Nash Equilibria in Reactive Strategies

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Motivation

- **Why infinitely repeated games?**
- **Why cooperation?**
- **Why stability?**
- Why complexity?

Related works and inspiration

Arkady Kryazhimskiy (2014)

Equilibrium stochastic behaviours

in repeated games, **2012**.

Main scope: infinitely repeated game of 2 players x N strategies.
Q: Existence of equilibrium for arbitrary subsets of 1-memory strategies.

Big Question

How does a tiny change in complexity of strategies influence properties of the Nash equilibrium?

What would you guess?

Strategies and payoff function

Infinitely repeated 2x2 game.

Payoff defined as **limit of averages.**

Reactive strategies = stochastic strategies defined only on the last opponents action.

Reactive strategies

2nd player **(columns)**

1st player **(rows)**

$$\begin{pmatrix} A_1 A_2 & B_1 B_2 \\ C_1 C_2 & D_1 D_2 \end{pmatrix}$$

p_1 = **P** (**1st** row | last opponent's action = **1st** column)

q_1 = **P** (**1st** row | last opponent's action = **2nd** column)

p_2 = **P** (**1st** column | last opponent's action = **1st** row)

q_2 = **P** (**1st** column | last opponent's action = **2nd** row)

Complexity of strategies

Increasing complexity in 2x2 repeated games

mixed strategies in [0,1]

reactive strategies in [0,1]x[0,1]

1-memory strategies in [0,1]x[0,1]x[0,1]x[0,1]

Rigorously answered questions

- ➡ **Q1.** What are all possible pairs of reactive strategies leading to an equilibrium?
- ➡ **Q2.** What are all possible symmetric games admitting equilibria? How common are these games?

Partly answered questions

- ➡ **Q3.** Are there new effects of interactions in equilibria caused by the increase of strategy complexity?
- ➡ **Q4.** If we replace reactive strategies with 1-memory ones, then what properties of equilibria are affected?

Payoff equivalence

For fixed strategies we observe Markov chain with stationary distribution on 4 states of one-shot game

$$s_1 = \frac{q_2(p_1 - q_1) + q_1}{1 - (p_1 - q_1)(p_2 - q2)} \qquad s_2 = \frac{q_1(p_2 - q_2) + q_2}{1 - (p_1 - q_1)(p_2 - q_2)}$$

$$\begin{array}{c} \\ s_1 \\ 1 - s_1 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & 1 - s_2 \end{array} \\ \begin{pmatrix} A_1 A_2 & B_1 B_2 \\ C_1 C_2 & D_1 D_2 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ s_1 \\ 1 - s_1 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & 1 - s_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{array}$$

Payoffs are Identical to one-shot game with mixed strategies

$$J_i = A_i s_1 s_2 + B_i s_1 (1 - s_2) + C_i (1 - s_1) s_2 + D_i (1 - s_1)(1 - s_2)$$

Sets of strategies

$$0 < p_i, q_i < 1$$

➡No Tit For Tat

➡Noise proof

➡First round does not matter

➡Stationary distribution always exists

Equilibria generated by SD

Theorem

(p_1, q_1) and (p_2, q_2) is a Nash equilibrium

with the corresponding SD (s_1, s_2) if

$$\begin{cases} q_1 & = \frac{c_2 s_1 + b_2 s_2 + 2a_2 s_1 s_2}{c_2 + a_2 s_2}, \ p_1 - q_1 = -\frac{b_2 + a_2 s_1}{c_2 + a_2 s_2}, \\ q_2 & = \frac{b_1 s_1 + c_1 s_2 + 2a_1 s_1 s_2}{c_1 + a_1 s_1}, \ p_2 - q_2 = -\frac{b_1 + a_1 s_2}{c_1 + a_1 s_1}, \\ 0 & \geq a_2(p_1 - q_1), \ 0 \geq a_1(p_2 - q_2), \\ 0 & < p_1, q_1, p_2, q_2 < 1. \end{cases}$$

