INTERACTIVE MULTIOBJECTIVE DECISION MAKING BY THE SEQUENTIAL PROXY OPTIMIZATION TECHNIQUE: SPOT

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PREFACE

Methodologies for decision making with conflicting multiple objectives has attracted increasing attention since the early period of IIASA activity. In the System and Decision Sciences Area of IIASA, decision making processes with conflicting objectives as well as multiobjective optimization are main projects and many techniques have been developed. This paper intends to provide a modest approach to such a research direction for decision sciences.

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ABSTRACT

In this paper, we propose a new interactive multiobjective decision making technique, which we call the Sequential Proxy Optimization Technique (SPOT), in order to overcome the drawbacks of the conventional multiobjective decision making methods. Our method combines the desirable features of both the Surrogate Worth Trade-off (SWT) method and the Multiattribute Utility Function (MUF) method. We can interactively derive the preferred solution of the decision maker efficiently by assessing his marginal rate of substitution and maximizing sequentially the local proxy preference function. A numerical example illustrates the feasibility and efficiency of the proposed method.

1. Introduction

The development of decision making methodologies under multiple conflicting objectives has been one of the most active areas of research in recent years. Several techniques have been developed ; among them two rival methods, namely, the multiattribute utility function (MUF) method [1] and the surrogate worth trade-off (SWT) method [2,3] use global and local utility (preference) modelling respectively.

The MUF method developed by Keeney et al., global utility function modelling, uses two assumptions of preference independence and utility independence to limit the utility function to specialized forms - additive or multiplicative. Once the form is selected, a few assessments determine the free parameters. These global functions are mathematically simple and convenient, but they have disadvantages. Their assumptions are reasonable locally, but when assumed globally, they are very restrictive and may force the decision maker (DM) to fit a function not truly representing his or her preferences.

The SWT method developed by Haimes et al., local utility function modelling, provides an alternative approach that avoids restrictive assumptions. Instead of specifying the utility function globally, their procedures construct a sequence of local preference approximations of it.

The general philosophy taken in the interactive approach using the local utility function modelling is that the multiobjective decision making process should follow the following 3-step procedure.

Step 1. Generate Pareto optimal solutions

Step 2. Obtain meaningful information to interact with the DM

Step 3. Use information obtained in step 2 to interact with the DM and select the final solution based on the DM's preference response.

The interactive Frank-Wolfe (IFW) method developed by Geoffrion et al. [4], put special emphasis on steps 2 and 3. In step 2, the DM is simply supplied with the current values of the objective functions to which the DM responds by providing the marginal rate of substitution (MRS) values between two objectives. This information is then used to modify the objective function for generating a new point in step 1 of the next iteration by applying Frank-Wolfe algorithm. Unfortunately, this method does not guarantee that the generated solution in each iteration will be Pareto optimal.

The SWT method uses the ε-constraint problem as a means of generating Pareto optimal solutions. Objective trade-offs, whose values can be easily obtained from the values of some strictly positive Lagrange multipliers from step 1 are used as the information carrier in step 2. And in step 3, the DM responds by expressing his degree of preference over the prescribed trade-offs by assigning numerical values to each surrogate worth function. This method guarantees the generated solution in each iteration to be Pareto optimal and the DM can select his preferred solution from among Pareto optimal solutions. However, the original version of the SWT method is noninteractive and some improvement, particularly in the way the information from the DM is utilized, must be made.

Recently, Chankong and Haimes [5,6] and Simizu et al. [7] independently proposed an interactive version of the SWT method on the basis of the SWT and the IFW methods. Their methods follow all the steps of the SWT method up to the point where all the surrogate worth values corresponding to the Pareto optimal solution are obtained from the DM. An interactive on-line scheme was constructed in such a way that the values of either the surrogate worth function or the MRS are used to determine the direction in which the utility function, although unknown, increases most rapidly. In their method, however, the DM must assess his preference at each trial solution in order to determine the step size. Such a requirement is very difficult for the DM, since he does not know the explicit form of his utility function.

On the other hand, in 1978, Oppenheimer proposed a proxy approach [8] to multiobjective decision making. He introduced the local proxy preference functions in the IFW method. In his procedure the local proxy preference function is updated at each iteration by assessing a new MRS vector. Then the proxy is maximized to find a better point. Unfortunately, this method, like the IFW method, does not guarantee the generated solution in each iteration to be Pareto optimal. Furthermore, the systematic procedure to maximize the proxies is not mentioned, so it seems to be very difficult to do so in practice.

In this paper, we propose a new interactive multiobjective decision making technique, which we call the sequential proxy optimization technique (SPOT) incorporating the desirable features of the conventional multiobjective decision making methods. In our interactive on-line scheme, after solving the ε-constraint problem, the values of the MRS assessed by the DM are used to determine the direction in which the utility function increases most rapidly and the local proxy preference function is updated to determine the optimal step size and Pareto optimality of the generated solution is guaranteed. A numerical example illustrates the feasibility and efficiency of the proposed method.

