{x}_i\}) = 1 \quad . \qquad (9)$$

Assuming the vector-coefficient $\{\ell_1\}$ from (6') to be normalized, the same result can be obtained for k_2 from (7). Then, the series (5) can be represented as follows:

$$|\lambda_i(\varepsilon) - \lambda_1| \leq |k_1|\varepsilon + |k_2|\varepsilon^2 \quad ,$$

because the powers of ε higher than 2 give very small magnitudes. Introducing for $|k_1| \leq 1$, and $|k_2| \leq 1$,

$$|\lambda_i(\varepsilon) - \lambda_i| \leq \varepsilon + \varepsilon^2 \quad . \qquad (10)$$

The same inequality can be derived for the eigenvectors from (5'):

$$|\{x_i(\varepsilon)\} - \{x_i\}| \leq \varepsilon + \varepsilon^2 \quad . \qquad (10')$$

The inequalities (10) and (10') are the basic result which will be used for further discussions. They show how the eigenvalues and the eigenvectors of the multiregional growth matrix depend on the magnitude of the migration flow. They show also that with diminishing the migrations, i.e. when $\underset{\sim}{G}$ converges to $\underset{\sim}{G}_r$, the eigenvalues and eigenvectors of $\underset{\sim}{G}$ converge to those of $\underset{\sim}{G}_r$. That is why (10) and (10') will be referred to in this paper as the theorem for the consistency of the multiregional and the single-regional growth matrices.

In Section II the consistency of some demographic notions will be studied. Before that, some inferences from (10) and (10') are necessary.

1. The series (10) can be deduced without using the modulus sign. For, when λ_i is real, then $\{x_i\}$ is real too, and the modulus signs introduced in (9) do not convert complex numbers into real. Therefore, (10) will give an upper limit for $\lambda_i(\varepsilon)$:

$$\lambda_i(\varepsilon) \leq \lambda_i + \varepsilon + \varepsilon^2 \quad .$$

This shows that when λ_i is real, $\lambda_i(\varepsilon)$ is necessarily real too. This result is shown also by the series (5). Studies of the single-regional growth matrix show that there exists exactly one positive eigenvalue (see for instance Keyfitz, 1968, Chapter 3.2). Therefore, $\underset{\sim}{G}$ has at least n positive eigenvalues (n = number of regions). The definition of $\{\ell_1\}$ in equation (6') suggests also that when x_i is a vector with complex elements, then $\{\ell_1\}$ is also complex. Then the series (5) shows that when λ_i is complex, one can expect $\lambda_i(\varepsilon)$ to also be complex. This inference can be proved theoretically but the proof is omitted here because a more sophisticated notation has to be introduced.

Briefly, the number of positive eigenvalues of the multiregional growth matrix is equal to the number of regions. Since they are "residuals" of the single-regional values, their magnitude will be close to unity. As for the eigenvectors of $\underset{\sim}{G}$, it may be inferred that the number of real eigenvectors is equal to the number of regions, exactly one being positive. This is a result of the famous Frobenius theorem (see for instance Gantmacher, 1959, vol. II). Real eigenvectors appear which correspond to negative eigenvalues, but they are not considered here.

The same inferences hold for the complex eigenvalues and their conjugates, which determine the length of the population wave in the single-regional case: their number in the multiregional case is equal to the number of regions. Later in this paper they will be referred to as the <u>dominant complex eigenvalues</u>. The eigenvalues of $\underset{\sim}{G}_r$ and $\underset{\sim}{G}$ for the example are presented in Table 3.

It appears to be quite difficult to state which eigenvalues of $\underset{\sim}{G}$ correspond to a specific region, because there is no criterion to be included in the computer programs. One way to solve such a problem is to simulate increasing migration flows from zeros (the case of $\underset{\sim}{G}_r$) to the observed ones (the case of $\underset{\sim}{G}$) and to follow the changes of the eigenvalues. In practice, this is not necessary.

2. The inequalities (10) and (10') show that the multiretional growth matrix is well-conditioned, i.e. small changes in its elements lead to small changes in its eigensystem.

Table 3. Eigenvalues of the single-region matrices* $\underset{\sim}{G}_1$, $\underset{\sim}{G}_2$, $\underset{\sim}{G}_3$ and of the three-regional matrix* $\underset{\sim}{G}$.

EIGENVALUES OF:		
G1 N.WEST	G2 N.EAST	G3 SOFIA
REAL IMAG.	REAL IMAG.	REAL IMAG.
1.01084	1.02900	0.98501
0.21446±0.79294	0.23516±0.79771	0.28113±0.73587
-0.43907±0.30493	0.03948±0.29029	-0.39118±0.31577
0.05208±0.29571	-0.39795±0.29338	-0.04660±0.40382
-0.25805±0.14343	-0.30238±0.20954	-0.14965±0.07866
-0.14967	-0.17762	-0.37242
EIGENVALUES OF G**		
0.99014	1.02245	0.95558
0.27464±0.72607	0.20854±0.76239	0.23357±0.79177
-0.04426±0.39905	0.05562±0.29104	0.03997±0.28974
-0.42465±0.29533	-0.38774±0.31121	-0.39483±0.29030
-0.24988±0.14482	-0.30068±0.21014	-0.14822±0.07816
-0.36738	-0.14787	-0.17709

*The population is considered until the end of the reproduction period, which is not a restriction because $\underset{\sim}{G}_{ij}$ and $\underset{\sim}{G}$ turn out to be indecomposable.

