WORKING PAPER

AN IMPLEMENTATION OF THE REFERENCE POINT APPROACH FOR MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT

This paper studies the reference point approach of Wierzbicki for multiobjective optimization. The method does not necessarily aim at finding an optimum under any utility function but rather it is used to generate a sequence of efficient solutions which are interesting from the decision maker's point of view. The user can interfere via suggestions of reference values for the vector of objectives. The optimization system is used to find (in a certain sense) the nearest Pareto solution to each reference objective.

The approach is expanded for adaptation of information which may accumulate on the decision maker's preferences in the course of the interactive process. In this case any Pareto point is excluded from consideration if it is not optimal under any linear utility function consistent with the information obtained. Thus, the pareto points being generated are the "nearest" ones among the rest of the Pareto points.

Wierzbicki's approach is implemented on an interactive mathematical programming system called SESAME and developed by Orchard-Hays. It is now capable of handling large practical multicriteria linear programs with up to 99 objectives and 1000 to 2000 constraints. The method is tested using a forest sector model which is a moderate sized dynamic linear program with twenty criteria (two for each of the ten time periods). The approach is generally found very satisfactory. This is partly due to the simplicity of the basic idea which makes it easy to implement and use.

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1. INTRODUCTION

In many practical decision situations there is a need to find a compromise between a number of conflicting objectives. Furthermore, the decision may involve several decision makers in partly conflicting, partly cooperative situations. Mathematically such decision problems can often be formulated as a multiobjective optimization problem or in the framework of game theory. In this paper we concentrate on the former approach for developing decision aid techniques for the problem. For an overview on various approaches, see, for instance Bell et al. (1977), Starr and Zeleny (1977), and Wierzbicki (forthcoming).

In our opinion, the reference point optimization method with penalty function scalarization (Wierzbicki 1979) is an appropriate tool for studying such problems. This approach has several desirable properties:

-- it applies to convex and nonconvex cases
-- it can easily check Pareto-optimality of a given decision
-- it can be easily supplemented by an *a posteriori* computation of weighting coefficients for the objectives
-- it is numerically well-conditioned and easy for implementation

-- the concept of reference point optimization makes it possible to take into account the opinions of a decision maker directly, without necessarily asking him questions about his preferences.

In this paper we will focus on the interactive use of reference point optimization for multiobjective linear programming with a single decision maker. However, we believe that the same approach proves to be useful for group decision problems as well. The reference point optimization will be reviewed first and some preliminary results will be given. Thereafter, we develop an approach for employing information which may be revealed on the decision maker's preferences in the course of the interactive process. The multiobjective method has been computerized in the SESAME-system, a large interactive mathematical programming system designed for IBM 370 under VM/CMS (Orchard-Hays 1978). A sample of numerical experiments will be reported at the end of the paper.

2. REFERENCE POINT OPTIMIZATION

Let A be in $R^{m\times n}$, C in $R^{p\times n}$, and b in R^m and consider the multicriteria linear program (MCLP):

(MCLP.1) $$Cx = q$$

(MCLP.2) $$Ax = b$$

(MCLP.3) $$x \geq 0 \quad ,$$

where the decision problem is to determine an n-vector x of decision variables satisfying (MCLP.2-3) and taking into account the p-vector q of objectives defined by (MCLP.1). We will assume that each component of q is desired to be as large as possible.

An objective vector value $q = \bar{q}$ is *attainable* if there is a feasible x for which $Cx = \bar{q}$. Let q_i^*, for i = 1,2,..., p, be the largest attainable value for q_i; i.e., $q_i^* = \sup \{q_i | q \text{ attainable}\}$. The point $q^* \equiv (q_1^*, q_2^*, \ldots, q_p^*)^T$ is the *utopia point.* If q^* is attainable, it is a solution for the decision problem. However, usually q^* is not attainable. A point $\bar{q}$ is *strictly pareto inferior* if there is an attainable point q for which $q > \bar{q}$. If there is an attainable q for which $q \geq \bar{q}$ and the inequality is strict at least in one component, then $\bar{q}$ is *pareto inferior*. An attainable point $\bar{q}$ is *weakly pareto-optimal* if it is not strictly pareto inferior and it is *pareto-optimal* if there is no attainable point q such that $q \geq \bar{q}$ with a strict inequality for at least one component. Thus a pareto optimal point is also weakly pareto optimal, and a weakly pareto optimal point may be pareto inferior. For brevity, we shall call a pareto optimal point sometimes a *pareto point* and the set of all such points the *pareto set*.

What we call a *reference point* or *reference objective* is a suggestion $\bar{q}$ by the decision maker (or the group of them) reflecting in some sense a "desired level" for the objectives. According to Wierzbicki (1979), we consider for a reference point $\bar{q}$ a penalty scalarizing function $s(q-\bar{q})$ defined over the set of objective vectors q. Characterization of functions s, which result in pareto optimal (or weakly pareto optimal) minimizers of s over attainable points q is given by Wierzbicki (1979).

If we regard the function $s(q-\bar{q})$ as the "distance" between the points q and $\bar{q}$, then, intuitively, the problem of finding such a minimum point means finding among the Pareto set the *nearest* point $\hat{q}$ to the reference point $\bar{q}$. (However, as it will be clear later, our function s is not necessarily related to the usual notion of distance). Having this interpretation in mind, the use of reference points optimization may be viewed as a way of guiding a sequence $\{\hat{q}^k\}$ of pareto points generated from the sequence $\{\bar{q}^k\}$ of reference objectives. These sequences will be generated in an interactive process and such interference should result in an interesting set of attainable points $\hat{q}^k$. If the sequence $\{\hat{q}^k\}$ converges, the limit point may be seen as a solution to the decision problem.

Initial information to the decision maker may be provided by maximizing all objectives separately. Let $q^i = (q^i_j)$ be the vector of objectives obtained when the i^{th} objective is maximized for all i. Then the matrix (q_j^i), i,j, = 1,..., p, yields information on the range of numerical values of objective functions, and the vector $q^* = (q_i^i)$ is the utopia point. It should be stressed, however, that such initial information is not a necessary part of the procedure and in no sense limits the freedom of the decision maker.

We denote $w \equiv q - \bar{q}$, for brevity. Then, a practical form of the penalty scalarizing function s(w), where minimization results in a linear programming formulation, is given as follows:

$$s(w) = -\min\{\rho \min_i w_i, \sum w_i\} - \varepsilon w \quad . \qquad (1)$$

Here ρ is an arbitrary penalty coefficient which is greater than or equal to p and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$ is a nonnegative vector of parameters. In the special case of $\rho = p$, (1) reduces to

$$s(w) = -\rho \min_i w_i - \varepsilon w \quad . \qquad (2)$$

So far in our experience, form (1) of the penalty scalarizing function has proven to be most suitable. Other practical forms have been given in Wierzbicki (1979a).