2. Multiobjective Decision Making Problem

2.1 Preliminaries

The multiobjective optimization problem (MOP) is represented as

MOP

$$\min_{x} \quad (f_1(x), f_2(x), \ldots, f_n(x)) \triangleq f(x) \tag{1}$$

subject to

$$x \in X = \{ x \mid x \in E^N,\ g_i(x) \leqq 0,\ i=1,2,\ldots,m \} \tag{2}$$

where x is an N-dimensional vector of decision variables, $f_1,\ldots,f_n$ are n district objective functions of the decision vector x, $g_1,\ldots,g_m$ are a set of inequality constraints and X is the constrained set of feasible decisions.

Such a vector-valued index usually induces a partial ordering on the set of alternatives, one cannot speak of optimal solutions. Fundamental to the MOP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MOP is one where any improvement of one objective function can be achieved only at the expense of another. Mathematically, a formal definition of a Pareto optimal solution is given below:

Definition 1.

A decision x* is said to be a Pareto optimal solution to the MOP, if and only if there does not exist another $\bar{x}$ so that $f_j(\bar{x}) \leqq f_j(x^*)$, j=1,2,...,n, with strict inequality holding for at least one j.

Usually, Pareto optimal solutions consist of an infinite number of points, and some kinds of subjective judgement should be added to the quantitative analyses by the DM. The DM must select his preferred solution from among Pareto optimal solutions.

Definition 2.

A preferred solution is a Pareto optimal solution which is chosen as the final decision through the preference ordering relation given by the DM.

Then we can state the multiobjective decision making problem (MDMP) we wish to solve

MDMP

$$\max_{x} \quad U\,(\, f_1(x), f_2(x), \ldots, f_n(x)\,) \tag{3}$$

$$\text{subject to} \quad x \in X^P \tag{4}$$

where X^P is the set of Pareto optimal solutions of the MOP and $U(\cdot)$ is the DM's overall utility function defined on $F \triangleq \{f(x) \mid x \in E^N\}$ and is assumed to exist and is known only implicitly to the DM.

One way of obtaining Pareto optimal solutions to the MOP is to solve ε-constraint problem $P_k(\varepsilon_{-k})$

$P_k(\varepsilon_{-k})$

$$\min \quad f_k(x) \tag{5}$$

$$\text{subject to} \quad x \in X \cap X_k(\varepsilon_{-k}) \tag{6}$$

$$\varepsilon_{-k} \in E_k \tag{7}$$

where

$$\varepsilon_{-k} \triangleq (\, \varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_n\,) \tag{8}$$

$$X_k(\varepsilon_{-k}) \triangleq \{x \mid f_j(x) \leqq \varepsilon_j \;,\; j=1,\ldots,n,\; j\neq k\,\} \tag{9}$$

$$E_k \triangleq \{\, \varepsilon_{-k} \mid X_k(\varepsilon_{-k}) \neq \phi\,\} \tag{10}$$

The following theorem is well known [5,6].

Theorem 1.

A unique solution $P_k(\varepsilon_{-k})$, for any $1 \leqq k \leqq n$, is a Pareto optimal solution of the MOP. Conversely, any Pareto optimal solution of the MOP solves (not necessarily uniquely) $P_k(\varepsilon_{-k})$ for some $\varepsilon_{-k} \in E_k$ and for all $k=1,\ldots,n$.

2.2 Multiobjective Decision Making in Objective Space

Let us assume that $x^*(\varepsilon_{-k})$, an optimal solution to the $P_k(\varepsilon_{-k})$, be unique for the given ε_{-k}. And let AE_k be a set of ε_{-k} such that all the ε-constraint (9) is active, that is

$$AE_k \triangleq \{ \varepsilon_{-k} \mid \varepsilon_{-k} \in E_k \ , \ f_j(x^*(\varepsilon_{-k})) = \varepsilon_j \ , \ j=1,\ldots,n, \ j \neq k \} \tag{11}$$

Define the following active ε-constraint problem $AP_k(\varepsilon_{-k})$.

$\underline{AP_k(\varepsilon_{-k})}$

$$\min_x \quad f_k(x) \tag{12}$$

$$\text{subject to} \quad x \in X \cap X_k(\varepsilon_{-k}) \tag{13}$$

$$\varepsilon_{-k} \in AE_k \tag{14}$$

If the Kuhn-Tucker condition for problem $AP_k(\varepsilon_{-k})$ is satisfied, the Lagrange multiplier $\lambda_{kj}(\varepsilon_{-k})$ associated with the jth active constraint can be represented as follows:

$$\lambda_{kj} = - \{ \partial f_k(\varepsilon_{-k}) \} / \{ \partial f_j(\varepsilon_{-j}) \} \qquad j=1,\ldots,n, \ j \neq k \tag{15}$$

The optimal values of the original decision variables, $x^*(\varepsilon_{-k})$, and the corresponding values of the primal objective, $f_k[x^*(\varepsilon_{-k})]$, determines the trade-off surface by the repeated solution of $AP_k(\varepsilon_{-k})$, with various values of the secondary objectives, ε_j, $j=1,\ldots,n$, $j \neq k$.

