A MULTIACTIVITY LOCATION MODEL WITH ACCESSIBILITY- AND CONGESTION-SENSITIVE DEMAND

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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Public Facility Location Task (formerly the Normative Location Modeling Task) which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

This paper develops some research issues outlined in a previous paper (WP-80-79), concerning the generalization of spatial-interaction activities. The descriptive model proposed in Section 2 is of special interest in its own right, since its applicability goes beyond location problems, and Section 3, although primarily theoretical in character, concludes with an outline of some operational algorithms whose application to real-world examples will constitute one of the next steps for research in the Public Facility Location Task.

A list of publications in the Public Facility Location Series appears at the end of this paper.

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ABSTRACT

This paper has two aims. The first one is to build a generalization of the doubly-constrained spatial interaction model, in order to account for sensitiveness of demand to accessibility and congestion and for possible multiple interacting activities. In Section 2 it is shown how this can be done by keeping an extremal representation for the model, which is closely related to the Neuburger's consumer surplus maximizing principle. The second aim is to embed the model developed in Section 2 in a multiactivity optimal location problem, and to develop operational tools to solve the resulting mathematical programming problems. This subject is treated in Section 3. Section 4 is devoted to the discussion of three possible applications: the urban system, the health care system, and the retail system.

CONTENTS

A MULTIACTIVITY LOCATION MODEL WITH ACCESSIBILITY- AND CONGESTION-SENSITIVE DEMAND

1. GENERAL INTRODUCTION

Although most location-allocation models deal with a single category of facilities at a time, in real urban areas different facilities, activities, and settlements are present at the same time. Usually interactions take place among them in terms of customers' journeys, exchange of goods, money flows, and information. These interactions tie all activities together, and have to be taken into account in model building of both a descriptive and a normative character.

Many descriptive models of multiactivity systems have been built since the well-known *Model of Metropolis* (Lowry, 1964), and their mathematical formulation and economic interpretation have recently been greatly improved (see MacGill and Wilson, 1979 for a review). The introduction of normative features (i.e., the optimal size and location of physical stocks) also appears in some works (such as Coelho and Williams, 1978; Boyce and LeBlanc, 1979; Leonardi, 1979).

In this paper a new class of models is proposed, which, it is felt, will improve existing ones in two respects.

a. *Sensitivity of demand to accessibility and congestion is taken into account.* Most existing models assume production and/or attraction constrained spatial interactions. It is, however, sensible to assume that total demand production is not given beforehand, but is itself an endogenous variable of the spatial-interaction system. Specifically, it is reasonable to assume that demand for activities increases with accessibility. Moreover, it is sensible to assume that demand is also sensitive to congestion, that is, the more a facility is crowded, the less attractive it will be for customers. It will be shown that both features can be introduced in spatial interaction models in a very natural way, with no significant changes in their general structure.

b. *Possible combinatorial and indivisibility features in the location and size of stocks are taken into account.* Most existing models assume simple linear costs for establishing and maintaining stocks, thus being unable to account for economies of scale, bounds on the number of facilities, and bounds on their feasible sizes. On the other hand, most of these features have been introduced very well in the so-called "plant location" models developed in Operations Research (OR) (for example Erklenkotter, 1978). The usual objective function of plant location models is not very useful in Regional Science applications, since it is linear and induces an unrealistic spatial interaction behavior (i.e., users always choose the nearest destination, with no possible cross substitution). The objective functions based on Neuburger's (1971) measure of consumer surplus, however, embed realistic spatial interactions very well. It is therefore natural to take advantage of the best parts of both approaches: namely, to use a Neuburger-type objective function and plant-location-type cost functions and constraints. It will be shown how this gives rise to a family of combinatorial optimization problems of a new kind.

Although this paper will focus on the general theoretical and computational problems posed by these new models, some possible applications are discussed in Section 4. They are:

i. *The application to Lowry-type systems.* In this case the optimization concerns the location of housing and services, taking the relationships among them into account. This is not new: it has been the subject of many previous works. A completely new insight, however, is given by introducing accessibility- and congestion-sensitive mechanisms. They introduce several interesting realistic features, such as the formation of unused housing stocks, of location-dependant service-attendance ratios, and of different levels of congestion across space. Another possible improvement results from introducing combinatorial structures in the cost functions and in the constraints, since it is well known that real urban planning and management problems are faced with indivisibilities, threshold-like constraints, scale effects, and the unrealistic spatial allocations obtained by continuous models.

ii. *The application to multilevel service systems.* Many services have many stages, or levels, which users will possibly go through in a given order. A typical example is given by a health care system, where users may enter the system at the lower level (usually made up of widespread small general-purpose facilities) and possibly be sent to higher, more localized levels (usually made up of larger facilities for specialized treatments). The introduction of sensitivity to accessibility and congestion is fully justified in these systems, of course, as well as indivisibilities and scale economies. Furthermore, in the example of the health care system, it is possible to have some stages in which transport between levels has the character of an emergency, rather than the normal spatial-interaction behavior. This will produce mixed models, where some stages behave according to spatial interaction, and others behave more like the OR plant location models (with possible maximum travel time constraints).

iii. *The application to multiple-destination service systems.* There are many instances where consumers make trips with many destinations, instead of home-return trips with a single destination. Most trips to retail activities are of the former kind, since usually customers have a shopping program made up of different goods, not necessarily available in the same place. But the usefulness of a round-trip scheme is not limited to shopping. Many generally different kinds of services have interactions within them, in the sense that part of the demand attracted by any one of the services may generate demand for another, depending on accessibilities. Apart from considerations similar to those for cases i and ii, it is of special interest for this case to have results on possible aggregations, that is, ways of building possible multipurpose facilities. This is surely relevant for retail activities, in which possible optimal patterns for shopping centers may be revealed, taking into account both spatial interaction and economies of scale.

Not all of the above problems can be solved easily, of course. Therefore, together with the general optimality conditions, approximate heuristic solution methods are developed here. Most of them are shown to be even more interesting and useful than the exact ones, since their general form is an approximate ranking rule, based on cost/benefit indicators, possibly to be improved by successive approximations. The cost/benefit indicators usually have intuitive interpretations, being made up of terms related to accessibility, congestion, demand potential, and so on.

A few general references will be given. The approach used in Section 2 to introduce accessibility and congestion sensitiveness is very closely related to the approach proposed by Walsh and Gibberd (1980), although it has been developed independently.

(Earlier related work is also found in Dacey and Norcliffe, 1976; and in Jefferson and Scott, 1979.) The general structure of the optimal location problem developed in Section 3 is related to the one proposed in Leonardi (1979), although it has been substantially revised and extended and findings of more recent research (Leonardi, 1980a and b) have been taken into account.

2. A GENERAL ACCESSIBILITY- AND CONGESTION-SENSITIVE MULTI-ACTIVITY SPATIAL INTERACTION MODEL

2.1 A Generalization of the Doubly-Constrained Spatial Interaction Model

According to the usual doubly-constrained spatial interaction model, the total number of trips for each origin-destination pair (i,j) and for a given trip purpose is determined by the set of equations

$$S_{ij} = u_i v_j f_{ij}$$

$$\sum_j S_{ij} = G_i$$

$$\sum_i S_{ij} = A_j$$

where

S_{ij} is the number of trips from origin i to destination j

G_i is the total number of trips generated from i

A_j is the total number of trips attracted in j

f_{ij} is a measure of the impedance to travel from i to j; usually, but not necessarily, $f_{ij} = e^{-\beta C_{ij}}$, where C_{ij} is the cost of a trip from i to j and β is a given nonnegative constant, called the space discount rate

u_i, v_j are balancing factors of biproportionality

In the models of the above type the total trip generations and attractions are usually assumed to be determined exogenously and independently of the spatial interaction process.

Let it now be assumed that the following quantities can be defined

P_i is the maximum number of trips which can be generated from i, so that $G_i \leq P_i$; P_i will be called the *potential demand* in i

Q_j is the maximum number of trips which can be attracted in j, so that $A_j \leq Q_j$; Q_j will be called the *total capacity* in j

$U_i = P_i - G_i$ is the difference between the potential demand in i and the trips generated from i; U_i will be called the *unsatisfied demand* in i

$V_j = Q_j - A_j$ is the difference between the total capacity in j and the number of trips attracted in j; V_j will be called the *unused capacity* in j

A natural assumption which can be made for endogenously determined generations and attractions is that they increase with unsatisfied demand and unused capacity, respectively. A simple mathematical translation of this assumption is given by the following equations

$$u_i = U_i/g_i \tag{1}$$

$$v_j = V_j/h_j \tag{2}$$

where g_i, h_j are given nonnegative constants. Equation (1) states that the balancing factor for the origin is proportional to the unsatisfied demand. Equation (2) states that the balancing factor for the destination is proportional to the unused capacity.

Furthermore, from the definition of U_i and V_j, the following equations hold

$$G_i + U_i = P_i \tag{3}$$

$$A_j + V_j = Q_j \tag{4}$$

Substitution of (1) and (2) in the spatial interaction model and addition of (3) and (4) to the list of equations yield the following new model

$$S_{ij} = U_i V_j \frac{f_{ij}}{g_i h_j} \tag{5}$$

$$\sum_j S_{ij} = G_i \tag{6}$$

$$\sum_j S_{ij} = A_j \tag{7}$$

$$G_i + U_i = P_i \tag{8}$$

$$A_j + V_j = Q_j \tag{9}$$

By means of some easy rearrangements and substitutions, model (5)-(9) can be given the following two alternative representations.

a. *The production-constrained representation*, in which the constraint on demand generation is evidenced. It is defined by the following equations

$$S_{ij} = G_i \frac{V_j f_{ij}/h_j}{\phi_i} \tag{10}$$

$$\phi_i = \sum_j V_j f_{ij}/h_j \tag{11}$$

$$G_i = P_i \frac{\phi_i}{\phi_i + g_i} \tag{12}$$

The variables ϕ_i defined by (11) can be interpreted as *accessibility measures* in the Hansen sense (Hansen, 1959), and the unused capacities play the role of attractiveness measures. Equations (12) give the generated demands as functions of accessibilities. The shape of the graph of these functions is shown in Figure 1. By means of (10), (11), and (12) sensitiveness of generated demand both to accessibility and to congestion has been introduced. The total generated demand is a nondecreasing function of accessibility, and it tends to the total potential demand as accessibility increases. On the other hand, the accessibilities increase with the unused capacities or, which is the same, decrease as the congestion in the destinations increase.