**The eigenvalues of $\underset{\sim}{G}$ and not distributed among the regions.

Therefore, slight changes in the demographic variables (for instance, population in the middle or at the end of the year) have a minor effect on results obtained through the eigensystem--the intrinsic growth rate, the stable population, the stable regional shares, the reproductive value, etc.

3. Wilkinson and Bellman have shown that series similar to (5) and (5') holds for multiple eigenvalues of $\underset{\sim}{A}$, too. Thus the problem of multiple eigenvalues of the multiregional growth matrix is of secondary importance: by slightly changing some of its elements, the eigenvalues will be simple and (5) and (5') will then hold.

4. The inequalities (10) and (10') can be called the theorem for consistency of the multiregional and the single-region model, because by diminishing the migrations ($\varepsilon \to 0$), the behavior of the regional populations can be studied. For instance, when $\varepsilon \to 0$, (10) shows that each eigenvalue of $\underset{\sim}{G}$ converges to an eigenvalue of $\underset{\sim}{G}_r$. Where the dominant unique positive eigenvalue is concerned, it can be seen that it converges to the dominant eigenvalue of some region. The same is true of the unique positive eigenvector - it defines the stables populations, i.e. the multiregional stable population converges to the stable population of the same region. Therefore, this region can be called dominant in the multiregional system as it determines the long-run behavior of the whole system.

The multiregional stable population, which is defined with the positive eigenvector of $\underset{\sim}{G}$, does not converge to the stable population of each region separately but to the stable population of one single region only. This is due to the spatial properties of the multiregional system. In Section II it will be shown that the multiregional stable and the stable equivalent populations are defined by the spatial distribution of the population of the dominant region.

Theoretically, it is difficult to define the dominant region except when simulations are carried out. The empirical results have shown that the dominant region should be the one with the greatest stable growth ratio in the system of single regions. If there are two or more regions with approximately the same growth ratio, then the migration movements will help to define the dominant one (the region with the lowest outmigration flow should be favoured). For instance, consider the biregional system of the urban-rural population of Bulgaria. For the observed population in 1975, the urban share is 52% of the total, but its stable growth ratio is less than unity and that of the rural is greater than unity. Under conditions of stability, the share of the urban population is near 80%, but the dominant region will be the rural one because it will supply both regions with its population. This has been proved by simulations, too. Note that the rural-urban migration flow is much stronger than the counterflow.

Simulations have shown that the dominant region in the system N. West - N. East - Sofia is the second one. Its single-region dominant eigenvalue is 1.02900 (Table 3), which drops to 1.02245 in the three-regional case. The difference between the two values is small because the migration flows to and from this region are small.

In Section II the effect of the eigenvalues on the population path towards stability will be studied.

II. PHASES OF THE MULTIREGIONAL POPULATION PROJECTION

The consistency theorem shows that the number of positive eigenvalues of $\tilde{G}$ is equal to the number of regions. It shows also that there is this same number of pairs of complex eigenvalues whose real and imaginary parts are of approximately equal magnitude. One should expect that these should effect the population waves and the regional distribution. The analysis of this problem is the topic of this section.

II.1. Linear Decomposition of the Observed Age Distribution

The growth matrix $\underset{\sim}{G}$ for the system N. West - N. East - Sofia has 3 positive, 3 negative and 24 complex and conjugate eigenvalues, presented in Table 3. Since these are different, the eigenvectors will be linearly independent. Then each 30-dimensional vector can be presented as a linear combination:

$$\{k_0\} = c_1\{k_1\} + c_2\{k_2\} + \cdots + c_{30}\{k_{30}\} \quad , \qquad (11)$$

where $\{k_i\}$ is the i-th right eigenvector-column, and c_i are constants ($i = 1,30$).

Suppose $\{k_0\}$ is a column-vector of the observed population. It can be presented as

$$\{k_0\} = \begin{Bmatrix} k_0^1 \\ k_0^2 \\ k_0^3 \end{Bmatrix} \qquad (12)$$

where $\{k_0^i\}$, $i = 1,2,3$, is a column-vector of the population in the i-th region.

Equation (11) can be represented as

$$\begin{Bmatrix} k_0^1 \\ k_0^2 \\ k_0^3 \end{Bmatrix} = c_1 \begin{Bmatrix} k_1^1 \\ k_1^2 \\ k_1^3 \end{Bmatrix} + \cdots + c_{30} \begin{Bmatrix} k_{30}^1 \\ k_{30}^2 \\ k_{30}^3 \end{Bmatrix} \qquad (13)$$

It is necessary to find the constants c_i. It can be easily proved that, if $[H_i]$ is the i-th row-eigenvector, $([H_i]\{k_j\}) = 0$, when $i \neq j$. Multiplying (11) on the left by $[H_i]$ gives

$$([H_i]\{k_0\}) = c_i([H_i]\{k_i\}) \quad ,$$

from where

$$c_i = \frac{([H_i]\{k_0\})}{([H_i]\{k_i\})} \quad .$$

Note that when $\{k_1\}$ is the unique positive eigenvector $([H_1]\{k_0\})$ represents the total reproductive value, and $[H_1]$ is the spatial distribution of the reproductive potential of the observed population (Willekens, 1977).