Substituting the optimal solutions of the $AP_k(\varepsilon_{-k})$, $x^*(\varepsilon_{-k})$, given desired levels of the secondary objectives, ε_j, $j=1,\ldots,n$, $j \neq k$, the MODM can be restated as follows:

$$\max_{\varepsilon_{-k}} \quad U(\varepsilon_1,\ldots,\varepsilon_{k-1}, f_k[x^*(\varepsilon_{-k})], \varepsilon_{k+1},\ldots,\varepsilon_n) \tag{16}$$

No constraints are involved in equation (16), since all constraints were considered in the solution of the active epsilon-constraint problem $AP_k(\varepsilon_{-k})$.

Restricting the MDMP to the Pareto optimal region simplifies the approach considerably. The decision variables are now the desired levels of the objectives, ε_j, j=1,...,n, j≠k, rather than the original decision variables, x. The optimization is carried on in the objective function space, E^{n-1}, not in the decision variable space, E^N. This is of course, a clear advantage since in most realistic problems, $N \gg n$.

Throughout this paper we make the following.

Assumption 1 : $U : F \to R$ exists and is known only implicitly to the DM. Moreover, it is assumed to be concave, a strictly decreasing and continuously differentiable function on F.

Assumption 2 : All f_i, i=1,...,n and all g_j, j=1,...,m are convex and twice continuously differentiable in their respective domains and constraint set X is compact.

Assumption 3 : For every feasible $\varepsilon_{-k} \in AE_k$ the solution to $AP_k(\varepsilon_{-k})$ exists and is finite.

Under Assumptions 1-3, the following theorem holds.

Theorem 2.

Under Assumptions 1-3, the utility function $U(\varepsilon_1,\ldots,\varepsilon_{k-1},f_k[x^*(\varepsilon_{-k})],\varepsilon_{k+1},\ldots,\varepsilon_n)$ is concave with respect to $\varepsilon_{-k} \in AE_k$.

Proof

By the convexity of f_i and X, the set E_k is convex and the function $f_k[x^*(\varepsilon_{-k})]$ is convex with respect to ε_{-k}. Furthermore, by the monotonicity and concavity of U with respect to f_k, the following relations hold for any ε_{-k}, $\bar{\varepsilon}_{-k} \in AE_k$ and $0 \leqq \theta \leqq 1$.

$$U(\theta\varepsilon_{-k} + (1-\theta)\bar{\varepsilon}_{-k}, f_k(\varepsilon_{-k} + (1-\theta)\bar{\varepsilon}_{-k}))$$

$$\geqq U\left(\theta\varepsilon_{-k} + (1-\theta)\bar{\varepsilon}_{-k} \;,\; \theta f_k(\varepsilon_{-k}) + (1-\theta) f_k(\bar{\varepsilon}_{-k}) \right)$$

$$= U\left(\theta(\varepsilon_{-k}, f_k(\varepsilon_{-k})) + (1-\theta)\, U(\bar{\varepsilon}_{-k}, f_k(\bar{\varepsilon}_{-k}) \right)$$

$$\geqq \theta\, U(\varepsilon_{-k}, f_k(\varepsilon_{-k})) + (1-\theta)\, U(\bar{\varepsilon}_{-k}, f_k(\bar{\varepsilon}_{-k}))$$

where

$$f_k(\varepsilon_{-k}) \triangleq f_k[x^*(\varepsilon_{-k})]$$

$$U(\varepsilon_{-k}, f_k(\varepsilon_{-k})) \triangleq U(\varepsilon_1, \ldots, \varepsilon_{k-1}, f_k[x^*(\varepsilon_{-k})], \varepsilon_{k+1}, \ldots, \varepsilon_n)$$

Thus $U(\varepsilon_{-k}, f_k(\varepsilon_{-k}))$ is concave with respect to $\varepsilon_{-k} \varepsilon AE_k$.

3. Marginal Rate of Substitution

Now, before formulating the gradient, $\partial U(\cdot)/\partial \varepsilon$, of utility function U, we introduce the concept of the marginal rates of substitution (MRS) of the DM.

Definition 3.

At any f, the amount of f_i that the DM is willing to sacrifice to acquire an additional unit of f_j is called the MRS. Mathematically, the MRS is the negative slope of the indifference curve at f:

$$m_{ij}(f) = [\partial U(f)/\partial f_j]/[\partial U(F)/\partial f_i] = -df_i/df_j \Big|_{dU=0, df_r=0, r\neq i,j} \quad (17)$$

where each indifference curve is a locus of points among which the DM is indifferent.

The decision analyst assesses MRS by presenting the following prospects to the DM

$$f = (f_1,\ldots,f_i,\ldots,f_j,\ldots,f_n),$$
$$f' = (f_1,\ldots,f_i-\Delta f_i,\ldots,f_j+\Delta f_j,\ldots,f_n)$$

for a small fixed Δf_j, small enough so the indifference curve is approximately linear but large enough so the increment is meaningful, the analyst varies Δf_i until the DM is indifferent between f and f'. At this level, $m_{ij}(f) \simeq \Delta f_i/\Delta f_j$: in Fig.1, $df_i = -\Delta f_i$ and $df_j = \Delta f_j$.

