A MATHEMATICAL BASIS FOR SATISFICING DECISION MAKING

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SUMMARY

This paper presents an analysis of the satisficing decision-making process in a simple organization under multiple objectives. The role of aspiration levels or reference objective levels is stressed and a conceptual model of this behavior is presented.

A specification or rather modification of the mathematical concept of a value (utility) function that describes the satisficing behavior is given; the modified value function, called the achievement scalarizing function, should not be only order preserving but also order approximating in a certain sense. It is shown that the notions of reference objective levels and achievement scalarizing functions form a mathematical basis not only for satisficing decision making but also for Pareto optimization; this basis is an alternative to or even stronger than the approaches based on weighting coefficients or typical value functions. This mathematical basis, which can also be considered as a generalization of the goal programming approach in multiobjective optimization, results in pragmatic approaches to many problems of multiobjective analysis, including the problem of interactive assessment of solutions to economic models for policy analysis and planning purposes.

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INTRODUCTION

This paper is aimed at providing a mathematical background for satisficing decision making. It is assumed that the reader is well-acquainted both with the methodological reflection leading to the idea of satisficing decision making (March and Simon 1958, Boulding 1955) as with the state of the art of optimizing decision making and multiattribute decision analysis as representd, for example, by Bell et al. (1977), Charnes and Cooper (1961), Cohn and Marks (1975), Fishburn (1970), Haimes et al. (1975), Keeney and Raiffa (1976), and Nash (1950), and that he has also encountered some of the vexing problems in the applications of this highly developed theory. It is worthwhile, however, to reflect briefly on some of the main points in the discussion between the optimizing versus satisficing approaches to decision making.

The basic questions in applications of multiattribute decision analysis or multiobjective optimization may take various forms (see, e.g., Ackoff 1979, Dreyfus and Dreyfus 1976, Wierzbicki 1979) but can be summarized as follows:

-- *Is the maximization of a value (utility) function an adequate model for typical decision-making processes? If the rationality of a decision is restricted by various external or institutional aspects, how do we best model the decision-making process mathematically?*

The development of the classic apparatus of multicriteria optimization, preference relations, utility, and value theory, beginning with Pareto in 1896 and culminating with Debreu in 1959, was strongly related to economic theory. However, *economic theory is concerned with averages of thousands of decisions* and the individual consumer in this theory is a mathematical construction which has averaged out externalities, institutional dependencies and other whims of the individual. More recent developments of this theory take into account persistent externalities by introducing additional constraints and examining the restricted rationality of decision making--see Arrow (1974).

On the other hand, most individual decisions are made in some organizational structures. Even when shopping in a supermarket, an individual consumer often has a list of items to buy, composed with the help of his family, and his own rationality of choice is partly restricted by this list. When buying some new equipment, a manufacturer is restricted by various environmental and safety standards. Expressing such externalities by additional constraints to utility maximization is certainly possible, though not necessarily the best way to account for them. Not all of them have the hard character of a mathematical inequality; some might be overcome by ingenuity or trade-offs in other resources and should, therefore, be expressed by softer mathematical tools. These tools have not been fully developed yet and *the existing utility and value theory does not fully explain how decisions are made in organizational structures*.

This fact has been recognized by many economic theorists--Boulding (1955), March and Simon (1958), and others. An alternative satisficing approach to decision making has been developed: decisions in organizations are made to satisfy certain aspiration levels, not to maximize a utility or value function. Much methodological reflection and analysis support this approach. However, the problems of a mathematical description of the satisficing approach and its relations to optimizing approaches have not been investigated in more detail.

More recently, further interest in decision analysis has been stimulated by system analytic problems which encompass economic, technological, sociological, and environmental objectives and constraints. When aggregating such goals, utility or value functions do not usually have a straightforward objective meaning but reflect rather subjective preferences of a decision maker or a group of experts. Although there have been attempts to apply a satisficing approach in systems analysis, most of the detailed studies (Bell et al. 1977, Fishburn 1970, Keeney and Raiffa 1976) on decision analysis were related strongly to preferences and utility theory. Identification methods have been developed for individual and group preferences described by utility and value functions; statistical approaches have been considered to take into account uncertainty and risks; and interactive procedures have been devised in order to involve a decision maker more directly into the decision process based on learning about his preferences. There have also been many successful applications of this highly developed theory, particularly if the compared alternatives are given explicitly, their number is not too large, and the difficulty of the problem is related to comparing various sociological, environmental and economic consequences of the alternatives; a psychometric experiment performed on a group of experts helps them to better understand their own preferences.

However, it has been realized that while evaluating given alternatives is an important task, an even more important problem in systems analysis is *generating alternatives*. For example, the mathematical models used in economic and sociological planning describe implicitly an infinite number of alternatives and their consequences, and the problem is how to generate a restricted number of explicit alternatives with the help of these models in a region of interest of the decision maker or a group of them. This problem is related to the satisficing rather than the optimizing approach to decision making, and many researchers in multiobjective optimization have realized the need of an appropriate mathematical formulation. Sakluvadze (1971, 1974), Yu and Leitmann, and others considered the use of utopia points representing some unattainable aspiration levels as reference points for generating alternatives. Charnes and Cooper (1961), Dyer (1972), Kornbluth (1973), Ignizio (1978) and others developed goal programming--the use of variable bounds on objective levels in the process of multicriteria optimization. Yet these and related works have not had the impact they deserve for several reasons.

First, although many partial results have been obtained, a mathematical basis for satisficing decisions and their relation to optimal decisions has not been fully developed. Thus, the approaches based on the use of reference objectives--that is, any desirable aspiration levels for objectives--were looked upon as somewhat less scientific, 'ad hoc' approaches. It was not clear whether it is possible to develop a consistent, basic theory of multiobjective optimization and decision making starting with the use of reference objectives rather than with weighting coefficients or value (utility) functions. In other words, the necessary and sufficient conditions, existence conditions, relations to preference orderings, etc., had to be formulated in terms of reference objectives. Some more abstract aspects of this question have been analyzed in earlier works of the author (Wierzbicki 1975, 1977a, 1977b, 1978, 1979); a synthesis of relevant results is presented in this paper.

Second, although many researchers realized the relations between satisficing decision making and such approaches as goal programming (see, e.g., Ignizio 1978), some basic methodological questions have not been sufficiently analyzed: *What can be logically assumed about the decision-making process in a simple organization*, whose preferences in this organization should be mathematically modelled? What is the relation between satisficing decision making and utility or value maximization, etc.? *The main prupose of this paper is to present an analysis of such methodological questions together with resulting mathematical development.*

A METHODOLOGICAL HYPOTHESIS

The following hypothesis describes a conceptual model for the decision making process in a simple organization. The organization consists of a top decision maker or a group of them aggregated here for simplicity in a single unit and called *the boss*, and of technical or professional staff, again aggregated here in a single unit and called *the staff*. *The boss* formulates a decision problem for the staff, asking them to prepare one or several plans of action to attain certain goals; he *formulates the goals in terms of aspiration levels for several objectives*. *The staff* examines possible actions in detail, checks attainability of aspiration levels and *proposes detailed plans of action*. The boss can either accept a proposed plan and decide to execute it, or change his requirements and let the staff prepare new plans.