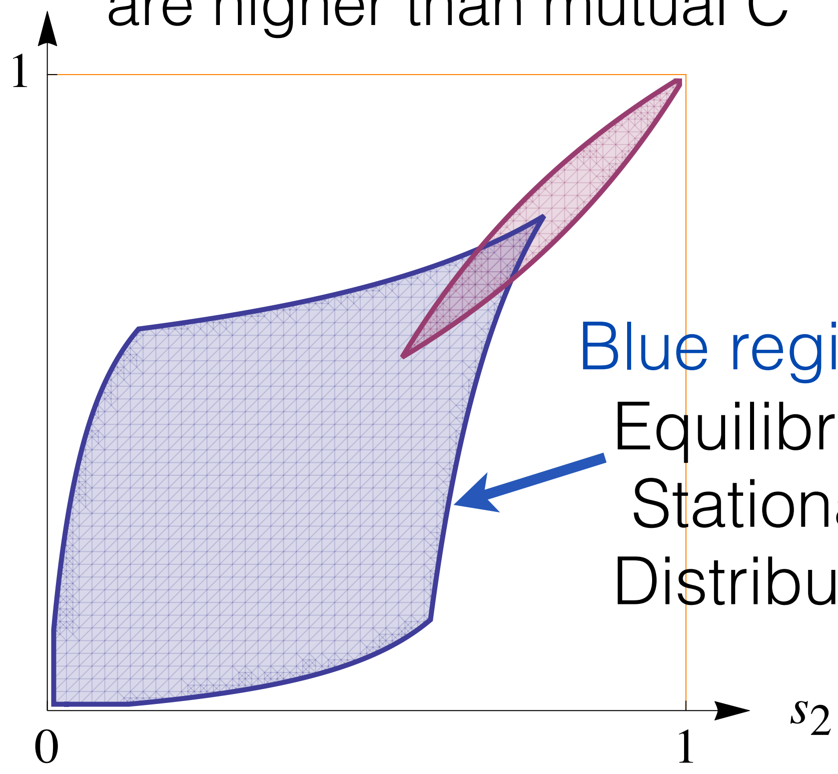
a_i, b_i, c_i are defined by one-shot game

Examples: Prisoners Dilemma

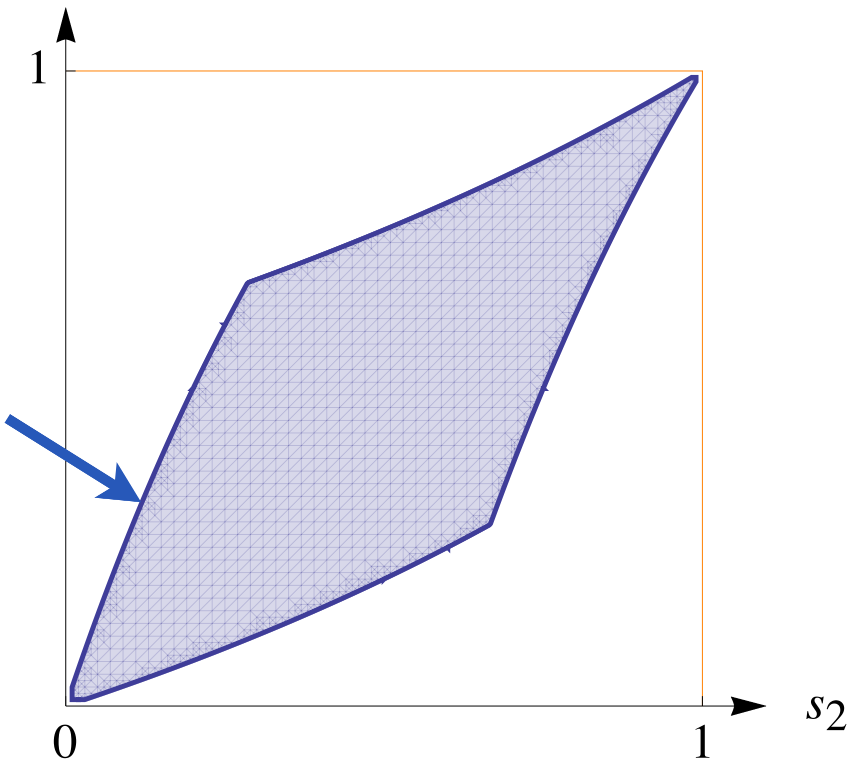
Game G^{P1}	C	D
C	5, 5	−1, 15
D	15, −1	0, 0

