Working Paper

THE ONE-YEAR - FIVE-YEAR MIGRATION PROBLEM

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May 1980
WP-80-81

International Institute for Applied Systems Analysis
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FOREWORD

Declining rates of national population growth, continuing differential levels of regional economic activity, and shifts in the migration patterns of people and jobs are characteristic empirical aspects of many developed countries. In some regions they have combined to bring about relative (and in some cases absolute) population decline of highly urbanized areas; in others they have brought about rapid metropolitan growth.

The objective of the Urban Change Task in IIASA's Human Settlements and Services Area is to bring together and synthesize available empirical and theoretical information on the principal determinants and consequences of such urban growth and decline.

The study of patterns of urban change is hampered, in many countries, by the inadequate availability of data on internal migration. These data often come in the form of five-year time intervals and have to be combined with other demographic and economic indicators that are reported annually. Thus there frequently exists a problem of reconciling one-year with five-year data. This is particularly difficult in the case of migration flow data. The authors of this paper consider this problem and outline an elegant mathematical procedure for dealing with it.

A list of publications in the Urban Change Series appears at the end of this paper.

Andrei Rogers
Chairman
Human Settlements
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ABSTRACT

A general problem in the analysis of mobility is caused by the comparison of data stemming from different time-period durations. Various methods for easing this problem have been suggested. In this paper, an extension of the mover-stayer model is discussed. A method for its solution is suggested by making use of matrix transformation and eigenvalue theory. The discussion is carried out in terms of migration tables and multiregional life tables, and data for three regions of Great Britain have been used for an illustration.

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CONTENTS

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I. THE PROBLEM

The analysis of mobility often is restricted by the unavailability of data. Frequently models are used to approximate longitudinal patterns with their cross-sectional data. Problems also arise because the cross-sectional data may refer to different periods of time.

In the case of migration, registration statistics in many countries can be used to produce origin-destination tables of migration flows over a period of one year. Censuses usually also provide such flow data, but over a five- or ten-year period. Statisticians are thus faced with two sets of data, which give different information that may be difficult to reconcile. Should priority be given to one of them, or do they reveal different patterns of migration?

In this paper, the discussion is carried out in the framework of multiregional demography. The above problem of mobility can be incorporated in the first steps in the construction of the simplest multiregional model: the multiregional life table.

Consider a multiregional population disaggregated by age and for which the necessary data on regional populations, births, deaths, and place-to-place migration are readily available. Assume that the width of the age group is five years and that the time periods of observation are, alternatively, of one-year and five-year durations, respectively. Then the multiregional life-table probabilities of migrating can be computed according to the formula (Rogers and Ledent 1976; Willekens and Rogers 1978).

$$\underset{\sim}{P}_5(x) = (\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}_1(x))^{-1} (\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}_1(x)) \qquad (1)$$

where $\underset{\sim}{P}_5(x)$ is the matrix of probabilities $p_{ij}(x)$ that a person at exact age x in region i will be in region j five years later; $\underset{\sim}{M}_1(x)$ is the matrix:

$$\underset{\sim}{M}_1(x) = \begin{bmatrix} M_{1\delta}(x) + \sum_{j\neq 1}^{n} M_{1j}(x) & -M_{21}(x) & \cdots\cdots & -M_{n1}(x) \\ \cdots\cdots\cdots & \cdots\cdots & & \cdots\cdots \\ -M_{1n}(x) & \cdots\cdots -M_{2n}(x) & \cdots & M_{n\delta}(x) + \sum_{\substack{j=1 \\ j\neq n}}^{n} M_{nj} \end{bmatrix}$$

where $M_{ij}(x)$ are the one-year observed gross migration rates, and $M_{i\delta}(x)$ is the death rate in region i for individuals aged x to x+4. The matrices $\underset{\sim}{M}_1(x)$ and $\underset{\sim}{P}_5(x)$ are of dimension nxn, where n = number of regions.

A factor of five in equation (1) is introduced to reconcile the one-year observed data with the five-year probability. This factor appears along with the assumption that the migrations are uniformly distributed over the five-year time period (Ledent 1978).

When the observed data refer to a five-year period, the above assumption is not necessary. In such a case, the following formula can be used:

$$\underset{\sim}{P}_5(x) = (\underset{\sim}{I} + \frac{1}{2}\underset{\sim}{M}_5(x))^{-1} (\underset{\sim}{I} - \frac{1}{2}\underset{\sim}{M}_5(x)) \qquad (2)$$

where $\underset{\sim}{M}_5(x)$ is a matrix constructed analogously to $\underset{\sim}{M}_1(x)$ from five-year observed gross migration and death rates.

If the assumption for the uniform distribution of the migrants were correct, the two equations (1) and (2) would give approximately equal results, and in such a case equation (1) would be a good approximation to equation (2). Computed results for three regions of Great Britain (East Anglia, South East, and the Rest of Britain) and for the period 1966-1971 and 1970 are presented in Table 1 for exact age 15. For other ages, the estimations are given in Appendix 3.

Table 1. Probabilities of dying and migrating at exact age 15 for the three regions of Great Britain.

1a: Time-period of observation 1970 [estimated with equation (1)].

	region of destination			
region of origin	1	2	3	death
1. East Anglia	0.838896	0.084048	0.073464	0.003591
2. South East	0.010098	0.917494	0.069230	0.003178
3. Rest of Britain	0.005401	0.047277	0.944153	0.003169

1b: Time-period of observation 1966-1971 [estimated with equation (2)].

region of origin	region of destination 1	2	3	death
1. East Anglia	0.898068	0.053417	0.044920	0.003595
2. South East	0.007041	0.948826	0.040965	0.003168
3. Rest of Britain	0.003073	0.030466	0.963210	0.003251

Obviously the probability of leaving the region of origin in Table 1a is substantially higher than the corresponding value in Table 1b. Therefore, equation (1) overestimates the probabilities of migrating and underestimates the probabilities of remaining in the region of origin five years later. The latter probabilities are represented by the main diagonal of each table. The same inferences hold for other ages too, as is shown by the estimates in Appendix 3.

That the two sets of probabilities are significantly different can best be evaluated by comparing the corresponding expectations of life given in Table 2 (see also Appendix 4).

Table 2. Expectations of life at age 15 for three regions of Great Britain, 1966-1971 and 1970 periods of observation.

(1) estimated with equation (1) and based on the 1970 period of observation.

(2) estimated with equation (2) and based on the 1966-1971 period of observation

region of origin		region of destination 1	2	3	total
1. East Anglia	(1)	18.46	17.76	23.16	59.38
	(2)	28.78	13.86	17.01	59.65
2. South East	(1)	2.82	34.46	22.14	59.32
	(2)	2.48	40.97	16.01	59.46
3. Rest of Britain	(1)	1.62	11.48	45.70	58.80
	(2)	1.29	8.22	49.25	58.76

The differences in the distribution of the expectation of life for an individual born in the first region (East Anglia) are too large to be neglected. Although not so large, the differences concerning the other two regions are also significant. The same holds true for other ages (Appendix 4).

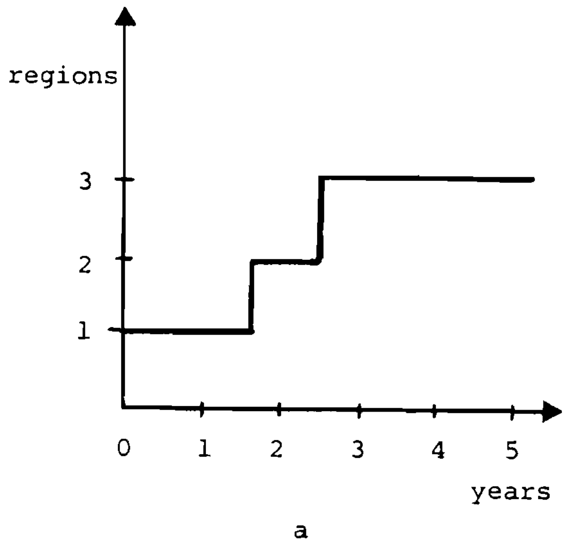
Now compare the probabilities for dying, exhibited in Tables 1a and 1b. They are obviously so close that the probabilities for dying from Table 1b are a good approximation to those from Table 1a. Their estimation, however, is based on the same assumption as for the migrants; namely, that the observed deaths are uniformly distributed over the five-year time period.

One and the same assumption gives different results: in the case of deaths it is valid, but in the case of migrations it is erroneous. The reason for this difference is that migration is a repetitive event, unlike death. Migrants are usually registered as such by comparing their places of residence at the beginning and at the end of the time-period of interest. Therefore, multiple moves within the same time-period are not identified.

An example is presented in Figure 1. Let the individual reside in region 1 at time 0. He will be a resident of the same region at the end of the first year, but at the end of the second he will be a resident of the second region. At the end of the third and fourth year he will be residing in region 3. By the end of the fifth year, he will be in region 3 according to the graph in Figure 1a, and back in region 1 according to Figure 1b.

In a one-year data collection system this individual would be registered as a migrant either three times (Figure 1a) or four times (Figure 1b). But if data are collected over a five-year period, the same individual would register one move in the case of Figure 1a and no move in the case of Figure 1b.

In the above example, the addition of a fourth move in the system of one-year observations produced one less move in the case of five-year observations. Hence, there exist some kinds of moves that are responsible for the erroneous results produced by the use of a multiplicative factor of five. A detailed description of the ideas outlined above may be found in Rees (1977).



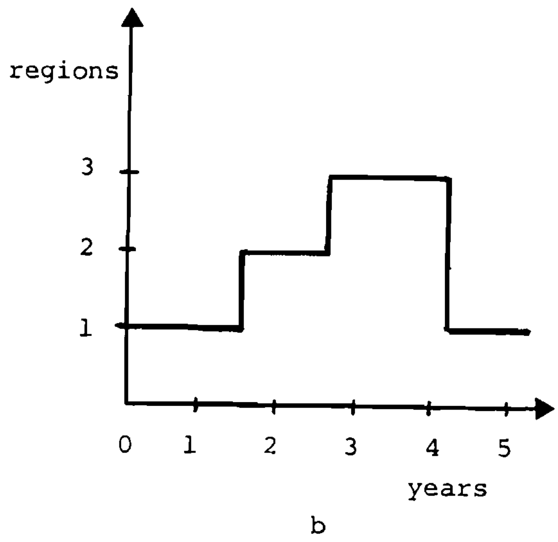


Figure 1. Moves of an individual among the three regions over a period of five years.

The above reflections suggest the representation of the individual's behavior as a stochastic process. If every move is independent from the others, and if the probability of a move does not depend on time, the process can be described as a homogeneous Markovian process.

The Markovian assumption gives rise to a new kind of an estimating procedure, represented by equation (3)

$$\underset{\sim}{P}_5(x) = [(\underset{\sim}{I} + \frac{1}{2}\underset{\sim}{M}_1(x))^{-1} (\underset{\sim}{I} - \frac{1}{2}\underset{\sim}{M}_1(x))]^5 \tag{3}$$

which is based on the following equality, typical for Markovian processes:

$$\underset{\sim}{P}_5(x) = [\underset{\sim}{P}_1(x)]^5 \tag{3a}$$

where $\underset{\sim}{P}_1(x)$ is a matrix with a typical element $p_{ij}(x)$ being the probability of an individual at exact age x in region i surviving in region j one year later. Thus defined, this probability has little demographic meaning, because of the inconsistency between the age-group width (5 years) and time-period of interest (1 year) but its formal definition is correct. If the Markovian assumption proves to be correct, then $[\underset{\sim}{P}_1(x)]^5$ is already demographically meaningful.*

For the three regions of Great Britain, equation (3) yields the results exhibited in Table 3. The probabilities in Table 3 are much the same as in Table 1b. Hence the Markovian assumption has not introduced any significant improvement. The same inference holds for the other ages too (Appendix 2).

Where interregional moves are concerned, the Markovian assumption has been suggested by Rogers (1965) and Rees (1977). Rees has applied the approach to two sets of data for Great Britain: for simple data referring to heads of households, and for census data for interregional migrations. In the former case, the results obtained are reported to be satisfactory but in the latter case, where 10 regions of Great Britain are involved, the model rates differ significantly from the observed ones. After a detailed examination of the problem the author concludes that "... a more complex [than the Markovian] process is involved when an interregional framework is employed". (p. 262)

Table 3. Probabilities of dying and migrating at exact age 15 for the regions of Great Britain (Markovian assumption).

	region of destination			
rest of origin	1	2	3	death
1. East Anglia	0.839297	0.083767	0.073345	0.003591
2. South East	0.010063	0.917623	0.069137	0.003178
3. Rest of Britain	0.005394	0.047212	0.944226	0.003169

*However, if the matrices $\underset{\sim}{P}_1(x)$ for x = 1,2,3,... are available $\underset{\sim}{P}_5(x)$ should be approximated by the matrix

$$\underset{\sim}{P}_1(x+4) \cdot \underset{\sim}{P}_1(x+3) \cdot \underset{\sim}{P}_1(x+2) \cdot \underset{\sim}{P}_1(x+1) \cdot \underset{\sim}{P}_1(x)$$

The Markovian assumption is theoretically better than the assumption for uniform distribution of the migrants, because it allows a consideration of return migration (see Figure 1b). Therefore, it can be thought of as distinguishing two different groups of population. Ideas of this kind have been explored by Blumen, Kogan, and McCarthy (1955) who gave rise to what is known today as the "mover-stayer" model. This model was later elaborated by Goodman (1961), Spilerman (1972), Boudon (1975), Bartholomew (1973) etc.