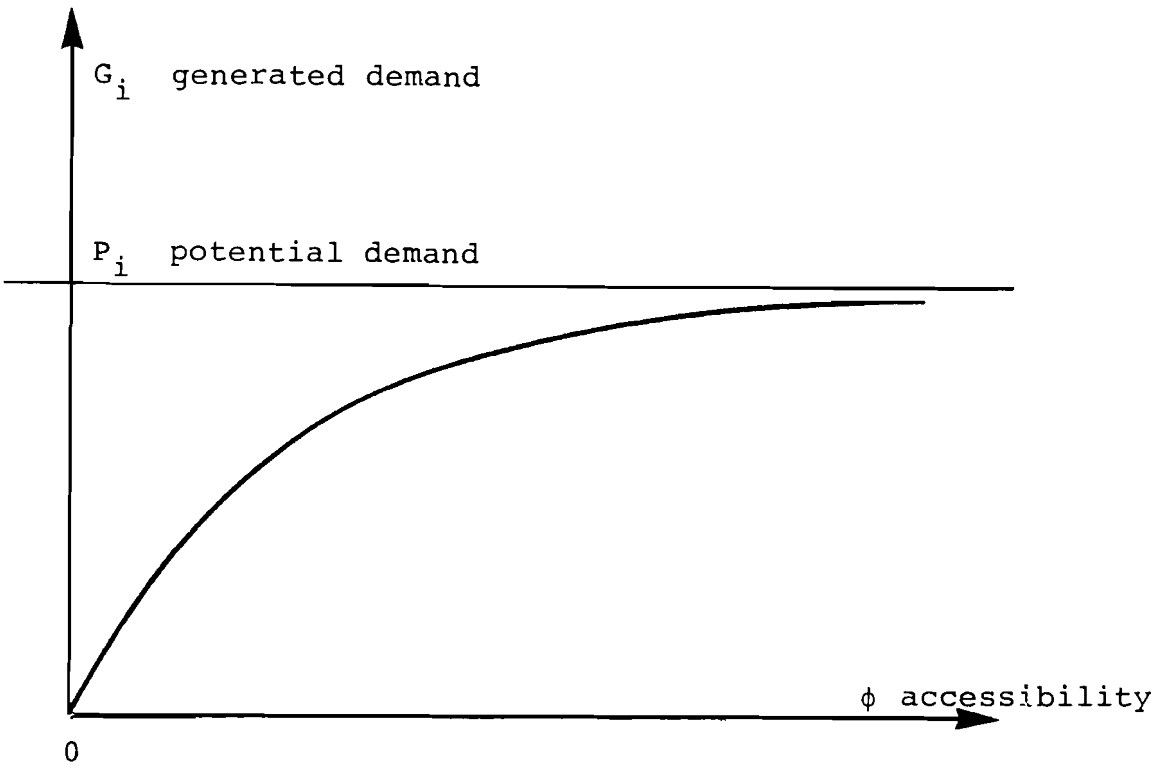

Figure 1. Generated demand as a function of accessibility.

b. *The attraction-constrained representation,* in which the constraint on demand attraction is evidenced. This representation is completely symmetrical with the production-constrained one, and is defined by the following equations

$$S_{ij} = A_j \frac{U_i f_{ij}/g_i}{\psi_j} \tag{13}$$

$$\psi_j = \sum_i U_i f_{ij}/g_i \tag{14}$$

$$A_j = Q_j \frac{\psi_j}{\psi_j + h_j} \tag{15}$$

In analogy with the interpretation given above for the ϕ_i, the variables ψ_j defined by (14) can be interpreted as *population potentials* in the Stewart sense (Steward, 1948), and the unsatisfied demands represent populations. Equations (15) give the attracted demands as functions of potentials. The shape of the graph of these functions is the same as for functions (12), and is shown in Figure 2.

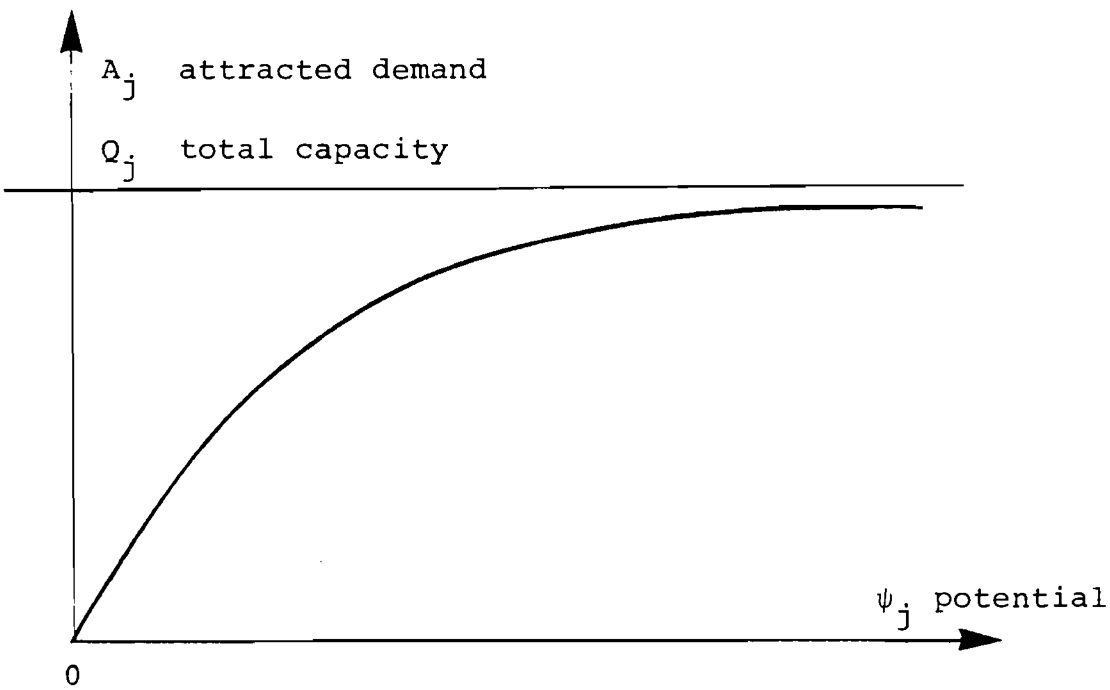

Figure 2. Attracted demand as a function of potential.

By means of (13), (14), and (15) sensitiveness of attracted demand both to potential and to unsatisfied demand has been introduced. The total attracted demand is a nondecreasing function of potential, and it tends to the total capacity as the potential increases. On the other hand, the potentials increase with the unsatisfied demand or, which is the same, decrease as the satisfied demand in the origins increases.

The above modified form of the classical spatial interaction model has been built by introducing the intuitive assumptions (1), (2), (3), and (4). It will be shown that this modified model can be derived by an extremal principle, which is closely related to Neuburger consumer's surplus maximization (Neuberger, 1971). Let the following function be defined

$$W(S,U,V) = -\sum_{ij} S_{ij}\left(\log\frac{S_{ij}}{f_{ij}} - 1\right) - \sum_{i} U_i\left(\log\frac{U_i}{g_i} - 1\right) - \sum_{j} V_j\left(\log\frac{V_j}{h_j} - 1\right)$$

then it can be shown that the solution to the mathematical program

$$\max_{S,U,V} \quad W(S,U,V) \tag{16}$$

$$\text{s.t.} \quad \sum_{j} S_{ij} + U_i = P_i \tag{17}$$

$$\sum_{i} S_{ij} + V_j = Q_i \tag{18}$$

is the spatial interaction model defined by (5)-(9). For the proof, it is first noted that the function W(S,U,V) is concave, because it is the sum of concave functions. Since constraints (17) and (18) are linear, (16)-(18) is a concave program, whose solution is unique. This solution must satisfy the Lagrange optimality conditions

$$\frac{\partial W}{\partial S_{ij}} - \nu_i - \mu_j = 0$$

$$\frac{\partial W}{\partial U_i} - \nu_i = 0$$

$$\frac{\partial W}{\partial V_j} - \mu_j = 0$$

or

$$-\log \frac{S_{ij}}{f_{ij}} - \nu_i - \mu_j = 0 \qquad (19)$$

$$-\log \frac{U_i}{g_i} - \nu_i = 0 \qquad (20)$$

$$-\log \frac{V_j}{h_j} - \mu_j = 0 \qquad (21)$$

where ν_i and μ_j are the Lagrange multipliers corresponding to constraints (17) and (18), respectively. From (19), (20), and (21) it follows that

$$S_{ij} = \bar{u}_i \bar{v}_j f_{ij} \qquad (22)$$

$$\bar{u}_j = U_i/g_i \qquad (23)$$

$$\bar{v}_j = V_j/h_j \qquad (24)$$

where

$$\bar{u}_i = e^{-\nu_i}$$

$$\bar{v}_j = e^{-\mu j}$$

But (23) and (24) are identical with assumptions (1) and (2), provided

$$u_i = \bar{u}_i \quad , \qquad v_j = \bar{v}_j$$

Moreover, (17) and (18) are equivalent to (3) and (4). Hence, the solution to (16), (17), (18) satisfies equations (5)-(9).

If, as a special case, the terms depending on unsatisfied demand U_i and unused capacity V_j are dropped, then W reduces to the Neuburger consumer's surplus (except for a multiplicative constant), provided the f_{ij} are of the form

$$f_{ij} = e^{-\beta C_{ij}}$$

2.2 Introducing Many Activities

The model discussed in Section 2.1 refers to just one trip purpose. Now let many trip purposes be introduced or, equivalently, let the trip attractors in each destination be many different activities. The following definitions will be needed.

P_i^k is the potential demand in i for activity k.

Q_j^k is the total capacity of activity k in j.

G_i^k is the demand for activity k generated in i, that is, the total number of trips from i which have an activity k as a destination.

A_j^k is the demand attracted by activity k in j, that is, the total number of trips having as a destination activity k which is located in j.

S_{ij}^k is the number of trips with purpose k (that is, having an activity k as a destination) from origin i to destination j.

f_{ij}^{k} is a measure of impedance to travel from i to j with purpose k; usually, but not necessarily, the measure of impedance is of the form $f_{ij}^{k} = e^{-\beta_k C_{ij}^{k}}$, where C_{ij}^{k} is the cost of traveling from i to j with purpose k and β_k are nonnegative space discount rates.

g_i^k, h_j^k are given nonnegative constants.

The following equations must hold

$$\sum_j S_{ij}^{k} = G_i^k \qquad (25)$$

$$\sum_i S_{ij}^{k} = A_j^k \qquad (26)$$

$$G_i^k + U_i^k = P_i^k \qquad (27)$$

$$A_j^k + V_j^k = Q_j^k \qquad (28)$$

The introduction of many activities is meaningful if interactions take place among them. Let it therefore be assumed that the potential demand for each activity k from each location i, P_i^k, is not a given constant, but a linear function of the demand attracted by all activities in i. This assumption is stated by the equations

$$P_i^k = Y_i^k + \sum_r A_i^r a_{rk}$$

or, after substitution from (28)

$$P_i^k = Y_i^k + \sum_r Q_i^r a_{rk} - \sum_r V_i^r a_{rk} \qquad (29)$$

where Y_i^k and a_{rk} are given nonnegative constants. The terms Y_i^k can be interpreted as exogenous inputs, while the coefficients a_{rk} are defined as follows

a_{rk} is the potential demand for activity k produced by a unit of attracted demand in activity r.