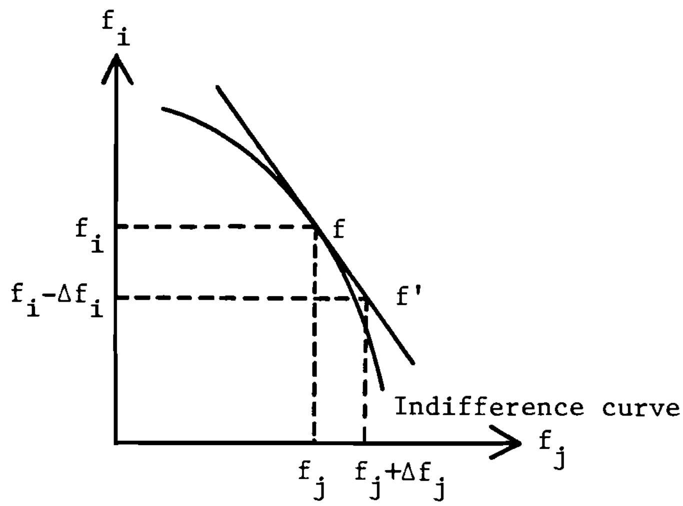

Figure 1. Assessing the Marginal Rate of Substitution.

4. Gradient Method in Objective Space

Now, we can formulate the gradient $\partial U(\cdot)/\partial \varepsilon_j$ of utility function $U(\cdot)$. Applying the chain rule

$$\partial U(\cdot)/\partial \varepsilon_j = \partial U(\cdot)/\partial \varepsilon_j + [\partial U(\cdot)/\partial f_k]\,[\partial f_k/\partial \varepsilon_j] \qquad j=1,\ldots,n,\ j\neq k \tag{18}$$

Using the relations (15) and (17), we have the following

$$\partial U(\cdot)/\partial \varepsilon_j = [\partial U(\cdot)/\partial f_k]\,(m_{kj} - \lambda_{kj}) \qquad j=1,\ldots,n,\ j\neq k \tag{19}$$

From the strict monotonicity of U with respect to f_k, $k=1,\ldots,n$, $\partial U(\cdot)/\partial f_k$ is always negative. Therefore $-(m_{kj} - \lambda_{kj})$ ($j=1,\ldots,n,\ j\neq k$) decide a direction improving the values of $U(\cdot)$ at a current point.

Under the assumptions 1-3, the optimality conditions for a maximization point ε_{-k} are $\partial U(\cdot)/\partial \varepsilon_{-k} = 0$, that is

$$m_{kj} = \lambda_{kj} \qquad j=1,\ldots,n,\ j\neq k \tag{20}$$

This is a well known result that at the optimum the MRS of the DM must be equal to the trade-off rate.

If the optimality condition (20) is not satisfied at the ℓth iteration, the optimal direction of search s_j^ℓ and the corresponding direction of Δf_k^ℓ are given by:

$$s_j^\ell = -(m_{kj}^\ell - \lambda_{kj}^\ell) = \Delta\varepsilon_j^\ell \quad j=1,\ldots,n,\ j\neq k \tag{21}$$

$$\Delta f_k^\ell = [\partial f_k(\varepsilon_{-k}^\ell)/\partial\varepsilon_{-k}^\ell]\ \Delta\varepsilon_{-k}^\ell = -\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda^\ell\ \Delta\varepsilon^\ell \tag{22}$$

Then, we must determine the optimal step size α which maximizes $U(\varepsilon_{-k}^\ell + \alpha\Delta\varepsilon_{-k}^\ell, f_k + \alpha\Delta f_k^\ell)$ along the direction $\Delta f^\ell = (\Delta\varepsilon_1^\ell,\ldots,\Delta\varepsilon_{k-1}^\ell,\Delta f_k^\ell,\Delta\varepsilon_{k+1}^\ell,\ldots,\Delta\varepsilon_n^\ell) \triangleq (\Delta\varepsilon_{-k}^\ell,\Delta f_k^\ell)$

To solve this linear search problem, the following two problems arise.

Problem 1.

The DM must assess his preference at each trial solution $(\varepsilon_{-k}^\ell + \alpha\Delta\varepsilon_{-k}^\ell, f_k^\ell + \alpha\Delta f_k^\ell)$ for several values of α, in order to determine the best step size. Such requirement is very difficult for the DM, since he does not know the explicit form of his utility function.

Problem 2.

Even if it is possible for the DM to assess the utility value, there remains a problem. In Fig. 2, new trial point $f^\ell + \alpha\Delta f^\ell$, where Δf^ℓ is a direction vector, is not a Pareto optimal solution for any α satisfying $0 \leqq \alpha \leqq \alpha_2$; that is, for $\alpha=\alpha_1$, there is a Pareto optimal point P_1 which is obviously better than T_1. Furthermore, for any α satisfying $\alpha>\alpha_2$, the trial point, like point T_3, becomes infeasible and the assessment at such a point is meaningless.

In order to resolve problem 2, we adopt $(\varepsilon_{-k}^\ell+\alpha\Delta\varepsilon_{-k}^\ell, f_k(\varepsilon_{-k}^\ell+\alpha\Delta\varepsilon_{-k}^\ell))$ as a trial point in the process of linear search instead of $(\varepsilon_{-k}^\ell+\alpha\Delta\varepsilon_{-k}^\ell, f_k^\ell + \alpha\Delta f_k^\ell)$.