The mover-stayer framework is based on the assumption that a certain part of the population has a zero probability of migration (<u>stayers</u>), and the remainder are those who make all of the moves (<u>movers</u>). The formal description of the model is:

$$\underset{\sim}{P}_5(x) = \alpha\, \underset{\sim}{\pi}_5(x) + (1 - \alpha)\underset{\sim}{I} \qquad (4)$$

where $0 < \alpha < 1$, $\underset{\sim}{P}_5(x)$ and $\underset{\sim}{\pi}_5(x)$ are matrices of probabilities for migrating (at exact age x), which are similarly defined, but differ in magnitude, and $\underset{\sim}{I}$ is the identity matrix.

The Markovian property now is assigned to the matrix $\underset{\sim}{\pi}(x)$, instead of $\underset{\sim}{P}(x)$. Therefore, if $\underset{\sim}{\pi}_1(x)$ were only available, a possible approximation of (2) would be:

$$\underset{\sim}{P}_5(x) = \alpha[\underset{\sim}{\pi}_1(x)]^5 + (1 - \alpha)\underset{\sim}{I} \qquad (5)$$

Note that for $\alpha = 1$, equation (5) reduces to equation (3a). Note also that according to this presentation of the mover-stayer model, α does not depend on the region of origin or destination.

Instead of elaborating on the last equation we shall proceed further by considering a possible extension.

II. THE HIGH- AND LOW-INTENSITY MOVERS MODEL

The mover-stayer model was based on the existence of two homogeneous groups of individuals—movers and stayers. In the demographic literature, however, migrants are often considered to consist of two groups with respect to the "parity" of the event. One group consists of those migrants who move only once during the period of observation, and the other consists of individuals who migrate more often. Sometimes the latter are referred to as "chronic" migrants. Long and Hansen (1975) report that the rates of return migration to the South of the USA are much higher than those for the first moves in the same direction. At the same time, the "returners" constitute a small part of the total number of migrants (10-20%).

Spilerman (1972) has considered the extension of the mover-stayer model by following the suggestions in the pioneering essay of Blumen et al. (1955), to allow for the existence for a continuum of intensities to move, and proposes a solution to the problem. However, this model is Markovian and cannot be used in the present case. Boudon (1975) suggests that two homogeneous populations should be considered, both with probabilities to move that are higher than zero. He focuses basically on intergenerational occupation tables. The methods for solution of the resulting model are based on the maximum likelihood principle, which brings about substantial computational difficulties when the dimension of the problem is large.

In the present paper, we shall assume, like Boudon, that the population consists of two groups with different intensities to move, but we propose a different method of solution (matrix diagonalization). It is believed that in this way the model will be closer to the demographic understanding of migration propensities, and will bring more theoretical insight into such empirical notions as returners or chronic migrants.

Let $p_{ij}(x)$ be the probability that an individual at exact age x in region i will be in region j one year later. Let $\sum_{j=1}^{n} p_{ij}(x) = 1$, where n = number of regions. The last equation states that the effect of mortality is not accounted for in the estimation of $p_{ij}(x)$. This assumption is made for convenience, since the matrix of the $p_{ij}(x)$ will be stochastic, hence its properties are easier to describe and understand.

Note that the probabilities $p_{ij}(x)$ as described here, are linked with the estimated probabilities $\hat{p}_{ij}(x)$ from:

$$\hat{\underset{\sim}{P}}(x) = (\underset{\sim}{I} + \frac{1}{2}\underset{\sim}{M}_1(x))^{-1}(\underset{\sim}{I} - \frac{1}{2}\underset{\sim}{M}_1(x))$$

where $\hat{\underset{\sim}{P}}(x) = \|\hat{p}_{ij}(x)\|$, with the following equality:

$$p_{ij}(x) = \frac{\hat{p}_{ij}(x)}{1 - \hat{p}_{i\delta}(x)}.$$

Having in mind that $\sum_{j=1}^{n} \hat{p}_{ij}(x) + \hat{p}_{i\delta}(x) = 1$, obviously $\sum_{j=1}^{n} p_{ij}(x) = 1$. The formal description of the extension of the mover-stayer model considered here is based on the following equality:

$$p_{ij}(x) = \alpha_{ij}(x)\ \pi_{ij}(x) + [1 - \alpha_{ij}(x)]\rho_{ij}(x) \qquad (6)$$

where π_{ij} and ρ_{ij} are probabilities with meaning analogous to that of p_{ij}, and $\alpha_{ij}(x)$ is a real parameter, $0 < \alpha < 1$. The equality shows that the probability $p_{ij}(x)$ which refers to the <u>total</u> population of region i at exact age x, is the weighted sum of two probabilities, which refer to subgroups of this regional population, with weights $\alpha_{ij}(x)$ and $[1 - \alpha_{ij}(x)]$ respectfully. The model defined by the above probabilities, will be called the <u>high-and low-intensity model</u>, to contrast it from the extension of Spilerman (1972).

In order to make use of this model, it is necessary to know the values of $\alpha_{ij}(x)$, $\pi_{ij}(x)$, and $\rho_{ij}(x)$, so that the estimation of $p_{ij}(x)$ would be possible. Unfortunately, these data are unavailable. That is why a number of further assumptions will be made in order to find a convincing method of solution for α, π, and ρ.

We shall first assume that the parameter $\alpha_{ij}(x)$ does not depend on the regions i and j, i.e., the delineation of the two groups of individuals with different intensities to migrate does not depend on the regionalization. The demographic meaning of this assumption is that some factors other than the regions (for instance, social status, economic occupation, etc.) determine the existence and the number of returners and chronic migrants. The validity of this and the following assumptions will be discussed later in the paper.

The matrix equivalent of (6) is:

$$\underset{\sim}{P}_1(x) = \alpha(x)\underset{\sim}{\pi}_1(x) + [1 - \alpha(x)]\underset{\sim}{\rho}_1(x) \tag{7}$$

where $\alpha(x)$ is a scalar depending on the age x. Note that $\alpha(x)$ and the elements of the two matrices $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ are all non-negative.

We shall further assume that the stochastic processes defined by the stochastic matrices $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ are Markovian. Thus we assume that these matrices fulfill the Kolmogoroff-Chapman equations (Chiang 1968, Karlin 1969). If so, the process, defined by $\underset{\sim}{P}(x)$, is a sum of two Markovian processes.

Generally, the sum of two Markovian processes is itself not a Markovian process. Since $\alpha(x) = 1$ reduces the process, defined by $\underset{\sim}{P}_1(x)$, the high- and low-intensity model is a non-Markovian extension of the Markovian model.

Equality (7) was based on a one-year time-period of observation. When the period of observation is τ years long, it can be represented as:

$$\underset{\sim}{P}_\tau(x) = \alpha(x)\underset{\sim}{\pi}_\tau(x) + [1 - \alpha(x)]\underset{\sim}{\rho}_\tau(x) \tag{8}$$

The Markovian assumption for $\underset{\sim}{\pi}_{\tau}$ and $\underset{\sim}{\rho}_{\tau}$ gives the following feedback between different values of τ ($\tau = 1$ and $\tau = 5$, say):

$$\begin{aligned} \underset{\sim}{\pi}_5(x) &= [\underset{\sim}{\pi}_1(x)]^5 \\ \underset{\sim}{\rho}_5(x) &= [\underset{\sim}{\rho}_1(x)]^5 \end{aligned} \tag{9}$$

With the expressions (9) in hand, equation (8) can be used to form the following system:

$$\begin{aligned} \underset{\sim}{P}_1(x) &= \alpha(x)\underset{\sim}{\pi}_1(x) + [1 - \alpha(x)]\underset{\sim}{\rho}_1(x) \\ \underset{\sim}{P}_5(x) &= \alpha(x)[\underset{\sim}{\pi}_1(x)]^5 + [1 - \alpha(x)][\underset{\sim}{\rho}_1(x)]^5 \end{aligned} \tag{10}$$

If the solution of this system, with respect to the unknowns $\alpha(x)$ and the elements of $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ is known, the one-year - five-year migration problem can be attacked in the light of the newly formulated model. Hence we proceed further to solve (10) with respect to $\alpha(x)$, $\pi_{ij}(x)$, and $\rho_{ij}(x)$ for each $i,j = 1,\ldots,n$. The unknowns in the system (10) are $2n^2 + 1$, where n = number of regions, while the number of the equations is 2n + 2n (the first 2n comes from the dimension of the matrices, and the second 2n comes from the restrictions $\sum_{j=1}^{n} \pi_{ij} = 1$ and $\sum_{j=1}^{n} \rho_{ij} = 1$).

In order to find the solution of the system in (10) we are faced with one basic problem: the number of unknowns is larger than the number of equations. For instance, for n = 3 the unknowns are 19, and the equations are 12, hence seven unknowns have to be exogenously defined. An additional problem is caused by the large number of equations in a system that is non-linear. The two problems will be considered further together.

Consider the system of Kolmogoroff's differential equations (Chiang 1968):

$$\underset{\sim}{P}'(\tau) = \underset{\sim}{P}(\tau)\ \underset{\sim}{\mu}$$

with the following initial condition:

$$\underset{\sim}{P}(0) = \underset{\sim}{I}$$

A typical element of the matrix $\underset{\sim}{\mu}$, is the intensity of μ_{ij} of migrating from region i to region j. The elements satisfy the conditions: $\sum_{j=1}^{n} \mu_{ij} = 0$, $\mu_{ij} \geq 0$ for $i \neq j$, and $\mu_{ii} \leq 0$. Some important properties of $\underset{\sim}{\mu}$ are given in Chiang (1968). In the demographic literature the intensity of migrating is often referred to as the "force" of migrating.

The formal solution of the system of Kolomogoroff differential equations is:

$$\underset{\sim}{P}(\tau) = e^{\underset{\sim}{\mu}\tau} \tag{11}$$

The definition of $e^{\underset{\sim}{\mu}\tau}$, as a function of matrices, can be found in Gantmacher (1959, Chapter V).

The matrices $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ are those of Markovian processes, hence they can also be represented as in (11), for $\tau = 1$:

$$\underset{\sim}{\pi}_1(x) = e^{\underset{\sim}{\mu}_1(x)} \tag{11a}$$

and

$$\underset{\sim}{\rho}_1(x) = e^{\underset{\sim}{\mu}_2(x)} \tag{11b}$$

Then the system (10), with $\alpha(x)$ set equal to α, transforms to:

$$\begin{aligned} \underset{\sim}{P}_1(x) &= \alpha\, e^{\underset{\sim}{\mu}_1(x)} + (1-\alpha)e^{\underset{\sim}{\mu}_2(x)} \\ \underset{\sim}{P}_5(x) &= \alpha\, e^{5\underset{\sim}{\mu}_1(x)} + (1-\alpha)e^{5\underset{\sim}{\mu}_2(x)} \end{aligned} \qquad (12)$$

Note that on the right-hand side, the probabilities of migrating are now replaced by the corresponding intensities of migrating.

We introduce next the following assumptions:

$$\underset{\sim}{\mu}_2(x) = k(x)\underset{\sim}{\mu}_1(x) \qquad 0 < k(x) < 1 \qquad (13)$$

and

$$k(x) = k \qquad \text{for all } x$$

Their demographic meaning is that the difference in the propensity to migrate for individuals from the two groups (weighted by the parameter α) is independent of the regions i and j. By introducing (13) into the system (12) we get, denoting $\underset{\sim}{\mu}_1$ by $\underset{\sim}{\mu}$:

$$\begin{aligned} \underset{\sim}{P}_1(x) &= \alpha\, e^{\underset{\sim}{\mu}(x)} + (1-\alpha)e^{k\underset{\sim}{\mu}(x)} \\ \underset{\sim}{P}_5(x) &= \alpha\, e^{5\underset{\sim}{\mu}(x)} + (1-\alpha)e^{5k\underset{\sim}{\mu}(x)} \end{aligned} \qquad (14)$$

By introducing the assumptions in (13), the number of unknowns reduces from $2n^2 + 1$ in (10) to $n^2 + 2$ in (14), the number of equations reducing to $2n + n$, the restrictions $\sum \pi_{ij} = 1$ and $\sum \rho_{ij} = 1$ being replaced by $\sum \mu_{ij} = 0$.

For $n = 2$, the number of equations will be equal to the number of unknowns. Hence, the solution can be sought directly from (14).

For n = 3, the number of unknowns will be 11, and the number of equations, 9. For n > 3, the number of unknowns will increasingly exceed the number of equations. Therefore, for n ≥ 3, the solution must be sought in an indirect manner. Here, the method of diagonalization will be utilized.

Let the eigenvalues of $\underset{\sim}{P}_1$ * be different and n in number. (This assumption is usual in social sciences and adequately reflects real world situations.) Then the transformation $\underset{\sim}{T}_1$ which diagonalized $\underset{\sim}{P}_1$, is defined by the n different right eigenvectors. Analogously let $\underset{\sim}{P}_5$ be diagonalized by $\underset{\sim}{T}_5$. By $\underset{\sim}{T}^{-1}$ we denote the inverse of the matrix $\underset{\sim}{T}$. Hence, $\underset{\sim}{T}_1^{-1}$ and $\underset{\sim}{T}_5^{-1}$ are constructed by the left eigenvectors of $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$, respectively. For more details about diagonalization see, for instance, Bellman (1960), Chiang (1968), or Gantmacher (1959).