The function W(S,U,V) introduced in Section 2.1 can be generalized as follows

$$W(S,U,V) = -\sum_{ijk} S_{ij}^k \left(\log \frac{S_{ij}^k}{f_{ij}^k} - 1\right) - \sum_{ik} U_i^k \left(\log \frac{U_i^k}{g_i^k} - 1\right) - \sum_{jk} V_j^k \left(\log \frac{V_i^k}{h_j^k} - 1\right)$$

If equations (29) are added to the list of constraints, the following generalization of (16)-(18) is obtained

$$\max_{S,U,V,P} \quad W(S,U,V) \tag{30}$$

$$\text{s.t.} \quad \sum_j S_{ij}^k + U_i^k = P_i^k \tag{31}$$

$$\sum_i S_{ij}^k + V_j^k = Q_j^k \tag{32}$$

$$P_i^k = Y_i^k + \sum_r Q_i^r a_{rk} - \sum_r V_i^r a_{rk} \tag{33}$$

The variables P_i^k have been added to the list of maximization variables in (30), since now they are no longer given constants.

The Lagrange optimality conditions for (30)-(33) are

$$\frac{\partial W}{\partial S_{ij}^k} - \nu_i^k - \mu_j^k = 0$$

$$\frac{\partial W}{\partial U_i^k} - \nu_i^k = 0$$

$$\frac{\partial W}{\partial V_j^k} - \mu_j^k - \sum_r \lambda_j^r a_{kr} = 0$$

$$\frac{\partial W}{\partial P_i^k} + \nu_i^k - \lambda_i^k = 0$$

or

$$- \log \frac{S_{ij}^k}{f_{ij}^k} - \nu_i^k - \mu_j^k = 0 \qquad (34)$$

$$- \log \frac{U_i}{g_i^k} - \nu_i^k = 0 \qquad (35)$$

$$- \log \frac{V_j^k}{h_j^k} - \mu_j^k - \sum_r \lambda_j^r a_{kr} = 0 \qquad (36)$$

$$\nu_i^k - \lambda_i^k = 0 \qquad (37)$$

where ν_i^k, μ_j^k, λ_i^k are the Lagrange multipliers corresponding to constraints (31), (32), and (33), respectively. From (34), (35), (36), and (37) it follows that

$$S_{ij}^k = u_i^k \, v_j^k \, f_{ij}^k \qquad (38)$$

$$u_i^k = U_i^k / g_i^k \qquad (39)$$

$$v_j^k = \prod_r \left(\frac{1}{u_j^r}\right)^{a_{kr}} V_j^k / h_j^k \qquad (40)$$

where

$$u_i^k = e^{-\nu_i^k}$$

$$v_j^k = e^{-\mu_j^k}$$

On the other hand, summation of (38) over j and equation (25) yield

$$G_i^k = u_i^k \phi_i^k \quad , \qquad \text{or } u_i^k = G_i^k/\phi_i^k \tag{41}$$

where

$\phi_i^k = \sum_j f_{ij}^k v_j^k$ can be interpreted as an accessibility measure, in analogy with (11)

Substitution of (41) into (40) gives for the v_j^k

$$v_j^k = \frac{V_j^k}{h_j^k} \prod_r \left(\frac{\phi_j^r}{G_j^r}\right)^{a_{kr}} \tag{42}$$

Equation (42) gives much insight in the way activities interact. If it is compared with its analogous (2) for the simple case, it is seen that the attractiveness of activity k in destination j is still proportional to the unused capacity V_j^k, but it is also proportional to the term

$$\prod_r \left(\frac{\phi_j^r}{G_j^r}\right)^{a_{kr}} \tag{43}$$

that is, the product of the ratios of accessibilities to generated demands for <u>all activities r</u> in j. These ratios are raised to the power a_{kr}, which is a measure of the intensity of interaction between activities k and r. Therefore, the value of (43) is mainly determined by the activities which have strong interactions with k. If, as a special case, $a_{kr} = 0$ for some r, then k and r have no interaction at all, and the corresponding factor in (43) reduces to 1. If, as a limiting case, all $a_{kr} = 0$, there is no interaction among activities, the value of (43) reduces to 1, and the value of (42) reduces to

$$v_j^k = V_j^k/h_j^k$$

which is the same as (2), except for the superscript k. In other words, the model with many activies reduces to a set of independent models with a single activity.

If (42) is substituted into ϕ_i^k, the following equations are obtained

$$\phi_i^k = \sum_j f_{ij}^k \frac{V_j^k}{h_j^k} \prod_r \left(\frac{\phi_j^r}{G_j^r}\right)^{a_{kr}} \qquad (44)$$

In equations (44) the multiplier effect of accessibilities on themselves, and hence on demand generation, is evidenced. All accessibilities from all locations and to all activities are tied together by (44), and these ties are stronger the higher the values of the coeffients a_{kr}.

2.3 An Example: The Lowry Model Revisited

The usefulness of model (30)-(33) is shown by the following example. Let an urban system be given, which is assumed to behave according to the classic economic base theory, as it has been introduced in the well known Lowry model (Lowry, 1964). In order to get a qualitative understanding for the structure of the model, some very crude simplifying assumptions will be introduced. These assumptions are the following.

a. The urban system has only two types of endogenous activities, housing and service, with no further breakdown.
b. Only one exogenous input is given, the basic sector, with no further breakdown.
c. The households are homogenous, and only the householder works.
d. The demand for housing arises only from the basic sector and the service sector.
e. The demand for services arises only from the housing sector.

Let $k = 1$ label the housing sector and $k = 2$ label the service sector, and introduce the following definitions.

S_{ij}^{1} is the number of households living in j, whose householders work in i.

S_{ij}^{2} is the number of daily trips made by households living in i to services located in j.

P_{i}^{1} is the potential demand for housing from i, that is, the total number of households whose householders work in i.

P_{i}^{2} is the potential demand for service from i, that is, the maximum number of daily trips to services which can be made by households living in i.

Q_{j}^{1} is the total capacity for housing in j, that is, the total number of dwelling units, or the housing stock, in j.

Q_{j}^{2} is the total capacity for services in j, that is, the total size of services, or the service stock, in j. (Q_{j}^{2} is assumed to be measured in terms of maximum number of daily customers that can be served).

G_{i}^{1} is the demand for housing generated in i, that is, $G_{i}^{1} = \sum_{j} S_{ij}^{1}$; in general $G_{i}^{1} < P_{i}^{1}$, that is, not all the potential demand for housing is necessarily satisfied.

G_i^2 is the demand for service generated in i, that is, $G_i^2 = \sum_j S_{ij}^2$; in general $G_i^2 < P_i^2$, that is, the maximum number of possible trips to services is not necessarily made.

A_j^1 is the demand for housing attracted in j, that is, $A_j^1 = \sum_i S_{ij}^1$; in general $A_j^1 < Q_j^1$, that is, not all the housing stock is necessarily used.

A_j^2 is the demand for service attracted in j, that is, $A_j^2 = \sum_i S_{ij}^2$; in general $A_j^2 < Q_j^2$, that is, not all the service capacity is necessarily used.

Y_i^1 is the number of households whose householder works in the basic sector.

a_{12} is the potential number of daily trips to services made by a household.

a_{21} is the ratio between workers in the service sector and total attracted service demand.

Assumptions b, d, and e imply that $Y_i^1 = 0$, $a_{11} = 0$, $a_{22} = 0$. Equations (29) assume the simple form

$$P_i^1 = Y_i^1 + A_i^2 a_{21} \tag{45}$$

$$P_i^2 = A_i^1 a_{12} \tag{46}$$

Equation (45) states that the total potential demand for housing is equal to the number of workers in the basic sector, plus the number of workers in the service sector. Equation (46) states that the total potential demand for service is equal to the maximum number of daily trips to services which can be made by households.

If equation (42) is applied to the housing sector, it takes the form

$$v_j^1 = \frac{V_j^1}{h_j^1} \left(\frac{\phi_j^2}{G_j^2} \right)^{a_{12}} \tag{47}$$

where

$V_j^1 = Q_j^1 - A_j^1$ is the unused housing stock in j

ϕ_j^2 is the accessibility to services from j

Therefore, the attractiveness of location j as a place of residence, as measured by v_j^1, increases both with the availability of dwelling units, V_j^1, and with the accessibility to services, ϕ_j^2. The main trade-off in residential choice is thus embodied in (47). The third trade-off term, the home-to-work travel cost, is introduced if the production-constrained representation (see Section 2.1) is used for the S_{ij}^1

$$S_{ij}^1 = G_i^1 \frac{V_j^1 \left(\frac{\phi_j^2}{G_j^2} \right)^{a_{12}} f_{ij}^1 / h_j^1}{\phi_i^1} \tag{48}$$

where $\phi_i^1 = \sum_j v_j^1 f_{ij}^1$, and v_j^1 is defined as in (47). Since f_{ij} depends on the cost of traveling from i to j, plus possibly some additional costs associated with location j (like the rent), (48) shows how availability of houses, accessibility to services, home-to-work travel cost and location costs determine the overall attractiveness for residential location. Other results easily derived from the production-constrained representation are

$$G_i^1 = P_i^1 \frac{\phi_i^1}{\phi_i^1 + g_i^1}$$ the demand for housing generated in j (49)

$$U_i = P_i^1 \frac{g_i^1}{\phi_i^1 + g_i^1}$$ the unsatisfied demand for housing in i (50)

$$A_j^1 = \frac{V_j^1}{h_j^1} \left(\frac{\phi_j^2}{G_j^2}\right)^{a_{12}} \sum_i \frac{G_i^1}{\phi_i^1} f_{ij}^1$$ the housing demand attracted in j (51)

Equation (51) can be given a more meaningful and simpler form. From (49) and (50) it follows that

$$U_i^1 = \frac{G_i^1}{\phi_i^1} g_i^1$$

therefore

$$\sum_i \frac{G_i^1}{\phi_i^1} f_{ij}^1 = \sum_i U_i^1 f_{ij}^1 / g_i^1 = \psi_j^1$$ is the unsatisfied housing demand potential, as defined in the attraction-constrained representation of Section 2.1

Substitution of this result into (51) yields

$$A_j^1 = \frac{V_j^1}{h_j^1} \left(\frac{\phi_j^2}{G_j^2}\right)^{a_{12}} \psi_j^1 \qquad (52)$$

Equation (52) embodies the spatial interaction process in the most synthetic and intuitive way. It says that the total housing demand attracted in j is proportional to the availability of houses in j, V_j^1, to the potential ψ_j^1, which is a measure of nearness of j to unsatisfied housing demand, and to the accessibility to services from j, raised to the power a_{12}, which is the maximum daily frequency of home-to-service trips. From (52) an equation for the unused housing stock is easily derived. If:

$$A_j^1 = Q_j^1 - V_j^1$$

is substituted for A_j^1 in (52), and the resulting equation is solved for V_j^1, it is found that

$$V_j^1 = Q_j^1 \frac{h_j^1}{\left(\frac{\phi_j^2}{G_j^2}\right)\psi_j^1 + h_j^1} \tag{53}$$

Equation (53) says that the formation of unused housing stock mainly takes place in locations far from both services and from places of work, where the housing demand arises. This is exactly what might be expected. However, from (52) it is seen that the unused housing stock is an attracting factor for new housing demand. Therefore, the housing demand is forced to trade off accessibility to services and nearness to the place of work (which would solely guide their choice, other things being equal) with availability of houses, which acts as a constraint. The resulting spatial pattern is a concentration of households in locations with highest accessibility to services and places of work, whose housing capacity is near to saturation, and a lower density of households in the less accessible locations, where unused housing stock may possibly be found. This is indeed very close to what actually happens in real urban systems, and it is also very similar to what the classic Lowry model predicts. However, when total demand grows faster than the housing stock, all locations tend to be saturated, whether their accessibility is high or low. This behavior is also very close to the actual behavior of congested urban systems, but it cannot be accounted for by the classic "unconstrained" Lowry model.