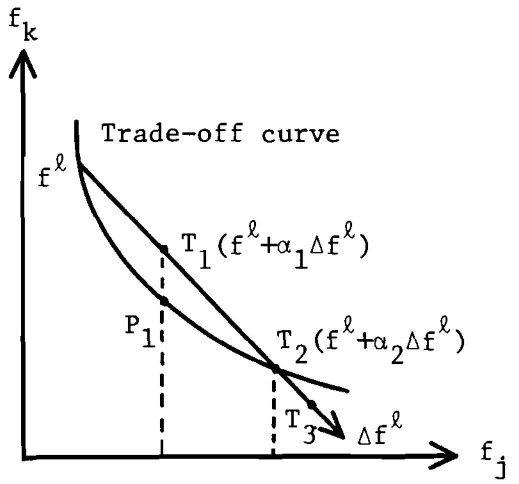

Figure 2. Gradient Method in Objective Space.

Our trial point becomes a Pareto optimal solution by solving the active epsilon constraint problem $AP_k(\varepsilon_{-k})$. Although it is necessary to solve $AP_k(\varepsilon_{-k})$ for several values of α, the generated solution in each iteration becomes a Pareto optimal solution and the DM can select his preferred solution from among Pareto optimal solutions.

Concerning the problem 1, it is necessary to construct some kind of utility (preference) function, so we introduce local proxy preference functions like Oppenheimer's method [8] as explained in the following.

5. Local Proxy Preference Functions

Using the deterministic additive independence condition $P(f) = \sum p_i(f_i)$ of Keeney et al. [1], together with assumptions about a marginal rate of substitution variation, Barrager and Keelin [9], [10] derive the following three global utility functions:

(1) sum-of-exponentials

$$\text{If} \quad [-\partial m_{ij}(f)/\partial f_j] \,/\, m_{ij}(f) = \omega_j \quad \text{then} \quad P(f) = -\sum a_i \exp(-\omega_i f_i) \tag{23}$$

It implies that if the DM is indifferent between any f^1 and f^2, then he is also indifferent between $f^1+\Delta$ and $f^2+\Delta$ where $\Delta = (\Delta, \Delta, \ldots, \Delta)$.

(2) sum-of-powers ($\alpha_j \neq 0$)

$$\text{If} \quad [-\partial m_{ij}(f)/\partial f_j] \,/\, m_{ij}(f) = (1+\alpha_j)/f_j \ \text{then} \ P(f) = -\sum a_i f_i^{\alpha_i} \tag{24}$$

It implies that as the DM accumulates more of each attribute, he becomes less sensitive to substitutions among them.

(3) sum-of-logarithms

$$\text{If} \quad [-\partial m_{ij}(f)/\partial f_j]/m_{ij}(f) = 1/(M-f_j) \ \text{then} \ P(f) = \sum a_i \ell_n (M-f_i) \tag{25}$$

where M is a sufficiently large positive number.

This utility function can be viewed as the additive form of the Cobb-Douglass function. It implies that if the DM is indifferent between any $M-f^1$ and $M-f^2$, then he is also indifferent between $b(M-f^1)$ and $b(M-f^2)$ for any positive constant b.

Although these utility functions are very restrictive globally, they are reasonable when assumed locally. We use these utility functions as local proxy preference functions to determine the optimal step size because they seem to be a very good model locally.

In the following, we deal with the sum-of-exponentials as an example of a local proxy preference function, but for other types of proxy functions similar discussions can be made.

For the sum-of-exponentials function, the constant a_1 can arbitrarily be set equal to one in $P(f) = -\sum a_i \exp(-\omega_i f_i)$. The remaining parameters, $a_2,\dots,a_n$, $\omega_1,\omega_2,\dots,\omega_n$ can be calculated from MRS assessment. At any f, there are n-1 MRS at each of two points plus a single MRS at third point are required to fit the 2n-1 parameters.

The numerical MRS actually assessed relate to the sum-of-exponential parameters a_i and ω_i in the following way;

$$m_{kj}(f) = [\partial P(f)/\partial f_j]/[\partial P(f)/\partial f_k] = [\omega_j a_j \exp(-\omega_j f_j)/(\omega_k a_k \exp(-\omega_k f_k)] \qquad (26)$$
$$j=1,\dots,n,\ j\neq k$$

By taking the logarithm and solving a set of linear equations, the proxy parameters a and ω are uniquely determined from the 2n-1 assessment of $m_{kj}(f)$. If the equations are linearly dependent, an additional assessment at the third point is required.

6. Consistency of Marginal Rate of Substitution Assessment

We have assumed up to this point that at each iteration the DM provides MRS consistent with a continuously differentiable deterministic utility function, But, it is a question whether the DM can respond precise and consistent values of MRS through the whole searching process. In this section, we relax the assumption and examine techniques for checking MRS consistency following Oppenheimer [8].

Two types of consistency tests are employed, the first testing MRS consistency at a single point, and the second testing consistency at successive points.