For any scalar $\hat{s}$ the set $S_{\hat{s}}(\bar{q}) \equiv \{q | s(w) \geq \hat{s}, w = q - \bar{q}\}$ is called a level set. Such sets have been illustrated for function (1) in Figure 1 for $\rho = p$, for $\rho > p$ and for a very large value for ρ. In each case, if $w \ngeq 0$, then s(w) is given by (2); i.e., the functional value is proportional to the worst component of w. If $\rho = p$, the same is true for $w \geq 0$ as well. If $w > 0$, then for large enough ρ (see the case $\rho >> p$) s(w) is given by $\sum w_i$. In the general case, when $\rho > p$, the situation is shown in the middle of Figure 1. When $w \geq 0$ and its components are close enough to each other (that is, $(\rho-1)w_1 \geq w_2$ and $(\rho-1)w_2 \geq w_1$, for p = 2), then s(w) is given by $\sum w_i$. Otherwise, formula (2) applies again.

For $\varepsilon = 0$, scalarizing function (1) guarantees only weak pareto optimality for its minimizer. However, as will be shown in Lemma 1 below, if $\varepsilon > 0$, then pareto optimality will be guaranteed.

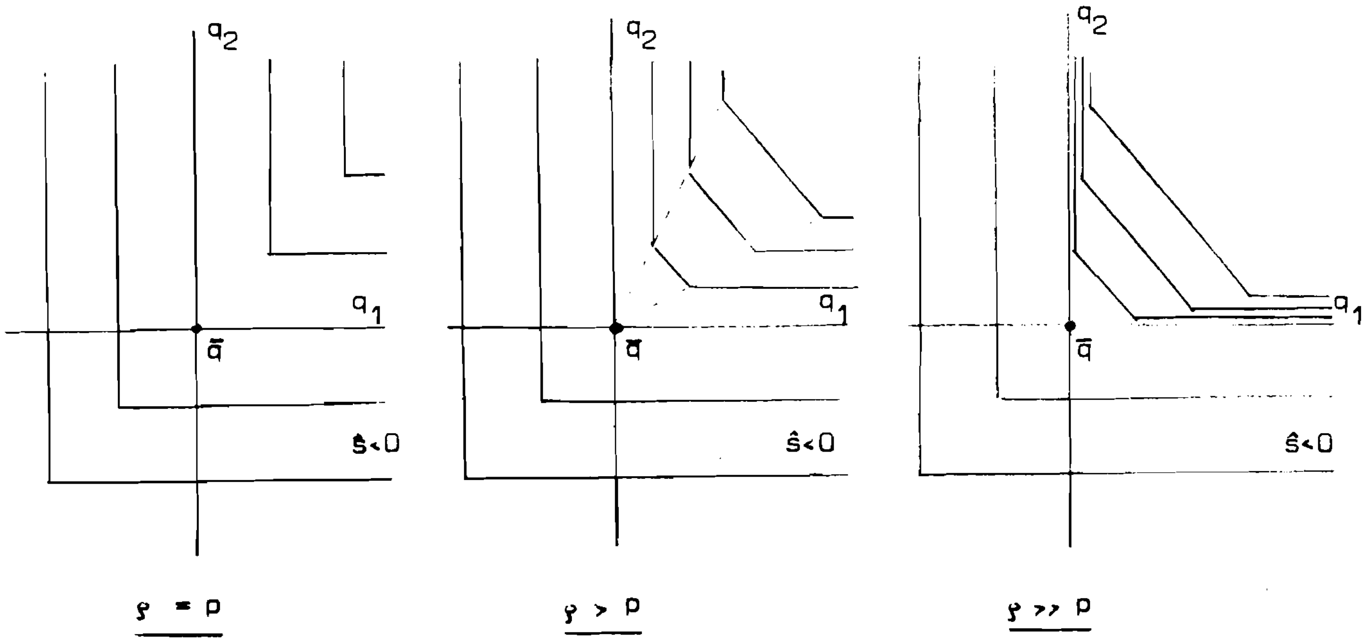

Figure 1. Level sets for penalty scalarizing functions (1) and (2) for $\varepsilon = 0$.

The problem of minimizing $s(q-\bar{q})$ defined by (1) over the attainable points q, can be formulated as a linear programming problem. In particular, if we again denote $w = q - \bar{q} = Cx - \bar{q}$ and introduce an auxiliary decision variable y, this minimization problem can be stated as the following problem (P):

find y, w, and x to

(P.1) $\quad \min \quad y - \varepsilon w$

(P.2) $\quad \text{s.t. } Ey + Dw \leq 0$

(P.3) $\quad -w + Cx = \bar{q}$

(P.4) $\quad Ax = b$

(P.5) $\quad x \geq 0$,

where E and D are appropriate vectors and matrices. Furthermore, $D \leq 0$, and if $w = \hat{w}$ and $y = \hat{y}$ are optimal for (P), then $\hat{s} = \hat{y} - \varepsilon\hat{w}$ is the minimum value attained for the penalty function s. The detailed formulation of (P) is given in the Appendix. The optimal solution for (P) will be characterized by the following result:

LEMMA 1. *Let* $(y,w,x) = (\hat{y},\hat{w},\hat{x})$ *be an optimal solution and* δ, μ, *and* π *the corresponding dual vectors related to constraints (P.2), (P.3), and (P.4), respectively. Denote by* $\hat{q} = C\hat{x}$ *the corresponding objective vector, and by* $\hat{s} = \hat{y} - \varepsilon\hat{w}$ *the optimal value for the penalty function, and by* Q *the attainable set of objective vectors* q. *Then* $\hat{q} \in Q \cap S_{\hat{s}}(\bar{q})$ *and the hyperplane* $H = \{q | \mu(\hat{q}-q) = 0\}$ *separates* Q *and* $S_{\hat{s}}(\bar{q})$. *Furthermore,* $\mu \geq \varepsilon$ *and* $q = \hat{q}$ *maximizes* μq *over* $q \in Q$; *i.e.,* $\hat{q}$ *is pareto optimal if* $\varepsilon > 0$, *and* $\hat{q}$ *is weakly pareto optimal if* $\varepsilon \geq 0$.

Remark. As illustrated in Figure 2, the hyperplane H approximates the pareto set in the neighborhood of $\hat{q}$. Thus the dual vector μ may be viewed as a vector of trade-off coefficients which tells roughly how much we have to give up in one objective in order to gain (a given small amount) in another objective.