The analysis which has been carried out for the housing sector applies to the service sector as well, and it will not be repeated here.

3. THE OPTIMAL LOCATION PROBLEM

3.1 The Primal Problem

In Section 2 the analysis of the descriptive process has been carried out. Now the problem of how to control the multi-activity spatial interaction system in some optimal way will be posed. That is, given that customers behave as if they were looking for the optimal solutions to problems (30)-(33), how can a public authority improve this optimizing behavior by suitably choosing the values for the physical stocks of activities, that is, the capacitites Q_j^k? The above question implies the assumption that the goal of the customers (maximizing the function W defined in Section 2.2) is in agreement with that of the public authority, so that no conflicting-goal problem arises between customers and public authority. The public authority is also assumed to pay the costs to establish the capacities Q_j^k. Let the cost functions be of the form

$$a_j^k + b_j^k Q_j^k$$

where a_j^k is a fixed-charge cost to be paid for establishing an activity k in location j, while b_j^k is a unit cost. Fixed charges have the effect of introducing economies of scale and threshold effects, as it will be shown later.

The optimization problem can be split in two steps:

a. choose a subset of locations and a subset of activities to be established for each chosen location;

b. given the result of step a, find the optimal size of the activities to be established in each chosen location.

While step a gives rise to a combinatorial problem, step b is a smooth mathematical programming problem. Let therefore step b be solved first, and step a be introduced in the next section. It will be thus assumed that the chosen locations and activities are given, and only the capacities Q_j^k have to be found. The resulting mathematical programming problem is the same as (30)-(33), the only difference being that establishing costs are subtracted from the objective function W, and the capacities Q_j^k are added to the list of decision variables:

$$\max_{S,U,V,P,Q} \quad W(S,U,V) - \lambda\left(\sum_{jk} a_j^k + \sum_{jk} Q_j^k b_j^k\right) \tag{54}$$

$$\text{s.t.} \quad \sum_j S_{ij}^k + U_i^k = P_i^k \tag{55}$$

$$\sum_i S_{ij}^k + V_j^k = Q_j^k \tag{56}$$

$$P_i^k = Y_i^k + \sum_r Q_i^r a_{rk} - \sum_r V_i^r a_{rk} \tag{57}$$

The parameter λ which multiplies the cost term in (54) is a trade-off parameter, weighting costs against benefits. Usually a sensitivity analysis has to be made on λ, in order to assess the appropriate trade-off level. Alternatively, λ may be interpreted as a Lagrange multiplier arising from the relaxation of a budget constraint.

Formulation (54)-(57) is somewhat redundant. First, since the locations and activities are assumed to be given, the sum of the fixed-charge costs is constant, and can be dropped from the objective function (54). Secondly, there is no use to keep P_i^k and Q_j^k in the list of decision variables, since by means of equations (55) and (56) they can be expressed in terms of the variables S_{ij}^k, U_i^k, V_j^k. Therefore, after some substitutions and rearrangements, problem (54)-(57) reduces to:

$$\max_{S,U,V} \quad W(S,U,V) - \lambda \sum_{jk} b_j^k \left(\sum_i S_{ij}^k + V_j^k\right) \tag{58}$$

$$\text{s.t.} \quad \sum_j S_{ij}^k + U_i^k - \sum_r a_{rk} \sum_j S_{ji}^r = Y_i^k \tag{59}$$

Problem (58)-(59) will be referred to as the "primal" problem.

3.2 Some Duality Results

The saddle-point problem equivalent to (58)-(59) is

$$\min_{\nu} \max_{S,U,V} \quad L(S,U,V,\nu)$$

where the Lagrangean function L is defined as:

$$L(S,U,V,\nu) = W(S,U,V) - \lambda \sum_{jk} b_j^k \left(\sum_i S_{ij}^k + V_j^k\right) +$$

$$+ \sum_{ik} \nu_i^k \left(Y_i^k - \sum_j S_{ij}^k - U_i^k + \sum_r a_{rk} \sum_j S_{ji}^r\right)$$

and the ν_i^k are the Lagrange multipliers, or dual variables, corresponding to constraints (59). The vanishing of the derivatives of L with respect to the primal variables yields the following equations

$$\left.\begin{aligned} &- \log \frac{S_{ij}^k}{f_{ij}^k} - \lambda b_j^k - \nu_i^k + \sum_r a_{kr} \nu_j^r = 0 \quad , \quad \text{or} \\ &\qquad S_{ij}^k = f_{ij}^k \, e^{-(\lambda b_j^k + \nu_i^k - \sum_r a_{kr} \nu_j^r)} \end{aligned}\right\} \tag{60}$$

$$- \log \frac{U_i^k}{g_i^k} - \nu_i^k = 0 \quad , \quad \text{or} \qquad U_i^k = g_i^k \, e^{-\nu_i^k} \tag{61}$$

$$- \log \frac{V_j^k}{h_j^k} - \lambda b_j^k = 0 \quad , \quad \text{or} \qquad V_j^k = h_j^k \, e^{-\lambda b_j^k} \tag{62}$$

By means of equations (60), (61), and (62) the primal variables can be expressed in terms of the dual variables in closed form. Substitution into *L* and some rearrangements yield the following unconstrained "dual" problem

$$\min_{\nu} \; D(\nu)$$

where the dual objective function D is defined as

$$D(\nu) = \sum_{ijk} S_{ij}^{k}(\nu) + \sum_{ik} U_{i}^{k}(\nu) + \sum_{jk} V_{j}^{k} + \sum_{ik} \nu_{i}^{k} Y_{i}^{k} \qquad (63)$$

and the functions $S_{ij}^{k}(\nu)$, $U_{i}^{k}(\nu)$, and the constants V_{j}^{k} are defined by equations (60), (61), and (62). The dual objective function can be given an intuitive interpretation. From equation (32), the total capacity of activity k in location j, Q_{j}^{k}, is given by

$$Q_{j}^{k} = \sum_{i} S_{ij}^{k} + V_{j}^{k}$$

so that

$$Q_{j}^{k}(\nu) = \sum_{i} S_{ij}^{k}(\nu) + V_{j}^{k} \qquad (64)$$

is the total capacity of activity k in location j, as a function of the dual variables

If (64) is substituted into (63), the dual function becomes

$$D(\nu) = \sum_{ik} U_{i}^{k}(\nu) + \sum_{jk} Q_{j}^{k}(\nu) + \sum_{ik} \nu_{i}^{k} Y_{i}^{k} \qquad (65)$$

The first two terms of (65) are the *total unsatisfied demand* and the *total capacity*, respectively. The philosophy behind minimization of $D(\nu)$ is therefore a balance between a welfare goal (minimizing unsatisfied demand) and an efficiency goal (minimizing the total capacity). Now let the combinatorial part of the problem (step a of Section 3.1) be introduced. Define the boolean variables

$$x_j^k = \begin{cases} 1, & \text{if an activity k is located in j} \\ 0, & \text{otherwise} \end{cases}$$

Further constraints may be introduced on the number of activities which can be established in the same location. For simplicity, it will be provisionally assumed that only one activity can be established in each location. The assumption seems restrictive, but it may be easily relaxed, as it will be done in later sections. In terms of the boolean variables, the assumption gives rise to the constraints

$$\sum_k x_j^k \leq 1$$

If the boolean variables and the sum of the fixed charges (which now is no longer a constant) are suitably introduced in (65), the following modified dual function is obtained

$$D(\nu,x) = \sum_{ik} \left[U_i^k(\nu) + \nu_i^k \ Y_i^k\right] + \sum_{jk} x_j^k \left[Q_j^k(\nu) - \lambda a_j^k\right] \tag{66}$$

This function has to be minimized with respect to the dual variables ν_i^k and maximized with respect to the variables x_j^k. The resulting problem is

$$\max_x \min_\nu \quad D(\nu,x) \tag{67}$$

$$\text{s.t.} \quad \sum_k x_j^k \leq 1 \tag{68}$$

$$x_j^k \in \{0,1\} \tag{69}$$

An upper bound to the optimal value of D is obtained by relaxing constraint (69) and replacing it with the weaker condition:

$$0 \leq x_j^k \leq 1 \tag{70}$$

where the variables x_j^k are allowed to assume any real value in the unit interval. But the right-hand side inequality in (70) is redundant, since it is already implied by constraints (68). Therefore, the relaxed version of (67)-(69) becomes

$$\max_x \min_\nu \; D(\nu,x) \tag{71}$$

$$\text{s.t.} \quad \sum_k x_j^k \leq 1 \tag{72}$$

$$x_j^k \geq 0 \tag{73}$$

Problem (71)-(73) is a saddle-point problem. It is therefore natural to look at $D(\nu,x)$ as the Lagrangean function associated with some "primal" problem, the variables x_j^k playing the role of Lagrange multipliers. It will be shown that such a "primal" problem indeed exists, and it is given by the following mathematical program

$$\min_{\nu,z} \; P(\nu,z) \tag{74}$$

$$\text{s.t.} \quad z_j \geq Q_j^k(\nu) - \lambda a_j^k \tag{75}$$

$$z_j \geq 0 \tag{76}$$

where the function $P(\nu,z)$ is defined as

$$P(\nu,z) = \sum_{ik} [U_i^k(\nu) + \nu_i^k Y_i^k] + \sum_j z_j$$

To show that (74)-(76) is equivalent to (71)-(73), the following "Lagrangean" function is introduced

$$\bar{D}(\nu,z,x,\varepsilon) = P(\nu,z) - \sum_{jk} x_j^k \left[z_j - Q_j^k(\nu) - \lambda a_j^k\right] - \sum_j \varepsilon_j z_j$$

where x_j^k and ε_j are the Lagrange multipliers corresponding to constraints (75) and (76), respectively. Problem (74)-(76) is equivalent to the following saddle-point problem

$$\max_{x,\varepsilon} \min_{\nu,z} \bar{D}(\nu,z,x,\varepsilon) \tag{77}$$

$$\text{s.t.} \quad x_j^k \geq 0 \tag{78}$$

$$\varepsilon_j \geq 0 \tag{79}$$

(The nonnegativity constraints on the multipliers are required because constraints (75) and (76) are inequalities.) The vanishing of the derivatives of $\bar{D}$ with respect to z_j implies

$$1 - \sum_k x_j^k - \varepsilon_j = 0$$

or

$$\sum_k x_j^k = 1 - \varepsilon_j \tag{80}$$

Equation (80) and constraints (78) and (79) imply

$$0 \leq 1 - \varepsilon_j \leq 1$$

therefore (80) is equivalent to (72). Substitution of (80) into

(77) yields:

$$\bar{D}(\nu,z,x,\varepsilon) = \sum_{ik}\left[U_i^k(\nu) + \nu_i^k\ Y_i^k\right] + \sum_j z_j -$$

$$- \sum_j (1-\varepsilon_j) z_j + \sum_{jk} x_j^k \left[Q_j^k(\nu) + \lambda a_j^k\right] - \sum_j \varepsilon_j z_j$$

and since the terms in z_j and ε_j cancel out a comparison with (66) shows that

$$\bar{D}(\nu,z,x,\varepsilon) = D(\nu,x)$$

It follows that problem (77)-(79) is equivalent to problem (71)-(73), and hence that problem (74)-(76) is equivalent to problem (71)-(73).