It is necessary now to make several idealizing assumptions that specify additionally some aspects of the decision making process and result in a relatively simple mathematical model of the organization.

First, *it is assumed that the goals of the action are clearly and completely preceived*. In other words, the boss and the staff must have the same objectives in mind, including those which might be more important for the staff but less so for the boss, and have a common understanding about what it means to improve each of the objectives. This does not mean that the boss and the staff should have the same preferences on various objectives; they need not agree on details, only on principles. Additionally, it might be required that the boss specifies aspiration levels for all objectives, even for those not so important for him. In particular, resources (budget, time, etc.) allocated by the boss for the planned action might be usefully treated also as objectives rather than as constraints, and the allocated levels of resources become then aspiration levels.

Mathematically, this assumption means that the boss and the staff have the same space of objectives and the same notion of a natural inequality in this space (the same partial preordering or quasiorder of the space) but not necessarily the same preference structure (not the same complete preordering). The aspiration levels given by the boss form a reference point in the objective space. To simplify the discussion, it might be agreed that all objectives are improved if their levels are enlarged, which corresponds to Pareto maximization or to the natural partial ordering generated by the natural positive cone in the objective space; however, more complicated situations can be also analyzed.

Second, *it is assumed that the boss is consistent.* This means that he cannot prefer plans in which one of the objectives has deteriorated, all others being the same. Mathematically, it means that his preference mapping (complete preordering) is strictly monotonic in the sense of the natural inequality in objective space (preserves the partial preordering of the space). Besides this requirement, his preferences might be arbitrary.

Third, *it is assumed that the staff is dedicated and efficient.* Dedication of the staff means the same as the consistency of the boss: the preferences of the staff must increase as the objectives of the planned action improve, although the detailed pattern of these preferences might be different than those of the boss. Efficiency means something more: the staff actually maximizes the preferences and proposes only nondominated plans, that is, such that no single objective can be improved without deteriorating others (the ther 'nondominated' is preferred here to the term 'Pareto optimal', which has a more specific meaning, or 'efficient' plans, which implies economic efficiency, while various objectives might also have noneconomic interpretation). Mathematically, this assumption means that the staff preference mapping not only strictly preserves the partial preordering of the objective space, but also is maximized during the preparation of plans.

Fourth, *it is assumed that the staff takes seriously the aspiration levels and strives to attain them.* This assumption is crucial for describing the satisficing behavior in the organization and the limited rationality of choice of the staff. To better understand what restrictions result from this assumption, consider three possible types of outcomes of the work of the staff.

If the aspiration levels given by the boss are attainable with some surplus, the staff is free to use its own preferences to choose the proposed plan; but the freedom is restricted to the surplus above the aspiration levels. The staff should not bother the boss with too many questions about how to allocate the surplus; one or several detailed plans should be presented for the boss' approval, and all plans should be nondominated according to the third assumption.

If the aspiration levels are not attainable, the staff must choose plans which have results that match these levels as closely as possible. The sense of closeness is left for the staff to decide; again they should not bother the boss too much. The staff could also propose several plans corresponding to to the aspiration levels, all plans being nondominated according to the third assumption.

The simplest but most important case is when the aspiration levels are just attainable without any surplus, that is, non-dominated by any other attainable outcomes. Here the staff rationality is most severely restricted: as implied by the fourth assumption, the staff must propose at least one plan with outcomes that precisely match the boss' wishes, although some alternative plans might be proposed as well. Since it is the boss' perogative to choose and accept plans or to ask for preparation of new plans with altered aspiration levels, the fourth assumption really implies that he fully controls the organization, no matter what other properties the preferences of the staff have.

A mathematical description of the fourth assumption must be chosen to reflect this particular restriction of the staff rationality that gives full control to the boss. It will be shown later in more mathematical detail that the fourth assumption can be represented by the following *axiom of order approximation: the set of objective outcomes preferred by the staff to the aspiration levels given by the boss must closely approximate the set of outcomes that are better than the aspiration levels in the natural inequality sense* (in the partial preordering sense). In other words, the preference ordering of the staff relative to the given aspiration levels must closely approximate the natural partial preordering, common to the boss and the staff. An interpretation of this axiom is perfectly straightforward: in order not to come into conflict with their own and the boss' preferences around the aspiration levels, the perfect staff should keep to the agreed principles of what is naturally better, not to guess or bargain about what might be marginally better.

Clearly, all the above assumptions describe a type of *ideal organization*, which does not occur in practice. Staff members do bargain with their bosses, bosses are not necessarily consistent in their decisions, etc. However, the above model of an ideal organization might serve as a starting point for introducing further aspects and deviations from the ideal model.

It might also be argued that this model is too ideal to describe satisficing decision making in organizations: a main logical reason for accepting satisficing decisions is that there is usually no time to really optimize them, and the assumption of efficiency of the staff might therefore be challenged. However, the time allocated for the staff to prepare the plan might be taken into account to define conditional efficiency. Moreover, the staff is not required to optimize a global value function for the entire organization; this task is reserved for the boss, and he can really do so by changing aspiration levels if he wishes. The assumption of efficiency means only that the staff would not propose dominated plans of action, with outcomes that can be clearly improved.

The main purpose of the analysis of such an ideal organization is to define a class of functions which would describe the preferences of the staff under its limited rationality of choice. These are, in a sense, modified value functions. However, these functions must express both the utility of achieving the aspiration levels with some surplus or the disutility of not achieving these levels. Moreover, these functions must reflect the specific order approximation axiom implied by the fourth assumption. Therefore, these functions depend explicitly and nonlinearly on the assumed aspiration levels. Following the tradition of goal progamming and reference point optimization (see Ignizio 1978, Wierzbicki 1977a) these functions will be called *achievement scalarizing functions*. As it will be shown later, main axiomatic requirements defining such a class of functions are order preservation and order approximation properties. There are several reasons for studying this class of functions.

First, although the boss can control the ideal organization no matter what particular achievement scalarizing function characterizes the staff--provided the basic axioms for this function are fulfilled--the shape of this function might influence the easiness of interaction between the boss and the staff. This subject requires further theoretical and experimental studies; in this paper, only several examples of such functions are described.