Let $\underset{\sim}{T}_1^{-1}\ \underset{\sim}{P}_1\ \underset{\sim}{T}_1 = \text{diag}\ (\underset{\sim}{P}_1) = \underset{\sim}{\Lambda}_1$, where $\underset{\sim}{\Lambda}_1$ is a diagonal matrix of the eignevalues of $\underset{\sim}{P}_1$. Correspondingly, let diag $(\underset{\sim}{P}_5) = \underset{\sim}{\Lambda}_5$. Introducing the diagonalization into (14) gives:

$$\underset{\sim}{\Lambda}_1 = \underset{\sim}{T}_1^{-1}\ (\alpha e^{\underset{\sim}{\mu}} + (1-\alpha)e^{k\underset{\sim}{\mu}})\ \underset{\sim}{T}_1$$

$$\underset{\sim}{\Lambda}_5 = \underset{\sim}{T}_5^{-1}\ (\alpha e^{5\underset{\sim}{\mu}} + (1-\alpha)e^{5k\underset{\sim}{\mu}})\ \underset{\sim}{T}_5$$

where

$$\underset{\sim}{\Lambda}_1 = \alpha\ \underset{\sim}{T}_1^{-1}\ e^{\underset{\sim}{\mu}}\ \underset{\sim}{T}_1 + (1-\alpha)\underset{\sim}{T}_1^{-1}\ e^{k\underset{\sim}{\mu}}\ \underset{\sim}{T}_1$$

$$\underset{\sim}{\Lambda}_5 = \alpha\ \underset{\sim}{T}_5^{-1}\ e^{5\underset{\sim}{\mu}}\ \underset{\sim}{T}_5 + (1-\alpha)\underset{\sim}{T}_5^{-1}\ e^{5k\underset{\sim}{\mu}}\ \underset{\sim}{T}_5 \qquad (14a)$$

Further, it will be necessary to use a certain class of matrices, which are defined as follows:

<u>Definition</u>. The matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are *related* if they can be diagonalized by the same transformation $\underset{\sim}{T}$.**

It is easy to show that if the matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are *related*, then the matrix:

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*To simplify the notation, age groups will no longer be denoted.

**The authors would like to thank A. Seifelnasr, who indicated that the word "similar" which was used here originally was inappropriate because this term is used in the literature to define another class of matrices.

$$\underset{\sim}{C} = \alpha\ f(\underset{\sim}{A}) + \beta\ g(\underset{\sim}{B})$$

where f(.) and g(.) are certain scalar functions and α and β are real numbers, is also related to $\underset{\sim}{A}$ and $\underset{\sim}{B}$ (see Gantmacher 1959, Chapter V). In particular, if $\underset{\sim}{v}$ is the diagonalized matrix diag($\underset{\sim}{\mu}$), then,

$$\text{diag}\ (e^{\underset{\sim}{\mu}}) = e^{\underset{\sim}{v}}$$

Consider now the system (14a). Since the left-hand side of each equation is a diagonal matrix, the same will be true of the sum of matrices on the right-hand side. But the matrices $\underset{\sim}{\mu}$ and $k\underset{\sim}{\mu}$ are *related*, therefore, according to the above definition, the matrices $e^{\underset{\sim}{\mu}}$ and $e^{k\underset{\sim}{\mu}}$ are also *related*. Hence they are diagonalized by one and the same transformation. Let $\underset{\sim}{U}$ be such a transformation. Then $\underset{\sim}{U}$ diagonalizes a linear combination of $e^{\underset{\sim}{\mu}}$ and $e^{k\underset{\sim}{\mu}}$, and hence diagonalizes $\underset{\sim}{P}_1$ as well. Then $\underset{\sim}{P}_1$ and $e^{\underset{\sim}{\mu}}$, or $e^{k\underset{\sim}{\mu}}$ are also undefined. If so, the transformation $\underset{\sim}{T}_1$ diagonalizes $e^{\underset{\sim}{\mu}}$ and $e^{k\underset{\sim}{\mu}}$.

Analogously, $\underset{\sim}{T}_5$ diagonalizes the *related* matrices $\underset{\sim}{P}_5$, $e^{5\underset{\sim}{\mu}}$, and $e^{5k\underset{\sim}{\mu}}$. Then (14a) can be represented as:

$$\underset{\sim}{\Lambda}_1 = \alpha\ e^{\underset{\sim}{v}} + (1 - \alpha)e^{k\underset{\sim}{v}}$$
$$\underset{\sim}{\Lambda}_5 = \alpha\ e^{5\underset{\sim}{v}} + (1 - \alpha)e^{5k\underset{\sim}{v}} \qquad (15)$$

having in mind the similarity between $\underset{\sim}{\mu}$, $k\underset{\sim}{\mu}$, $5\underset{\sim}{\mu}$, and $5k\underset{\sim}{\mu}$, and applying successively the property of matrix functions cited above.

Note that $e^{\underset{\sim}{\mu}}$ and $e^{5\underset{\sim}{\mu}}$ (or, $e^{k\underset{\sim}{\mu}}$ and $e^{5k\underset{\sim}{\mu}}$) are *related*, hence the transformations $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_5$ should diagonalize them both. This implies that the matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$ should also be *related*, and be diagonalized by either $\underset{\sim}{T}_1$, or $\underset{\sim}{T}_5$. Transformations are however unique, hence $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_5$ should be equal. This condition is too rigid to be met by the practical implications, but we can relax it a little, by assuming that $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_5$ are empirically close enough to meet the theoretical requirements: i.e., that when applied to the diagonal matrices $\underset{\sim}{\Lambda}_1$ and $\underset{\sim}{\Lambda}_5$, they yield the initial matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$. That is, the following expressions:

$$\hat{\underset{\sim}{P}}_1 = \underset{\sim}{T}_5 \, \underset{\sim}{\Lambda}_1 \, \underset{\sim}{T}_5^{-1} \doteq \underset{\sim}{P}_1 \qquad (16a)$$

and

$$\hat{\underset{\sim}{P}}_5 = \underset{\sim}{T}_1 \, \underset{\sim}{\Lambda}_5 \, \underset{\sim}{T}_1^{-5} \doteq \underset{\sim}{P}_5 \qquad (16b)$$

must be true.

If the expressions (16a) and (16b) do not hold, the whole theory developed up to now will not hold. This would mean that the Markovian assumptions or some of the assumptions made for the matrices $\underset{\sim}{\pi}$ and $\underset{\sim}{\rho}$ are not valid. Therefore, the approximations in (16a) and (16b) provide a measure of the validity of the model considered here.* The authors consider such a measure to be only an empirical one, i.e., the numerical expressions for $\hat{\underset{\sim}{P}}$ and $\underset{\sim}{P}$ have to be compared, and this will be done in the next section, where the numerical application of the model is considered. For the time being, we shall just mention that the numerical results show that $\hat{\underset{\sim}{P}}$ and $\underset{\sim}{P}$ are close enough in order to accept the validity of the model. For convenience to the reader, the estimated results are given in Appendix 2.

Let $\lambda_i(\underset{\sim}{P}_1)$ be the i-th eigenvalue of $\underset{\sim}{P}_1$ and $\lambda_i(\underset{\sim}{P}_5)$ of $\underset{\sim}{P}_5$. Let ν_i the i-th eigenvalue of $\underset{\sim}{\mu}$. Then the system of matrix equations (15) can be presented as the following non-linear system of equations:

*Some theoretical aspects of this approximation are considered in Appendix 1.

$$\left.\begin{aligned}\lambda_i(\underset{\sim}{P}_1) &= \alpha\, e^{\nu_i} + (1-\alpha)e^{k\nu_i}\\ \lambda_i(\underset{\sim}{P}_5) &= \alpha\, e^{5\nu_i} + (1-\alpha)e^{5k\nu_i}\end{aligned}\right\}\quad , \quad i = 1,\ldots,n \qquad (17)$$

The matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$ are stochastic. Therefore, their largest eigenvalue is equal to unity and the corresponding eigenvalue of matrix intensity is equal to zero. Hence, two of the equations in (17) will turn into equalities, and must be excluded from the system. Then the number of the equations will decrease to 2n - 2. At the same time, the number of unknowns is n + 1 (since for some i, $\nu_i = 0$), which is a substantial decrease if compared with (14).

Now let n = 2. Then there will be 2 equations, and 3 unknowns. In (14) for n = 2 the number of equations was equal to the number of unknowns, however; that is why after (14) only $n \geq 3$ will be considered.

Let n = 3. The equations are 4, the unknowns also 4. Therefore, the system is well defined.

Let n > 3. Then the equations will be more than the unknowns. Therefore, if the system is consistent, the method of solution for n = 3 can be applied. We proceed further toward this solution, assuming n = 3.

In order to simplify the notation, let $z_i = e^{\nu_i}$. Let also $\lambda_1(\underset{\sim}{P}_1)$ and $\lambda_1(\underset{\sim}{P}_5)$ be equal to unity, hence $\underset{\sim}{\mu}_1 = 0$. Then (17) can be rewritten as:

$$\left.\begin{aligned}\lambda_i(\underset{\sim}{P}_1) &= \alpha\, z_i + (1-\alpha)z_i^k\\ \lambda_i(\underset{\sim}{P}_5) &= \alpha\, z_i^5 + (1-\alpha)z_i^{5k}\end{aligned}\right\}\quad , \quad i = 2,3 \qquad (18)$$

Let k be held fixed. The last system can be rearranged as:

$$\alpha = \frac{\lambda_i(\underset{\sim}{P}_1) - z_i^k}{z_i - z_i^k} \qquad i = 2,3 \qquad (19)$$

$$\alpha = \frac{\lambda_i(\underset{\sim}{P}_5) - z_i^{5k}}{z_i^5 - z_i^{5k}}$$

and hence,

$$\frac{\lambda_i(\underset{\sim}{P}_1) - z_i^k}{z_i - z_i^k} = \frac{\lambda_i(\underset{\sim}{P}_5) - z_i^{5k}}{z_i^5 - z_i^{5k}} \qquad i = 2,3$$

Note that the above equations are well defined, since the exclusion of the eigenvalue $\nu_1 = 0$ assures that all the denominators be non-zero.

In the last equation above, there are three unknowns: k, z_2, and z_3. An additional restriction is provided by the assumption that α does not depend on the regions. Therefore, the solutions for z_2 and z_3 must be such that (19) would yield the same value for α. The last condition is utilized to construct the following algorithm for solving (18):

Step 1. Fix an arbitrary value for k, $0 < k < 1$.

Step 2. Form the function:

$$f(z_i) = (\lambda_i(\underset{\sim}{P}_1) - z_i^k)(z_i^5 - z_i^{5k}) - (\lambda_i(\underset{\sim}{P}_5) - z_i^{5k})(z_i - z_i^k)$$

for the given value of k.

Step 3. Find the roots of $f(z_i) = 0$, by applying the method of Newton-Raphson:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

starting with $z_0 = 0.01$, say. Recall that z_i is bounded in the interval (0,1) because $z_i = e^{\nu_i}$, and $\nu_i < 0$.

Step 4. With the roots for z_i, estimate α from (19). Let z_i provide an estimated α denoted by α_i.

Step 5. If $\alpha_2 \neq \alpha_3$, go back to Step 1. If $\alpha_2 = \alpha_3$ (up to a predefined tolerance level), the solution is found.

The small initial value for z_0 is assumed in order to avoid the finding of the trivial root $z_i = 1$, which gives $1 = z_i = e^{\nu_i}$, i.e., $\nu_i = 0$.

By finding the solution of (18) and with the found values for α, k, ν_2, and ν_3, we can construct the matrices $\underset{\sim}{\pi}$ and $\underset{\sim}{\rho}$. Thus the initial system (10) can be numerically constructed.

It is possible to find an approximate solution by minimizing a function F of four variables:

$$F(z_2, z_3, \alpha, k) = \sum_{i=2}^{3} \left\{ [\lambda_i(\underset{\sim}{P}_i) - \alpha z_i - (1-\alpha) z_i^k]^2 + [\lambda_i(\underset{\sim}{P}_5) - \alpha z_i^5 - (1-\alpha) z_i^{5k}]^2 \right\}$$

This method of solution was found to give the same results as the one described above, and is to be preferred where library nonlinear-optimization routines are available.

III. NUMERICAL VERIFICATION

Consider the two matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$ for the age-group 15-19 of the three regions of Great Britain considered in the first section. Let the effect of mortality be eliminated, so that the two matrices are stochastic, that is, with row elements summing up to unity. Their numerical expressions then are:

$$\underset{\sim}{P}_1 = \begin{Vmatrix} 0.96614 & 0.01829 & 0.01556 \\ 0.00220 & 0.98320 & 0.01460 \\ 0.00114 & 0.00997 & 0.98889 \end{Vmatrix}$$

and

$$\underset{\sim}{P}_5 = \begin{Vmatrix} 0.90131 & 0.05361 & 0.04508 \\ 0.00706 & 0.95184 & 0.04109 \\ 0.00308 & 0.03056 & 0.96635 \end{Vmatrix}$$

The eigenvalues are: $\lambda_1(\underset{\sim}{P}_1) = 1,\ \lambda_2(\underset{\sim}{P}_1) = 0.96405,\ \lambda_3(\underset{\sim}{P}_1) = 0.97419$

$\lambda_1(\underset{\sim}{P}_5) = 1,\ \lambda_2(\underset{\sim}{P}_5) = 0.89477,\ \lambda_3(\underset{\sim}{P}_5) = 0.92473.$

The eigenvalues of each matrix are different, hence the eigenvectors are also different, and they define the diagonalization transformations.