A more detailed description of the general structure of the solution to (74)-(76) will now be given. The way the function $P(\nu,z)$ has been built always forces the variables z_j to assume the lowest possible value in the optimal solution of (74)-(76). From constraints (75) and (76) it follows that it must be either

$$z_j = \max_k \left[Q_j^k(\nu) - \lambda a_j^k\right] \tag{81}$$

or

$$z_j = 0 \tag{82}$$

or both, which ever is greater. When only (81) holds, a k* which maximizes its right-hand side exists such that

$$z_j = Q_j^{k*}(\nu) - \lambda a_j^{k*} > 0 \tag{83}$$

while for every $k \neq k^*$ it must be

$$z_j > Q_j^k(\nu) - \lambda a_j^k \tag{84}$$

Therefore, constraint (75) is binding for activity k* only, and nonbinding for all other activities. It follows that the multipliers of constraints (75) are

$$x_j^{k*} \geq 0 \qquad (85)$$

$$x_j^k = 0 \qquad \text{for} \qquad k \neq k* \qquad (86)$$

On the other hand, since (82) does not hold, constraint (76) is nonbinding, therefore it must be

$$\varepsilon_j = 0 \qquad (87)$$

Substitution of (85), (86) and (87) into (80) yields

$$x_j^{k*} = 1 \qquad (88)$$

that is, the location j is chosen to establish an activity k*. It is important to notice that (88) yields a natural integer solution, that is, one which is feasible for the original combinatorial problem.

When only (82) holds, then it follows that

$$Q_j^k(\nu) - \lambda a_j^k < 0 \qquad \text{for all } k \qquad (89)$$

that is, constraints (75) are nonbinding for all activities, and the corresponding multipliers will be

$$x_j^k = 0 \qquad \text{for all } k \qquad (90)$$

In other words, no activity will be established in location j. Again it is important to notice that (90) yields a natural integer solution.

When both (81) and (82) hold, then (85) and (86) hold as well, but instead of (87) it must be

$$\varepsilon_j \geq 0$$

and from (80) it follows that

$$x_j^{k*} = 1 - \varepsilon_j \leq 1 \tag{91}$$

That is, a natural integer solution is no longer assured and fractional values for x_j^{k*} may be (and usually are) introduced.

Equations (81) and (82) suggest a new possible formulation of problem (74)-(76). From (81) and (82) it follows that

$$z_j = \max\{\max_k [Q_j^k(\nu) - \lambda a_j^k] , 0\}$$

and if this result is substituted in $P(\nu,z)$ the following non-smooth optimization problem is obtained:

$$\min_\nu G(\nu) \tag{92}$$

where the function $G(\nu)$ is defined as

$$G(\nu) = \sum_{ik} [U_i^k(\nu) + \nu_i^k Y_i^k] + \sum_j \max \{\max_k [Q_j^k(\nu) - \lambda a_j^k],0\}$$

Problem (92) is computationally attractive, since it is unconstrained and contains only the variables ν_i^k. The price to be paid for this simplicity is the nonsmoothness of the function G.

A summary of the main duality and equivalence results is useful. If $\bar{x}$, $\bar{\nu}$, $\bar{z}$ denote the optimal values for the corresponding arrays of variables, the following equalities hold

$$D(\bar{\nu},\bar{x}) = G(\bar{\nu}) = P(\bar{\nu},\bar{z})$$

For general nonoptimal values x,ν,z the following inequalities hold

$$D(\nu,x) \leq G(\nu) \leq P(\nu,z) \tag{93}$$

If, as a special case, x is the optimal integer solution, it is seen from (85) that $G(\nu)$ provides the tighter upper bound to $D(\nu,x)$. Anyway, both G and P can be used to compute upper bounds to D, depending on computational convenience. Problem (92) is simple, but nonsmooth, as already stated. Problem (74)-(76) is a smooth convex programming problem but has the nonlinear constraints (75). If an algorithm to solve either (92) or (74)-(76) is available, it can be used to find the optimal relaxed values for the x_j^k. If all the x_j^k assume natural integer values, then the original combinatorial problem is solved, and no further refinement is needed. If some x_j^k assume fractional values, a branch-bound refinement procedure may be started.

3.3 Heuristic Approximations

The problems introduced in Section 3.2 may be hard to solve exactly, and the requirement of integer values for the variables x_j^k makes the task even harder.

However, there are many reasons why applications to real problems should not be obsessed with finding exact solutions. First, the input data and the definition of the physical setting are always less precise than an exact algorithm seems to imply. The set of possible locations, for instance, is usually a set of zones in which a given area is subdivided, and there is much arbitrariness in this subdivision. An exact algorithm would possibly be very sensitive to changes in the subdivision, but in the real world such changes are meaningless. Secondly, finding an exact solution corresponding to a given set of input data is much less interesting and useful than having a whole spectrum of solutions corresponding to different sets of input data. A sensitivity analysis has typically to be carried out on parameters like the space discount rate, the travel costs, the tradeoff between benefits and costs, the elasticity of demand to accessibility, the minimum feasible size for the activities, and so on. Finding an exact solution for all the possible combinations of different values for these parameters is usually prohibitive. Third, producing numerical solutions is not the only aim of optimal location models, nor is it necessarily the main one. Qualitative understanding of the relationships among the main factors affecting location patterns is often a much more interesting goal, both in theory and in applications.

The reasons listed above suggest that fast and easy heuristic approximations could be a useful tool for optimal location problems.

A heuristic approach to solving (74)-(76), subject to the integrality conditions on the multipliers x_j^k, may be developed starting from equations (81)-(91), which can be summarized as follows

let k^* maximize

$$Q_j^k(\nu) - \lambda a_j^k$$

then

$$\text{if} \quad Q_j^{k^*}(\nu) - \lambda a_j^{k^*} > 0 \ , \qquad x_j^{k^*} = 1 \tag{94}$$

$$\text{if} \quad Q_j^{k^*}(\nu) - \lambda a_j^{k^*} = 0 \ , \qquad 0 \le x_j^{k^*} \le 1 \tag{95}$$

$$\text{if} \quad Q_j^{k^*}(\nu) - \lambda a_j^{k^*} < 0 \ , \qquad x_j^{k^*} = 0 \tag{96}$$

The above result refers to general (noninteger) values for x_j^k. Since, however, an integer solution is looked for, it will be assumed that only integer values will be introduced in the trial solutions. Therefore equation (95) can be dropped and the following heuristic optimality conditions are obtained:

$$\text{if} \quad Q_j^{k*}(\nu) \geq \lambda a_j^{k*} \ , \qquad x_j^{k*} = 1 \tag{97}$$

$$\text{if} \quad Q_j^{k*}(\nu) < \lambda a_j^{k*} \ , \qquad x_j^{k*} = 0 \tag{98}$$

Conditions (97) and (98) state a very reasonable efficiency principle. If the total size required by the activity k* in location j is greater than, or equal to, the fixed charge a_j^{k*} to be paid for establishing it (multiplied by the trade-off parameter λ), than j is a good location for activity k*, and it is worth establishing it there. If the total size required by the activity k* in location j is less than the fixed charge term, then j is a bad location for activity k*, which will not be established there. From (97) and (98) a very simple interpretation of the fixed-charge term follows: λa_j^k *is the minimum feasible size for an activity k in location j.* This interpretation is very useful in applications, since it is often easier to assess the values for the minimum feasible sizes, rather than for the fixed costs. Conditions (97) and (98) can be rephrased in the following first rule of thumb (ROT1):

> *ROT1* *choose only those locations where at least one activity requires a capacity at least as great as the minimum feasible size; establish in each of these locations only the activity with the highest difference between required capacity and minimum size*

The rationale behind ROT1 is that only those activities will be established that attract enough demand to justify at least the minimum feasible size. The reason why only one activity is possibly established in each location is because of constraints (72). But now these constraints can be easily relaxed, and the more general case, in which many different activities can be established in the same location, can be introduced. The efficiency conditions for this case are stated in the following, second rule of thumb (ROT2):

ROT2	*step 1*	*for each possible location, rank the activities according to the difference between required capacity and minimum feasible size; if this difference is negative, drop the corresponding activities from the list*
	step 2	*establish in each location as many activities as possible, choosing them according to the ranking obtained in step 1*

There is some vagueness in step 2, since the precise meaning of "as many activities as possible" has not been defined. However, it is felt that it is better to keep this vagueness, and leave the decision maker some freedom on judging by inspection when to stop picking up activities from the list. This freedom is needed because the constraints imposed in each zone by the limited availability of space and by the already existing physical stocks act in a nonsmooth and hardly quantifiable way. The tools provided by ROT2 do not solve the problem of how to meet these constraints in the best way, which is left to town designers and architects. ROT2 simply yields a set of indicators by means of which activities and locations can be ranked for a possible choice.

It is worth recalling that, although the indicators and the ranking produced by ROT2 are very simple and intuitive, they are rooted in a rigorous ground, since ROT2 has been obtained by suitably approximating and generalizing the exact optimality conditions (94), (95), and (96).

Both ROT1 and ROT2 can be used to generate improved values for the variables x_j^k. The general structure of a possible iterative algorithm is shown in the block diagram of Figure 3.