Second, the notion of the ideal organization can be also used as a blueprint for devising interactive systems composed of a model user (an economist, a system analyst, a decision maker) interpreted as the boss and of a model (of econometric, system analytic, etc., nature) augmented with an achievement scalarizing function and an optimization procedure, interpreted as the staff. In a preparatory stage, it is necessary to define the model outputs that are interesting for the user, the sense of a natural inequality in the space of outputs, and also the model inputs (parameters, scenarios, etc.) that might be changed in optimization; moreover an achievement scalarizing function and an optimization procedure that maximizes this function are chosen. Then the user simply specifies desirable model outputs as aspiration levels; the system responds whether these outputs are attainable or not and proposes one or several alternatives of outputs, close to the desired in the nonattainable case, better than the desired in the attainable (with surplus) case, and matched to the desired in the just attainable case. By changing his requirements, the user can obtain various alternatives from the model. Such a system might be advantageous to interactive use of planning models, for including human judgement in formal modelling, even for devising hierarchical structures of models with various degrees of aggregation (when the upper-level model is interpreted as the boss) and in many other modelling situations traditionally approached by trial and error procedures.

Third, a detailed study of the ideal organization might serve as a starting point for various extensions: hierarchical organizations when the boss is himself part of a staff of a higher-level manager; negotiations of aspiration levels between groups of decision makers; inclusion of additional objectives by the staff; uncertainty either in the boss' requirements or in the staff's responses, etc.

Finally, observe that the above hypothesis on the decision making process in a simple, idealized organization serves several purposes. By applying notions of modified utility and value maximization when describing satisficing decision making, it provides for a bridge between these theories. On the other hand, since the boss might optimize a global value function for the entire organization but is not necessarily required to do so, the above hypothesis changes the traditional sense of optimization. If the aspiration levels represent the intuition, experience, and judgement of the boss, not formalized into a value function, then the optimization in this model of an organization is represented by the efficient work of the staff, generating alternatives that are in a sense best relative to the boss' wishes. However, these wishes are not interpreted as hard inequality cosntraints; if they are not attainable, then some alternatives that are close to them might be found. Thus, *the above hypothesis also describes a method of soft inclusion of human judgement in optimization procedures*.

MATHEMATICAL FOUNDATIONS

To represent the above hypothesis mathematically, a modification of the value or utility function concept is needed; the modified functions are called achievement scalarizing functions.

Let $E_0 \subset E$ be a set of admissible decisions or alternatives to be evaluated. Let G be a (linear topological) space of objectives or performance indices or outcomes. Let a mapping $Q: E_0 \to G$ be given, defining numerically the consequences of each alternative. Denote by $Q_0 = Q(E_0)$ the set of attainable objectives. Let a natural inequality, that is, a partial preordering in G be given; to simplify the presentation, assume that the preordering is transitive and can be expressed by a *positive cone* (any closed, convex, proper cone) $D \subsetneq G$:

$$q_1, q_2 \in G, \quad q_1 \leqslant q_2 \iff q_2 - q_1 \in D \quad . \tag{1}$$

A corresponding strong partial preordering is

$$q_1, q_2 \in G, \quad q_1 < q_2 \iff q_2 - q_1 \in \tilde{D} \stackrel{df}{=} D \setminus (D \cap -D) \, . \tag{2}$$

If the cone D has a nonempty interior $\mathring{D}$, it is possible also to introduce a strict partial preordering:

$$q_1, q_2 \in G, \quad q_1 \ll q_2 \iff q_2 - q_1 \in \mathring{D} \quad . \tag{3}$$

Suppose that we maximize all objectives (gains, etc.). A generalized Pareto (nondominated) objective is a *D-maximal element of Q_0*:

$$\hat{q} \in Q_0 \text{ is D-maximal} \iff Q_0 \cap (\hat{q} + \tilde{D}) = \emptyset \quad . \tag{4}$$

A slightly weaker notion, admitting a few more than only nondominated points is that of *weak D-maximal elements:*

$$\hat{q} \in Q_0 \text{ is weakly D-maximal} \iff Q_0 \cap (\hat{q} + \mathring{D}) = \emptyset \quad . \tag{5}$$

For a normed space G, we can define also a stronger notion of *D_ε-maximal elements*, admitting a few less than all nondominated points:

$$\hat{q} \in Q_0 \text{ is } D_\varepsilon\text{-maximal} \iff Q_0 \cap (\hat{q} - \tilde{D}_\varepsilon) = \emptyset \quad , \tag{6}$$

where D_ε is an ε-conical neighborhood of D:

$$D_\varepsilon = \{q \in G: \text{dist}(q,D) < \varepsilon \|q\|\} \; ; \; \tilde{D}_\varepsilon = D_\varepsilon \setminus (D_\varepsilon \cap -D_\varepsilon) . \quad (7)$$

An *achievement scalarizing function* (shortly, a *scalarizing function*) is a function $s: G \to R$, with argument $q - \bar{q}$ where $q = Q(x) \in Q_0$ is an attainable objective ($x \in E_0$ is an admissible decision) and $\bar{q} \in G$ is an *arbitrary* reference objective *(aspiration level, not constrained* to Q_0 nor otherwise); G is assumed to be a normed space. A scalarizing function is defined, moreover, by the following requirements:

a) it should be *strictly order-preserving* in q:

$$q_1 \ll q_2 \Rightarrow s(q_1 - \bar{q}) < s(q_2 - \bar{q}) \quad , \qquad (8)$$

or, if possible, *strongly order-preserving*

$$q_1 \prec q_2 \Rightarrow s(q_1 - \bar{q}) < s(q_2 - \bar{q}) \quad , \qquad (9)$$

where, clearly, strong order preservation implies strict order preservation;

b) it should be *order representing*

$$S_0 \stackrel{\text{df}}{=} \{q \in G: s(q - \bar{q} \geq 0\} = \bar{q} + D \; ; \; s(0) = 0 \quad , \qquad (10)$$

or, at least, *order approximating* for some small $\varepsilon > 0$,

$$\bar{q} + D \subset S_0 \stackrel{\text{df}}{=} \{q \in G: s(q - \bar{q}) \geq 0\} \subseteq \bar{q} + D_\varepsilon ; \; s(0) = 0, \quad (11)$$

where, clearly, order representation implies order approximation;

c) if $\bar{q} \in Q_0 - D$, then the maximization of $s(q - \bar{q})$ over $q \in Q_0$ should represent a concept of either allocation or maximization of the surplus $q - \bar{q} \in D$; if $\bar{q} \notin Q_0 - D$, then the maximization of $s(q - \bar{q})$ over $q \in Q_0$ should represent a concept of distance minimization between $\bar{q}$ and the D-maximal set $\hat{Q}_0 = \{\hat{q} \in Q_0: Q_0 \cap (\hat{q} + \tilde{D}) = \emptyset\}$.