The system (15) now will be:

$$\begin{Vmatrix} 1.0 & 0 & 0 \\ 0 & 0.96405 & 0 \\ 0 & 0 & 0.97419 \end{Vmatrix} = \alpha\, e^{\begin{Vmatrix} 0 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{Vmatrix}} + (1-\alpha)\, e^{k\begin{Vmatrix} 0 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{Vmatrix}}$$

$$\begin{Vmatrix} 1.0 & 0 & 0 \\ 0 & 0.89477 & 0 \\ 0 & 0 & 0.92473 \end{Vmatrix} = \alpha\, e^{5\begin{Vmatrix} 0 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{Vmatrix}} + (1-\alpha)\, e^{5k\begin{Vmatrix} 0 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{Vmatrix}}$$

The equivalent of (18), after removing the two trivial equalities, is:

$$\begin{aligned} 0.96405 &= \alpha\, e^{\nu_2} + (1-\alpha)e^{k\nu_2} \\ 0.97419 &= \alpha\, e^{\nu_3} + (1-\alpha)e^{k\nu_3} \\ 0.89477 &= \alpha\, e^{5\nu_2} + (1-\alpha)e^{5k\nu_2} \\ 0.92473 &= \alpha\, e^{5\nu_3} + (1-\alpha)e^{5k\nu_3} \end{aligned} \tag{18a}$$

For the system (18a) we search a solution for α, k, ν_2, and ν_3. Further, denoting $z_2 = e^{\nu_2}$ and $z_3 = e^{\nu_3}$, (18a) yields:

$$\alpha_2 = \frac{0.96405 - z_2^k}{z_2 - z_2^k} \qquad \alpha_3 = \frac{0.97419 - z_3^k}{z_3 - z_3^k}$$

$$\alpha_2 = \frac{0.89477 - z_2^{5k}}{z_2^5 - z_2^{5k}} \qquad \alpha_3 = \frac{0.92473 - z_3^{5k}}{z_3^5 - z_3^{5k}} \tag{19a}$$

The algorithm at the end of the previous section was then applied. A unique value for k was found, such that $\alpha_2 = \alpha_3$, and that was $k = 0.01$. For this k, $\alpha = 0.0233$, and $\nu_2 = -1.6848$, $\nu_3 = -1.0051$ ($\nu_i = \ln z_i$).

The values for α and k allow us to state that a subgroup which is 2.3% of the total population of Great Britain, aged 15-19, has an intensity to migrate one hundred times as large as that for the remaining population. Note that this large difference in the intensities does not imply the same differences in the probabilities to migrate! Recalling that the matrices $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ from (11a) and (11b) are represented as:

$$\underset{\sim}{\pi}_1(x) = e^{\underset{\sim}{\mu}} \tag{20a}$$

and

$$\underset{\sim}{\rho}_1(x) = e^{k\underset{\sim}{\mu}} \tag{20b}$$

it is possible to estimate them and find the numerical expression for (10).

Note at first that if $\underset{\sim}{P}_1(x)$ is diagonalized with the transformation $\underset{\sim}{T}_1(x)$, $\underset{\sim}{\pi}_1(x)$ and $\underset{\sim}{\rho}_1(x)$ are diagonalized with the same transformation ($\underset{\sim}{\pi}_1$ and $\underset{\sim}{\rho}_1$ are *related*). Then (20a) yields:

$$\text{diag}(\underset{\sim}{\pi}_1) = \underset{\sim}{T}_1^{-1}\ \underset{\sim}{\pi}_1\ \underset{\sim}{T}_1 = \underset{\sim}{T}_1^{-1}\ e^{\underset{\sim}{\mu}}\ \underset{\sim}{T} = e^{\underset{\sim}{v}}$$

and similarly from (20b):

$$\text{diag}(\underset{\sim}{\rho}_1) = \underset{\sim}{T}_1^{-1} \underset{\sim}{\rho}_1 \underset{\sim}{T}_1 = e^{k\underset{\sim}{v}}$$

Then,

$$\text{diag}(\underset{\sim}{\pi}_1) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & e^{\nu_2} & 0 \\ 0 & 0 & e^{\nu_3} \end{Vmatrix}$$

and

$$\text{diag}(\underset{\sim}{\rho}_1) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & e^{k\nu_2} & 0 \\ 0 & 0 & e^{k\nu_3} \end{Vmatrix}$$

From the last two expressions, $\underset{\sim}{\pi}_1$ and $\underset{\sim}{\rho}_1$ can be found by applying the reverse transformations.

$$\underset{\sim}{\pi}_1 = \underset{\sim}{T}_1 \, \text{diag}(\underset{\sim}{\pi}_1) \, \underset{\sim}{T}_1^{-1}$$

and

$$\underset{\sim}{\rho}_1 = \underset{\sim}{T}_1 \, \text{diag}(\underset{\sim}{\rho}_1) \, \underset{\sim}{T}_1^{-1}$$

The estimated values for $\underset{\sim}{\pi}_1$ and $\underset{\sim}{\rho}_1$ are:

$$\underset{\sim}{\pi}_1 = \begin{Vmatrix} 0.23138 & 0.38506 & 0.38360 \\ 0.04575 & 0.58615 & 0.36809 \\ 0.02863 & 0.25083 & 0.72054 \end{Vmatrix}$$

and

$$\underset{\sim}{\rho}_1 = \begin{Vmatrix} 0.98373 & 0.00952 & 0.00675 \\ 0.00116 & 0.99271 & 0.00614 \\ 0.00048 & 0.00421 & 0.99531 \end{Vmatrix}$$

While $\underset{\sim}{\rho}_1$ is structured similarly to $\underset{\sim}{P}_1$, this is not the case with $\underset{\sim}{\pi}_1$. The elements on the main diagonal of $\underset{\sim}{\pi}_1$ reflect the probabilities for the high-intensity movers to remain in the region of origin, over a period of one year. They are by far lower than usual. Note that these kinds of probabilities depend substantially on the size of the regional populations. That is why the comparatively small region of East Anglia is connected with outmigration probabilities.

For $(\underset{\sim}{\pi}_1)^5$ and $(\underset{\sim}{\rho}_1)^5$, the following expressions may be derived:

$$(\underset{\sim}{\pi}_1)^5 = \underset{\sim}{T}_1 \text{ diag}[(\underset{\sim}{\pi}_1)^5] \underset{\sim}{T}_1^{-1}$$

and

$$(\underset{\sim}{\rho}_1)^5 = \underset{\sim}{T}_1 \text{ diag}[(\underset{\sim}{\rho}_1)^5] \underset{\sim}{T}_1^{-1}$$

where

$$\text{diag}[(\underset{\sim}{\pi}_1)^5] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & e^{5\nu_2} & 0 \\ 0 & 0 & e^{5\nu_3} \end{Vmatrix}$$

and

$$\text{diag}[(\underset{\sim}{\rho}_1)^5] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & e^{5k\nu_2} & 0 \\ 0 & 0 & e^{5k\nu_3} \end{Vmatrix}$$

The final numerical estimation for $\underset{\sim}{P}_5(x)$ using

$$\underset{\sim}{P}_5(x) = \alpha(\underset{\sim}{\pi}_1(x))^5 + (1 - \alpha)(\underset{\sim}{\rho}_1(x))^5 \qquad x = 15$$

is:

$$\begin{Vmatrix} 0.90092 & 0.05367 & 0.04541 \\ 0.00645 & 0.95098 & 0.04257 \\ 0.00332 & 0.02909 & 0.96760 \end{Vmatrix} = 0.0233 \begin{Vmatrix} 0.04468 & 0.38844 & 0.56692 \\ 0.04450 & 0.38908 & 0.56640 \\ 0.04397 & 0.38425 & 0.57179 \end{Vmatrix}$$

$$+ 0.9767 \begin{Vmatrix} 0.92142 & 0.04566 & 0.03292 \\ 0.00554 & 0.96443 & 0.03003 \\ 0.00234 & 0.02058 & 0.97707 \end{Vmatrix}$$

and

$$\underset{\sim}{P}_1(x) = \alpha\underset{\sim}{\pi}_1(x) + (1 - \alpha)\underset{\sim}{\rho}_1(x) \qquad x = 15$$

is

$$\begin{Vmatrix} 0.96614 & 0.01830 & 0.01556 \\ 0.00220 & 0.98320 & 0.01460 \\ 0.00114 & 0.00997 & 0.98889 \end{Vmatrix} = 0.0233 \begin{Vmatrix} 0.23138 & 0.38506 & 0.38360 \\ 0.04575 & 0.58615 & 0.36809 \\ 0.02863 & 0.25083 & 0.72054 \end{Vmatrix}$$

$$+ 0.9767 \begin{Vmatrix} 0.98373 & 0.00952 & 0.00675 \\ 0.00116 & 0.99271 & 0.00614 \\ 0.00048 & 0.00421 & 0.99531 \end{Vmatrix}$$

Note that the estimated matrix $\underset{\sim}{P}_5(15)$ is very close to the observed one, given on page 20, while $\underset{\sim}{P}_1(15)$ is exactly the same.

What deserves special attention in the last numerical equality, is the matrix $[\underset{\sim}{\pi}_1(15)]^5$. Its columns have approximately equal numbers. This is a consequence of the fact that $\underset{\sim}{\pi}_1$ refers to the group having approximately a one hundred times larger intensity for moving than the other group. Since $[\underset{\sim}{\pi}_1]^{\tau} = e^{\underset{\sim}{\mu}\tau}$, and $[\underset{\sim}{\rho}_1]^{\tau} = e^{k\underset{\sim}{\mu}\tau}$, both processes tend to one and the same asymptotes, but the first approaches it much quicker (see Figure 2, whereby $[\underset{\sim}{a}]_{ij}$ is denoted an element from the i-th row any j-th column of a matrix $\underset{\sim}{a}$).

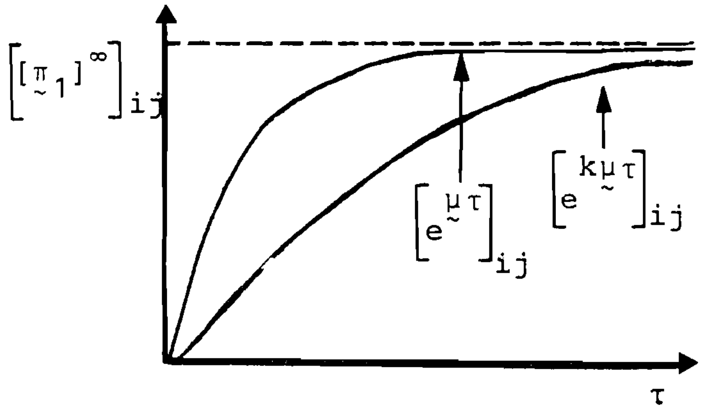

Figure 2. Asymptotic behavior of $e^{\mu\tau}$ and $e^{k\mu\tau}$.

Therefore, $[\pi_1]^5$ is very close to the asymptotic distribution, say denoted by $[\pi_1]^\infty$. But $[\pi_1]^\infty$ defines the stable state of the high-intensity movers, hence, even if this part of the population is not stable in the initial period of time, it will reach spatial stability over a period of 5-10 years. Taking into account that real demographic processes are sufficiently homogeneous over such a small period of time, it is reasonable to suppose that the spatial distribution of the high-intensity movers is approximately stable at the initial point of time.

Because the matrices P_1, π_1 and ρ_1 are related, $P_1^\infty = \pi_1^\infty = \rho_1^\infty$. Therefore, we get a proof that the process described by the high- and low-intensity movers model retains the important demographic properties of stabilization and ergodicity, although the model is not Markovian.

Until now, one age group (15-19) was considered. All the above estimates were repeated for the other age groups (15 in all). The solution with respect to α, k, ν_2, and ν_3, of the system (18) was sought using the same algorithm. The method of solution failed twice, for the age groups 55-59, and the last one 70-74. Generally, the solutions are not satisfactory for the ages beyond 50. The results are exhibited in Table 4.

It is believed that the suggested procedure gives bad results for the aged population primarily because of the solution method. During the search for solution for the age groups beyond 50, it could be observed that α and k tend to zero. Since as $k \rightarrow 0$ the model tends to the mover-stayer one, and $\alpha \rightarrow 0$ reduces it to the Markovian, it is possible that the more sophisticated estimation procedures of the high- and low-intensity movers model are more inaccurate than those which would be more suitable for the easier-to-solve models mentioned above, when the migration movements are very low. The above reflections explain to some extent the gaps between the solutions for the age groups 45-49 and 50-54.

Consider now the first ten age groups in Table 4. The following inferences can be made:

Table 4. Values of α, k, ν_2, ν_3, for different age groups.

age group	α	k	ν_2	ν_3
0-4	0.03156	0.01279	-1.35706	-0.62757
5-9	0.02014	0.01280	-1.11350	-0.59498
10-14	0.01519	0.01311	-0.92648	-0.54148
15-19	0.02338	0.01045	-1.67540	-1.04744
20-24	0.04147	0.00787	-2.72179	-1.62251
25-29	0.04166	0.01436	-1.35777	-0.78607
30-34	0.02259	0.01248	-1.33867	-0.82405
35-39	0.02244	0.01286	-0.91052	-0.53732
40-44	0.01020	0.01000	-1.02088	-0.66838
45-49	0.01601	0.01350	-0.51931	-0.34236
50-54*	-	-	-	-
55-59	0.00288	0.001830	-2.05199	-3.57241
60-64	0.00336	0.002410	-1.53867	-3.01742
65-69	0.00972	0.007235	-0.44394	-0.78785
70-74*	-	-	-	-

*Solution not found.

1) The values for k are quite similar, the mean being 0.01202.

2) The values for α generate a curve which is very much like a migration curve. [Different migration schedules for Great Britain are exhibited in Rees (1969)].