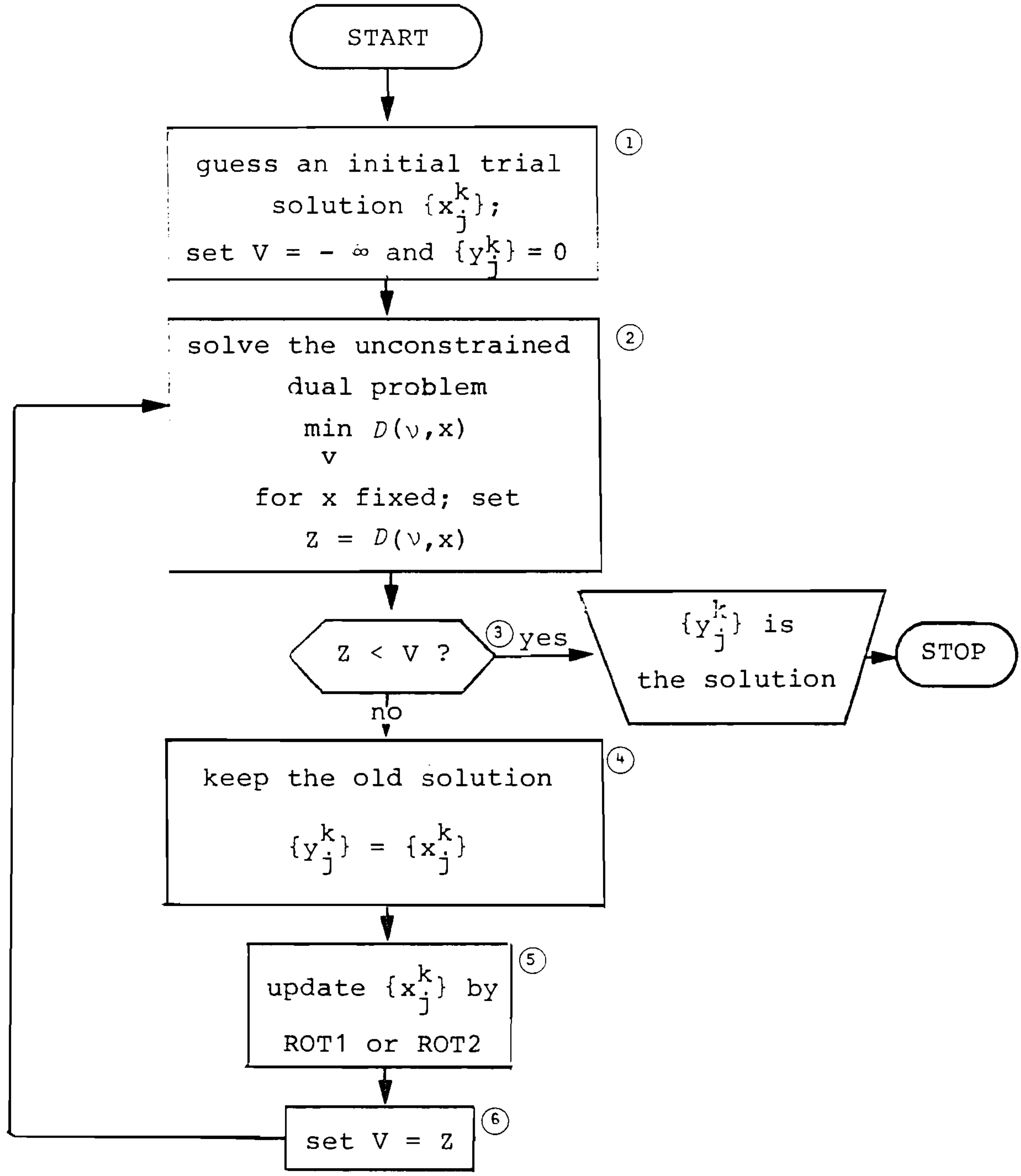

Figure 3. An iterative heuristic algorithm for the multi-activity location problem.

The diagram is self-explanatory and only a few comments are needed. The function $D(\nu,x)$ is the one defined by (66). Step 1 is quite arbitrary and any initial guess can be used. However, since the algorithm is a heuristic one, independence of the final solution from the initial guess may not be assured. Therefore, it is worth putting some effort in finding a good initial guess. When no better assumption is available, two possible starts are:

1. all activities are established in all locations;
2. no activity is established in any location.

Although assumption 1 may seem more reasonable, assumption 2 has some definite computational advantages. This can be shown by performing the first iteration of the algorithm. If all $x_j^k = 0$, then

$$\min_{\nu} \; D(\nu,x)$$

reduces to

$$\min_{\nu} \sum_{ik} \left(g_i^k \, e^{-\nu_i^k} + \nu_i^k \, Y_i^k \right) \tag{99}$$

[(99) follows from (61) and (66)]. Standard calculus yields the solution to (99)

$$e^{-\nu_i^k} = \frac{Y_i^k}{g_i^k} \quad , \qquad \text{or} \quad e^{\nu_i^k} = \frac{g_i^k}{Y_i^k} \tag{100}$$

and substitution of (100) into (64) yields the total capacities required in each location

$$Q_j^k = e^{-\lambda b_j^k} \left[\sum_i f_{ij}^k \, \frac{Y_i^k}{g_i^k} \, \prod_r \left(\frac{g_j^r}{Y_j^r} \right)^{a_{kr}} + h_j^k \right] \tag{101}$$

The right-hand side of (101) depends only on given constant terms, and can be computed beforehand. Going now to step 5 of the algorithm, the differences

$$Q_j^k - \lambda a_j^k \tag{102}$$

are computed and used to update the x_j^k, either by ROT1 or by ROT2. A sensible updating can be performed only if some of the differences (102) are positive. If, however, it is found at this step that all the differences (102) are negative, the algorithm stops after two iterations, and the final solution is the same as the starting guess 2, that is doing nothing. When this happens, no location problem exists. It is therefore suggested to use the start 2, in order to let the algorithm check for this possibility from the very beginning. It is also suggested that, when the algorithm has such a stop, input data should be carefully checked for possible mistakes.

The method suggested for step 2 is computationally the best one, since the mathematical program

$$\min_{\nu} \; D(\nu, x)$$

for x fixed reduces to

$$\min_{\nu} \; D(\nu) \tag{103}$$

where $D(\nu)$ is the function defined by (63) or (65). Problem (103) is a simple convex unconstrained minimization problem, which can be solved by standard techniques. However, it might be felt that it is rather abstract, since it is defined in terms of the dual variables, to which no immediate intuitive meaning can be given. It may therefore be useful to reformulate the resulting capacities Q_j^k in terms of the more "physical" quantities and indicators introduced in Section 2.2. Some rearrangements of

(32), (38), (39), and (42) yield

$$Q_j^k = V_j^k \left[\frac{1}{h_j^k} \prod_r \left(\frac{\phi_j^r}{G_j^r} \right)^{a_{kr}} \psi_j^k + 1 \right] \qquad (104)$$

where all the variables have been already defined in Sections 2.1 and 2.2. The main definitions will be briefly restated

V_j^k is the unused capacity of activity k in location j

ϕ_j^r is the accessibility to activities r from location j

ψ_j^k is the potential of unsatisfied demand for activity k in location j

G_j^r is the demand for activity r generated in location j

Equation (104) gives an intuitive interpretation of the mechanism implied by (103), and indicators like accessibilities and potentials give further elements to evaluate the resulting location pattern. It might be argued that some iterative scheme based on (104), together with (31), (32), (33), could be devised, without resorting to the dual formulation (103). Such a scheme could actually be built, in close analogy with the original method proposed by Lowry (1964). However, it would be computationally very poor, compared to the efficiency of (103).

In step 3 of Figure 3 a stopping rule based on the value of the dual objective function is suggested. A stronger stopping rule could be based on the array $\{x_j^k\}$, that is: *stop when the same* $\{x_j^k\}$ *is obtained in two successive steps.* However, such a condition may possibly never be met, since the x_j^k can assume only integer values. Cycling may occur, indicating that possible multiple solutions exist, or that the algorithm is not able to give any further improvement. The rule based on the value of the dual objective function seems therefore more stable.

The algorithm of Figure 3 assumes that all the parameters are held constant. It will now be shown how it can be generalized to perform some sensitivity analysis. As an example, let it be assumed that a sensitivity analysis on the trade-off parameter λ is needed. A possible algorithm is shown in the block diagram of Figure 4. In this algorithm it has been assumed that only ROT2 is used. A few explanations will be given. The functions $D(\nu)$ and $\mathcal{D}(\nu,x)$ are defined by equations (65) and (66), respectively. The sensitivity analysis starts with $\lambda = 0$, that is no costs are paid to establish the activities (alternatively, no constraint is put on the minimum feasible size of the activities). $\mathcal{D}(\nu,x)$ reduces $D(\nu)$, and all activities are established in all locations.* This is the meaning of the initial steps 1 and 2. In step 3 a nonzero trade-off paramter is introduced, and its initial value is set equal to a given step size τ. Steps 5-11 closely resemble the routine 2-6 of the algorithm of Figure 3. There are two major differences, however. First, the initial trial solution is replaced by the solution produced in the last iteration over λ. It is argued that small changes in λ produce small changes in the array $\{x_j^k\}$, so that the optimal $\{x_j^k\}$ for a given λ should be a good start to find the optimal $\{x_j^k\}$ for $\lambda + \tau$. In this way, the information gained at each iteration over λ is used to speed up the convergence at the next iteration. Secondly, the new stopping rule 7 has been introduced. Its meaning has already been discussed in connection with the problem of the choice of an initial start for the algorithm of Figure 3. When the value of λ is such that

$$Q_j^k(\nu) < \lambda\, a_j^k \qquad \text{for all } j \text{ and } k \tag{105}$$

*Here it is assumed for simplicity that ROT2 is used with no constraints on the number of activities to be established in each location. A modified version of the algorithm which takes such constraints into account can be easily developed.

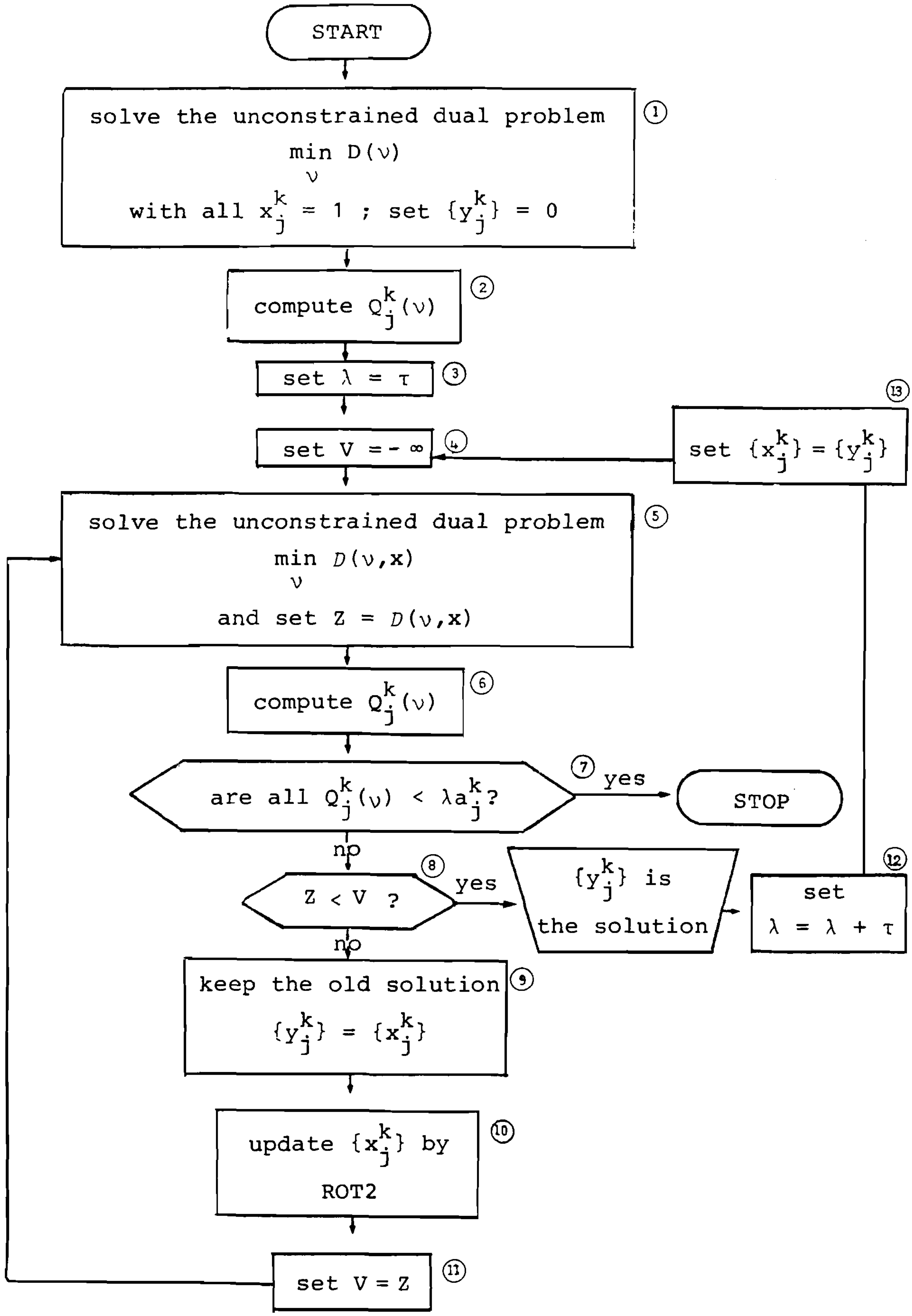

Figure 4. An iterative heuristic algorithm for the multi-activity location problem with sensitivity analysis on the trade-off parameter.