Proof. Clearly $\hat{q}$ is attainable (i.e., $\hat{q} \in Q$) and by definition $\hat{q} \in S_{\hat{s}}(\bar{q})$. In order to prove the separability assertion we show that (i) $\hat{q}$ minimizes μq over $S_{\hat{s}}(\bar{q})$ and that (ii) $\hat{q}$ maximizes μq over Q. Noting that $q = w + \bar{q} = Cx$, these two problems may be stated as follows:

$$\text{P(i)} \quad \begin{aligned} &\text{minimize } \mu w + \mu\bar{q} \\ &\text{st.} \\ &\quad y - \varepsilon w \geq \hat{s} \\ &\quad Ey + Dw \leq 0 \quad , \end{aligned}$$

and

$$\text{P(ii)} \quad \begin{aligned} &\text{maximize } \mu Cx \\ &\text{st.} \\ &\quad Ax = b \\ &\quad x \geq 0 \end{aligned}$$

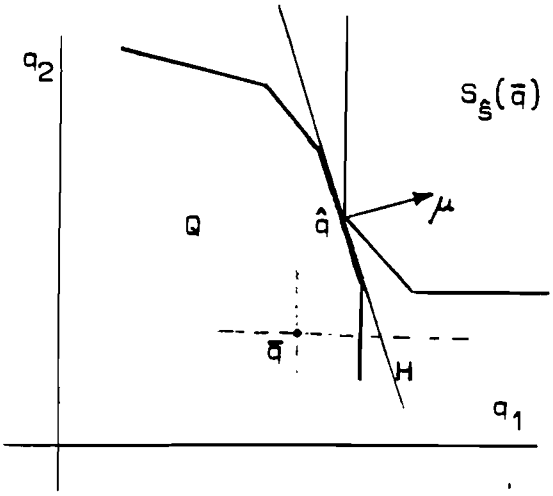

Figure 2. An illustration of Lemma 1.

Letting the dual multipliers for the first constraint of P(i) be equal to -1, we can readily check, based on the optimality conditions for (P), that $\hat{y}$, $\hat{w}$, $\hat{x}$, δ, μ, and $-\pi$ satisfy the optimality conditions for P(i) and P(ii). Based on dual feasibility, we have $\mu = \varepsilon - \delta D$ and $\delta \leq 0$. Because $D \leq 0$, we have $\mu \geq \varepsilon$. Thus, if $\varepsilon > 0$ ($\varepsilon \geq 0$), then $\hat{q}$ is (weakly) pareto optimal.||

3. EMPLOYING INFORMATION ON PREFERENCES

While applying the reference point optimization a sequence $\{\bar{q}^k\}$ of reference points and the corresponding sequence $\{\hat{q}^k\}$ pareto points will be generated. Usually these sequences reveal partially the decision makers preferences. For instance, after obtaining a pareto point $\hat{q}^{k-1}$, a new reference point $\bar{q}^k$ may be chosen so that $\bar{q}^k$ is preferred to $\hat{q}^{k-1}$. In the following we intend to exploit such information. In such a procedure we shall not necessarily generate the nearest pareto point to a reference point. We will restrict the pareto points being generated to those which are consistent (in the sense defined below) with the information gained from the interactive process.

Initially, we will assume a linear utility function $\lambda^* q$, where λ^* is a vector such that q is preferred to q' if and only if $\lambda^* q > \lambda^* q'$, for all q and q'. The vector λ^* is not known explicitly. However, because each objective q_i is to be maximized, we have $\lambda^* \geq 0$; i.e., $\lambda^* d^i \geq 0$ for each unit vector d^i. Furthermore, other information concerning λ^* may be obtained during the interactive procedure. As above, if the decision maker prefers $\bar{y}^k$ to $\hat{y}^{k-1}$, then, denoting $d = \bar{y}^k - \hat{y}^{k-1}$, we have $\lambda d > 0$. In general let d^i, for $i = 1,2,\ldots, I_k$, be the vectors of *preferred directions* (including the unit vectors) being revealed by iteration k of the procedure. This implies that

$$\lambda^* \in \Lambda^k \equiv \{\lambda \mid \lambda d^i \geq 0 \text{ , for } i = 1,2,\ldots, I_k\} \quad , \qquad (3)$$

i.e., λ^* is in the dual cone of the cone spanned by the vectors d^i. (Actually, λ^* is in the interior of Λ^k.) See also Zionts and Wallenius (1976).

Let Q^k be the set of pareto points which are *consistent* with respect to Λ^k in the sense that $\hat{q} \in Q^k$ if and only if there is $\lambda \in \Lambda^k$ such that $\lambda\hat{q} \geq \lambda q$, for all attainable $q \in Q$. We shall now discuss an approach to provide a pareto point $\hat{q} \in Q^k$ related to a reference point $\bar{q}$. For this purpose we rewrite (P.3) as

$$(\bar{P}.3) \qquad -w + Cx - \sum_{i=1}^{I_k} d^i z_i = \bar{q} \quad ,$$

where the scalars z_i are nonnegative decision variables. This revised problem will be referred to as problem $(\bar{P})$. An interpretation of this problem is to find the nearest pareto point (among all pareto points) to the cone, which is spanned by the vectors d^i of preferred directions and whose vertex is at the reference point $\bar{q}$. Another characterization of the revised problem $(\bar{P})$ is given as follows:

LEMMA 2. *If $\varepsilon > 0$, $w = \hat{w}$ is optimal for the revised problem $(\bar{P})$, and $\hat{q} = \bar{q} + \hat{w}$, then $\hat{q} \in Q^k$; i.e., $\hat{q}$ is a pareto point which is consistent with respect to the information obtained in Λ^k.*

Proof. Let $(y,w,x,z_i) = (\hat{y},\hat{w},\hat{x},\hat{z}_i)$ be optimal for $(\bar{P})$ and, as before, δ, μ, and π the optimal dual solution. Define $\bar{\bar{q}} = \bar{q} + \sum_i d^i z$. Then the above also solves (P) with the reference point $\bar{\bar{q}}$. Thus, by Lemma 1, $\mu \geq \varepsilon > 0$ and $\hat{q}$ maximizes μq over attainable points q. By the optimality condition for z_i, we have $\mu d^i \geq 0$, for all i. Thus $\mu \in \Lambda^k$, and therefore, $\hat{q}$ is a pareto point consistent with Λ^k.||

In practice, the decision makers utility function normally is not linear. However, in the neighborhood of his most desired solution the utility function normally has a satisfactory linear approximation and, therefore, the above procedure may still be useful. Because of nonlinearity, the vectors d^i of preferred directions may appear conflicting to a linear utility function; i.e., the set Λ^k reduces to a single point (the origin) and the vectors d^i span the whole space. Of course, this may occur also for reasons other than the nonlinearity. For instance, lack of training in using the approach may easily result in conflicting statements on preferences. In either case, such conflict results in an unbounded optimal solution for the revised problem $(\overline{P})$. In such a case, we suggest that the oldest vectors d^i (the ones generated first) will be deleted as long as boundedness for $(\overline{P})$ is obtained. This approach seems appealing in accounting both for the learning process of the user (decision maker) and for his possible nonlinear utility function.