3) The absolute values for each of the ν_i also generate a curve that resembles a migration curve, although not so closely as the curve generated by α.

These regularities can be used in the implementation of the model, which is the topic of discussion in the next section.

IV. IMPLEMENTATION OF THE MODEL

The previous two sections set out the mathematical and numerical descriptions of the high- and low-intensity movers model. The numerical results verify the assumptions made, thus also verifying the model itself. They were derived, however, on the basis of two sets of data--from one-year and five-year observations--both disaggregated by age.

In order to make use of the model, we must suppose that only one set of data is available, and then implement it to obtain approximations for the other set. Since one-year data are usually available from vital statistics in most countries, they will be supposed to be given. Before considering the numerical results once again the theoretical background will be further developed.

In Section II it is shown that starting with the following matrix equation:

$$\underset{\sim}{P}_1(x) = \alpha(x)\underset{\sim}{\pi}_1(x) + [1 - \alpha(x)]\underset{\sim}{\rho}_1(x)$$

the following system of scalar equations can be constructed:

$$\lambda_2(\underset{\sim}{P}_1) = \alpha\, e^{\nu_2} + (1-\alpha)e^{k\nu_2}$$
$$\lambda_3(\underset{\sim}{P}_1) = \alpha\, e^{\nu_3} + (1-\alpha)e^{k\nu_3} \qquad (21)$$

where the age subscript x is again omitted. In the last system, the equations are two, while the unknowns are four: α, k, ν_2, and ν_3. Therefore, two of them must be specified exogenously. This is in fact the basic point in the implementation of the model.

Recalling the inferences from Table 4 at the end of the previous section, it seems reasonable to search for values of α and k which might refer even to the aggregated-by-age population, α_{TOT} and k_{TOT} say. Then two approaches are possible: keep these values constant for all ages or disaggregate them, in accordance with the results from Table 4 [i.e., k_{TOT} may be kept constant, and α_{TOT} may be used to generate a set $\alpha(x)$ for all x, such that $\alpha(x)$ form a curve similar to that of the observed migration rates, and the arithmetic mean of $\alpha(x)$ be equal to α_{TOT}].

In either case, values for α_{TOT} and k_{TOT} are sufficient. How these are derived will be discussed later in this section, but now suppose they are somehow available. If so, k_{TOT} and α_{TOT}, or $\alpha(x)$, can be entered in the system (21) to solve for $\nu_2(x)$ and $\nu_3(x)$. After that, the following system can be solved with respect to the unknowns $\lambda_2(\underset{\sim}{P}_5)$ and $\lambda_3(\underset{\sim}{P}_5)$ (the age subscript being omitted):

$$\lambda_2(\underset{\sim}{P}_5) = \alpha\, e^{5\nu_2} + (1-\alpha)e^{5k\nu_2}$$
$$\lambda_3(\underset{\sim}{P}_5) = \alpha\, e^{5\nu_3} + (1-\alpha)e^{5k\nu_3} \qquad (22)$$

Thus the diagonalized matrix $\underset{\sim}{\Lambda}_5 = \mathrm{diag}(\underset{\sim}{P}_5)$ becomes available. [Recall that $\lambda_1(\underset{\sim}{P}_5) = 1$]. In order to find $\underset{\sim}{P}_5$ it is necessary to have its diagonalizing transformation. But the discussion here suggests that $\underset{\sim}{P}_5$ is a function of $\underset{\sim}{P}_1$, $\underset{\sim}{P}_5 = f(\underset{\sim}{P}_1)$, where the function f(.) is specified in (10). Therefore, $\underset{\sim}{T}_1$ must diagonalize $\underset{\sim}{P}_5$. Hence,

$$\underset{\sim}{P}_5 = \underset{\sim}{T}_1 \underset{\sim}{\Lambda}_5 \underset{\sim}{T}_1^{-1} \tag{23}$$

Note that equation (23) implies $\underset{\sim}{T}_1 = \underset{\sim}{T}_5$. This equality was discussed on page 16, and it was inferred that it should be approximately true (Appendix 2). This then implies that (23) is also an approximation. According to the structure of the model, this approximation should yield better results than those discussed in the first section.

What remains unclear is how values for α and k even for the aggregated-by-age population, could become available. One way to find them is to look at sociological studies: α can be inferred from statements on what part of the population is moving more frequently, and k can be inferred from statements on how much larger this frequency is, keeping in mind that k indicates intensity and not probability differences.

Here another, much more preferable, way of deriving α and k will be discussed. In many countries data are available on interregional migration flows aggregated by age (the migration flow matrix) stemming from censuses or enquiries which are usually held each five or ten years. Since the mid-period multiregional population data are usually available, one can estimate then an age-aggregated matrix of the origin-destination migration rates. Let this matrix be denoted $\underset{\sim}{M}_5(TOT)$. For Great Britain, it was estimated as follows:

$$\underset{\sim}{M}_5(TOT) = \begin{bmatrix} 0.92659 & 0.03618 & 0.03724 \\ 0.00694 & 0.95628 & 0.03678 \\ 0.00267 & 0.01750 & 0.97982 \end{bmatrix} \tag{24a}$$

The same matrix, for a one-year time period, is as follows:

$$\underset{\sim}{M}_1(\mathrm{TOT}) = \begin{bmatrix} 0.97494 & 0.01290 & 0.01217 \\ 0.00214 & 0.98606 & 0.01180 \\ 0.00075 & 0.00581 & 0.99344 \end{bmatrix} \qquad (24b)$$

Note that these matrices have the same structure as those from page 20. Their eigenvalues are: $\lambda_1(\underset{\sim}{M}_5) = 1$; $\lambda_2(\underset{\sim}{M}_5) = 0.91973$; $\lambda_3(\underset{\sim}{M}_5) = 0.94296$; $\lambda_1(\underset{\sim}{M}_1) = 1$; $\lambda_2(\underset{\sim}{M}_1) = 0.97286$; $\lambda_3(\underset{\sim}{M}_1) = 0.98159$. Repeating further the procedures from Section 3, one receives the following values for the unknown parameters:

$$\alpha_{\mathrm{TOT}} = 0.02198, \quad k_{\mathrm{TOT}} = 0.01049, \quad \nu_2(\mathrm{TOT}) = -1.1735, \quad \nu_3(\mathrm{TOT}) = -0.7092 \qquad (25)$$

These values will be used to derive the age-specific migration-rate matrices, $\underset{\sim}{M}_5(x)$. This can be done in two different ways. First, for each x the parameters α and k from (25) are kept constant. Consider further the case when x = 15. New values for ν_2 and ν_3 may be estimated from (21). Then, values for $\lambda_2\ [\underset{\sim}{M}_5(15)]$ and $\lambda_3\ [\underset{\sim}{M}_5(15)]$ are found correspondingly equal to 0.89003 and 0.92254, by making use of the system (22). Thus the diagonalized matrix $\underset{\sim}{\Lambda}_5(15) = \mathrm{diag}[\underset{\sim}{M}_5(15)]$ becomes available, bearing in mind that $\lambda_1[\underset{\sim}{M}_5(15)] = 1$. Then the transformation $\underset{\sim}{T}_1(15)$, which diagonalizes $\underset{\sim}{M}_1(15)$ may be used to obtain:

$$\underset{\sim}{M}_5(15) = \underset{\sim}{T}_1(15)\underset{\sim}{\Lambda}_5(15)\underset{\sim}{T}_1^{-1}(15) = \begin{bmatrix} 0.89647 & 0.05640 & 0.04714 \\ 0.00679 & 0.94911 & 0.04411 \\ 0.00343 & 0.03015 & 0.96642 \end{bmatrix} \qquad (26a)$$

The second way of deriving the matrices $\underset{\sim}{M}_5(x)$ for all x is to keep k constant once again, but to use α_{TOT} and the observed migration schedules to yield values for $\alpha(x)$ for each x. Suppose the migration schedule is given by the age-specific rates $m_1(x)$ which can be estimated at the national level. Let n be the number of age groups. Then, from the expressions for the means:

$$\frac{\sum_x m_1(x)}{n} = m_1 \qquad \frac{\sum_x \alpha(x)}{n} = \alpha$$

$\alpha(x)$ can be derived as $\alpha(x) = m_1(x)\frac{2}{m_1}$.

For x = 15, $\alpha(15)$ was estimated to be equal to 0.03404. This value of α, together with k_{TOT} from (25), were used to derive the following matrix:

$$\underset{\sim}{M}_5(15) = \begin{bmatrix} 0.91063 & 0.04545 & 0.04294 \\ 0.00555 & 0.95367 & 0.04078 \\ 0.00317 & 0.02783 & 0.96900 \end{bmatrix} \qquad (26b)$$

Each one of the matrices (26a) or (26b) can be rearranged accordingly (see the expression on page 2) and then set into equation (2), which then yields the desired matrix $\underset{\sim}{P}_5(15)$. These procedures yielded the following results (Table 5).

Table 5. Approximated probabilities of dying and migrating at exact age 15 for three regions of Great Britain.

5a: Derived by making use of (26a)

region of origin	region of destination 1	2	3	death
1. East Anglia	0.898531	0.052791	0.045082	0.003595
2. South East	0.006347	0.948149	0.042336	0.003168
3. Rest of Britain	0.003291	0.028926	0.964532	0.003251

5b: Derived by making use of (26b)

region of origin	region of destination 1	2	3	death
1. East Anglia	0.911296	0.043880	0.041226	0.003598
2. South East	0.005237	0.952348	0.039248	0.003167
3. Rest of Britain	0.003050	0.026778	0.966927	0.003251

Both tables are very close to the observed probabilities, exhibited in Table 1a, and show much better results than Table 1b. It is worth noting that α_{TOT} gives better results than $\alpha(x)$ although their numerical values are substantially different. This proves the insensitivity of the high- and low-intensity movers model with respect to its parameters.

For consistency, Table 6 gives the life expectancies at age 15 estimated by making use of (26a) and (26b) as described above.

Table 6. Expectations of life at age 15 for three regions of Great Britain, derived by making use of (26a) and (26b):

(1) derived by making use of (26a)
(2) derived by making use of (26b).

region of origin		region of destination 1	2	3	total
1. East Anglia	(1)	27.69	14.69	17.25	59.63
	(2)	30.40	13.03	16.25	59.68
2. South East	(1)	2.33	40.65	16.47	59.45
	(2)	2.11	41.91	15.45	59.47
3. Rest of Britain	(1)	1.17	8.52	49.07	58.76
	(2)	1.11	7.86	49.77	58.74

Again, in both cases, the results are very close to those in Table 2(2), and α_{TOT} yields better results than $\alpha(x)$.

These numerical results refer to the age 15, but the inferences hold for all other ages. For convenience to the reader, the complete set of expectations of life are given in Appendix 4, together with the levels of migration. The latter are given at the age 0, and represent a measure of the goodness-of-fit, for the different approximations (see the introductory remarks in Appendix 4).

Thus the model suggested here gives a sufficiently good approximation to the problem considered. Recalling that a number of assumptions were made in order to find a solution, it turns out that these assumptions are plausible. They refer to the independency of certain models' variables, α and k, on the regionalization, and hence may be used to show that the inhomogeneity of the population with respect to the interpretations of α and k does not depend on the regionalization.

The fact that the transformations $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_5$ are approximately equal, may be interpreted as a preserved ranking in the attractiveness of the regions with respect to the migrations. That is, over these periods of time, the magnitude of the migration flows may change, but only proportionally for each direction.

Finally, the fact that α and k barely depend on the age groups was unexpected, but it has its demographic or social interpretation: it shows that the differences in the age-specific migration curves of the "chronic" migrants and "all" migrants are insignificant with respect to the one-year - five-year migration problem.

APPENDIX 1

In the text it was shown that the empirical transition matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_5$ can be diagonalized by almost the same matrices $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_5$, such that

$$\underset{\sim}{P}_1 \doteqdot \underset{\sim}{T}_5^{-1}(\underset{\sim}{T}_1\underset{\sim}{P}_1\underset{\sim}{T}_1^{-1})\underset{\sim}{T}_5 \qquad (A1)$$

and

$$\underset{\sim}{P}_5 \doteqdot \underset{\sim}{T}_1^{-1}(\underset{\sim}{T}_5\underset{\sim}{P}_5\underset{\sim}{T}_5^{-1})\underset{\sim}{T}_1 \qquad (A2)$$

This empirical fact led to the conclusion that the n(n-1)-dimensional problem of estimating the five-year transition matrix from the one-year matrix (or vice versa) can be reduced to the (n-1)-dimensional problem of estimating the eigenvalues $\lambda_i(\underset{\sim}{P}_5)$ [or $\lambda_i(\underset{\sim}{P}_1)$], $i = 2,\ldots,n$; $\lambda_1 = 1$. Further we will consider for simplicity only the case when all the λ_i are real and positive. For simplicity let also n = 3. This case is presented graphically in Figure A1.