Game G^{P2}	C	D
C	5, 5	−2, 8
D	8, −2	0, 0

Red region - both payoffs are higher than mutual C



Any level of C is possible



Blue region = Equilibrium Stationary Distribution

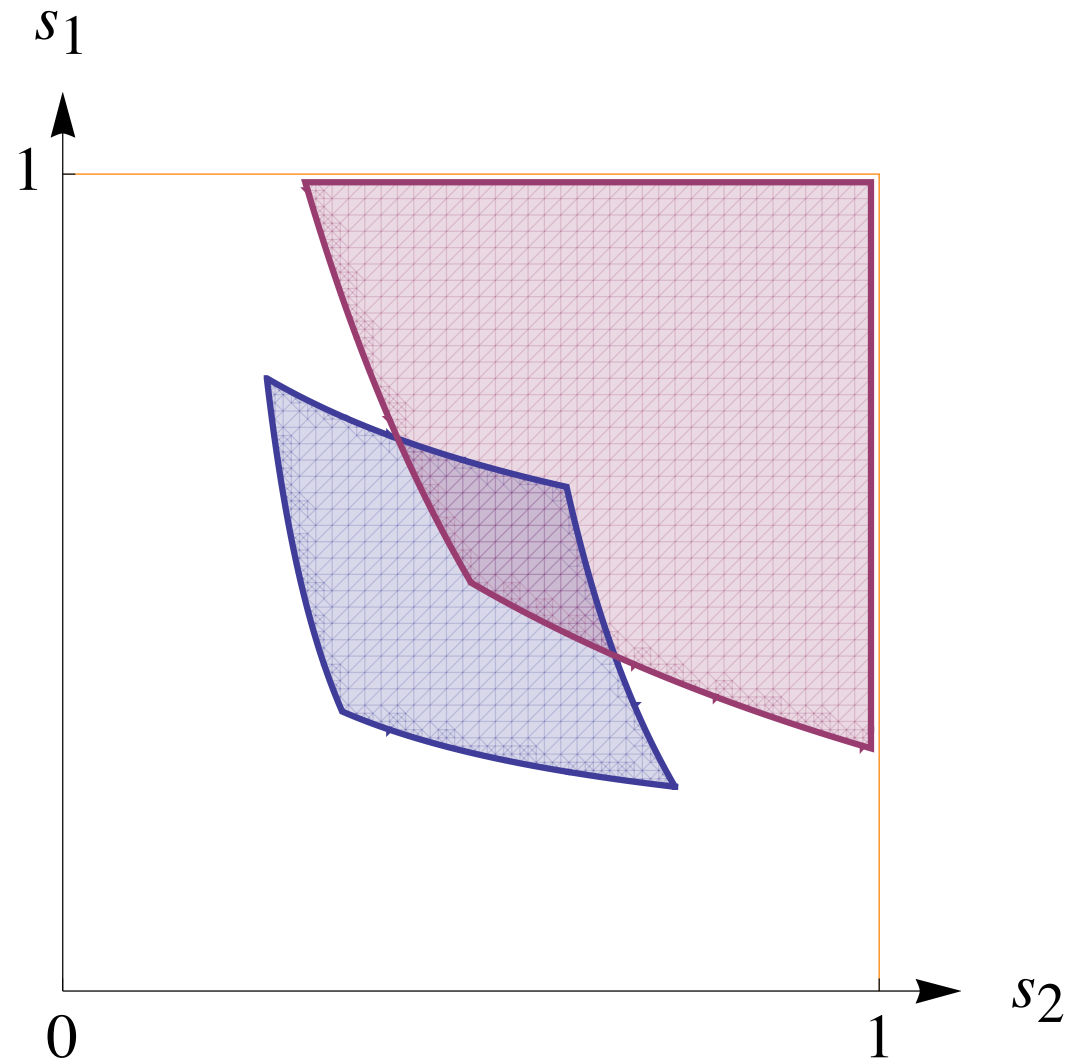
No brain game

Game with Pareto efficient pure equilibria

Game G^{NB}	I	II
I	19, 19	2, 7
II	7, 2	0, 0

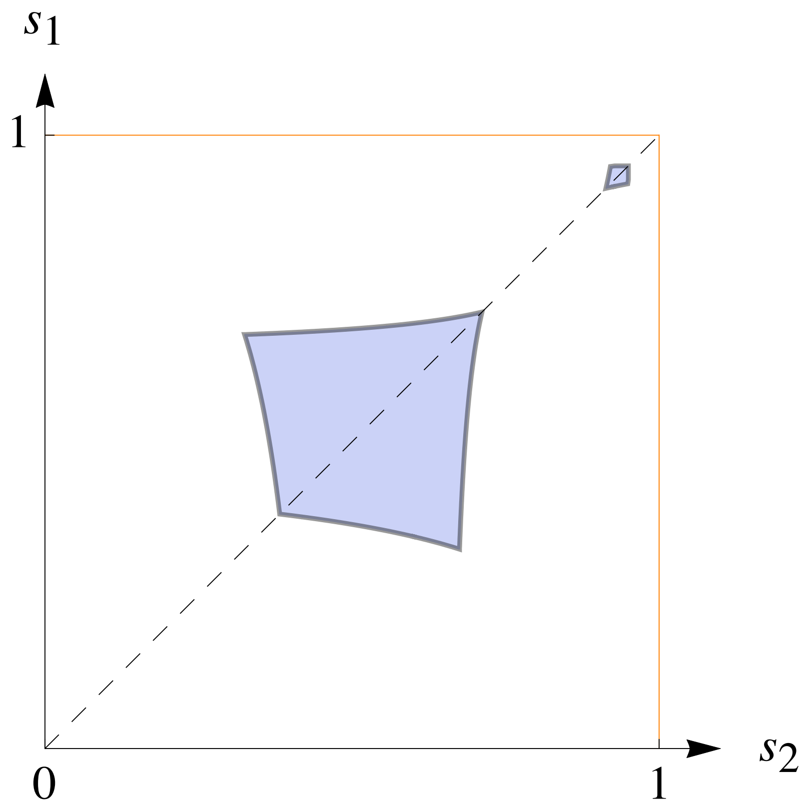