then no location problem exists, and no further sensitivity analysis over λ is needed. However, it should be noticed that the logic behind the algorithm of Figure 3 here is turned upside-down. For the algorithm of Figure 3 a start with no activity in no location has been suggested. In the algorithm of Figure 4 a start with all activities in all locations is required, and conditions (105) are used as a final stopping rule. This is because in the algorithm of Figure 4 conditions (105) are reached in a natural way, and have nothing to do with possible mistakes or inconsistencies in the input data.

In step 12 of Figure 4 the value of λ is updated by incrementing it with the step size τ, and the whole process is repeated.

The above approach can be extended to include the sensitivity analysis on other meaningful parameters, like the space discount factor. The details of these extensions will not be developed here.

4. SOME APPLICATIONS

4.1 The Urban System

The descriptive model of a simplified urban system has been already outlined in Section 2.3. Here it will be shown how this model can be embedded in a mathematical program which optimizes the location of housing and services. The same notation and assumptions of Section 2.3 will be kept, and the following further assumptions will be introduced.

a. The unsatisfied demand for housing is always zero, that is, the housing demand is accessibility insensitive. This assumption implies that the housing stock will always be made big enough to satisfy all the demand. It also implies

$$g_i^1 = 0 \qquad \text{for all } i$$

b. The unused capacity for services is always zero, that is, the service demand is congestion insensitive. This assumption implies

$$h_j^2 = 0 \qquad \text{for all } j$$

c. The costs for establishing the housing and the service facilities consists of the fixed charge term only. Otherwise stated, instead of establishing costs, minimum size requirements are introduced. Moreover, the minimum feasible sizes are the same for all locations.

d. No constraints are placed on the number of different activities which may be established in each location, that is, housing and services may always be located in the same place.

The above assumptions have been introduced for sake of simplification, although there is no real computational obstacle to solve the problem in its more general form.

Assumption a prevents the formation of unsatisfied housing demand. The existence of such a demand implies introducing phenomena like cohabitation, overcrowding, and formation of slums, which are surely realistic features of many real urban systems. However, it is felt that such phenomena need the introduction of variables other than accessibility and available capacity to be fully explained, and this would go beyond the scope of this simplified example. Assumptions b and c imply that all the service facilities are equally attractive, and only their accessibility determines the customers' choice. This is clearly an oversimplification in the more realistic case where the service sector is disaggregated in many subsectors. Assumption d is indeed very realistic in an aggregate model, although it is no longer so when services and possibly housing are disaggregated.

In spite of the above limitations, it is felt that the analysis of such a simplified model will be useful to understand the basic structure of the relationship between location patterns and space, without overshadowing it with other social and economic details.

Application of equations (60) yields

$S^1_{ij} = f^1_{ij}\ e^{-(\nu^1_i - a_{12}\nu^2_j)}$ the households working in i and living in j

$S^2_{ij} = f^2_{ij}\ e^{-(\nu^2_i - a_{21}\nu^1_j)}$ the trips made by housholds living in i to services in j

$U^2_i = g^2_i\ e^{-\nu^2_i}$ the unsatisfied demand for services in i

$V^1_j = h^1_j$ the unused housing capacity in j

and substitution of the above results into (64) gives

$$Q^1_j = e^{a_{12}\nu^2_j} \sum_i f^1_{ij} - e^{-\nu^1_i} + h^1_j \qquad (106)$$

the total housing capacity required in j

$$Q^2_j = e^{a_{21}\nu^1_j} \sum_i f^2_{ij}\, e^{-\nu^2_i} \qquad (107)$$

the total service capacity required in j

Equations (106) and (107) can be introduced in the algorithms of Figure 3 and 4; if the following quantities are defined

z_1 the minimum feasible size for a housing facility

z_2 the minimum feasible size for a service facility then the heuristic optimality conditions (97) and (98) become

if $Q_j^1 \geq z_1$ establish a housing stock of size Q_j^1 in j

if $Q_j^2 \geq z_2$ establish a service stock of size Q_j^2 in j

Equations (106) and (107) are very simple, and express the required capacities in terms of the dual variables ν_i^1 and ν_i^2. If, however, a more "physical" representation is preferred, then equation (104) may be used, and the result is

$$Q_j^1 = \left(\frac{\phi_j^2}{G_j^2}\right)^{a_{12}} \psi_j^1 + h_j^1 \tag{108}$$

$$Q_j^2 = \left(\frac{\phi_j^1}{G_j^1}\right)^{a_{21}} \psi_j^2 \tag{109}$$

where

ϕ_j^1 is the accessibility to the housing stock for householders working in j

ϕ_j^2 is the accessibility to services for households living in j

ψ_j^1 is the potential of housing demand in j

ψ_j^2 is the potential of service demand in j

G_j^1 is the demand for housing generated in j, that is, the number of householders working in j

G_j^2 is the demand for services generated in j, that is, the number of trips to services made by households living in j

While the interpretation of equation (108) is straightforward, equation (109) requires some explanation. It may seem strange that the total service capacity in j, Q_j^2, is proportional to a power of the accessibility to the housing stock from j, ϕ_j^1. But it must be recalled that services need workers, and thus generate housing demand. Equation (109) states a simple balancing principle, by means of which service location is determined both by nearness to demand (by means of demand potential ψ_j^2), and by nearness to housing facilities required for workers in the service sector (by means of the accessibility ϕ_j^1).

4.2 The Health Care System

Let a health care system be given which satisfies the following assumptions.

a. The system consists of N types of facilities, each type numbered from 1 to N. A facility belongs to level k if it is of type k.

b. Patients go to a facility of level $k \neq 1$ either from their residence or from a facility of level $k-1$. Patients go to a facility of level 1 only from their residence.

c. The service capacity is fully used in all levels, that is, the health care demand is congestion insensitive.

d. The minimum feasible size of the facilities of each level is given.

The ordering of health care facilities into levels is usually associated with different degrees and specializations of treatments. For instance, the first level might include general-purpose day-care facilities, usually fairly scattered and accessible; the second level might include urban hospitals, where more specialized and infrequent treatments are available, usually localized in few places; the third level might include regional hospitals, where very specialized treatments are available, usually very localized. The number of levels may vary with different health care organizations in different countries. Assumption b should be relaxed if further disaggregations of specialities within the same level are introduced. In this case a tree structure, rather than a simple ordering, might be more appropriate. However, here only the aggregate case will be considered, in order to keep the example as simple as possible.

The following definitions will be needed

$a_{k,k+1}$ is the fraction of patients in the facilities of level k which require a treatment in a facility of level k+1. It will be assumed that $0 < a_{k,k+1} < 1$ for all $k = 1,\ldots,N-1$, and $a_{N,N+1} = 0$

Y_i^k is the demand for facilities of level k from the residences in i

z_k is the minimum feasible size for a facility of level k

Application of equations (60) and (61) yields

$$S_{ij}^k = f_{ij}^k \, e^{-(\nu_i^k - a_{k,k+1}\nu_j^{k+1})} \quad , \qquad \text{for } k \neq N$$

$$S_{ij}^N = f_{ij} \, e^{-\nu_i^N}$$

$$U_i^k = g_i^k \, e^{-\nu_i^k}$$

and substitution of the above results into (64) gives for the required capacities

$$Q_j^k = e^{a_{k,k+1}\nu_j^{k+1}} \sum_i f_{ij}^k \, e^{-\nu_i^k} \quad , \qquad \text{for } k \neq N \tag{110}$$

$$Q_j^N = \sum_i f_{ij}^N \quad e^{-\nu_i^N} \tag{111}$$

The heuristic optimality conditions (97) and (98) become

if $Q_j^k \geq z_k$ establish a facility of level k in location j

A reinterpretation of (110) and (111) by means of equation (104) is also possible. The required service capacities, expressed in terms of accessibilities, potentials, and generated demands, are

$$Q_j^k = \left(\frac{\phi_j^{k+1}}{G_j^{k+1}}\right)^{a_{k,k+1}} \psi_j^k \quad , \qquad \text{for } k \neq N \tag{112}$$

$$Q_j^N = \psi_j^N \tag{113}$$

Equation (112) states that the size of the facility of level k required in location j depends both on the demand potential for level k (a term depending on the residences and the levels below k) and on the accessibility to the facilities of level k+1. Equation (113) states that the size of the facility of the highest level, N, required in location j depends only on the demand potential for level N, that is, from the residences and the levels below N.