The single point test requires a second set of assessments at each point and checks whether the MRS of the DM satisfies the chain rule, i.e. $m_{kj} = m_{ki} m_{ij}$ $i,j=1,\dots,n$, $i \neq k, k \neq i, k \neq j$. Since only n-1 unique MRS among the objectives exist at any point, the second set can be used to measure the discrepancy E:

$$E = [(\Delta f_k/\Delta f_j) - (\Delta f_k/\Delta f_i)(\Delta f_i/\Delta f_j)] \,/\, (\Delta f_k/\Delta f_j) \quad (\%) \qquad (27)$$

Certainly we would not expect exact agreement. Instead, we set a reasonable tolerance level; if the discrepancy exceeds the tolerance, the analyst should explain the inconsistency to the DM and reassess the MRS until the discrepancy is resolved.

The second test checks for decreasing marginal rates of substitution of the proxy. In assumption 1, we assumed the utility function U(f) is strictly concave, satisfying a strictly decreasing marginal rate of substitution. So, we must check the concavity and monotonicity of the proxy P(f). The necessary and sufficient condition for the three types of proxy P(f) to be strictly decreasing and concave can be shown using the parameter values condition.

The following theorem can be easily proven by constructing the Hessian matrix of P(f).

Theorem 3.

(1) The sum-of-exponentials proxy P(f) is strictly decreasing and concave if and only if all the parameters a_i and ω_i are strictly positive, i.e.,

$$a_i > 0 \text{ and } \omega_i > 0, \quad i=1,\ldots,n \tag{28}$$

(2) The sum-of-powers proxy P(f) is strictly decreasing and concave if and only if

$$a_i > 0; \quad \alpha_i > 1 \quad i=1,\ldots,n \tag{29}$$

(3) The sum-of-logarithms proxy P(f) is strictly decreasing and concave if and only if

$$a_i > 0 \quad i=1,\ldots,n \tag{30}$$

7. Algorithm of the SPOT

Following the above discussions, we can now construct the algorithm of the sequential proxy optimization technique (SPOT) in order to obtain the preferred solution of the DM for the MDMP.

Step 1 Choose initial point $\varepsilon_{-k}^{1} \varepsilon AE_k$ and set $\ell = 1$.

Step 2 Set $\varepsilon_{-k} = \varepsilon_{-k}^{\ell}$, solve an active ε-constraint problem $AP_k(\varepsilon_{-k}^{\ell})$ for ε_{-k}^{ℓ} and obtain a Pareto optimal solution $x^*(\varepsilon_{-k}^{\ell})$, a Pareto optimal value $f^{\ell} = (\varepsilon_{-k}^{\ell}, f_k^{\ell}[x^*(\varepsilon_{-k}^{\ell})])$ and corresponding Lagrange multiplier λ_{kj}^{ℓ} (j= 1,...,n, $j \neq k$).

Step 3 Assess the MRS of the DM at f^{ℓ}, where Δf_j (j=1,...,n, $j \neq k$) must be fixed small enough so the indifference curve is approximately linear but large enough so the increment is meaningful.

Step 4 For MRS at f^{ℓ}, evaluate discrepancy E. If $E < \delta_1$ go to step 5, where the tolerance δ_1 is a prescribed sufficiency small positive number. If E exceeds the tolerance, the DM reassess the MRS until the tolerance condition is satisfied.

Step 5 If $| m_{kj}^{\ell} - \lambda_{kj}^{\ell} | < \delta_2$ for j=1,...,n, $j \neq k$, stop, where the tolerance δ_2 is a prescribed sufficiency small positive number. Then a Pareto optimal solution $(\varepsilon_{-k}^{\ell}, f_k^{\ell}[x^*(\varepsilon_{-k}^{\ell})])$ is the preferred solution of the DM. Otherwise, determine the direction vector $\Delta\varepsilon_{-k}^{\ell}$ by

$$s_j^{\ell} = -(m_{kj}^{\ell} - \lambda_{kj}^{\ell}) = \Delta\varepsilon_j^{\ell} \quad (j=1,\ldots,n,\ j \neq k)$$

Step 6 Obtain two Pareto optimal points ${}^1f^\ell$ and ${}^2f^\ell$ in the neighbourhood of f^ℓ and assess n-1 MRS m^ℓ_{kj} at a point ${}^1f^\ell$ plus a single MRS at a third point ${}^2f^\ell$. If the consistency check at step 4 is passed, select the form of the proxy function that will be used at each iteration by the measure about MRS variation. If the parameter value conditions of theorem 3 are passed go to the next step. Otherwise, the DM reassesses the MRS until the parameter value conditions are satisfied.

Step 7 Determine the step size α which maximizes the proxy preference function $P\,(\,\varepsilon^\ell_{-k}+\alpha\Delta\varepsilon^\ell_{-k}, f^\ell_k[x^*(\varepsilon^\ell_{-k}+\alpha\Delta\varepsilon^\ell_{-k})]\,) \triangleq P(\alpha)$ as follows. Change the step size, obtain corresponding Pareto optimal values and search for three α values α_A, α_B and α_C which satisfy

$$\alpha_A < \alpha_B < \alpha_C$$

$$P(\alpha_A) < P(\alpha_B) > P(\alpha_C)$$

Then a local maximum of $P(\alpha)$ is in the neighbourhood of $\alpha = \alpha_B$. If $U(f^{\ell+1}) > U(f^\ell)$ where $f^{\ell+1} = (\varepsilon^\ell_{-k}+\alpha^\ell_B\Delta\varepsilon^\ell_{-k}, f^\ell_k[x^*(\varepsilon^\ell_{-k}+\alpha^\ell_B\Delta\varepsilon^\ell_{-k})]\,)$, set $\ell = \ell+1$ and return to step 2. Otherwise reduce α_B to be $\frac{1}{2}, \frac{1}{4}\ldots$ until improvement is achieved.