Observe that requirements *a* and *b* are axiomatic, although though formulated alternatively: it is easy to show that (9) and (10) cannot be satisfied simultaneously, hence we require either (8) and (10) or (9) and (11). Requirement *c* is descriptive and partly follows from *a* and *b*.

Requirement *a* results directly in a sufficient condition of Pareto-maximality. In fact, the following well-known lemma holds (Debreu 1959, see also Da Cunha and Polak 1967, Wierzbicki 1977a):

LEMMA 1. *If s is strongly order preserving then its maximal points in* $q \in Q_0$ *are D-maximal:*

$$\hat{q} = \arg\max_{q \in Q_0} s(q - \bar{q}) \Rightarrow Q_0 \cap (\hat{q} - \tilde{D}) = \emptyset \quad . \qquad (12)$$

If s is strictly order preserving, then its maximal points are weakly D-maximal.

Requirement *b* results in a necessary condition of Pareto-maximality, much stronger than the known conditions based on weighting coefficients. The following lemma was given first in Wierzbicki (1977a), in a less general formulation:

LEMMA 2. *If s is both order preserving* ($q_1 \gtrsim q_2 \Rightarrow s(q_1 - \bar{q}) \geq s(q_2 - \bar{q})$ *for any* q_1, q_2, $\bar{q}$) *and order representing and if* $\bar{q} = \hat{q}$ *is (weakly) D-maximal, then the maximum of s over* $q \in Q_0$ *is attained at* $\bar{q} = \hat{q}$ *and is equal to zero*

$$Q_0 \cap (\hat{q} + \mathring{D}) = \emptyset \Rightarrow \hat{q} \in \operatorname{Arg}\max_{q \in Q_0} s(q - \hat{q}) \; ; \; \max_{q \in Q_0} s(q - \hat{q}) = 0. \qquad (13)$$

If s is order preserving and order approximating for a given $\varepsilon > 0$ *and if* $\bar{q} = \hat{q}$ *is* D_ε*-maximal, then the maximum of s over* Q_0 *is also attained at* $\bar{q} = \hat{q}$ *and is equal to zero, so that (13) holds with* $\mathring{D}$ *substituted by* $\tilde{D}_\varepsilon$.

The proof of Lemma 2 for an order approximating function s is as follows. Suppose $\hat{q} \notin \operatorname{Arg}\max_{q \in Q_0} s(q - \hat{q})$; then there is such

$\tilde{q} \in Q_0$ that $s(\tilde{q}-\hat{q}) > s(\hat{q}-\hat{q}) = 0$. In other words, $\tilde{q} \in \tilde{S}_0 = \{q \in G: s(q-\hat{q}) > 0\}$. Clearly, $\tilde{S}_0 \subset \hat{q} + D_\varepsilon$ by the assumption of order approximation. However, $\tilde{q} \notin \hat{q} + (D_\varepsilon \cap -D_\varepsilon)$, since $\tilde{q} \in \hat{q} + (D_\varepsilon \cap -D_\varepsilon) = (\hat{q}+D_\varepsilon) \cap (\hat{q}-D_\varepsilon)$ would imply $s(\tilde{q}-\hat{q}) = 0$ by the assumption of order preservation. Thus, $\tilde{q} \in \hat{q} + \tilde{D}_\varepsilon$ and $\tilde{q} \in Q_0$, which contradicts the assumption that $Q_0 \cap (\hat{q}+\tilde{D}_\varepsilon)$ is empty. The modification of the proof for an order representing function s is obvious. Clearly, a strictly or strongly order preserving function is order preserving, hence the assumptions of Lemma 2 are satisfied for all achievement scalarizing functions.

Observe that Lemma 2 is a necessary condition for D-maximality (or D_ε-maximality) *even for nonconvex sets* Q_0; the geometrical interpretation of this condition is that of separation of sets Q_0 and $\hat{q} + \tilde{D}$ at $\hat{q}$ by a cone S_0, see Figure 1.

Observe also that it is really requirement *b* that distinguishes mathematically a scalarizing function from a value function; the latter is usually supposed to satisfy requirement *a*. We conclude that, with the help of requirements *a* and *b* and the resulting Lemmas 1 and 2, even stronger fundamental theoretical results on multiobjective optimization are obtained than the known results based on weighting coefficients; thus, the reference objectives are not only an equivalent, but an even stronger theoretical tool than weighting coefficients. Lemma 2 can be used, for example, for checking the attainability and Pareto-optimality of a given $\bar{q} \in G$. If an order representing and order preserving function $s(q-\bar{q})$ is maximized, and $\bar{q}$ is not attainable, then $\max_{q \in Q_0} s(q-\bar{q}) < 0$; if $\bar{q}$ is attainable and weakly Pareto-optimal, then $\max_{q \in Q_0} s(q-\bar{q}) = 0$; if $\bar{q}$ is attainable but not weakly Pareto-optimal, then $\max_{q \in Q_0} s(q-\bar{q}) > 0$. This cannot be achieved when using weighting coefficients or typical value or utility functions.

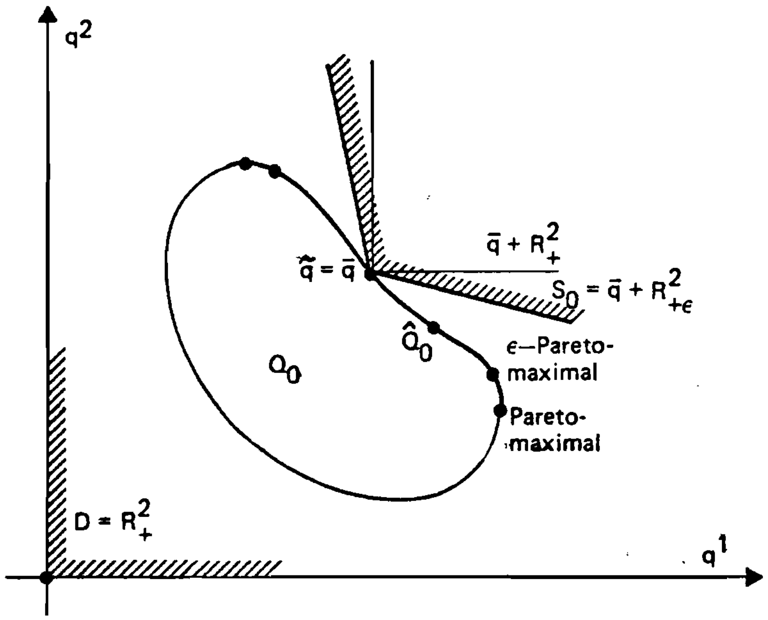

Figure 1. The separation of Q_0 and $\hat{q} + \tilde{D} = \bar{q} + \tilde{R}^2_+$ by $S_0 = \bar{q} + R^2_{+\varepsilon}$.