When the vectors $\{k_i\}$ are normalized, and the left eigenvectors are computed as the inverse of the matrix $[\{k_i\}]_{i=1}^{30}$ (i.e. the left eigenvectors are placed one next to the other), $([H_i]\{k_i\}) = 1$, from which

$$c_i = ([H_i]\{k_0\}) \quad . \tag{14}$$

Equation (13) can be rewritten for each region separately. Substituting for c_i from (14) gives for normalized vectors: $\{k_i\}$ and $[H_i]$:

$$\left.\begin{aligned}
\{k_0^1\} &= \sum_{i=1}^{30} ([H_i]\{k_0\})\{k_i^1\} \quad , \\
\{k_0^2\} &= \sum_{i=1}^{30} ([H_i]\{k_0\})\{k_i^2\} \quad , \\
\{k_0^3\} &= \sum_{i=1}^{30} ([H_i]\{k_0\})\{k_i^3\} \quad .
\end{aligned}\right\} \tag{15}$$

The constants c_1 to c_9, which will be used for further discussions, are presented in Table 4.

Table 4. Constants c_1 to c_9 estimated from (14).

c_1 = 1346452	$c_{4,5}$ = 129104 ± 101234.i
c_2 = 1533637	$c_{6,7}$ = 11363 ± 8711.i
c_3 = -211645	$c_{8,9}$ = 42999 ± 208.i

Equation (12) will be used to study the projections of the three populations. Namely, multiplying on the left with the three-regional growth matrix $\underset{\sim}{G}$, and considering $\underset{\sim}{G}\{k_i\} = \lambda_i\{k_i\}$ gives

$$\underset{\sim}{G}\{k_0\} = c_1\lambda_1\{k_1\} + \cdots + c_{30}\,\lambda_{30}\{k_{30}\} \quad .$$

Multiplying t times with $\underset{\sim}{G}$:

$$\underset{\sim}{G}^t\{k_0\} = \sum_{i=1}^{30} c_i\lambda_i^t\{k_i\} \quad .$$

The last equation can be presented as in (15):

$$\left.\begin{aligned} \{k_0^1(t)\} &= \sum_{i=1}^{30} c_i\lambda_i^t\{k_i^1\} \quad , \\ \{k_0^2(t)\} &= \sum_{i=1}^{30} c_i\lambda_i^t\{k_i^2\} \quad , \\ \{k_0^3(t)\} &= \sum_{i=1}^{30} c_i\lambda_i^t\{k_i^3\} \quad . \end{aligned}\right\} \qquad (16)$$

The equations (16) can be used to project multiregional populations. They will be used further by separating the components on the right side into three groups: first, the quantities due to the three positive eigenvalues; second, those due to the six dominant complex eigenvalues; and third, those due to the remaining eigenvalues. The computer results are presented in Table 5, the discussion of which is the topic of the next section.

II . 2. Effect of the Eigenvalues and Eigenvectors on the Population Projection

Table 5 presents the equations (16) at different time periods. When t = 0, the decomposition of the observed population is presented as expressed in (15). The first three columns represent the quantities $c_i \lambda_i^t \{k_i^j\}$, where i,j = 1,2,3, and i denotes the positive eigenvalues, j denotes the region. The second three columns represent the quantities $c_i \lambda_i^t \{k_i^j\} + \bar{c}_i \bar{\lambda}_i^t \{k_i^j\}$, which are due to the i-the dominant complex eigenvalue and its conjugate. The last three columns give the quantities: $\sum_{i=1}^{g} c_i \lambda_i^t \{k_i^j\}$, $\sum_{i=1}^{3} c_i \lambda_i^t \{k_i^j\}$, and $\sum_{i=1}^{30} c_i \lambda_i^t \{k_i^j\}$. All the numbers are presented for aggregated-by-age populations (the age interval is 50 years). The length of the time period is five years.

Table 5 makes it possible to trace the effect of the eigenvalues on the population projection. When columns 7 and 9 are compared, it can be seen that the effect of the secondary eigenvalues (i.e. λ_{10} to λ_{30}) is negligible after six-seven periods of projection.

The first three columns represent the effect of the positive eigenvalues. The quantities due to these are the largest in size of population. The quantities due to λ_1 and λ_3 (columns 1 and 3) are constantly decreasing, as they are less than unity, while those due to λ_2 (the dominant eigenvalue, column 2) are increasing.

The quantities due to λ_2 at time t = 0 represent the size of the stable equivalent population of each region (Rogers, 1976).

Table 5. Quantities due to the eigenvalues.