4. COMPUTER IMPLEMENTATION

A package of SESAME/DATAMAT programs has been prepared for automating the use of the multicriteria optimization technique utilizing user-specified reference points. The scalarizing function defined in (1) with $\varepsilon = 0$ was adopted for this implementation. A model revision into the form of (P) is carried out and a *neutral solution* corresponding to a reference point $\bar{q} = 0$ is computed and recorded first. Each time a new reference point $\bar{q}$ is given, the optimal solution for (P) is found starting with the neutral solution and using parametric programming, that is, parametrizing the reference point as $\theta\bar{q}$ with θ increasing from 0 to 1. Although the exploitation of preferences (as described in Section 3) has not yet been implemented, we are already able to design experiments for studying the influence of employing such cumulative information. Some optional algorithmic devises have been implemented to force the sequence of pareto points to converge. As it will be clear later, such a procedure does not guarantee an optimal solution (under any utility function) but often it is expected to be useful for generating interesting pareto points.

There is no explicit limit to the size of model which can be handled except that the number of objectives cannot exceed 99. The limiting factor is likely to be disk space since a model is effectively duplicated on the user's disk.

The package of programs is referred to as the MOCRIT Package, or simply MOCRIT. The standard package consists of three files: a SESAME RUN file, a DATAMAT program file, and a dummy data file which exists merely for technical reasons. There are essentially four programs in MOCRIT: (1) REVISION, which reformulates the model into the form of (P) and creates the neutral solution, (2) START, which initializes the system for a interactive session, (3) SESSION, which utilizes the standard technique of reference point optimization, and (4) CONVERGE, which forces the sequence of parato points to converge. The use of REVISION and SESSION is mandatory. START is a convenience to obviate the need to enter various SESAME parameters for each session. CONVERGE is an option; it cannot be used meaningfully before SESSION has been executed at least once. CONVERGE is actually a prologue to SESSION which it activates as a terminal step.

These "programs" are really RUN decks consisting of appropriate SESAME commands. There are corresponding decks (DATAMAT programs) which are executed automatically by the RUN decks. All four MOCRIT programs terminate by returning to the SESAME environment in manual mode. Regular SESAME commands and procedures can be interspersed manually from the terminal at such times. (For details, see Orchard-Hays 1977).

4.1 The REVISION Program

The purpose of this program is to revise an existing linear programming model containing two or more functional rows into a form suitable for multiobjective optimization. The existing model file must have been previously created with DATAMAT (or

CONVERT) in standard fashion. This file is not altered; a new file containing the revised model is created instead.

After creating the new model, REVISION further solves the model with a reference point of all zero, and obtains thereby the neutral solution. This initial solution must be obtained only once and the optimal basis is recorded on a disk file for further use.

REVISION also creates another file containing two tables. One is used to record selected results form the neutral solution. The other is used by the START program to set the various SESAME parameters for the revised model, i.e., model name, model file name, RHS name, name of RANGE set if any, and name of BOUND set. Thus it is unnecessary to set these for subsequent sessions.

The reference point $\bar{q}$ as well as the model parameters dependent on the penalty coefficient ρ are specified initially in the revised model as symbolic names. When their values are decided on, they are specified numerically at run time without generating the whole model over agian. For instance, to obtain the neutral solution, REVISION requires penalty coefficient ρ. Its value is obtained via an interactive response. If it is subsequently changed (see the SESSION program) the neutral solution will, in general, no longer be feasible. This may not be done normally but, if necessary, a new neutral solution can be obtained as shown in Orchard-Hays (1979).

4.2 The START and SESSION Programs

After a model has been revised and the neutral solution obtained and recorded, the model is ready for use with the interactive multiobjective procedure. Such use is referred to as a *session*. A session is initiated by executing the START program. All this does is define the necessary SESAME parameters unique to the model.

After executing START but before executing SESSION, the reference point must be defined. This is done with the SESAME procedure VALUES which is quite flexible with respect to formats and functions. If necessary, also the value of the penalty coefficient ρ may be changed at this point. After the reference point has been defined, execution of SESSION results in the following sequence of events:

(i) Any existing solution file is erased.

(ii) The problem set-up procedure is called and the existing reference point is incorporated for use in parametric programming.

(iii) The basis of the neutral solution is recalled.

(iv) The simplex procedure is called. After a basis inversion and check of the solution, the neutral solution is recovered.

(v) The parametric programming procedure is called to parametrize the reference point $\theta\bar{q}$ over the parameter values $\theta \in [0,1]$.

(vi) A SESAME procedure is called to record selected portions of the solution.

(vii) DATAMAT is called to execute a program to display results at the terminal (and to print off-line) and also to record necessary information for possible subsequent use by CONVERGE.

(viii) The control is returned to SESAME in manual mode.

If it is desired to try another reference point, we call the procedure VALUES again and then rerun SESSION. This may be done repeatedly.

If it is desired to get a print-out of the full solution (or selected portions) in standard LP solution format after return from SESSION, it can be obtained using the SESAME procedures in the usual way (see Orchard-Hays 1977). An example of part of the results displayed at the terminal is given in Figure 4. Each row carrying user-defined labels F1 to I10 refers to an objective. The column REFER.PT defines the reference point $\bar{q}$, column SUB.FN yields the pareto point $\hat{q}$ obtained, and column W is just the difference $\hat{q} - \bar{q}$ of the above two columns. Column DUAL is the (negative of the) vector μ of trade off coefficients defined in Lemma 1.

		REFER.PT	SUB.FN	W	DUAL
F1	=	2048	2670	622	-.99
F2	=	1398	2020	622	-.56
F3	=	688	1310	622	-.63
F4	=	508	1130	622	-.65
F5	=	358	980	622	-.65
F6	=	-161	461	622	-.63
F7	=	1489	2111	622	-.57
F8	=	2599	3221	622	-.49
F9	=	4709	5331	622	-1.12
F10	=	5849	6471	622	-.67
I1	=	2035	2657	622	-1.33
I2	=	2889	3511	622	-.40
I3	=	2328	2950	622	-.76
I4	=	3348	3970	622	-.98
I5	=	4368	4990	622	-1.17
I6	=	4328	4950	622	-1.28
I7	=	5349	5971	622	-1.29
I8	=	5859	6481	622	-1.24
I9	=	7339	7961	622	-2.81
I10	=	7849	8471	622	-1.68

Figure 3. An example of results displayed in a session. (The reference point is $\bar{q}^s$ of Section 5.2).