However, every order preserving function--a value or utility function or a scalarizing function--defines at its maximal points $\hat{q}$ the corresponding weighting coefficients $\hat{\lambda}$, if it is differentiable

$$\hat{\lambda} = \frac{\partial s(\hat{q}-\bar{q})}{\partial q} / \left\| \frac{\partial s(\hat{q}-\bar{q})}{\partial q} \right\| \quad ; \; \hat{\lambda} \in D^* = \{\lambda \in G^* : \langle \lambda, q \rangle \geq 0, \forall q \in D\} \quad , \qquad (14)$$

where the norm used in (14) is the norm of the dual space G^* to the objective space, D^* is the dual cone to D and $\langle \cdot , \cdot \rangle$ denotes the duality relation. If $G = R^n$, then it is typically assumed that weighting coefficients sum up to one, which implies the sum of absolute values norm in (14) and the maximum norm for the objective space. If s is only subdifferentiable, any of its subgradients at $\hat{q}$ can be used to define $\hat{\lambda}$ similarly as in (14).

There are two important corollaries to Lemmas 1 and 2.

COROLLARY 1. *Suppose a scalarizing function s is strictly or strongly order preserving and upper semicontinuous in a topology in G. Suppose there is* $\bar{q} \in G$ *such that the set* $(\bar{q}+D) \cap Q_0$ *is compact in the same topology. Then there exist (possibly weakly) D-maximal points of set* Q_0.

The proof of the corollary is immediate: the Weierstrass' theorem implies the existence of a maximum point $\hat{q}$ of $s(q-\bar{q})$ in the set $(\bar{q}+D) \cap Q_0$. By Lemma 1, this point is a (possibly weakly) D-maximal point of $(\bar{q}+D) \cap Q_0$. It is easy to check that it is also a (possibly weakly) D-maximal point of Q_0.

The following corollary establishes the fact that the boss can fully control the organization if the staff preferences are described by an achievement scalarizing function.

COROLLARY 2. *Suppose a scalarizing function s is order preserving and order representing. Define the mapping* $\hat{\hat{q}}: G \to \hat{Q}_0 = \{\hat{q} \in Q_0: Q_0 \cap (\hat{q}+\tilde{D}) = \emptyset\}$ *by* $\hat{\hat{q}}(\bar{q}) = \arg\min \|\hat{q}-\bar{q}\|$ *for* $\hat{q} \in \underset{q \in Q_0}{\mathrm{Arg\,max}}\, s(q-\bar{q})$. *Then the mapping is onto. If a scalarizing function s is order preserving and order approximating and the mapping* $\hat{\hat{q}}$ *is defined similarly but with* $\hat{\hat{q}}: G \to \hat{Q}_{0\varepsilon} = \{\hat{q} \in Q_0: Q_0 \cap (\hat{q}+\tilde{D}_\varepsilon) = \emptyset\}$, *then the mapping is also onto.*

The proof is also immediate: it is necessary to show that for every $\hat{q} \in \hat{Q}_0$ or $\hat{q} \in \hat{Q}_{0\varepsilon}$ there exists a $\bar{q} \in G$ such that $\hat{\hat{q}}(\bar{q}) = \hat{q}$. Lemma 2 implies that it is sufficient to choose $\bar{q} = \hat{q}$ to obtain $\hat{\hat{q}}(\hat{q}) = \hat{q}$. This immediate result has, however, important interpretation: *any desired nondominated and attainable point* $\hat{q} \in \hat{Q}_0$ *or, at least,* $\hat{q} \in \hat{Q}_{0\varepsilon}$ *can be obtained by moving the reference point (aspiration level)* $\bar{q}$ *only, no matter what the other properties of the achievement scalarizing function are (which particular notions of distance minimization or surplus allocation have been assumed in this function).*

A further conclusion that can be derived from Corollary 2 and from the possibility of determining marginal *a posteriori* information $\hat{\lambda}$ as given by equation (14) is that the boss or decision maker can change $\bar{q}$ in such a way that $\hat{q} = \hat{q}(\bar{q})$ finally converges to a maximum point of his own value or utility function--under some assumptions concerning the reasonability of his strategy in changing $\bar{q}$, see Wierzbicki 1979a.

Consider finally another interpretation of an achievement scalarizing function $s(q-\bar{q})$: let it represent a value function of a consumer under various externalities expressed by $\bar{q}$ and let these externalities have a probability distribution $p(\bar{q})$. After averaging over these externalities, the consumer value or utility function can be obtained by:

$$u(q) = \int_G s(q-\bar{q})p(\bar{q})d\bar{q} \quad . \tag{15}$$

This function is order preserving, since it is a generalized convex combination of order preserving functions. This represents another possible link between value optimization and satisficing decision making.

EXAMPLES OF ACHIEVEMENT SCALARIZING FUNCTIONS

To show that the above theory is applicable for satisficing decision making and multiobjective optimization problems, we must first present some examples of functions satisfying the axiomatic requirements *a* and *b* as well as the descriptive requirement *c*.

Assume that $G = R^n$, $D = R^n_+$. Let a utility (value) function $u(q)$ be defined for $q \in R^n_+$; assume the utility function is nonnegative, $u(q) \geq 0$ for $q \in R^n_+$, zero on the boundary of R^+_n, $u(q) = 0$ for $q \in \partial R^n_+$, and strictly order preserving (not necessarily strongly, since this is impossible for $q \in \partial R^n_+$). Now suppose a threshold $\bar{q} \in R^n$ is defined, and the origin of the space shifted to the threshold; therefore, the utility function $u(q-\bar{q})$ is defined only for $q \in \bar{q} + R^n_+$. To define, additionally, the function for $q \notin \bar{q} + R^n_+$, one can choose the following expression:

$$s(q-\bar{q}) = u((q-\bar{q})_+) - \rho\|(\bar{q}-q)_+\| \quad , \tag{16}$$

where $(\cdot)_+$ denotes the positive part of a vector, $||(\bar{q}-q)_+|| = \mathrm{dist}(q,\bar{q}+R^+_n)$, and $\rho > 0$ is a penalty coefficient. The function $s(q-q)$ has here two interpretations.

First, it is an *extended (beyond) threshold utility function:* it might describe the behavior of an average consumer both above and below a threshold $\bar{q}$ of subsistence. Above the threshold, the average consumer maximizes his utility u; below the threshold, his disutility corresponds to a distance from satisfying all basic needs.