TIME	λ1	λ2	λ3	λ4,L5	λ6,λ7	λ8,λ9	∑λ1,λ9	∑λ1,λ3	∑λ1,λ30
REGION N,WEST									
T= 0	463718,	112937,	70747,	-7716,	4972,	622,	645280,	647402,	680228,
T= 1	459146,	115473,	67604,	-5940,	742,	-704,	636321,	642222,	656994,
T= 2	454619,	118065,	64601,	1388,	-2798,	-752,	635122,	637285,	638699,
T= 3	450136,	120715,	61731,	4342,	-1630,	128,	635422,	632583,	633127,
T= 4	445698,	123425,	58989,	1548,	1068,	572,	631301,	628113,	633227,
T= 5	441303,	126196,	56369,	-1766,	1464,	180,	623747,	623869,	622145,
T= 6	436952,	129029,	53865,	-1902,	-56,	-306,	617581,	619847,	617984,
T= 7	432644,	131926,	51472,	18,	-938,	-266,	614857,	616042,	615046,
T= 8	428378,	134888,	49186,	1156,	-356,	84,	613337,	612452,	613169,
T= 9	424154,	137916,	47001,	624,	438,	220,	610353,	609071,	610400,
REGION N,EAST									
T= 0	-145255,	1232124,	59,	-2116,	-476,	11854,	1096189,	1086928,	1112681,
T= 1	-143823,	1259786,	57,	-514,	154,	-6214,	1109443,	1116019,	1135420,
T= 2	-142405,	1288068,	54,	994,	362,	-10982,	1136091,	1145717,	1155649,
T= 3	-141001,	1316985,	52,	856,	56,	-894,	1176052,	1176036,	1187696,
T= 4	-139611,	1346551,	49,	-128,	-202,	7066,	1213724,	1206990,	1217826,
T= 5	-138234,	1376781,	47,	-586,	-118,	3910,	1241799,	1238594,	1239253,
T= 6	-136871,	1407690,	45,	-244,	78,	-2988,	1267709,	1270864,	1265552,
T= 7	-135521,	1439292,	43,	218,	106,	-4060,	1300079,	1303814,	1301675,
T= 8	-134185,	1471604,	41,	268,	-4,	140,	1337864,	1337460,	1337830,
T= 9	-132862,	1504642,	39,	16,	-68,	2832,	1374598,	1371819,	1374062,
REGION SOFIA									
T= 0	737479,	188591,	-140558,	4988,	1410,	1542,	793453,	785513,	809389,
T= 1	730207,	192825,	-134314,	-62750,	1494,	-52,	727412,	788719,	814643,
T= 2	723007,	197154,	-128348,	-37472,	-258,	-1076,	753008,	791814,	819374,
T= 3	715878,	201580,	-122647,	17230,	-1042,	-468,	810534,	794812,	825330,
T= 4	708820,	206106,	-117199,	32046,	-272,	514,	830014,	797727,	822986,
T= 5	701831,	210733,	-111993,	7218,	536,	558,	808885,	800571,	804337,
T= 6	694911,	215464,	-107018,	-15346,	394,	-90,	788316,	803357,	788716,
T= 7	688059,	220301,	-102264,	-12778,	-170,	-422,	792724,	806096,	794497,
T= 8	681275,	225247,	-97722,	2228,	-318,	-136,	810574,	808800,	810315,
T= 9	674558,	230303,	-93381,	8924,	-26,	224,	820603,	811480,	819847,

When the observations above are compared with similar observations for a single-region projection (for instance, Keyfitz, 1968, chapter 3), it can be seen that both in the multiregional and in the single-regional population projections, the secondary eigenvalues cause waves with a short length, and therefore their influence disappears within a few time periods. There are differences, however, when the positive and the complex dominant eigenvalues are concerned as their number is equal to the number of regions.

II.2.1. Complex Dominant Eigenvalues

In the single-regional theory, it is the greatest (in absolute value) complex eigenvalue which determines the waves due to the age distribution of the observed population. Usually logarithms are taken and the formula of De Moivre is then used:

$$(u + iv)^t = e^{5(x+iy)t} = e^{5x}(\cos 5ty + i\sin 5ty) \quad , \quad (17)$$

where $u+iv$ is a complex eigenvalue, and 5 denotes the 5-year time period of projection.

X and y are estimated by:

$$e^{5x} = \sqrt{u^2 + v^2} \quad , \qquad 5y = \mathrm{arctg}\ \frac{v}{u} \quad .$$

When t increases by $2\pi/y$, cos 5yt and sin 5yt go back to the initial point, so $2\pi/y$ is the length of the wave, caused by u + iv. The damping of the wave is given by e^{5xt}, because in practise x is always negative.

According to the consistency theorem, in the multiregional case there exist exactly n pairs (n = number of regions) of complex conjugate eigenvalues of approximately the same absolute value. The empirical results suggest that when the real or the imaginary parts are compared separately, they are approximately of the same value, too (see Table 3). Each of these eigenvalues will cause a wave with similar lengths and magnitudes.

Table 5 shows that a unique complex eigenvalue, together with its conjugate, determines the wave for the population of the N. East region (λ_8 and λ_9) or of the Sofia region (λ_4 and λ_5). This is not the case for the N. West region, the wave of which is determined by two pairs: λ_4, λ_5 and λ_6, λ_7 (Table 5).

The empirical results show that, in general, the waves of a regional population are represented as the sum of several periodic motions. The theory of vibrations suggests that the sum of several periodic motions is a periodic motion too, when the parameters specifying the vibrations are rational numbers. In the multiregional model this clearly is the case, because the eigenvalues are estimated until the 6th decimal place, say. There are difficulties, however, due to the fact that the new vibration is periodic in the sense that a finite aperiodic motion is repeating itself. For the N.West region, the first three lengths of the wave are 25, 23 and 27.6 years long, respectively.

The behavior of the N. West's population is the most general in multiregional studies. Therefore, the waves will usually be aperiodic as we are concerned with the first few periods only. Unlike the single-region case (Keyfitz, 1968), it cannot be stated that the length of the wave caused by the dominant complex eigenvalue and its conjugate is equal to the mean length of generation. Moreover, the simulation approach appears to be the only practical way to find the waves of a regional population.

However, some rough approximations can be made. The simulations have shown that λ_4 is a "residual" of the dominant complex eigenvalue of region Sofia, and λ_8 is of N. East. Moreover, the migration flow from N. West to Sofia is much stronger than the other flows in the system. It can be stated that if the migration flow from region i to region j and/or k is strong, the wave of the population of region i can be determined as the sum of the waves of the populations in i, j and/or k. The length of the summed wave can be taken as equal to the arithmetic mean of the two (or three, etc.) lengths. If the migration flows from region i to all the other regions are small, then its wave can be defined by one eigenvalue, as in the single-region case.