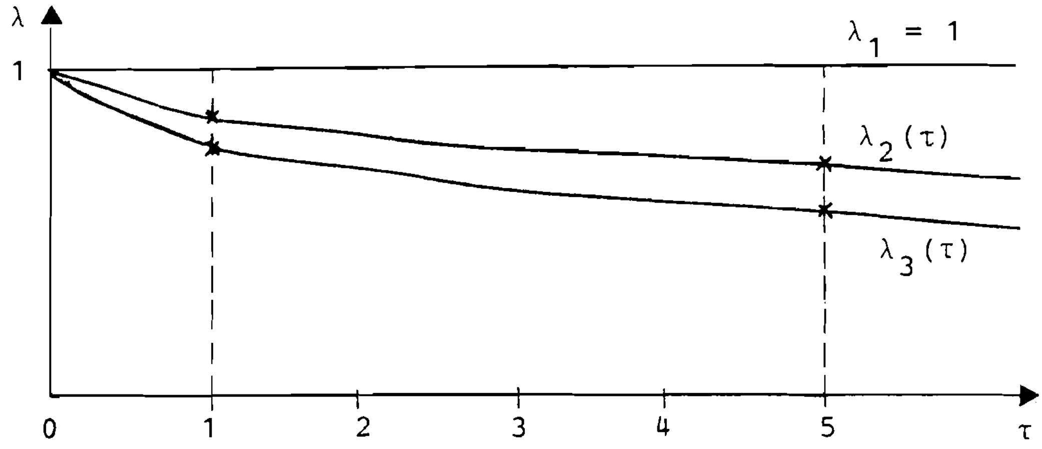

Figure A1. Dependence of eigenvalues of a transition matrix on time.

If the matrices $\tilde{P}_1$ and $\tilde{P}_5$ are known, it is necessary to fit the empirical points $[1, \lambda_2(1), \lambda_2(5)]$ and $[1, \lambda_3(1), \lambda_3(5)]$ as functions of time.

In this paper it was suggested instead to make use of the following approximating function:

$$\lambda_i(\tau) = \alpha\, e^{\nu_i \tau} + (1 - \alpha) e^{k\nu_i \tau} \tag{A3}$$

where α and k are known (or can be found from aggregate data, in which case they will also be approximated).

As usual the decreasing of the dimension [from n(n-1) to n] inevitably presents additional theoretical problems. In this case, they are the following ones:

1. Is it always possible to solve the equation (A3) for $\tau = 1$ if α and k are given?
2. Are 1, e^{ν_1}, and e^{ν_2} eigenvalues of any stochastic matrix?
3. Are 1, e^{ν_1}, and e^{ν_2} eigenvalues of any continuous-time Markovian transition matrix?

4. Are 1, e^{ν_1}, and e^{ν_2} eigenvalues of any stochastic matrix which can be diagonalized by a given transformation T_1?

The answers to these questions follow next.

1. <u>Equation (A3) has a unique non-negative solution.</u> It is easy to see that the function

$$f(\nu) = \alpha e^{\nu_\tau} + (1 - \alpha)e^{k\nu_\tau}$$

is a monotonically decreasing one, $f(0) = 1$, $\lim_{\nu\to\infty} f(\nu) = 0$, and, hence, for $0 < \lambda \leq 1$, the equation $f(\nu) = \lambda$ has a unique non-negative solution.

2. <u>Theorem.</u> (Suleimanova, 1949). The set of n+1 real numbers $\{1, \lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $|\lambda_i| < 1$ for $i = 1,2,\ldots,n$ is a set of eigenvalues of a positive stochastic matrix provided that the sum of the modulus of the negative numbers of the set is less than unity.
3. The problem of representing some stochastic matrix as a continuous time Markovian transition matrix (imbedding problem) can be avoided by considering an integer m and discrete time. The necessary conditions for such imbedding can be found in the paper of Singer and Spilerman (1976).
4. If the transformation $\underset{\sim}{T}_1$ of the matrix $\underset{\sim}{P}_1$ is such that λ_1 is equal to 1, then it is easy to show that the matrix

$$\underset{\sim}{\pi} = \underset{\sim}{T}_1^{-1} \begin{pmatrix} 1 & & & \\ & e^{\nu_1} & & \\ & & e^{\nu_2} & \\ & & & \ddots \\ & & & & e^{\nu_n} \end{pmatrix} \underset{\sim}{T}_1 \qquad \text{(A4)}$$

has the property $\sum_{i=1}^{n} \pi_{ij} = 1$. This is so because the eigenvector, corresponding to the eigenvalue equal to 1, is up to a scalar always equal to (1, 1, 1,...,1). It is necessary only to check if $\underset{\sim}{\pi} > \underset{\sim}{0}$.

Empirical results show that in our case $\underset{\sim}{\pi}$ is always positive and hence stochastic. In the general case it is necessary to prove that the transformation

$$\rho^{-1}(\underset{\sim}{P}) \quad , \text{ where}$$

$$\rho(x) = \alpha\, x + (1 - \alpha)x^k$$

conserves the positiveness of the matrix

$$\underset{\sim}{\pi} = \rho^{-1}(\underset{\sim}{P})$$

and this problem is still an unresolved question.

APPENDIX 2

This Appendix presents the estimations which verify the validity of the assumption that the transition matrices $\underset{\sim}{P}_1(x)$ and $\underset{\sim}{P}_5(x)$ can be diagonalized by one and the same transformation matrix $\underset{\sim}{T}(x)$ for each age group. The following matrices are compared:

$\underset{\sim}{P}_5$ (five-year observed)

$\hat{\underset{\sim}{P}}_5 = \underset{\sim}{T}_1 \, \underset{\sim}{\Lambda}_5 \, \underset{\sim}{T}_1^{-1}$ (five-year estimated)

and

$\underset{\sim}{P}_1$ (one-year observed)

$\hat{\underset{\sim}{P}}_1 = \underset{\sim}{T}_5 \, \underset{\sim}{\Lambda}_1 \, \underset{\sim}{T}_5^{-1}$ (one-year estimated)

The comparison with the "Markovian" approximation

$$\underset{\sim}{P}_5 = (\underset{\sim}{P}_1)^5 \qquad \text{and} \qquad \underset{\sim}{P}_1 = (\underset{\sim}{P}_5)^{\frac{1}{5}}$$

is also given.

age group 1
****

five-year obs.			five-year est.			fifth degree		
0.88975	0.05502	0.05523	0.88912	0.05138	0.05950	0.82383	0.08312	0.09305
0.00893	0.94485	0.04623	0.00916	0.94525	0.04559	0.01490	0.91527	0.06983
0.00447	0.01899	0.97654	0.00314	0.02010	0.97677	0.00484	0.03087	0.96430

one-year obs.			one-year est.			fifth root		
0.96181	0.01835	0.01984	0.96203	0.01975	0.01823	0.97671	0.01091	0.01238
0.00331	0.98213	0.01456	0.00318	0.98200	0.01481	0.00195	0.98868	0.00937
0.00102	0.00646	0.99253	0.00149	0.00606	0.99244	0.00065	0.00414	0.99522

age group 2
****

five-year obs.			five-year est.			fifth degree		
0.91464	0.03916	0.04620	0.91401	0.04079	0.04520	0.87660	0.05936	0.06404
0.00720	0.95475	0.03805	0.00639	0.95703	0.03658	0.00932	0.93954	0.05114
0.00259	0.01564	0.98178	0.00277	0.01710	0.98013	0.00391	0.02394	0.97216

one-year obs.			one-year est.			fifth root		
0.97393	0.01269	0.01338	0.97413	0.01222	0.01365	0.98215	0.00854	0.00932
0.00200	0.98745	0.01056	0.00226	0.98676	0.01098	0.00134	0.99118	0.00748
0.00081	0.00495	0.99424	0.00075	0.00453	0.99472	0.00057	0.00350	0.99593

age group 3

five-year obs.

0.92914	0.03458	0.03628
0.00570	0.96346	0.03084
0.00227	0.01593	0.98181

five-year est.

0.92893	0.03424	0.03683
0.00540	0.96311	0.03149
0.00233	0.01530	0.98237

fifth degree

0.90449	0.04674	0.04877
0.00740	0.95134	0.04127
0.00307	0.02007	0.97686

one-year obs.

0.98008	0.00984	0.01008
0.00156	0.98997	0.00847
0.00063	0.00412	0.99525

one-year est.

0.98014	0.00993	0.00993
0.00164	0.99006	0.00830
0.00061	0.00429	0.99509

fifth root

0.98534	0.00711	0.00755
0.00112	0.99245	0.00642
0.00048	0.00312	0.99640

age group 4

five-year obs.

0.89593	0.05704	0.04704
0.00755	0.94978	0.04267
0.00318	0.03178	0.96504

five-year est.

0.89551	0.05717	0.04733
0.00688	0.94890	0.04421
0.00344	0.03023	0.96634

fifth degree

0.83971	0.08572	0.07457
0.01030	0.91949	0.07021
0.00548	0.04795	0.94657

one-year obs.

0.96553	0.01869	0.01578
0.00225	0.98296	0.01479
0.00115	0.01010	0.98874

one-year est.

0.96567	0.01866	0.01566
0.00246	0.98326	0.01427
0.00107	0.01062	0.98831

fifth root

0.97812	0.01208	0.00981
0.00146	0.98941	0.00913
0.00071	0.00625	0.99305

age group 5
****

five-year obs.			five-year est.			fifth degree		
0.86049	0.07308	0.06643	0.86276	0.07901	0.05823	0.74953	0.13111	0.11936
0.01060	0.92992	0.05948	0.01089	0.92545	0.06366	0.01763	0.85576	0.12662
0.00410	0.04145	0.95445	0.00345	0.03990	0.95665	0.00768	0.07882	0.91350

one-year obs.			one-year est.			fifth root		
0.94359	0.03039	0.02601	0.94287	0.02865	0.02849	0.97080	0.01709	0.01211
0.00412	0.96806	0.02782	0.00405	0.96979	0.02617	0.00236	0.98432	0.01332
0.00164	0.01735	0.98101	0.00188	0.01810	0.98002	0.00071	0.00836	0.99093

age group 6
****

five-year obs.			five-year est.			fifth degree		
0.87004	0.06207	0.06790	0.86996	0.06733	0.06271	0.78704	0.11137	0.10159
0.01085	0.92755	0.06160	0.00967	0.92711	0.06322	0.01602	0.88154	0.10244
0.00384	0.03156	0.96460	0.00404	0.03085	0.96512	0.00654	0.05000	0.94346

one-year obs.			one-year est.			fifth root		
0.95297	0.02530	0.02173	0.95298	0.02318	0.02384	0.97244	0.01451	0.01306
0.00365	0.97442	0.02193	0.00411	0.97457	0.02132	0.00209	0.98474	0.01317
0.00139	0.01071	0.98790	0.00130	0.01098	0.98772	0.00084	0.00643	0.99273

age group 7
****

five-year obs.

0.90142	0.04568	0.05290
0.00842	0.94319	0.04839
0.00293	0.02177	0.97530

five-year est.

0.90127	0.04661	0.05212
0.00825	0.94423	0.04751
0.00295	0.02263	0.97441

fifth degree

0.85469	0.06812	0.07719
0.01204	0.91742	0.07054
0.00440	0.03357	0.96203

one-year obs.

0.96897	0.01478	0.01625
0.00263	0.98262	0.01476
0.00091	0.00703	0.99205

one-year est.

0.96903	0.01449	0.01648
0.00268	0.98230	0.01503
0.00090	0.00677	0.99233

fifth root

0.97938	0.00984	0.01079
0.00175	0.98846	0.00979
0.00061	0.00467	0.99472

age group 8
****

five-year obs.

0.92492	0.03457	0.04051
0.00659	0.95708	0.03634
0.00223	0.01636	0.98141

five-year est.

0.92414	0.03547	0.04039
0.00582	0.95937	0.03481
0.00246	0.01763	0.97990

fifth degree

0.88834	0.05373	0.05793
0.00890	0.94221	0.04890
0.00347	0.02485	0.97168

one-year obs.

0.97654	0.01141	0.01205
0.00190	0.98801	0.01009
0.00072	0.00513	0.99415

one-year est.

0.97681	0.01118	0.01200
0.00217	0.98730	0.01054
0.00063	0.00478	0.99459

fifth root

0.98432	0.00738	0.00830
0.00121	0.99167	0.00712
0.00050	0.00361	0.99589

age group 9
****

five-year obs.

0.94373	0.02775	0.02853
0.00541	0.96619	0.02839
0.00224	0.01291	0.98485

five-year est.

0.94308	0.02762	0.02930
0.00504	0.96742	0.02753
0.00189	0.01383	0.98428

fifth degree

0.92505	0.03651	0.03844
0.00669	0.95723	0.03608
0.00249	0.01813	0.97939

one-year obs.

0.98451	0.00761	0.00788
0.00140	0.99122	0.00739
0.00051	0.00371	0.99578

one-year est.

0.98469	0.00766	0.00765
0.00149	0.99089	0.00762
0.00060	0.00347	0.99594

fifth root

0.98833	0.00569	0.00597
0.00104	0.99335	0.00560
0.00038	0.00282	0.99680

age group 10
****

five-year obs.

0.95535	0.02340	0.02125
0.00482	0.97118	0.02401
0.00238	0.01038	0.98724

five-year est.

0.95530	0.02250	0.02222
0.00444	0.96985	0.02572
0.00138	0.01000	0.98862

fifth degree

0.94190	0.03100	0.02710
0.00614	0.96158	0.03228
0.00166	0.01257	0.98576

one-year obs.

0.98808	0.00642	0.00550
0.00127	0.99215	0.00658
0.00034	0.00256	0.99710

one-year est.

0.98806	0.00666	0.00528
0.00136	0.99251	0.00613
0.00060	0.00264	0.99675

fifth root

0.99089	0.00462	0.00450
0.00091	0.99387	0.00522
0.00028	0.00203	0.99769

age group 11
●●●●●●●●●●●●

five-year obs.

0.96875	0.01678	0.01448
0.00441	0.97358	0.02201
0.00227	0.00797	0.98976

five-year est.

0.96408	0.01508	0.02084
0.00242	0.97688	0.02070
0.00110	0.00777	0.99113

fifth degree

0.93340	0.03557	0.03103
0.00586	0.96359	0.03055
0.00153	0.01158	0.98689

one-year obs.