4.3 The COVERGE Program

The CONVERGE program may be used instead of SESSION after the latter has been executed at least once. The VALUES procedure must be executed first, as usual, to define a new reference point. However, this reference point, denoted by $\bar{\bar{q}}$, is not actually used. Let $\hat{q}^k$ be the last pareto point obtained (by either SESSION or CONVERGE). A new reference point is computed from $\bar{\bar{q}}$ in two stages as follows. First $\bar{\bar{q}}$ is projected on the hyperplane H defined in Lemma 1, passing through $\hat{q}^k$ and orthogonal to the dual vector μ. This projection q^* is given by

$$q^* = \bar{\bar{q}} + [\mu(\hat{q}^k - \bar{\bar{q}})/\mu\mu^T]\mu^T \quad . \tag{4}$$

The new reference point $\bar{q}^{k+1}$ is then chosen from the line segment $[q^*, \hat{q}^k]$; i.e., a point $\bar{q}^{k+1} = q^* + \theta(\hat{q}^k - q^*)$ is chosen for some $\theta \in [0,1]$. The following options have been considered: (i) choose $\theta = 0$ (i.e., choose $\bar{q}^{k+1}$ as the projection q^*), or (ii) choose the smallest $\theta \in [0,1]$ so that $\max_i(\bar{q}^{k+1} - \hat{q}_i^k) \leq y^k$, where y^k is a user-specified tolerance. The value for y may either be entered directly or it may be specified as a percentage of the "distance" between the previous reference point $\bar{q}^k$ and the pareto point $\hat{q}^k$; i.e., $y^k = \beta^k \max_i (\bar{q}_i^k - \hat{q}_i^k)$, where β^k is a coefficient entered by the user. This latter option may be used meaningfully only if the reference point $\bar{q}^k$ is not a pareto inferior point, for instance, a point obtained by CONVERGE in the preceeding session. For an illustration of the modified reference point, see Figure 4.

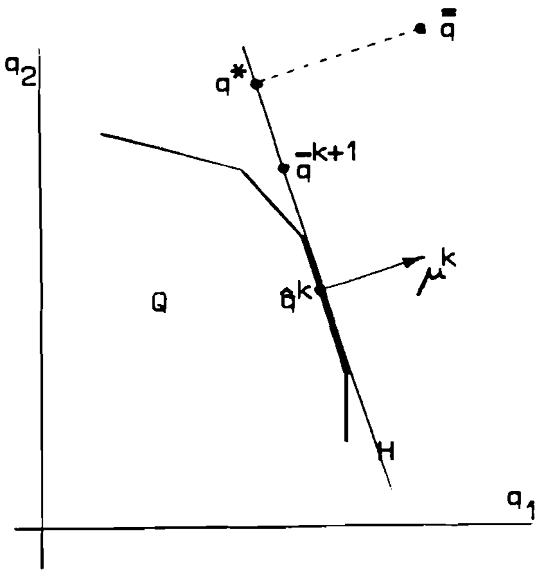

Figure 4. Modification of the reference point in CONVERGE.

Note that

$$y^k \geq \max_i (\bar{q}_i^{k+1} - \hat{q}_i^k) \geq \max_i (\bar{q}^{k+1} - \hat{q}^{k+1}) \geq 0 \quad . \qquad (5)$$

Thus, if $y^k \geq 0$ and the sequence $\{y^k\}$ converges to zero, then the sequence of optimal values for (P) converges to zero.

Remark. A limit point of $\{\hat{q}^k\}$ is not necessarily a solution to the multicriteria optimization problem, because the convergence is mechanically forced without taking the decision maker's preferences properly into account. The only purpose of the CONVERGENCE routine is to provide some algorithmic help to converge to a, hopefully, interesting pareto point.

5. COMPUTATIONAL EXPERIENCE

For testing purposes we used a ten period dynamic linear programming model developed for studying long-range development alternatives of forestry and forest based industries in Finland (Kallio et al. 1978). This model comprises two subsystems, the forestry and the industrial subsystem, which are linked to each other through raw wood supply. The forestry submodel describes the development of the volume of different types of wood and the age distribution of different types of trees in the forests within the nation. In the industrial submodel various production activities, such as saw mill, panels production, pulp and paper mills, as well as further processing of primary wood products, are considered. For a single product, alternative technologies may be employed so that the production process is described by a small Leontief model with substitution. Besides supply of raw wood and demand for wood products, production is restricted through labor availability, production capacity, and financial resources. All production activities are grouped into one financial unit and the investments are made within the financial resources of this unit. Similarly, the forestry is considered as a single financial unit.

A key issue between forestry and industry is the income distribution which is determined through raw wood price. Consequently, we have chosen two criteria: (i) the profit of the wood processing industries, and (ii) the income of forestry from selling the raw wood to industry. These objectives are considered separately for each time period of the model. Thus, the problem in consideration has 20 criteria altogether.

Of course, both the average raw wood price and quantity of wood sold must be implicit in such a model. In order to handle this in a linear programming framework, we use interpolation. We consider two exogeneously given wood prices for each type of raw wood and for each period. The quantities sold at each price are endogeneous and the average wood price results from the ratio of these quantities. The complete model after REVISION consists of 712 rows and 913 columns.

We experiment first with different values for the penalty coefficient ρ. Then, fixing $\rho = p$ (the number of objectives) we generate a sequence $\{\bar{q}^k\}$ of reference points and compute the corresponding sequence $\{\hat{q}^k\}$ of pareto points as solutions to (P). The influence of accumulated information on preferences will be experimented with thereafter. Finally, we try out the procedures of forcing convergence.

5.1 Influence of the Penalty Coefficient

Using the scalarizing function (1) we experimented with different values of the penalty coefficient ρ and with different reference points $\bar{q}$. As pointed out in Section 2, unless the reference point $\bar{q}$ is pareto inferior, the pareto point $\hat{q}$ obtained as a solution of (P) is independent of ρ, namely the one corresponding to the max min criterion of the scalarizing function (2). On the other hand, if $\bar{q}$ is pareto inferior, then $\hat{q}$ in general depends on ρ. In the extreme case of $\rho = p$, we again obtain the max min solution.