Second, it is an achievement scalarizing function. It is clearly strictly order preserving: any norm in R^n is strictly order-preserving for positive components (not strongly, if the maximum norm is used). It is also order representing: $S_0 \overset{\mathrm{df}}{=} \{q \in R^n : s(q-\bar{q}) \geq 0\} = \bar{q} + R^n$, since $u((q-\bar{q})_+)$ might be positive only for $q \in \bar{q} + R^n$ (if any component of the vector $q - \bar{q}$ is negative or zero, then the corresponding component of the vector $(q-\bar{q})_+$ is zero, and $u((q-\bar{q})_+) = 0$ for $(q-\bar{q})_+ \in \partial R^n_+$). It also expresses a notion of surplus allocation resulting from utility maximization if $q - \bar{q} \in R^n_+$, and a notion of distance minimization, if $q - \bar{q} \notin R^n_+$. In fact, $\mathrm{Arg}\min_{q\in\hat{Q}_0} ||q-\bar{q}|| \subset \mathrm{Arg}\max_{q\in Q_0} s(q-\bar{q})$, if $\bar{q} \notin Q_0 - R^n_+$.

Various norms in R^n and various utility functions can be used to define a specific form of (16) (see Wierzbicki 1979b). One of the most useful is the following convex, piecewise linear function:

$$s(q-\bar{q}) = \min(\rho \min_{1\leq i\leq n} (q^i-\bar{q}^i) \quad , \quad \sum_{i=1}^{n} (q^i-\bar{q}^i)) \quad ; \quad \rho \geq n \quad , \qquad (17)$$

where upper indices denote vector components. The maximization of this function is equivalent to the following linear programming problem (provided the set E_0 of admissible decisions x is described by linear inequalities and all objective functions $q_i = Q_i(x)$ are also linear)

$$\text{maximize } y,\ q \in Q_0 = Q(E_0)\ ,\ y \in Y_0(q-\bar{q}) = \\ = \{y \in R^1 : y \leq \rho(q^i-\bar{q}^i)\ ,\ i = 1,\ldots,n\ ;\ y \leq \sum_{i=1}^{n}(q^i-\bar{q}^i)\}\ . \tag{18}$$

After solving this problem, the weighting coefficients $\hat{\lambda}$ can be a posteriori determined from the dual program.

Another class of achievement scalarizing functions are *penalty scalarizing functions*. Their construction is based upon simple reasoning: if $q \in \bar{q} + R^n_+$, we maximize a norm or a component of $q - \bar{q}$; if $q \notin \bar{q} + R^n_+$, we penalize for the distance between q and $\bar{q} + R^n_+$. An example of this class is the following function

$$s(q-\bar{q}) = \|q-\bar{q}\| - \rho\|(\bar{q}-q)_+\|\ ,\quad \rho > 1\ , \tag{19}$$

which is strictly order preserving (strongly for all norms in R^n but for the maximum norm) and order approximating with $\varepsilon \geq 1/\rho$ (see Wierzbicki 1978). This function expresses also a specific notion of distance minimization, if $q \notin \bar{q} + R^n_+$: if $\bar{q} \notin Q_0 - R^n_{+\varepsilon}$ and $\text{Arg}\max_{q \in Q_0} s(q-\bar{q}) \subset \hat{Q}_{0\varepsilon}$, then $\text{Arg}\min_{q \in \hat{Q}_{0\varepsilon}} \|q-\bar{q}\| \subset \text{Arg}\max_{q \in Q_0} s(q-\bar{q})$. However, $\text{Arg}\max_{q \in Q_0} s(q-\bar{q})$ is not always contained in $\hat{Q}_{0\varepsilon}$, although it is always contained in (weak) $\hat{Q}_0$, since the function $s(q-\bar{q})$ is R^n_+-order preserving, not $R^n_{+\varepsilon}$-order preserving. Depending on the norm chosen, this function possesses various further properties (see Wierzbicki 1979a, 1979b).

Another example is the penalty function resulting from a maximization of the component q^1 under (soft) constraints $q^2 \geq \bar{q}^2,\ldots,\ q^n \geq \bar{q}^n$

$$s(q-\bar{q}) = q^1 - \bar{q}^1 - \rho\|(\bar{q}^r-q^r)_+\|_{R^{n-1}}\ ;\ q^r = (q^2,\ldots,q^n) \in R^{n-1}\ . \tag{20}$$

This function has been frequently used in various approaches to scanning the Pareto set in multiobjective optimization; however, it is less known that this function is (strictly or strongly, depending on the norm) order preserving for $\rho > 0$ and order-approximating with $\varepsilon > 1/\rho$. Thus, any maximal point of this function, not necessarily satisfying the constraints, is a Pareto-maximal point, and any ε-Pareto-maximal point $\bar{q} = \hat{q}$ is maximal for this function.

The penalty function (19) is easily generalized for the case when G is a Hilbert space--for example, the space of time trajectories of solutions of a time-continuous dynamic economic model. The corresponding formula is

$$s(q-\bar{q}) = \| q-\bar{q} \| - \rho \| (\bar{q}-q)^{D^*} \| \quad , \tag{21}$$

where $(\cdot)^{D^*}$ denotes the operation of projection on the dual cone $D^* = \{q^* \in G^*: \langle q^*, q \rangle \geq 0,\ \forall q \in D\}$ (see Wierzbicki and Kurcyusz 1977). This function is strongly order preserving, if $\rho > 0$ and $D \subseteq D^*$, and order approximating with $\varepsilon \leq 1/\rho$ (see Wierzbicki 1977a).

Thus, we have many possible forms of scalarizing functions, some of them (17), (20) being rather simple and easily applicable.

Consider now in more detail the dependence of a maximal point $\hat{q}$ of a scalarizing function $s(q-\bar{q})$ on various factors: on the reference objective $\bar{q}$, on the choice of norm, on the choice of penalty coefficient ρ, on the concept of surplus allocation or the utility used in extended threshold utility functions. All these factors influence the maximal point $\hat{q}$. However, as it was explained in the previous section, the influence of the reference objective $\bar{q}$ is of primary importance, and the influence of other factors can be considered as secondary. If a mathematical model is used for aiding the decision making process, other factors can be specified by an optimization specialist: he can choose the norm in correspondence to the nature of the mathematical model (for example, if the model is linear, he might choose the maximum or the sum of absolute values norm; if the model is nonlinear, he might prefer the Euclidean norm); he can choose the penalty coefficient ρ to obtain a problem which is not too badly conditioned, but with reasonable violations of soft constraints; he can make his own guesses how to allocate a possible surplus $q - \bar{q} \in R^n_+$, etc. These decisions are important for the optimization expert in the sense of computational efficiency; however, they are clearly not essential for the decision maker who can choose any $\hat{q} \in \hat{Q}_0$ (or, at least, any $\hat{q} \in \hat{Q}_{0\varepsilon}$) by specifying and changing his wishes $\bar{q}$.