II.2.2. Positive Eigenvalues and Eigenvectors

The number of positive eigenvalues, as pointed out earlier in this paper, is equal to the number of regions, and their magnitude is close to unity. The case when at least one of them is greater than unity will be discussed here.

Exactly one eigenvector is positive; all the others have negative elements. Recalling that each eigenvector corresponding to a positive eigenvalue converges toward some single region's dominant positive eigenvector when the migration flows are diminished, it can be observed that the elements which correspond to that same region have to be positive, and the negative elements correspond to some other regions. This is due to the fact that the perturbations are very small, as shown by (5') and (10') in Section I.

The three real eigenvectors for the system N. West - N. East - Sofia are presented in Table 6. According to the above observations, it can be traced that the first eigenvector is the perturbed dominant eigenvector of N. West, the second of N. East, and the third of Sofia.

In the three-regional case the vector-function, which reflects the growth of the population of region i due to the positive eigenvalues and their corresponding eigenvectors, is the following one:

$$\{f(t)\} = c_1\lambda_1^t\{k_1^i\} + c_2\lambda_2^t\{k_2^i\} + c_3\lambda_3^t\{k_3^i\} \quad . \tag{18}$$

When the elements of the vectors in (18) are summed, f(t) will describe the trajectory of the total regional population. The trajectories of the three regions from the example are presented in Figure 1. These reflect some of the most common growth processes which can appear in practice.

Table 6. Eigenvectors corresponding to the positive eigenvalues.

EIGENVECTOR CORRESPONDING TO λ_1 *)

0.03669	0.03654	0.03670	0.03553	0.03364	0.03295	0.03297	0.03310	0.03322	0.03306
-0.01248	-0.01208	-0.01184	-0.01132	-0.01072	-0.01030	-0.01005	-0.00987	-0.00970	-0.00952
0.05298	0.05242	0.05262	0.05384	0.05561	0.05630	0.05637	0.05628	0.05594	0.05536

EIGENVECTOR CORRESPONDING TO λ_2 *)

0.00801	0.00784	0.00782	0.00775	0.00749	0.00725	0.00711	0.00697	0.00681	0.00659
0.09378	0.08991	0.08748	0.08443	0.08107	0.07830	0.07591	0.07349	0.07096	0.06807
0.01210	0.01174	0.01154	0.01201	0.01290	0.01315	0.01292	0.01261	0.01223	0.01177

EIGENVECTOR CORRESPONDING TO λ_3 *)

-0.03551	-0.03536	-0.03582	-0.03483	-0.03279	-0.03167	-0.03155	-0.03185	-0.03225	-0.03264
-0.00017	-0.00006	-0.00007	-0.00022	-0.00028	-0.00014	-0.00001	0.00011	0.00024	0.00032
0.06303	0.06363	0.06535	0.06582	0.06523	0.06539	0.06654	0.06814	0.06975	0.07124

*) Each vector is presented as transposed, i.e. the coordinates are presented in rows. The second row is a continuation of the first, and the third one is a continuation of the second. Therefore, the i-th row (i = 1,2,3) gives the coordinates which correspond to the i-th region, because Rogers' notation is used.

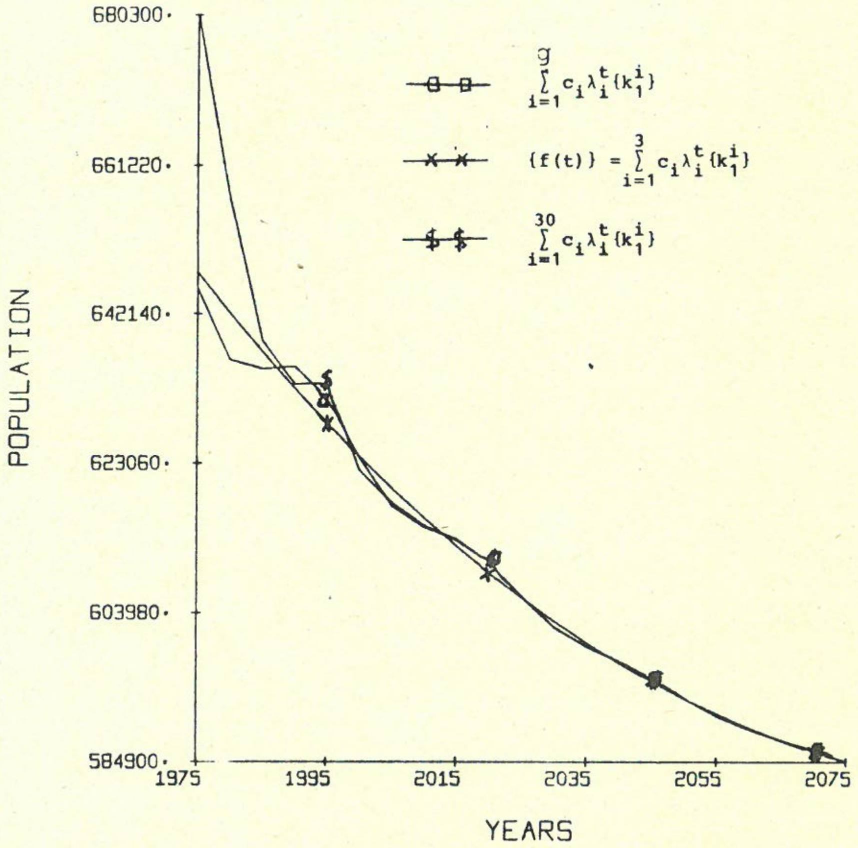

Figure 1a. Region N. West.