A special case is worth being mentioned, which is relevant for the health care example. Although it has always been assumed that customers behave according to a spatial interaction model, it may be possible that the transport of patients between some levels assumes an emergency character. In this case, the choice of the destination is no longer left to the customer; moreover, it is reasonable to assume that the accessibility sensitiveness disappears, since all emergency cases must be served. One possible approach to introduce emergency trips in the optimization model is as follows. If r is the level to which emergency trips are made, the corresponding terms

$$- \sum_{ij} S_{ij}^{r} \left(\log \frac{S_{ij}^{r}}{f_{ij}^{r}} - 1 \right) - \sum_{i} U_{i}^{r} \left(\log \frac{U_{i}^{r}}{g_{i}^{r}} - 1 \right) - \sum_{j} V_{j}^{r} \left(\log \frac{V_{j}^{r}}{h_{j}^{r}} - 1 \right)$$

in the primal objective function (30) are replaced by the single term

$$- \sum_{ij} S_{ij}^{r} t_{ij}$$

where t_{ij} is the travel time between locations i and j. Moreover, the inequality

$$S_{ij}^{r} \geq 0$$

is added to the list of constraints. It may be easily shown that the terms $U_{i}^{r}(\nu)$ and $Q_{j}^{r}(\nu)$ disappear from the dual objective function (65), and the dual problem is no longer unconstrained, since the constraints

$$t_{ij} \geq \sum_{k} a_{rk} \nu_{j}^{k} - \nu_{i}^{r} \qquad (114)$$

must be met. It may also be shown that the choice of the destination in this case reduces to the nearest-facility rule. Therefore, the required capacity for a facility of level r in location j is no longer given by (64), but by

$$Q_{j}^{r} = \sum_{i} (Q_{i}^{r-1} a_{rk} + Y_{i}^{r}) \delta_{ij} \qquad (115)$$

4.3 The Retail System

Let the multilevel assumption of the health care example be relaxed, so that trips between each pair of activities are possible. The resulting model is an appropriate one for systems where customers make trips with multiple destinations. A typical example is given by a retail system, where customers go shopping for different goods, not necessarily available in the same location. The behavioral model for such a system assumes the most general form discussed in Section 2.2, if the following definitions are introduced

Y_i^k is the number of trips originating from households living in i and having a retail activity k as a first destination

a_{rk} is the fraction of customers served in a retail activity r which looks for a retail activity k as the next destination

If no new assumptions are introduced, the resulting conditions for the optimal sizes are analogous to the ones obtained for the preceding examples. However, the retail system example poses a new interesting problem on the supply side, which is worth being discussed. So far it has been assumed that each activity gives rise to different facilities, and no common costs are shared among them. But in the retail case it may be sensible to lump a subset of different activities together, so as to reach overall economies of scale which each single activity could not reach. This problem may be paraphrased as the problem of the *optimal location and composition of shopping centres*. A slight generalization of the assumptions on establishing costs is required. Let the total cost for establishing a shopping centre in location j be given by the sum of

$a_j^k + b_j^k Q_j^k$ a linear-plus-fixed charge cost to be paid for each activity k where

$$\delta_{ij} = \begin{cases} 1 \quad , & \text{if } j \text{ is the nearest location of a facility of level } r \\ 0 \quad , & \text{otherwise} \end{cases}$$

Another approach to introduce emergency trips is as follows. A very steep function f(t) of travel time may be defined, like the one shown in the graph of Figure 5, and the impedance factor for level r may be given the value:

$$f_{ij}^{r} = f(t_{ij})$$

This approach has the advantage of requiring no changes to the original formulation of the problem. However, computational problems may arise from the small numbers introduced by the function f(t) defined above.

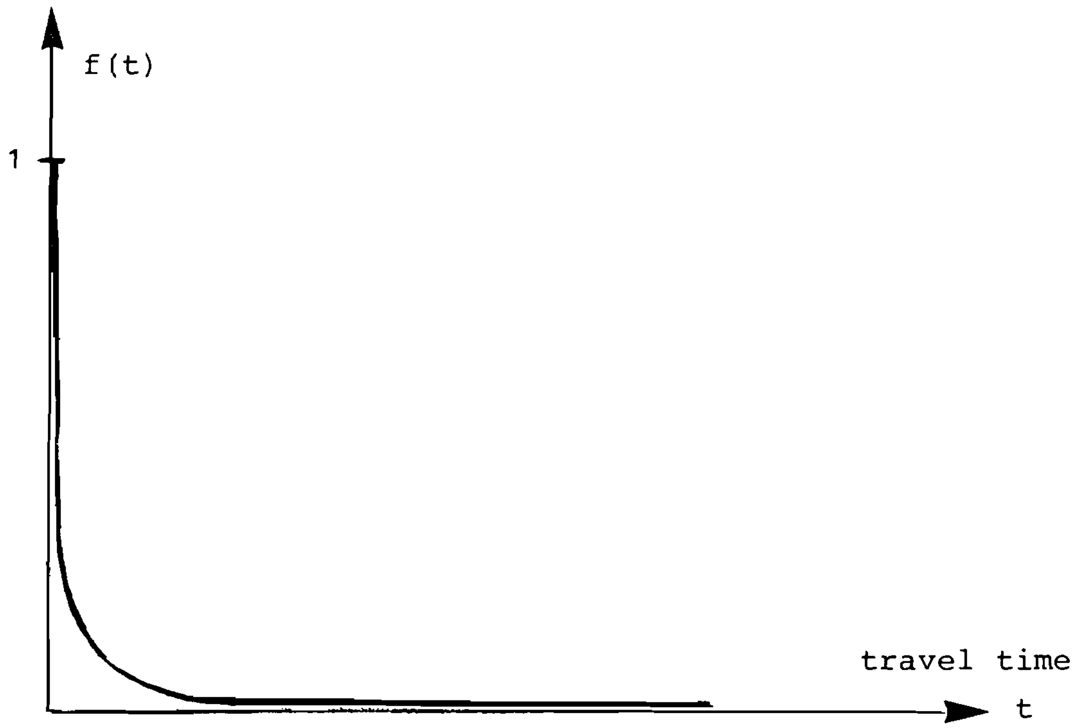

Figure 5. A very steep decay function of travel time.

q_j an overall fixed charge cost to be paid for establishing a shopping centre in j, independently of its size and of the retail sectors it is composed of

By means of the above assumptions the modified dual objective function (66) can be generalized as follows

$$D(\nu,x,y) = \sum_{ik} \left[U_i^k(\nu) + \nu_i^k Y_i^k\right] + \sum_j \{\sum_k x_j^k \left[Q_j^k(\nu) - \lambda a_j^k\right] - \lambda q_j y_j\} \quad (116)$$

where the new variables y_j are defined as

$$y_j = \begin{cases} 1 \; , & \text{if a shopping centre is established in location j} \\ 0 \; , & \text{otherwise} \end{cases}$$

The resulting saddle-point problem, analogous to (67)-(69), is

$$\max_{x,y} \min_{\nu} \; D(\nu,x,y) \quad (117)$$

$$\text{s.t.} \quad 0 \le x_j^k \le y_j \quad (118)$$

$$x_j^k \in \{0,1\} \quad (119)$$

The constraints (118) have the following meaning. When at least one activity is established in j, the way the function (116) has been built forces y_j to assume the value 1, that is, a shopping centre is open in j. When no activity is established in j, the $y_j = 0$ and no shopping centre is open in j. A comparison of (117)-(119) with (67)-(69) shows that the constraints (68), requiring no more than one activity in each location, have been dropped. These constraints have already been relaxed in Section

3.3, and for the shopping centre problem they are clearly meaningless, since an optimal combination of <u>many</u> different activities in the same location is looked for.

Problem (117)-(119) may be called a "nested" fixed charge problem, since fixed costs have to be paid at two levels, the activity level and the location level. Arguing as for (97)-(98), the following heuristic optimality conditions are obtained

if the subset A_j *of activities for which*

$$Q_j^k \geq \lambda a_j^k$$

is nonempty, and

$$\sum_{k \in A_j} (Q_j^k - \lambda a_j^k) - \lambda q_j \geq 0 \tag{120}$$

a shopping centre composed of the activities $k \in A_j$ *is established in* j*;*

if A_j *is empty, no shopping centre is established in* j

The above conditions may be looked at as a special form of ROT2, where the expression "as many activities as possible" of step 2 has been given a very precise meaning, which is: all activities in the subset A_j, provided it is nonempty and (120) is satisfied. Otherwise stated, activities are required to cover not only their own costs, but also the overall fixed cost for the location they share in common.

5. CONCLUDING COMMENTS AND ISSUES FOR FURTHER RESEARCH

As it has been stated many times in this paper, practical tools for urban planning should not place a disproportionate effort in looking for exact solutions to optimization problems. The ill-defined nature of many data and assumptions make it hard to give a realistic meaning to such an exactness. In the field of location problems, it is felt that a better understanding of customer behavior is by far more important than superimposing an "optimal" solution on poor behavioral assumptions. On the contrary, the main effort in the existing literature on location models has been placed on developing hundreds of algorithms to solve problems based on empirically untenable behavioral assumptions. The generalized multiactivity spatial interaction model proposed in Section 2 is not necessarily the best possible one, but it is felt that the model is able to introduce some commonly neglected features of customer behavior, like sensitiveness to accessibility and congestion, in a realistic way. When such features are embedded in an optimization framework, like the one developed in Section 3, the resulting mathematical programs are usually hard to solve. However, there are good reasons to believe that heuristic solutions to a problem based on sound behavioral assumptions are possibly better than exact solutions to a problem based on an over-simplified model of customer behavior. This spirit has guided the development of the methods suggested in Section 3.3 and further specialized in the examples of Section 4. It has also been shown that all the steps of these methods have intuitive interpretations. This may sometimes be dangerous, since common sense and mathematical optimality do not necessarily agree all the time. However, this danger is more than compensated for by the deeper insight which is gained in the structure of the location problem.

Another major goal of this paper has been to provide flexible methods. The outputs of the algorithms proposed in Section 3.3 should be used in a qualitative way, rather than in a quantitative one. The resulting rules for ranking the activities and the locations are much more important than the specific facility sizes resulting from a given set of input data. Sensitivity analysis has also been suggested as a standard approach, and possibly as the only sensible one to solve goal assessment problems.

The above remarks should not be interpreted as an underestimation of the importance of analyzing exact mathematical programming formulations. It has been shown in Section 3.2 how much insight can be gained simply by looking at the optimality conditions and at some duality relationships. Indeed, the exact optimality conditions and the duality results are the roots of the proposed heuristics. Further exploration of the formal properties of the exact formulations is therefore an issue for future research, as long as it will provide better economic interpretations and implementable algorithms.

Many other issues for future research can be listed. Some of them are obviously important, like the development of dynamic versions and the introduction of more complex cost functions and constraints. These themes will be developed in forthcoming papers in this series. Two of them deserve special attention for short term applied developments.

The first one is calibration. The whole framework developed in Section 2 may become useless without an efficient technique for calibrating the many parameters involved in it. The main difficulty lies in the introduction of hardly observable quantities, like the potential demand or the unused capacity. Such calibrating problems have already been solved in the simple case of a single activity (Walsh and Gibberd, 1980), and generalizations to the multiactivity case are under study.

The second one is the introduction of the transport network. If the proposed model is applied to the whole urban system, as suggested in Sections 2.3 and 4.1, then also the traffic conditions are significantly affected, and this must be accounted for in benefit and cost evaluation. Descriptive models combining multiactivity spatial interaction sytems and traffic assignment have been already developed by some authors, among them Evans (1976). A normative approach to the same problem, but with the transport network assumed as given, has been proposed by Boyce and LeBlanc (1979). The natural next step is therefore introducing the transport network in the list of decision variables. The complexity of the resulting optimization problem may be discouraging, if an exact algorithm is looked for. However, there are good reasons to believe that easily interpretable heuristics can be developed, along the lines suggested in this paper for the location problem. The use of flexible and qualitative decision rules is even more sensible in this case, since usually a planning authority does not disrupt the existing network. Tools for managing the existing network, and possibly indicating the required changes, are therefore needed, and easily interpretable benefit cost indicators and ranking rules seem to be well suited for this purpose.

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