AN INTERACTIVE TECHNIQUE OF SATISFICING DECISION MAKING VIA MULTIOBJECTIVE OPTIMIZATION

Consider now a practical interactive procedure for choosing a Pareto-maximal point, where the actual decisions are made by a decision maker and the mathematical model of a given problem and the optimization techniques serve only as a tool to help him to recognize quickly a relevant part of the Pareto-maximal set. This procedure can be interpreted as a technique for satisficing decision making with the help of a mathematical model.

At the beginning, the decision maker is presented with all the information about the model of the problem he desires--for example, with the maximal and minimal levels of objective functions when maximized separately, and with the corresponding decisions. After that, he is asked to specify the vector of the desired levels for all objective functions, $\bar{q}^0 = (\bar{q}^1_0, \ldots, \bar{q}^n_0) \in R^n$ (only the finite-dimensional case is considered here, although generalizations to the infinite-dimensional case are possible).

For each desired reference objective vector $\bar{q}_i$, the mathematical model and the optimization technique respond with:

1) The Pareto-maximal attainable objective vector $\hat{q}_i$, obtained through a maximization of a scalarizing function, and the corresponding weighting coefficients and decision variables;

2) n other Pareto-maximal attainable objective vectors $\hat{q}_{i,j}$, $j = 1, \ldots, n$, obtained through maximization of the scalarizing function with perturbed reference points:

$$\bar{q}_{i,j} = \bar{q}_i + d_i e_j; \; e_j = (0, \ldots 1_j, \ldots 0); \; d_i = \| \bar{q}_i - \hat{q}_i \|, \qquad (22)$$

where d_i is the distance between the desired objective vector $\bar{q}_i$ and the attainable one $\hat{q}_i$; e_j is the jth unit basis vector. The advantage of the reference point perturbation (22) is that if the point $\bar{q}_0$ is far from the Pareto set, the decision maker obtains a global description of the Pareto set by the points $\hat{q}_{0,j}$; if $\bar{q}$ is close to the Pareto set, then $\hat{q}_{i,j}$ describes finely the Pareto set in a neighborhood of the desired point $\bar{q}_i$, see Figure 2.

The decision maker can now either choose one of the proposed alternatives, or modify his reference objective point to $\bar{q}_{i+1}$. There are various further refinements of this procedure. The modifications $\bar{q}_{i+1}$ relative to $\hat{q}_i$ can be additionally constrained to provide for the convergence of the procedure to a point that maximizes a utility or value function. The differences $\bar{q}_{i+1} - \hat{q}_i$ can also be used to identify a utility or value function of the decision maker. These refinements, however, have only secondary importance for practical applications of the procedure since decision makers do use the satisficing approach and choose one of the generated alternatives soon.

The distinction in the interpretation of a model solution as a reasonable alternative responding to the wishes of the decision maker rather than as a normative 'optimal'solution is a very important one. Models that generate only one 'best' solution cannot be easily used for decision aiding in organizational structures, while models that respond to the wishes of a decision maker by generating various alternatives and proposing detailed decisions corresponding to these alternatives can be easily incorporated in organizational structures at any level.

This fact has been observed by researchers working on goal programming in multiobjective optimization, see Dyer (1972), Ignizio (1978) and Kornbluth (1973); however, the properties and possibilities of achievement scalarizing functions have not been fully investigated in goal programming, where the questions of a priori defined weighting coefficients, of the use of a lexicographic order, etc., are still predominant. Thus, the interactive procedure presented here can also be considered as a generalization of the goal programming approach.

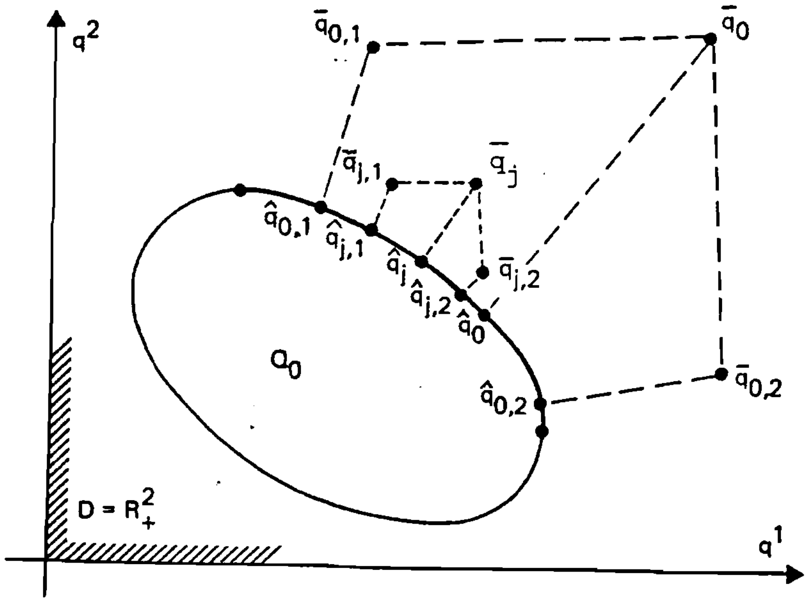

Figure 2. Interpretation of the interactive procedure.

OTHER APPLICATIONAL FIELDS OF REFERENCE OBJECTIVE SCALARIZATION

Scanning the Pareto set

When building a multiobjective optimization model, the analyst must experiment with it and, at least, attempt to scan the Pareto-set, that is, obtain a representation of it. Naturally, he should start by maximizing independently various objectives; after doing it, several methods of scanning the Pareto set can be applied, related to weighting coefficients, directional maximization, reference objectives, etc. Reference objectives with penalty scalarization result here in most robust and efficient techniques--see, e.g., Wierzbicki (1978), and Wierzbicki (1979). However, a scanning of the Pareto-set can be performed reasonably only when the number of objectives is small--say, not larger than three. For a larger number of objectives, an interactive technique as described in the preceeding section is much more reasonable. This applies particularly if the number of objectives is very large--say, in trajectory optimization.

Trajectory optimization

In typical formulations of dynamic optimization, single or multiple objectives are obtained through a normative aggregation of dynamic trajectories by integral functionals. However, experienced analysts, economist, and decision makers often evaluate intuitively entire trajectories, functions of time, better than aggregate integral indices. A decision maker, experienced in evaluating trajectories, can easily state his requirements in terms of a *reference trajectory* $\bar{q}(t)$, a scalar- or vector-valued function of time; it would be a quite impractical task, however, to identify his preference relation over the space of trajectories. Therefore, we should rather construct ad hoc a scalarizing functional, for example, of a form similar to (21) with $G = L^2[0;T]$ and $D = \{q \in L^2[0;T] : q(t) \geq 0$ a.e. on $[;T]\}$:

$$s(q-\bar{q}) = \int_0^T ((q(t)-\bar{q}(t))^2 - \rho(\bar{q}(t)-q(t))_+^2)dt \quad . \qquad (23)$$

If the time is discretized, then the sum replaces the integral; the problem becomes finite-dimensional, but it is still more convenient to think in terms of trajectories than in terms of separate objectives. This technique can be applied, for example, to any economic model in order to obtain feasible and (generalized) Pareto-optimal trajectories that are either close to or better than any given desired trajectories, see Figure 3.