0.98629	0.00739	0.00632
0.00122	0.99256	0.00622
0.00031	0.00236	0.99733

one-year est.

0.98900	0.00864	0.00236
0.00207	0.99038	0.00755
0.00060	0.00258	0.99682

fifth root

0.99271	0.00308	0.00422
0.00049	0.99532	0.00419
0.00022	0.00157	0.99821

age group 12
●●●●●●●●●●●●

five-year obs.

0.97825	0.01158	0.01016
0.00516	0.97224	0.02260
0.00153	0.00628	0.99220

five-year est.

0.97524	0.01531	0.00945
0.00491	0.97452	0.02058
0.00066	0.00640	0.99294

fifth degree

0.96624	0.01810	0.01565
0.00574	0.96667	0.02759
0.00119	0.00854	0.99027

one-year obs.

0.99314	0.00371	0.00314
0.00118	0.99321	0.00561
0.00024	0.00174	0.99803

one-year est.

0.99386	0.00286	0.00328
0.00129	0.99268	0.00604
0.00048	0.00168	0.99784

fifth root

0.99499	0.00312	0.00189
0.00100	0.99484	0.00417
0.00013	0.00130	0.99857

age group 13
****

five-year obs.			five-year est.			fifth degree		
0.97819	0.00968	0.01213	0.97438	0.01610	0.00952	0.96503	0.01935	0.01562
0.00570	0.96726	0.02704	0.00664	0.97065	0.02272	0.00793	0.96163	0.03044
0.00116	0.00601	0.99283	0.00080	0.00596	0.99324	0.00138	0.00796	0.99066

one-year obs.			one-year est.			fifth root		
0.99289	0.00398	0.00313	0.99379	0.00245	0.00376	0.99481	0.00328	0.00190
0.00163	0.99217	0.00620	0.00143	0.99135	0.00722	0.00136	0.99404	0.00461
0.00027	0.00162	0.99810	0.00037	0.00160	0.99803	0.00016	0.00121	0.99863

age group 14
****

five-year obs.			five-year est.			fifth degree		
0.97596	0.00906	0.01498	0.97453	0.01192	0.01355	0.96548	0.01767	0.01684
0.00514	0.96885	0.02601	0.00472	0.96985	0.02543	0.00702	0.95750	0.03549
0.00160	0.00599	0.99240	0.00114	0.00601	0.99285	0.00140	0.00839	0.99020

one-year obs.			one-year est.			fifth root		
0.99299	0.00364	0.00338	0.99343	0.00272	0.00385	0.99485	0.00243	0.00273
0.00144	0.99132	0.00724	0.00155	0.99100	0.00744	0.00096	0.99388	0.00516
0.00028	0.00171	0.99801	0.00040	0.00172	0.99788	0.00023	0.00122	0.99855

age group 15

five-year obs.

0.97468	0.00927	0.01605
0.00461	0.97831	0.01708
0.00172	0.00571	0.99257

five-year est.

0.97160	0.01105	0.01735
0.00199	0.98045	0.01756
0.00093	0.00556	0.99351

fifth degree

0.95386	0.02401	0.02213
0.00440	0.97284	0.02276
0.00114	0.00725	0.99161

one-year obs.

0.99059	0.00494	0.00448
0.00091	0.99449	0.00461
0.00023	0.00147	0.99830

one-year est.

0.99212	0.00390	0.00399
0.00214	0.99322	0.00464
0.00031	0.00166	0.99803

fifth root

0.99425	0.00225	0.00350
0.00040	0.99605	0.00354
0.00019	0.00112	0.99869

APPENDIX 3

In this Appendix, the probabilities of dying and migrating at exact age x are presented as estimated according to four different models. Appendix 3.1 concerns individuals born in region East Anglia; Appendix 3.2 - individuals born in region South East; and Appendix 3.3 - individuals born in region Rest of Britian.

The four different models of estimation are represented as follows:

Table A - estimated with equation (2). These estimates are the correct ones.

Table B - estimated with the parameters α and k, α being disaggregated by age such that the schedule is of the observed migration schedule for Britain, and the area under the curve is equal to α_{TOT}.

Table C - estimated with the parameters α and k, α aggregated by age.

Table D - estimated with equation (1).

The results in Tables B and C are approximately equal. Both give much better approximations to the results in Table A, than those from Table D.

Appendix 3.1 probabilities of dying and migrating

option 3

region e.anglia

Table A

age	death	migration from e.anglia to e.anglia	s.east	r.brit
0	0.019077	0.878803	0.050263	0.051856
5	0.001796	0.916665	0.037008	0.044531
10	0.001555	0.930246	0.033027	0.035172
15	0.003595	0.898068	0.053417	0.044920
20	0.004013	0.866501	0.067041	0.062445
25	0.003356	0.875430	0.057047	0.064167
30	0.004319	0.902390	0.042690	0.050601
35	0.006141	0.922117	0.032730	0.039012
40	0.010595	0.935395	0.026427	0.027583
45	0.018549	0.938719	0.022245	0.020486
50	0.028144	0.942074	0.015878	0.013903
55	0.048522	0.931112	0.010774	0.009592
60	0.079041	0.901212	0.008709	0.011038
65	0.129096	0.850309	0.007746	0.012849
70	1.000000	0.000000	0.000000	0.000000

Table B

age	death	migration from e.anglia to e.anglia	s.east	r.brit
0	0.019073	0.883325	0.045025	0.052577
5	0.001796	0.922943	0.035134	0.040127
10	0.001555	0.936659	0.029015	0.032771
15	0.003598	0.911296	0.043880	0.041226
20	0.004018	0.882393	0.056432	0.057157
25	0.003353	0.890947	0.053094	0.052606
30	0.004316	0.915103	0.036296	0.044285
35	0.006138	0.925940	0.031534	0.036388
40	0.010592	0.939213	0.023136	0.027059
45	0.018549	0.940257	0.020454	0.020740
50	0.028197	0.925367	0.023514	0.022921
55	0.048548	0.926070	0.012639	0.012743
60	0.079067	0.895656	0.013026	0.012251
65	0.129099	0.847444	0.010987	0.012469
70	1.000000	0.000000	0.000000	0.000000

Table C

age	death	migration from e.anglia to e.anglia	s.east	r.brit
0	0.019085	0.873151	0.051594	0.056170
5	0.001796	0.924027	0.034317	0.039859
10	0.001555	0.939917	0.026843	0.031685
15	0.003595	0.898531	0.052791	0.045082
20	0.004000	0.826082	0.094781	0.075137
25	0.003356	0.858027	0.076874	0.061743
30	0.004317	0.907994	0.040963	0.046726
35	0.006138	0.926855	0.030814	0.036193
40	0.010590	0.941873	0.021251	0.026286
45	0.018546	0.942748	0.018184	0.020521
50	0.028188	0.929261	0.020337	0.022214
55	0.048547	0.926934	0.011493	0.013026
60	0.079066	0.896416	0.012007	0.012511
65	0.129101	0.847997	0.010167	0.012736
70	1.000000	0.000000	0.000000	0.000000

Table D

age	death	migration from e.anglia to e.anglia	s.east	r.brit
0	0.018940	0.810806	0.080174	0.090080
5	0.001843	0.876374	0.058454	0.063329
10	0.001657	0.903852	0.046153	0.048338
15	0.003591	0.838896	0.084048	0.073464
20	0.004028	0.751251	0.127714	0.117006
25	0.002720	0.788667	0.108686	0.099927
30	0.003434	0.853717	0.066789	0.076059
35	0.006096	0.884107	0.052742	0.057055
40	0.011283	0.915184	0.035784	0.037749
45	0.018095	0.925257	0.030197	0.026451
50	0.029244	0.906652	0.034231	0.029872
55	0.047840	0.920240	0.017128	0.014792
60	0.077441	0.890556	0.017734	0.014269
65	0.132126	0.838018	0.015340	0.014516
70	1.000000	0.000000	0.000000	0.000000

Appendix 3.2 probabilities of dying and migrating
option 3

region s.east

Table A

age	death	migration from e.anglia	s.east	s.east to r.brit
0	0.020863	0.008171	0.927202	0.043764
5	0.001684	0.006789	0.954557	0.036970
10	0.001458	0.005433	0.963036	0.030073
15	0.003168	0.007041	0.948826	0.040965
20	0.003254	0.009658	0.930742	0.056346
25	0.003467	0.009916	0.928048	0.058569
30	0.004590	0.007838	0.941089	0.046483
35	0.006757	0.006217	0.951901	0.035125
40	0.011733	0.005153	0.955671	0.027443
45	0.019975	0.004582	0.952380	0.023062
50	0.031934	0.004181	0.942966	0.020920
55	0.053023	0.004801	0.921191	0.020985
60	0.085046	0.005125	0.885635	0.024193
65	0.133732	0.004395	0.839813	0.022060
70	1.000000	0.000000	0.000000	0.000000

Table B

age	death	migration from e.anglia	s.east	s.east to r.brit
0	0.020860	0.008018	0.930649	0.040473
5	0.001684	0.005486	0.959888	0.032942
10	0.001458	0.004559	0.965485	0.028498
15	0.003167	0.005237	0.952348	0.039248
20	0.003254	0.007474	0.929418	0.059853
25	0.003467	0.007611	0.935941	0.052982
30	0.004590	0.006329	0.948042	0.041040
35	0.006756	0.005160	0.956572	0.031513
40	0.011733	0.004162	0.958370	0.025735
45	0.019977	0.004030	0.952138	0.023855
50	0.031938	0.003855	0.941627	0.022581
55	0.053023	0.003991	0.922484	0.020503
60	0.085028	0.005328	0.887998	0.021647
65	0.133744	0.004353	0.838682	0.023222
70	1.000000	0.000000	0.000000	0.000000

Table C

age	death	migration from e.anglia	s.east	s.east to r.brit
0	0.020860	0.009287	0.928479	0.041374
5	0.001684	0.005357	0.959990	0.032969
10	0.001458	0.004196	0.966404	0.027942
15	0.003168	0.006347	0.948149	0.042336
20	0.003263	0.012957	0.902510	0.081270
25	0.003468	0.011095	0.923058	0.062379
30	0.004590	0.007229	0.945432	0.042749
35	0.006756	0.005030	0.956674	0.031540
40	0.011732	0.003780	0.959332	0.025156
45	0.019976	0.003563	0.953380	0.023081
50	0.031938	0.003311	0.942816	0.021935
55	0.053020	0.003606	0.923694	0.019680
60	0.085024	0.004903	0.889201	0.020873
65	0.133736	0.004016	0.839950	0.022298
70	1.000000	0.000000	0.000000	0.000000

Table D

age	death	migration from e.anglia	s.east	s.east to r.brit
0	0.018586	0.014359	0.899220	0.067835
5	0.001533	0.009175	0.938596	0.050695
10	0.001381	0.007299	0.950357	0.040962
15	0.003178	0.010098	0.917494	0.069230
20	0.003520	0.017154	0.855275	0.124051
25	0.003489	0.015630	0.880170	0.100711
30	0.004472	0.011791	0.914194	0.069542
35	0.006149	0.008730	0.936894	0.048227
40	0.011434	0.006551	0.946564	0.035451
45	0.019270	0.005977	0.943300	0.031453
50	0.031272	0.005636	0.933706	0.029386
55	0.052188	0.005428	0.916422	0.025961
60	0.083825	0.007268	0.881265	0.027643
65	0.133965	0.006090	0.829459	0.030486
70	1.000000	0.000000	0.000000	0.000000

Appendix 3.3 probabilities of dying and migrating
option 3

region r.brit

Table A

age	death	migration from e.anglia	migration from s.east	r.brit to r.brit
0	0.022136	0.004181	0.017998	0.955684
5	0.001871	0.002506	0.015177	0.980446
10	0.001633	0.002209	0.015518	0.980641
15	0.003251	0.003073	0.030466	0.963210
20	0.003800	0.003934	0.039181	0.953085
25	0.003777	0.003682	0.029948	0.962593
30	0.004903	0.002833	0.020878	0.971386
35	0.007725	0.002162	0.015796	0.974317
40	0.013437	0.002167	0.012477	0.971920
45	0.023507	0.002290	0.009976	0.964227
50	0.036279	0.002165	0.007582	0.953974
55	0.061317	0.001438	0.005829	0.931416
60	0.097875	0.001059	0.005380	0.895686
65	0.152017	0.001377	0.005082	0.841524
70	1.000000	0.000000	0.000000	0.000000

Table B

age	death	migration from e.anglia	migration from s.east	r.brit to r.brit
0	0.022139	0.002782	0.017830	0.957248
5	0.001871	0.002470	0.015382	0.980277
10	0.001633	0.002095	0.013823	0.982449
15	0.003251	0.003050	0.026778	0.966921
20	0.003801	0.003803	0.037141	0.955255
25	0.003778	0.003401	0.025834	0.966988
30	0.004904	0.002594	0.019444	0.973058
35	0.007725	0.002228	0.015956	0.974091
40	0.013437	0.001815	0.012860	0.971888
45	0.023511	0.001297	0.009262	0.965931
50	0.036282	0.001155	0.008531	0.954033
55	0.061318	0.000999	0.006328	0.931356
60	0.097873	0.001103	0.005652	0.895372
65	0.152017	0.001051	0.005482	0.841449
70	1.000000	0.000000	0.000000	0.000000

Table C

age	death	migration from e.anglia	migration from s.east	r.brit to r.brit
0	0.022138	0.002882	0.018335	0.956644
5	0.001871	0.002461	0.015392	0.980277
10	0.001633	0.002038	0.013536	0.982793
15	0.003251	0.003291	0.028926	0.964532
20	0.003797	0.004592	0.050819	0.940792
25	0.003777	0.003927	0.030489	0.961807
30	0.004903	0.002671	0.020336	0.972089
35	0.007725	0.002237	0.015947	0.974091
40	0.013437	0.001795	0.012538	0.972229
45	0.023511	0.001305	0.008951	0.966232
50	0.036282	0.001136	0.008269	0.954313
55	0.061319	0.001035	0.006063	0.931584
60	0.097874	0.001138	0.005441	0.895547
65	0.152019	0.001084	0.005267	0.841631
70	1.000000	0.000000	0.000000	0.000000

Table D

age	death	migration from e.anglia	migration from s.east	r.brit to r.brit
0	0.021738	0.004690	0.029974	0.943598
5	0.001798	0.003871	0.023720	0.970611
10	0.001556	0.003043	0.019920	0.975481
15	0.003169	0.005401	0.047277	0.944153
20	0.003588	0.007540	0.077211	0.911661
25	0.003647	0.006435	0.049150	0.940768
30	0.004800	0.004348	0.033083	0.957769
35	0.007492	0.003422	0.024503	0.964583
40	0.013599	0.002444	0.017808	0.966149
45	0.023844	0.001625	0.012251	0.962281
50	0.038534	0.001478	0.011135	0.948853
55	0.061789	0.001125	0.008033	0.929053
60	0.098387	0.001260	0.007228	0.893125
65	0.153253	0.001212	0.007210	0.838325
70	1.000000	0.000000	0.000000	0.000000

APPENDIX 4

In this Appendix, the expectations of life at age x are represented. The structure of this Appendix is the same as of Appendix 3. The expectations of life are given here, because they give a better empirical verification of the discussions in the text.