Remark. Under the assumption of ideal DM, the proposed SPOT algorithm is nothing besides a feasible direction method to solve MDMP. Thus, the convergence of the SPOT can be demonstrated by the convergence of the modified feasible direction method.

8. An Illustrative Example

We now demonstrate the interaction processes of the SPOT by means of an illustrative example which is designed to test the method under the assumption of an ideal DM.

Consider the following multiobjective decision making problem.

$$\min_{x} \quad f(x) = (f_1(x), f_2(x), f_3(x)) \tag{31}$$

subject to

$$x \in X = \{x \mid x_1^2 + x_2^2 + x_3^2 \leqq 100,\ 0 \leqq x_1, x_2, x_3 \leqq 10\} \tag{32}$$

where

$$f_1(x) = 565\,(x_1^2 + x_2^2 + 10x_2 + x_3^2 - 120x_3 + 800) \tag{33}$$

$$f_2(x) = (x_1+40)^2 + (x_2-224)^2 + (x_3+40)^2 \tag{34}$$

$$f_3(x) = (x_1-224)^2 + (x_2+40)^2 + (x_3+40)^2 \tag{35}$$

For illustrative purposes, we shall assume that the DM's structure of preference can be accurately represented by the utility function $U(f_1, f_2, f_3)$ where

$$U(f) = -180f_1 - (f_2-40000)^2 - (f_3-45000)^2 \tag{36}$$

However, it should be stressed that the explicit form of utility function as in (36) is used in this example purely for simulating values of MRS.
To be more specific, m_{kj} will be obtained through the following expression:

$$m_{kj}(f) = [\partial U(f)/\partial f_j] \,/\, [\partial U(f)/\partial f_k] \quad j=1,\ldots,n,\ j \neq k \tag{37}$$

m_{kj} obtained this way are as if they had been obtained from the ideal DM directly.

Let us now choose $f_1(x)$ as our primary objective and formulate the corresponding ε-constraint problem $P_1(\varepsilon_{-1})$.

$\underline{P_1(\varepsilon_{-1})}$

$$\min_x \quad f_1(x) \tag{38}$$

$$\text{subject to} \quad x \in X \cap X_1(\varepsilon_{-1}) \tag{39}$$

$$\text{where} \quad X_1(\varepsilon_{-1}) = \{x \mid f_j(x) \leqq \varepsilon_j,\ j=2,3\} \tag{40}$$

In applying the SPOT, (36) will be used to simulate our imaginary DM. To be more specific, m_{12} and m_{13} at the ith iteration will be obtained through (37).

To understand how the SPOT actually work, we give the following description of the iterations.

Iteration 1

Choose initial $\varepsilon^1_{-1}=(\varepsilon^1_2,\varepsilon^1_3) = (54000,50000)$ and solving $P_1(\varepsilon_{-1})$ by the generalized reduced gradient (GRG) algorithm [11] yields

$$(f^1_2,f^1_3) = (54000,50000),\ f^1_1 = 203889.082 \text{ and } \lambda^1_{12} = 76.321,\ \lambda^1_{13} = 206.654.$$

From this information, the DM, by giving values $(m^1_{12},m^1_{13}) = (155.555,55.555)$ (as simulated by (36)) determines the direction of search (s^1_2,s^1_3) to be

$$s^1_2 = -(m^1_{12}-\lambda^1_{12}) = -79.234, \qquad s^1_3 = -(m^1_{13}-\lambda^1_{13}) = 151.098$$

Update $(\varepsilon_2,\varepsilon_3)$ by the formula

$$\varepsilon^2_2 \leftarrow \varepsilon^1_2 + \alpha s^1_2 = 54000 - 79.234\alpha, \quad \varepsilon^2_3 \leftarrow \varepsilon^1_3 + \alpha s^1_3 = 50000 + 151.098\alpha$$

Construct the proxy preference function P(f) to determine the optimal step size. Adopt the following sum-of-exponentials.

$$P(f) = \sum_{i=1}^{3} P_i(f_i) = \sum_{i=1}^{3} -a_i \exp(\omega_i f_i)$$

Obtain two Pareto optimal solutions ${}^{1}f^{1}$, ${}^{2}f^{1}$ in the neighbourhood of f^{1} and assess MRS yields

$${}^{1}f^{1} = (179858.513,53920.770,50151.088),\ m_{12}({}^{1}f^{1})=154.675,\ m_{13}({}^{1}f^{1})=57.234$$

$${}^{2}f^{1} = (157905.452,53841.532,50302.186),\ m_{13}({}^{2}f^{1})=153.795$$

Using these values together with f^{1}, m_{12}^{1} and m_{13}^{1} , P(f) becomes

$$P(f) = -\exp(0.156881\times10^{-7}f_{1}) - 0.518417\times10^{-3}\exp(0.763776\times10^{-4}f_{2})$$
$$- 0.268061\times10^{-6}\exp(0.194543\times10^{-3}f_{3})$$

where all the parameters are positive, the parameter values condition is satisfied. Calculate Pareto optimal solutions and corresponding P(f) for each step size α = 1,2,4,8,16,24 yields the following.