In the first runs, we set the reference point $\bar{q}$ to zero, and applied the values 20(= p), 25, 50, and 100 for ρ. The results have been plotted in Figure 5. As $\bar{q} = 0$ appears to be

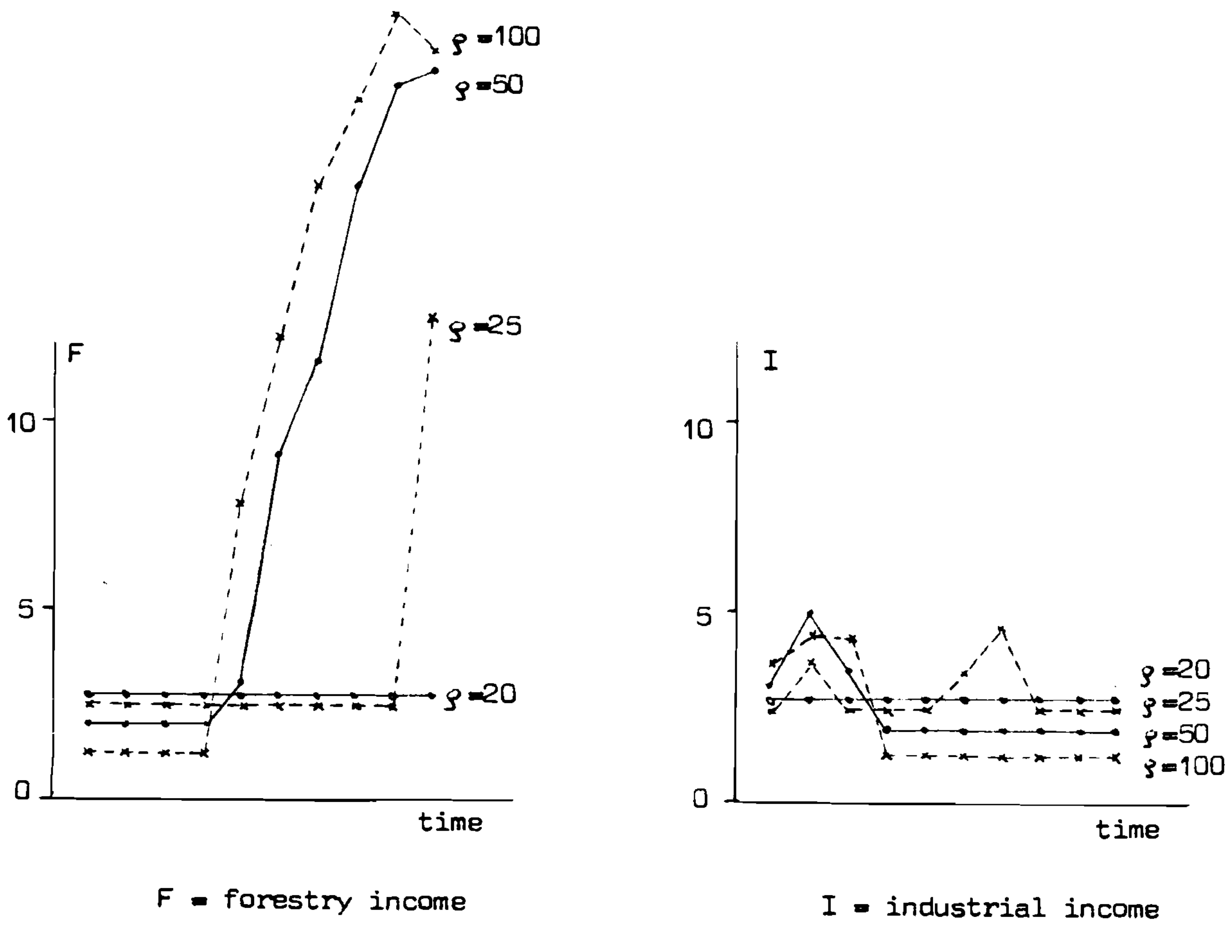

Figure 5. Experiment with different penalty coefficients ρ and with reference point $\overline{q} = 0$.

pareto inferior, the pareto optimal trajectories obtained are dependent on ρ. For $\rho = p$ a constant deviation $\hat{w}_i = \hat{q}_i - \overline{q}_i = 2.7$ is obtained for each objective i. When ρ is increased the minimum guaranteed for each w_i decreases, being 1.2 for $\rho = 100$. Simultaneously as ρ increases, the behavior of the $\hat{q}$ trajectories gets worse. Even for $\rho = 25$, there is a very high spike at the end of the trajectory of the forestry income.

Figure 6 shows a similar experiment where the reference point is moved from zero towards the pareto set. Actually in this case, $\overline{q}$ is about 90 percent of a pareto-optimal solution. Again, the case $\rho = p$ results in a constant deviation of $\hat{w}_i = 0.4$, for all i. For larger values of ρ, the same behavior of the trajectories was obtained as in the previous case $\overline{q} = 0$. However, as one might expect, the behavior of the trajectories does

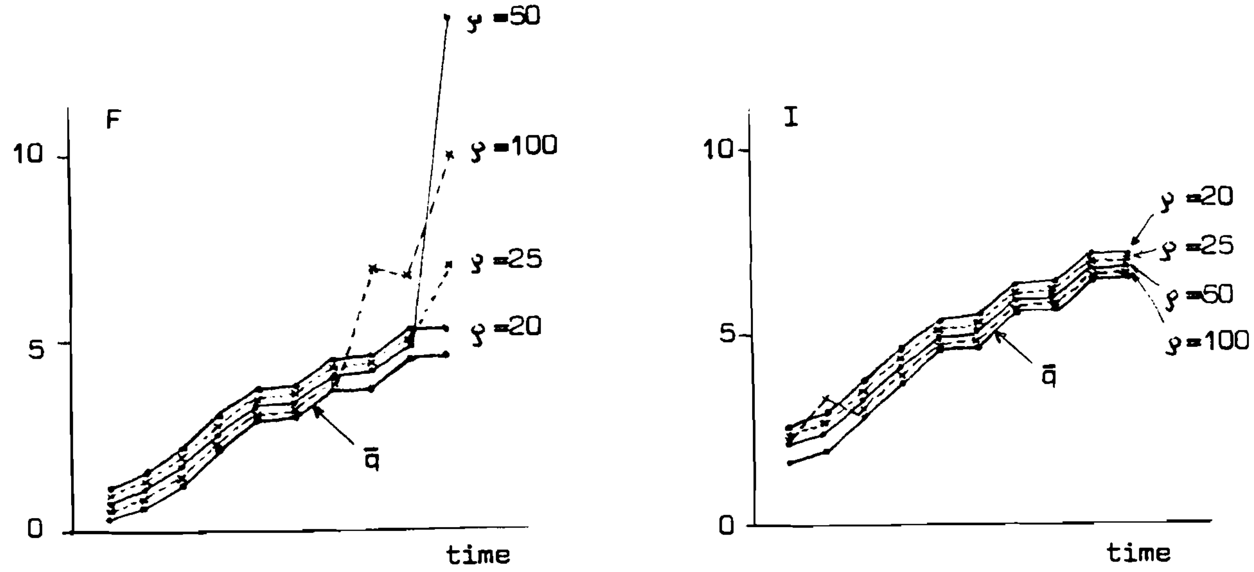

Figure 6. Experiments with different penalty coefficients and with the reference point about 90 percent of a pareto point.