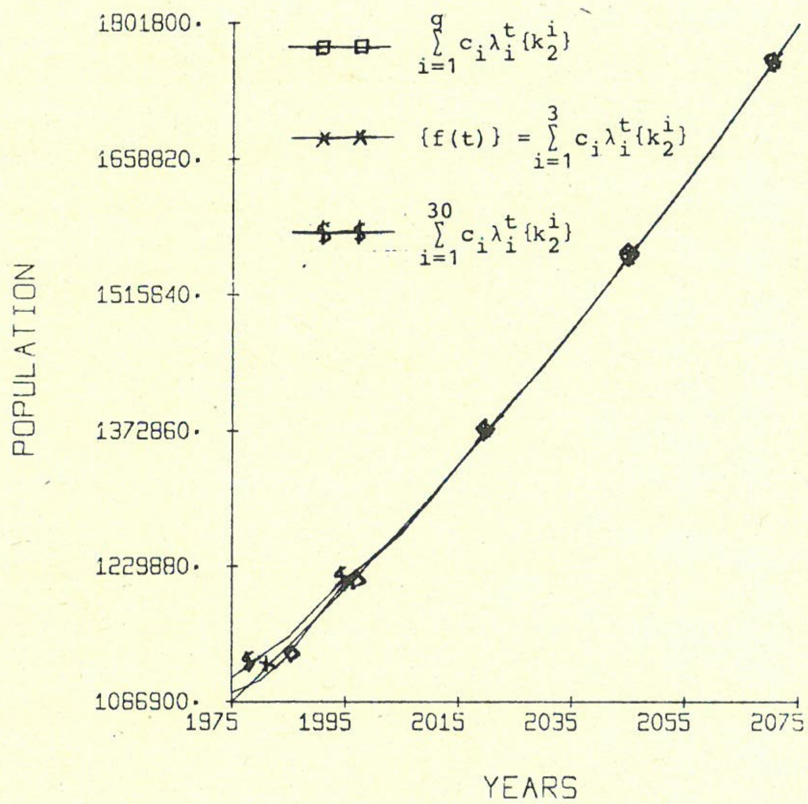

Figure 1b. Region N. East.

Figure 1. Trajectories of the population of three regions of Bulgaria.

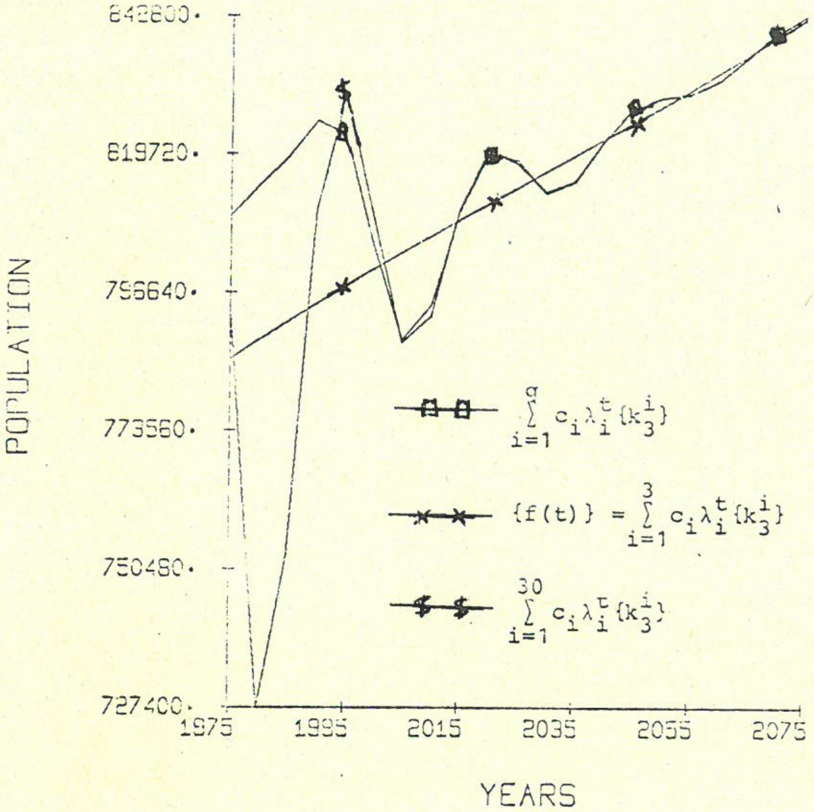

Figure 1c. Region Sofia.

The trajectory of the N. West region is typical for a region exhibiting high outmigration rates. The trajectory of the N. East region is typical for a region with a high fertility level, and low in and outmigration rates. That of Sofia is typical for a region with fertility below the replacement level and high in-migration rates.

In practice, more than one positive eigenvalue can be greater than unity. When the number of births in the three-regional example were increased 1.5 times (which causes a rise of the crude birth rate from about 18‰ to 27‰, i.e. to a completely reasonable value), the three positive eigenvalues were all greater than unity.