α	Pareto optimal solutions	P(f)
1	(179858.513,53920.770,50151.098)	- 1.03931
2	(157905.452,53841.532,50302.196)	- 1.03891
4	(119234.030,53683.064,50604.040)	- 1.03821
8	(58456.447,53366.128,51208.786)	- 1.03714
16	(-14457.998,52732.255,52147.572)	- 1.03606
24	(-35933.028,52098.384,53626.358)	- 1.03626

This result shows the maximization point of P(f) is in the neighbourhood of $\alpha = 16$, set optimal step size $\alpha^{1} = 16$ and go to the next iteration where

$$\varepsilon_{2}^{2} = \varepsilon_{2}^{1} + \alpha^{1}s^{1} = 52732.255, \quad \varepsilon_{3}^{2} = \varepsilon_{3}^{1} + \alpha^{1}s_{3} = 52417.572.$$

The same procedure continues in this manner. In this example, at the 5th iteration optimality condition is satisfied.

In the following, we roughly show the main results for each iteration.

Iteration 2

$$\varepsilon_{-1}^{2} = (\varepsilon_{2}^{2},\varepsilon_{3}^{2}) = (52732.255,52417.572),\ f_{1}^{2} = -14457.998$$

$s_2^2 = -(m_{12}^2-\lambda_{12}^2) = -55.28652,\ s_3^2 = -(m_{13}^2-\lambda_{13}^2) = 0.684$

$$P(f) = -\exp(0.167094\times10^{-6}f_1)-0.123431\times10^{-1}\exp(0.643129\times10^{-4}f_2)$$
$$-0.40928\times10^{-31}\exp(0.129212\times10^{-2}f_3)$$

$\alpha^2 = 16$

Iteration 3

$\varepsilon_{-1}^3 = (\varepsilon_2^3,\varepsilon_3^3) = (51847.671,52428.516),\ f_1^3 = 76325.022$

$s_2^3 = -(m_{12}^3-\lambda_{12}^3) = -8.82668,\quad s_3^2 = -(m_{13}^3-\lambda_{13}^3) = 14.2901$

$$P(f) = -\exp(0.181785\times10^{-6}f_1) - 0.24701\times10^{-2}\exp(0.904197\times10^{-4}f_2)$$
$$-0.122909\times10^{-3}\exp(0.130716\times10^{-3}f_3)$$

$\alpha^3 = 32$

Iteration 4

$\varepsilon_{-1}^4 = (\varepsilon_2^4,\varepsilon_3^4) = (51565.217,52885.796),\ f_1^4 = 70717.521$

$s_2^4 = -(m_{12}^4-\lambda_{12}^4) = -0.5598,\quad s_3^4 = -(m_{13}^4-\lambda_{13}^4) = -5.0054256$

$$P(f) = -\exp(0.726055\times10^{-7}f_1) - 0.136163\times10^{0}\exp(0.220701\times10^{-4}f_2)$$
$$-0.959876\times10^{-4}\exp(0.119558\times10^{-3}f_3)$$

$\alpha^4 = 12$

Iteration 5

$\varepsilon_{-1}^5 = (\varepsilon_2^5,\varepsilon_3^5) = (51558.499,52825.731),\ f_1^5 = 76622.752$

$s_2^5 = -(m_{12}^5-\lambda_{12}^5) = 0.93814,\quad s_3^5 = -(m_{13}^5-\lambda_{13}^5) = -1.713756$

Optimality test ($\delta_2 = 2$) is satisfied at this iteration at which $f^5 = (76622.752,51558.499,52825.731)$, $x^5 = (3.785588,6.185648,6.885281)$, $U^5 = -2.0863306\times10^8$. This result compares favorable with the true optimum which is $x^* = (3.870271,6.136885,6.881835)$ and $U^* = -2.08624446\times10^8$.

9. Conclusion

In this paper, we have proposed an interactive multiobjective decision making technique, which we call SPOT, incorporating the desirable features of both the SWT and the MUF method. In our interactive on-line scheme, after solving the epsilon constraint problem, the values of MRS assessed by the DM were used to determine the direction and the local proxy preference function was updated to determine the optimal step size. Of course, Pareto optimality of the generated solution in each iteration is guaranteed in our technique. An illustrative example demonstrated the feasibility and efficiency of the SPOT.

Although we have assumed the convexity of the objective functions and the constraint set, it is possible to extend our technique to the nonconvex problems by introducing the concept of local Pareto optimality. Furthermore, extentions to the non-smooth Pareto surface is also possible by utilizing the directional derivatives. Applications of the SPOT to environmental systems will be reported elsewhere.

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