Red region = players' payoffs > 7

Blue region = {all ESD}

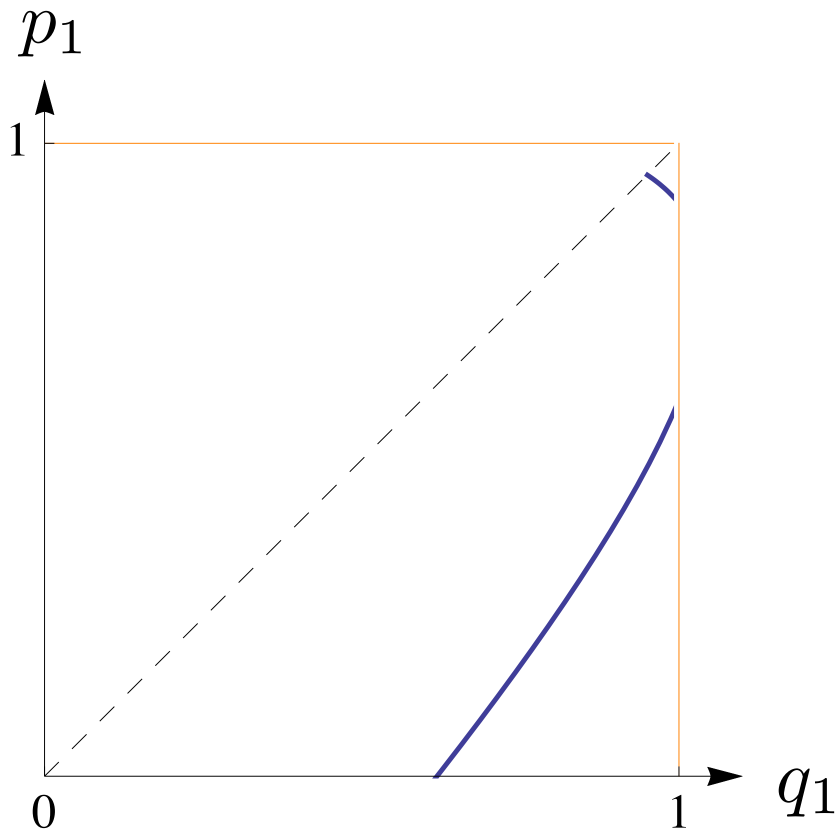


Discontinuous equilibrium regions

Game G^{D2}	C	D
C	$-12.5, -12.5$	$-9.5, -13$
D	$-13, -9.5$	$0, 0$



All symmetric Nash equilibria



Main properties

- Existence of equilibrium in games without mixed Nash equilibrium.