Since the trajectories of solutions to economic models are very often chosen judgementably from a set of possible trajectories, this technique can have wide applications and provide for a methodological tool of experimenting with such models. The concept of trajectory optimization via reference trajectories has been applied by Kallio et al. (1980) in a study of alternative policies for the Finnish forest industrial sector.

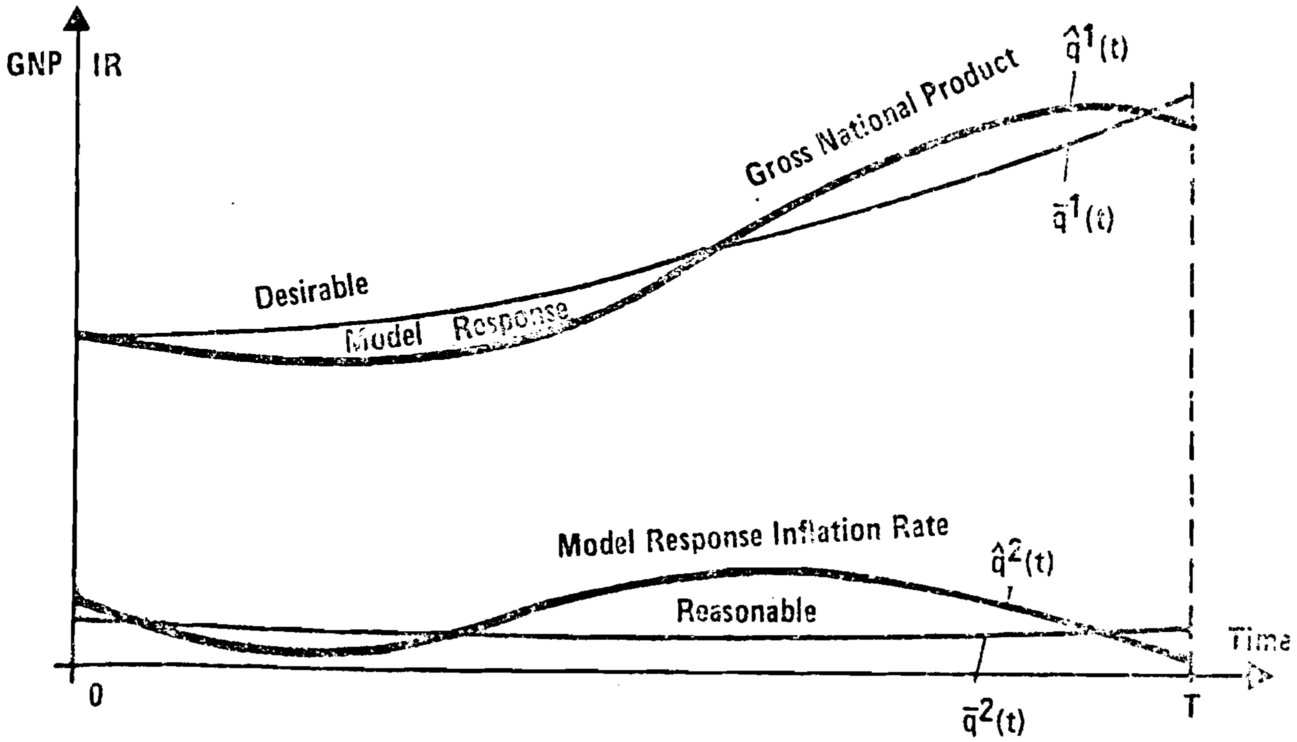

Figure 3. Functions of time or trajectories as reference objectives.

Semi-regularization of solutions of mathematical models

Any model that possesses many solutions or quasisolutions can be *Tikhonov-regularized* (Tikhonov 1975) by choosing a solution that is the closest one to a given reference point. Achievement scalarizing functions represent, in fact, a generalization of this idea: the principle of a *semiregularization*. Consider function (20) and suppose that $\bar{q}^r = (\bar{q}^s, \bar{q}^t)$, where $\bar{q}^s$ denotes reference objective components which should be either kept close to, or if possible, exceeded, and $\bar{q}^t$ denotes reference objective components which should be kept close to, independently from the sign of $\bar{q}^t - q^t$. The following penalty scalarizing function

$$s(q-\bar{q}) = q^1 - \bar{q}^1 - \rho^s || (\bar{q}^s - q^s)_+ || - \rho^t || \bar{q}^t - q^t || \quad , \tag{24}$$

is both order preserving and order approximating, if we consider the partial ordering defined by the cone $D = \{q \in R^n : q^1 \geq 0, \bar{q}^{s,i} \geq 0, \bar{q}^{t,j} = 0\}$. Therefore, we can use scalarizing functions also for objective components that should be kept close to a reference level from both sides.

CONCLUSIONS AND POSSIBLE EXTENSIONS

The main idea in constructing a mathematical basis for satisficing decision making is to accept the wishes of the decision maker in the form of aspiration or reference objective levels as a basic a priori information and then to build achievement scalarizing functions which not only depend strongly on this a priori information but also express the restricted rationality of a technical staff (or a mathematical model) helping the decision maker by proposing attainable alternatives corresponding in some sense to the desired aspiration levels. This restricted rationality can be expressed abstractly by introducing the order approximation property of an achievement scalarizing function, besides the natural property of order preservation which is common with typical value functions. The order approximation property results also in a necessary condition of Pareto optimality, applicable for nonconvex problems and stronger than other known necessary conditions. Thus, the mathematical basis for satisficing decision making corresponds to an alternative basic approach to multiobjective optimization, generalizing goal programming and utopia point approaches. This basis is related to some other important problems in the methodology of model evaluation, such as the problem of trajectory optimization or the problem of regularization of solutions of badly defined mathematical models. However, this abstract basis is also eminently pragmatical: the main idea of responding to the wishes of a decision maker rather than telling him what his wishes should be results in practical procedures for interactive decision making with institutional interpretations.

Many further problems--related to the use of reference objectives under uncertainty, hierarchical structures of decision making, etc., are still to be investigated. Much remains to be done, moreover, in a wider testing of a reference objective approach in many application fields.

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