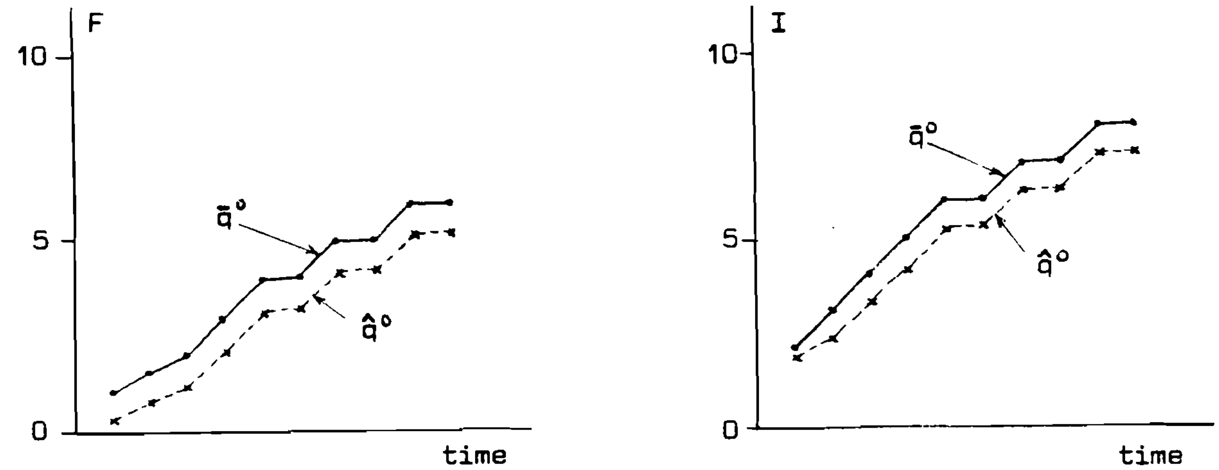

Figure 7. A nonattainable reference point $\bar{q}$ and its nearest pareto point.

not get as bad as in the previous case. As the reference point is closer to the pareto set, the variation in deviation $\hat{w}_i$ is smaller, yet still intolerable for very large ρ.

In the next case, a nonattainable point was chosen as a reference trajectory. The resulting trajectories (corresponding to the nearest pareto point) $\hat{q}$ have been illustrated in Figure 8. As $\hat{q}$ now corresponds to the max min criterion (2), we may expect the deviations $\hat{w}_i$ to be rather constant; in fact, w_i is equal to -0.8 except for the industrial profit at the first period (in which case the deviation is slightly more favorable).

5.2 Experiments with a Sample of Reference Points

For further tests we set $\rho = p$, generated a sequence of eight reference points and the corresponding pareto solutions. The results have been illustrated in Figures 8 and 9 , where the continuous trajectories refer to the reference point, and those drawn in broken lines refer to the pareto point. As an overall observation we may conclude, as expected, that the trajectory of the pareto solution tends to be the reference trajectory shifted up or down. (See also Figures 5 and 6 , for $\rho = 20$, and Figure 7.) However, this is not always the case. For extreme cases, see Figure 8(b) and 8(c), where the pareto trajectories have a very large spike. Such undesirable unsmoothness may be due to a multiplicity of optimal solution which are very different from each other. In our dynamic case, for instance, the first periods may totally determine the optimal objective function value for (P) and the multiple optimal solutions result from the variety of alternatives left for the later periods.

Next, the influence of the accumulated information on preferences was experimented. Let $\bar{q}^0$, and $\hat{q}^0$ be the reference point and the pareto point, respectively, in Figure 7, and let $\bar{q}^k$ and $\hat{q}^k$, for $k = 1,2,\ldots, 8$, be those defined in Figures 8 and 9 . For the purpose of our numerical tests we assume that the differences $d^k = \bar{q}^k - \hat{q}^{k-1}$ reveal the decision makers preferences in a way that d^k is a preferred direction, for $k = 1,2, \ldots, 8$. All vectors d^k, for $i \leq k$, will be made available when

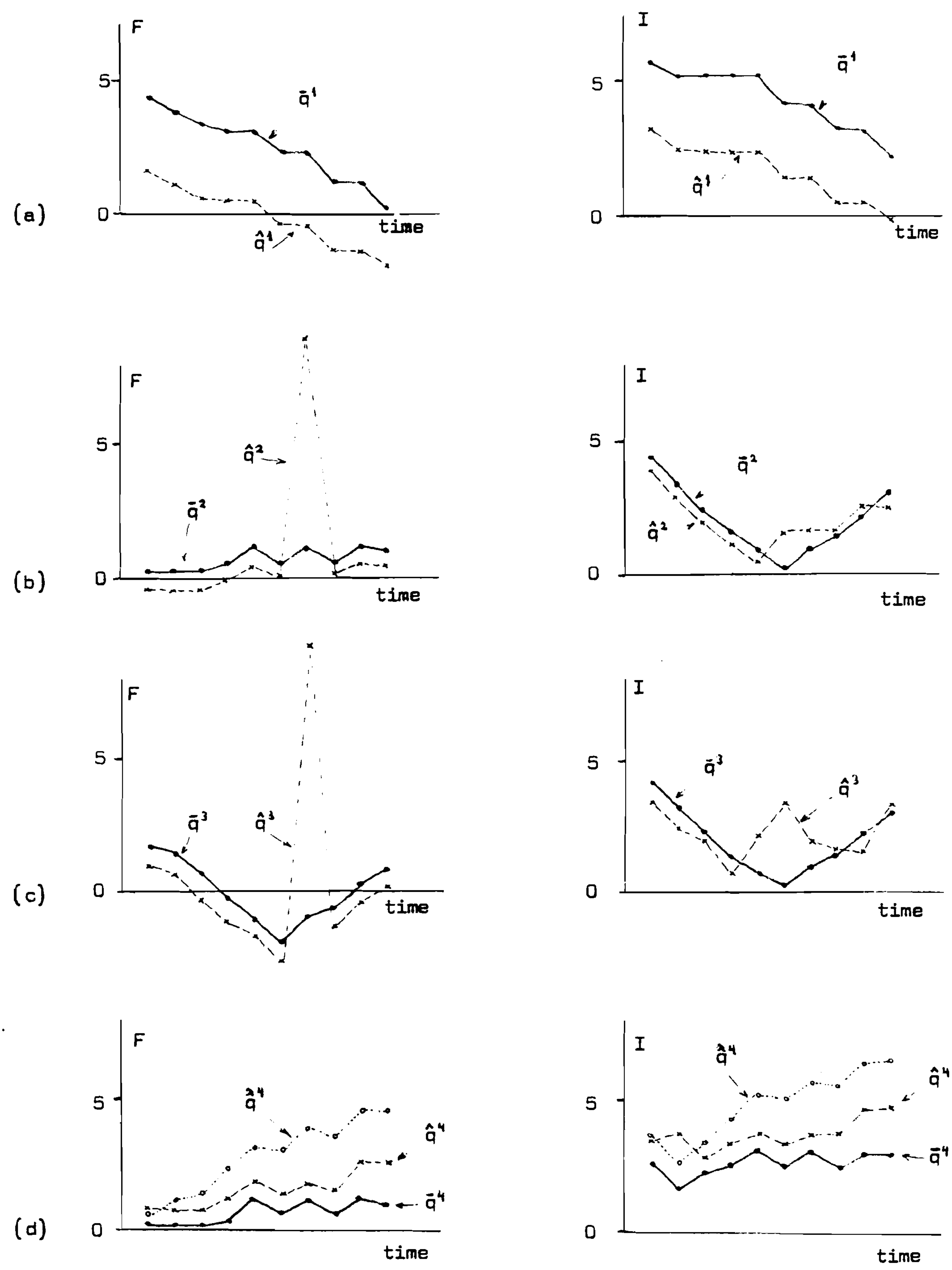

Figure 8. A sample of sessions.