If more than one positive eigenvalue is greater than unity, the growth of any regional population will depend on them in a diverse way, since (18) shows that the signs of the constants and of the elements of the eigenvectors determine the population growth, too. When the births in the example were increased,

$\lambda_3 > 1$ not being the dominant eigenvalue, c_3 was negative and $\{k_3^1\}$ had negative elements. Therefore, the quantity $c_3\lambda_3^t\{k_3^1\}$ should increase when t is increasing. $\{k_3^3\}$ had positive elements however. Therefore, $c_3\lambda_3^t\{k_3^3\}$ should decrease. Hence, λ_3 will cause a constant increase for one part of the population of N. West, and a constant decrease of the population of Sofia.

It appears quite difficult to state exactly when the positive nondominant eigenvalue greater than unity will increase or decrease the population of a specific region. The practical results show that the eigenvalue will decrease the regional population when it is one of the smallest in magnitude, since the eigenvector will have a large number of negative elements. This is proved by the results in Table 6.

When the eigenvalue tends to increase a regional population, its effect will not die away in the long run. In such cases, the trajectory will be of the kind represented in Figure 1b. It can usually be observed when the fertility level is very high and the outmigration level is low.

The trajectory (18) is evidently characteristic of the multiregional case only. In the single-regional case, it should describe the stable growth of the population. It is evidently due to the migrations which cause perturbations in the eigenvectors, so that each region is affected by a number of eigenvalues. Therefore, (18) describes a specific phase of the multiregional growth process, which is closely connected with the spatial properties of the model. It begins right from the start of the projection and continues until stabilization takes place. It is obvious that this phase affects the age distribution, too.

The three phases of the multiregional growth process can then be outlined as follows:

1. Stabilization of the single-region age distribution. During this phase, discrepancies in the age composition disappear. It is described by the dominant complex eigenvalues which cause aperiodic population waves. This phase is long as in the single-region case.

2. Stabilization of the spatial distribution. During this phase, the regional shares suggested by the positive eigenvector are reached. At the same time, the age composition continues to change smoothly according to the quantities $c_i\lambda_i^t\{k_i^j\}$, i.e. according to the magnitude of $c_i\lambda_i^t\{k_i^j\}$, and to the migration schedule of $\{k_i^j\}$, $i \neq j$. Note that in the absence of migrations, $\{k_i^j\} = \{0\}, i \neq j$.

The changes of the age structure which appear during this phase come sometimes to an age structure which is not typical for the single-region case. Figure 2 gives the stable age structure of the Sofia region from a seven-region study of Bulgaria (Philipov, 1978). This kind of age composition can be observed for regions with high inmigration rates and low fertility level.

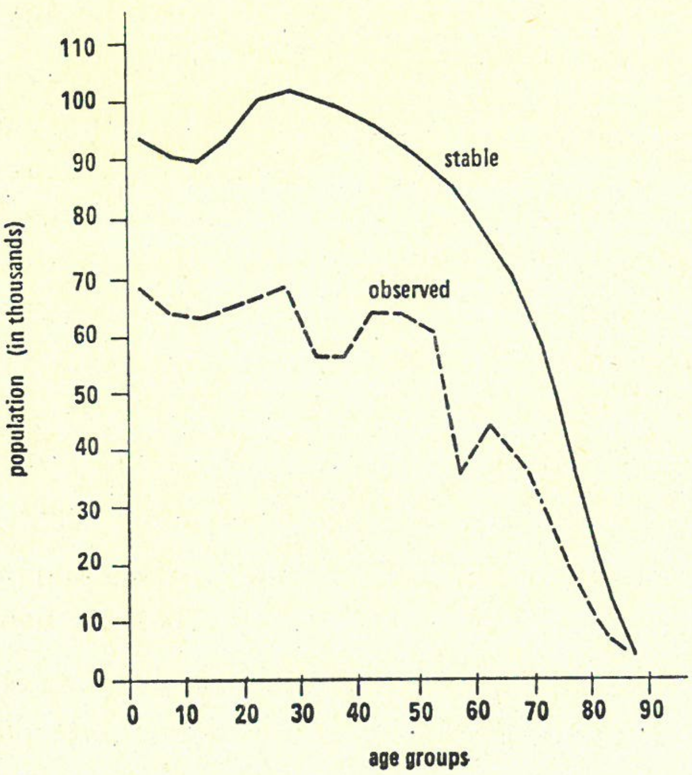

Figure 2. Observed (1975) and stable equivalent population of Region Sofia.

Source: Philipov (1978).

The length of this phase is strongly dependent on the strength of the migration movements. The weaker the moves are, the longer the phase is, because more time will be necessary to extinguish the quantities due to the nondominant positive eigenvalues. Practically, the stage is 500-1000 years long.

3. Stable age and spatial distribution. It is outlined by the positive eigenvector. This phase is easily studied with the stable equivalent population, which is the topic of the next section.

II.3. Stable Equivalent Population

This is the population with a constant age and spatial distribution, the projection of which will in the long run give the same numbers as the observed initial population. Its growth ratio is given by the unique dominant eigenvalue, and the age and spatial distribution is given by the unique positive eigenvector (λ_2 and $\{k_2\}$, say). The number of people is given by $c_2\{k_2\}$, where c_2 is from (14).