- Reactive Nash equilibria yield same or higher payoffs for both players than traditional mixed Nash.

- Continuum of equilibria is typical.

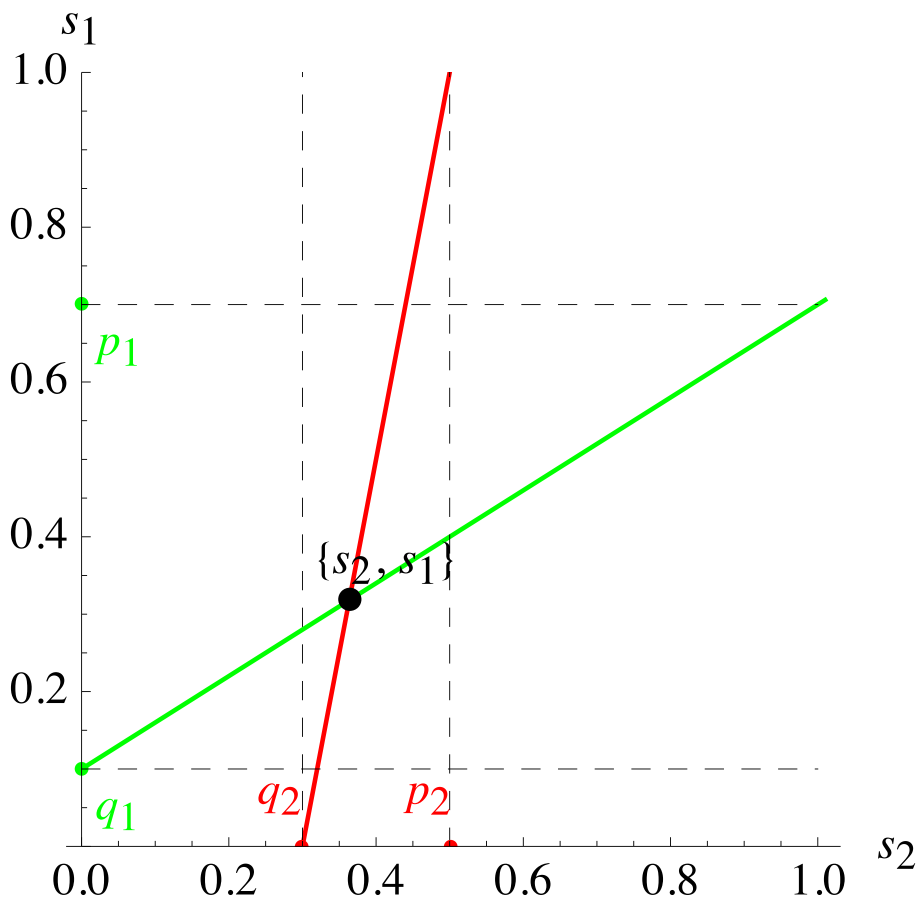
Main properties

- Existence of equilibrium in games with Pareto efficient dominant pure Nash (no brain games).
- Non-symmetric equilibria in games with symmetric payoff matrix, symmetric ESD in games with non-symmetric payoff matrix.