Appendix 4.4 is most helpful in this respect. It gives the regional distribution of the life expectancies at age zero, as a percentage to the total. This measure of migration is called the migration level.

Appendix 4.1 expectations of life by place of birth

initial region of cohort e.anglia

Table A

age	total	e.anglia	s.east	r.brit
0	73.06435	40.93980	14.47626	17.64828
5	69.43669	36.94765	14.62970	17.85934
10	64.55702	32.71232	14.32012	17.52458
15	59.65361	28.78300	13.86263	17.00798
20	54.85471	25.22608	13.26733	16.36129
25	50.05791	22.07278	12.46714	15.51799
30	45.22318	19.29715	11.48181	14.44422
35	40.41728	16.82754	10.39531	13.19442
40	35.67406	14.58276	9.25866	11.83264
45	31.06776	12.52962	8.11201	10.42613
50	26.66482	10.65217	6.98460	9.02805
55	22.45778	8.91441	5.88743	7.65594
60	18.59584	7.34651	4.87481	6.37453
65	15.12872	5.95201	3.97142	5.20529
70	12.15285	4.76494	3.20632	4.18159

Table B

age	total	e.anglia	s.east	r.brit
0	73.09180	42.62563	13.58164	16.88453
5	69.46439	38.65459	13.73097	17.07884
10	64.58482	34.38632	13.44905	16.74944
15	59.68147	30.39919	13.02908	16.25321
20	54.88312	26.75449	12.49320	15.63542
25	50.08735	23.47217	11.78339	14.83179
30	45.25232	20.53234	10.90085	13.81913
35	40.44584	17.87718	9.91503	12.65363
40	35.70110	15.44244	8.87327	11.38538
45	31.09241	13.20630	7.81328	10.07283
50	26.68634	11.15388	6.76988	8.76258
55	22.47504	9.26905	5.74510	7.46089
60	18.60916	7.58867	4.78855	6.23193
65	15.13769	6.09829	3.92913	5.11027
70	12.15793	4.82587	3.19806	4.13401

Table C

age	total	e.anglia	s.east	r.brit
0	73.03786	39.87083	15.27450	17.89253
5	69.41029	35.87259	15.44019	18.09751
10	64.53058	31.64669	15.13105	17.75284
15	59.62715	27.69015	14.69195	17.24505
20	54.82837	24.08595	14.12933	16.61309
25	50.03131	20.95878	13.31690	15.75563
30	45.19673	18.30097	12.25106	14.64470
35	40.39109	15.95318	11.06757	13.37034
40	35.64829	13.80612	9.84316	11.99900
45	31.04238	11.82975	8.61960	10.59303
50	26.63955	10.01059	7.43094	9.19802
55	22.43355	8.33453	6.27926	7.81976
60	18.57315	6.83635	5.21392	6.52289
65	15.10771	5.50704	4.26024	5.34044
70	12.13194	4.37217	3.44912	4.31064

Table D

age	total	e.anglia	s.east	r.brit
0	72.80315	29.77078	18.77510	24.25727
5	69.16043	25.73113	18.93327	24.49603
10	64.28154	21.89120	18.44443	23.94591
15	59.38100	18.45856	17.76056	23.16188
20	54.57787	15.47465	16.90228	22.20094
25	49.77751	13.07009	15.76754	20.93987
30	44.92973	11.18631	14.39406	19.34936
35	40.11012	9.62420	12.92924	17.55667
40	35.36221	8.25938	11.44214	15.66070
45	30.77094	7.04332	9.97619	13.75143
50	26.37483	5.93988	8.56031	11.87464
55	22.21351	4.94600	7.21045	10.05706
60	18.37084	4.06105	5.96118	8.34861
65	14.91985	3.27413	4.85012	6.79560
70	11.98053	2.60649	3.92115	5.45289

Appendix 4.2 expectations of life by place of birth

initial region of cohort s.east

Table A

age	total	e.anglia	s.east	r.brit
0	72.76063	2.57926	53.69096	16.49041
5	69.25768	2.61335	49.91430	16.73003
10	64.37093	2.56135	45.36145	16.44813
15	59.46222	2.48366	40.96908	16.00947
20	54.64418	2.38796	36.80819	15.44803
25	49.81932	2.26641	32.85591	14.69700
30	44.98650	2.11804	29.13494	13.73352
35	40.18504	1.95048	25.63597	12.59859
40	35.44996	1.77230	22.32324	11.35442
45	30.85555	1.59060	19.20091	10.06404
50	26.46224	1.40968	16.27994	8.77262
55	22.28231	1.23027	13.55335	7.49869
60	18.44316	1.05817	11.09296	6.29203
65	14.99926	0.89583	8.93211	5.17132
70	12.02812	0.75029	7.11128	4.16656

Table B

age	total	e.anglia	s.east	r.brit
0	72.76698	2.19881	54.69272	15.87545
5	69.26402	2.22518	50.92850	16.11033
10	64.37720	2.17621	46.34756	15.85343
15	59.46841	2.10607	41.91064	15.45170
20	54.65029	2.02167	37.69820	14.93042
25	49.82510	1.91699	33.69706	14.21106
30	44.99222	1.79054	29.92350	13.27818
35	40.19072	1.64767	26.35215	12.19090
40	35.45534	1.49471	22.95306	11.00757
45	30.86038	1.33819	19.73841	9.78378
50	26.46599	1.18213	16.72888	8.55498
55	22.28550	1.02862	13.92323	7.33364
60	18.44527	0.88313	11.39010	6.17204
65	14.99966	0.74464	9.15718	5.09784
70	12.02564	0.61683	7.26773	4.14108

Table C

age	total	e.anglia	s.east	r.brit
0	72.74550	2.43525	53.43427	16.87598
5	69.24203	2.46342	49.64874	17.12988
10	64.35522	2.40892	45.07616	16.87014
15	59.44641	2.33465	40.64504	16.46672
20	54.62826	2.24561	36.44219	15.94046
25	49.80346	2.12610	32.50816	15.16919
30	44.97081	1.97334	28.86644	14.13102
35	40.16946	1.80167	25.44020	12.92759
40	35.43467	1.62196	22.17944	11.63327
45	30.84056	1.44174	19.09189	10.30692
50	26.44775	1.26500	16.19757	8.98518
55	22.26844	1.09366	13.49517	7.67961
60	18.43030	0.93320	11.05284	6.44426
65	14.98720	0.78243	8.89777	5.30700
70	12.01601	0.64498	7.07244	4.29859

Table D

age	total	e.anglia	s.east	r.brit
0	72.79819	3.00049	46.87308	22.92462
5	69.12948	3.02074	42.92278	23.18596
10	64.23332	2.93499	38.53592	22.76242
15	59.32033	2.81985	34.36416	22.13633
20	54.50202	2.68534	30.48161	21.33507
25	49.68726	2.52106	26.96223	20.20396
30	44.85318	2.32707	23.80667	18.71943
35	40.04630	2.11663	20.90770	17.02197
40	35.29754	1.89919	18.18192	15.21644
45	30.70520	1.68388	15.62465	13.39666
50	26.31229	1.47354	13.23545	11.60331
55	22.15569	1.27213	11.02200	9.86156
60	18.32352	1.08362	9.01872	8.22118
65	14.88562	0.90651	7.25817	6.72094
70	11.94291	0.74655	5.79005	5.40631

Appendix 4.3 expectations of life by place of birth

initial region of cohort r.brit

Table A

age	total	e.anglia	s.east	r.brit
0	71.96190	1.33051	8.36979	62.26160
5	68.53434	1.34994	8.51325	58.67115
10	63.65786	1.32549	8.40152	53.93085
15	58.75750	1.29045	8.21698	49.25008
20	53.94091	1.24672	7.94257	44.75162
25	49.13517	1.19011	7.52361	40.42144
30	44.31017	1.11931	6.97281	36.21805
35	39.51413	1.03806	6.34748	32.12859
40	34.79619	0.95081	5.68466	28.16071
45	30.22595	0.86057	5.00967	24.35571
50	25.87470	0.76876	4.34338	20.76256
55	21.73273	0.67466	3.69290	17.36516
60	17.95303	0.58378	3.09287	14.27637
65	14.58047	0.49951	2.55763	11.52333
70	11.68486	0.42570	2.10235	9.15681

Table B

age	total	e.anglia	s.east	r.brit
0	71.94492	1.13994	8.01793	62.78704
5	68.51715	1.15864	8.15388	59.20464
10	63.64065	1.14079	8.04166	54.45820
15	58.74026	1.11224	7.86004	49.76799
20	53.92361	1.07444	7.60031	45.24886
25	49.11797	1.02334	7.20830	40.88632
30	44.29305	0.95847	6.69441	36.64017
35	39.49707	0.88367	6.11145	32.50195
40	34.77940	0.80264	5.48742	28.48933
45	30.20950	0.71847	4.84514	24.64590
50	25.85888	0.63433	4.20931	21.01524
55	21.71770	0.55159	3.58729	17.57882
60	17.93922	0.47417	3.00984	14.45521
65	14.56795	0.40186	2.49125	11.67485
70	11.67320	0.33647	2.04715	9.28958

Table C

age	total	e.anglia	s.east	r.brit
0	71.96233	1.19575	8.66171	62.10488
5	68.53492	1.21545	8.81093	58.50854
10	63.65845	1.19724	8.69739	53.76382
15	58.75809	1.16848	8.51504	49.07458
20	53.94149	1.13023	8.25159	44.55966
25	49.13568	1.07687	7.82424	40.23458
30	44.31060	1.00737	7.24297	36.06027
35	39.51449	0.92694	6.58950	31.99805
40	34.79636	0.84035	5.89883	28.05717
45	30.22581	0.75099	5.19515	24.27967
50	25.87410	0.66206	4.50344	20.70860
55	21.73193	0.57490	3.83035	17.32668
60	17.95193	0.49349	3.20769	14.25076
65	14.57885	0.41759	2.64964	11.51162
70	11.68218	0.34908	2.17223	9.16087

Table D

age	total	e.anglia	s.east	r.brit
0	72.04422	1.67049	11.76890	58.60482
5	68.58958	1.69563	11.95382	54.94012
10	63.70807	1.66608	11.76848	50.27349
15	58.80295	1.62132	11.48491	45.69672
20	53.98223	1.56295	11.07999	41.33929
25	49.16759	1.48281	10.44365	37.24113
30	44.33633	1.38078	9.60925	33.34629
35	39.53420	1.26460	8.69119	29.57841
40	34.80239	1.14097	7.73416	25.92726
45	30.23069	1.01562	6.77442	22.44065
50	25.87479	0.89212	5.84091	19.14176
55	21.76488	0.77351	4.94922	16.04214
60	17.97826	0.66222	4.12198	13.19406
65	14.59352	0.55830	3.38529	10.64993
70	11.69842	0.46493	2.76441	8.46908

Appendix 4.4 migration levels

Table A

	e.anglia	s.east	r.brit
e.anglia	0.560325	0.035449	0.018489
s.east	0.198130	0.737912	0.116309
r.brit	0.241544	0.226639	0.865202
total	1.000000	1.000000	1.000000

Table B

	e.anglia	s.east	r.brit
e.anglia	0.583179	0.030217	0.015845
s.east	0.185816	0.751615	0.111445
r.brit	0.231004	0.218168	0.872710
total	1.000000	1.000000	1.000000

Table C

	e.anglia	s.east	r.brit
e.anglia	0.545893	0.033476	0.016616
s.east	0.209131	0.734537	0.120364
r.brit	0.244976	0.231987	0.863019
total	1.000000	1.000000	1.000000

Table D

	e.anglia	s.east	r.brit
e.anglia	0.408922	0.041217	0.023187
s.east	0.257889	0.643877	0.163357
r.brit	0.333190	0.314906	0.813456
total	1.000000	1.000000	1.000000

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