It has been pointed out before that in the absence of migration, the vector $\{k_2^2\}$ gives the stable population of region N. East, and $\{k_2^1\}$ and $\{k_2^3\}$ would be equal to $\{0\}$. When migration movements are introduced, $\{k_2\}$ will be subject to perturbations. The smaller the migrations are, the smaller the perturbations will be, and the less $\{k_2^1\}$ and $\{k_2^3\}$ will differ from $\{0\}$. Table 5 shows that the observed population of the N. East region in 1975 is 680,228, while the stable equivalent population is equal to 112,937, or six times smaller. When the migration movements are diminished by a half by simulation, the stable equivalent will be 64,152, and diminishing them by ten gives 15,298 - a very unreasonable number.

It can be shown, by making use of (5'), that the outmigration from the dominant region determines the stable population of the other regions. Use will be made of Bellman's presentation of the constants d_j from (6'):

$$d_j = -(\{x_j\}, \underset{\sim}{B}\{x_i\}) \quad .$$

By λ_{ij} and $\{k_{ij}\}$ will be denoted the j-th eigenvalue and the j-th eigenvector of the i-th matrix of $\underset{\sim}{G}_r$ from (4). Suppose the second region to be the dominant one. Then (6') can be re-written for the first region as:

$$\{k_2^1\} = \{k_{21}^1\} + \varepsilon \sum_{j=1}^{10} \frac{(\{k_{ij}\}, \underset{\sim}{B}\{k_{21}\})\{k_{ij}^1\}}{\lambda_{ij} - \lambda_{21}} \quad ,$$

where $\{k_{ij}^1\}$ is the part of $\{k_{ij}\}$ which corresponds to region 1. Obviously, $\{k_{21}^1\} = \{0\}$. Therefore, the magnitude of $\{k_2^1\}$ - the stable equivalent population of region 1 when region 2 is dominant - will depend on ε and on $d_j = (\{k_{ij}\}, \underset{\sim}{B}\{k_{21}\})$, which reflect the strength of the migration flows. The magnitude of the constant d_j depends only on the migrations from region 2 to region 1, i.e. on $\underset{\sim}{B}_{21}$ when the matrix $\underset{\sim}{B}$ is represented as $\underset{\sim}{G}$ from (1), because $\{k_{1j}^2\} = \{k_{1j}^3\} = \{0\}$, and $\{k_{21}^1\} = \{k_{21}^3\} = \{0\}$. Therefore, the smaller the elements of $\underset{\sim}{B}$, the smaller the elements of $\{k_2^1\}$ will be. When $\underset{\sim}{B}_{21} = \underset{\sim}{0}$, $\{k_2^1\} = \{0\}$. This shows that if there are no migrations from the dominant region to some other region in the multiregional model, this region's stable equivalent population will be equal to zero.

This fact may be interpreted as follows. Suppose that region 2 is the dominant one and that there are no migrations from region 2 to region 1 in a three-regional system. Consider equation (16.1). When t is so large that the projection is in its second phase, then it reduces to:

$$\{k_0^1(t)\} = c_1\lambda_1^t\{k_1^1\} + c_3\lambda_3^t\{k_3^1\} \quad ,$$

because $\{k_2^1\} = \{0\}$. The magnitude of $\{k_0^1(t)\}$ will depend in the long run on the signs of the constants and on the magnitudes of λ_1 and λ_3. If $\lambda_1 < 1$ and $\lambda_3 < 0$, the population will disappear;

if, for instance, $\lambda_1 > 1$ and $c_1 > 0$, it will constantly increase, but when t is big enough it can be neglected when compared with the populations of the other two regions.

From what has been discussed about the stable multiregional population, the notion of a dominant region can be clarified. This is the region whose population in the long run will dominate the whole system. Its spatial distribution, which depends on the region's outmigration rates, defines the system's spatial distribution.

III. CONCLUSION

The discussions in Sections I and II show that when space is introduced into the study of regional population, the population projection will differ considerably. First, three phases of the projection process can be observed, the most important for practical reasons being the second one, as it affects considerably the total number of the regional population. The population waves which are due to discrepancies in the age distribution are practically aperiodic, but every period which could be observed has a length close to that in the single-region case. The long-run behavior of the multiregional population is determined by the dominant region and the outmigration rates of its population.

The three phases were analyzed on the basis of the fact that the multiregional growth matrix has n positive eigenvalues close in magnitude to unity, where n is equal to the number of regions. This fact was proved by treating the migrations as perturbations. It has been discussed that the smaller the perturbations, the closer the multiregional model would be to its single-region analogue in the short and middle-run. As for the long-run, the notion of a dominant region was used, which makes no sense in the single-region case. These remarks make it possible to follow the behavior of a multiregional system when the migration movements are supposed to diminish or increase. When the long-run is concerned, the outmigrations from the dominant region only are significant, while in the short and middle-run any migration flow is significant.

When the number of regions is very large, the projection is very difficult because of the large size of the multiregional growth matrix. In such a case it is necessary to carry out some aggregations or decompositions. Rogers (1976) has suggested that first the multiregional population should be aggregated by age and then projected (i.e. going directly to phases 2 and 3) afterwards the age distribution of the stable single-region population should be considered (going back to phase 1). This procedure unfortunately misses the impact of the migrations on the age composition. Therefore, age compositions like the one in Figure 2 will never be reached. The author was unable to find any pattern that would help in deriving this kind of age distributions without estimating the eigenvectors. Use might be made only of the empiricial reflection that it is typical for regions with a low fertility levels if the fertility of the dominant region is above the replacement level.

Another approach to solving problems of this kind might be the usage of model multiregional stable populations. These are described in Rogers (1975). Instead of going back to phase 1, it might be more convenient to select a particular multiregional stable population according to specific fertility and migration schedules.

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