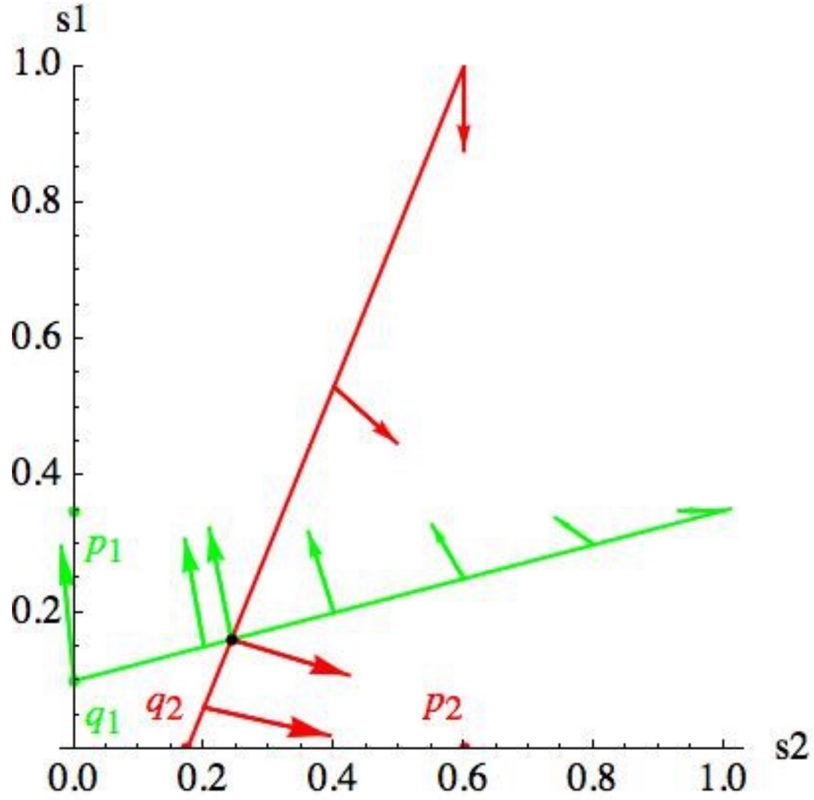
Attainability sets and stationary distributions

All feasible stationary distribution for a fixed opponent's strategy

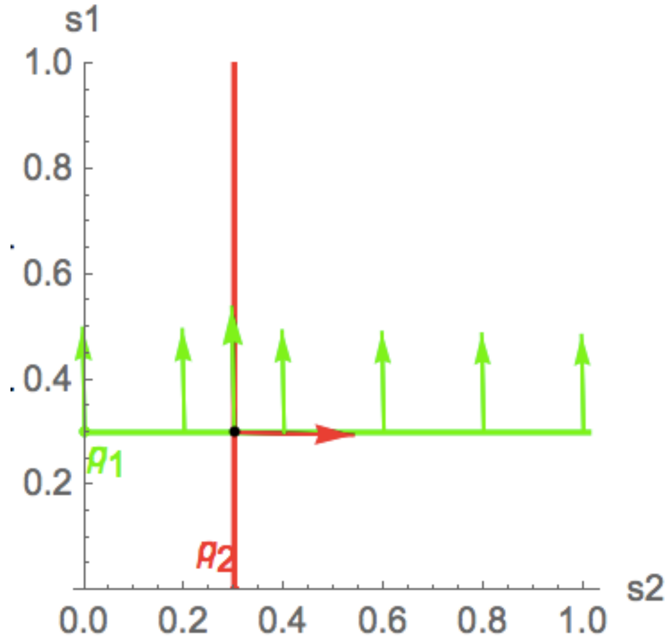
AS for 1 player = red line
AS for 2 player = green line



Necessary and sufficient conditions



Mixed strategies



Mutual indifference

$$J_i = A_i s_1 s_2 + B_i s_1 (1 - s_2) + C_i (1 - s_1) s_2 + D_i (1 - s_1)(1 - s_2)$$

Comparison

Dutta,P.K. & Siconolfi,P.	Presented work
For high discount factor there is a **simple criterion** for the existence of Nash equilibrium (reverse dominance)	Even for symmetric games the corresponding criterion requires much more tedious calculations. Reverse dominance is not necessary.
Simple lower and upper bounds for equilibrium payoffs	There exist equilibria leading to higher payoffs than the upper bound for 1-memory strategies
Chance to have an equilibrium equals to 1/3	Chance to have an equilibrium equals to 31/96 **(1/96 less)** compare to 24/96 in one-shot game

Comparison

Dutta,P.K. & Siconolfi,P.	Presented work

Payoff relevant indeterminacy
holds true
(continuum of distinct equilibrium payoffs)

There is no folk theorem