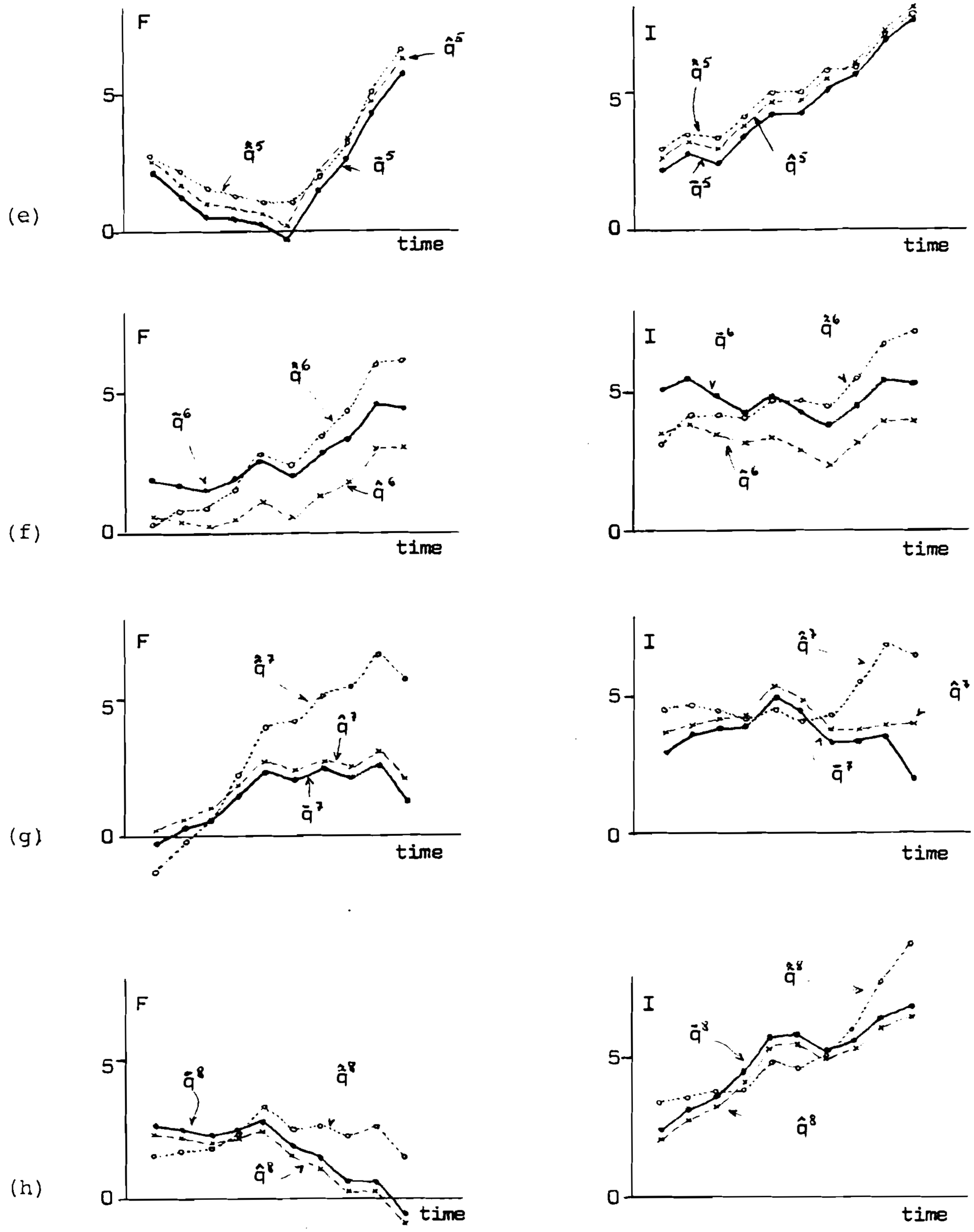

Figure 9. A sample of sessions (continued).

applying the reference point $\bar{q}^k$ in the revised problem $(\bar{P})$. Thus, all information gained on preferences is being used. The pareto points resulting as optimal solutions for $(\bar{P})$ have been illustrated in dotted lines in Figures 8 and 9. For k = 1,2, and 3, the additional information did not have any influence on the pareto point; i.e., the same solutions $\hat{q}^k$ were obtained as before. However, thereafter a significant change was observed in most cases, and in addition, the obtained revised pareto point seems to be better than the one obtained from problem (P) (see Figures 8(d), 9(f), and 9(g), for instance). On the other hand, we may observe that the revised trajectories usually resemble the shape of the reference trajectory to a lesser degree than do the trajectories obtained form problem (P). These observations suggest that perhaps in practice both pareto trajectories ought to be computed in each session.

5.3 Forcing Convergence

In Section 4 we developed procedures for modifying the users suggested sequence of reference points in such a way that the pareto points obtained are forced to converge. One of these procedures was controlled by a sequence $\{\beta^k\}$ of percentages, and another by a sequence $\{y^k\}$ of tolerances. Both of them were tested using the above sequece $\{\bar{q}^k\}_{k=0}^{8}$ as a sequence reference points suggested by the user.

First we discuss the case of using the β-factors. After obtaining the initial solution $\hat{q}^0$, the CONVERGE program was applied for each suggestion $\bar{q}^k$. The results obtained, when a constant value $\beta^k = .5$ (for all k) was used, are illustrated in Figure 10. It also describes the results when the experiment was repeated for $\beta^k = .9$. In both cases, practically no change in $\hat{q}^k$ was obtained after $k \geq 2$. Thus the convergence proved to be extremely fast; in fact, for many applications probably undesirably fast. An explanation for this phenomenon may be found from the fact that the hyperplane (on which the reference points are projected) is close to the pareto set in the neighborhood of the last pareto point obtained. This in turn is likely to result in a sequence of objective function values for (P), which converges fast to zero.

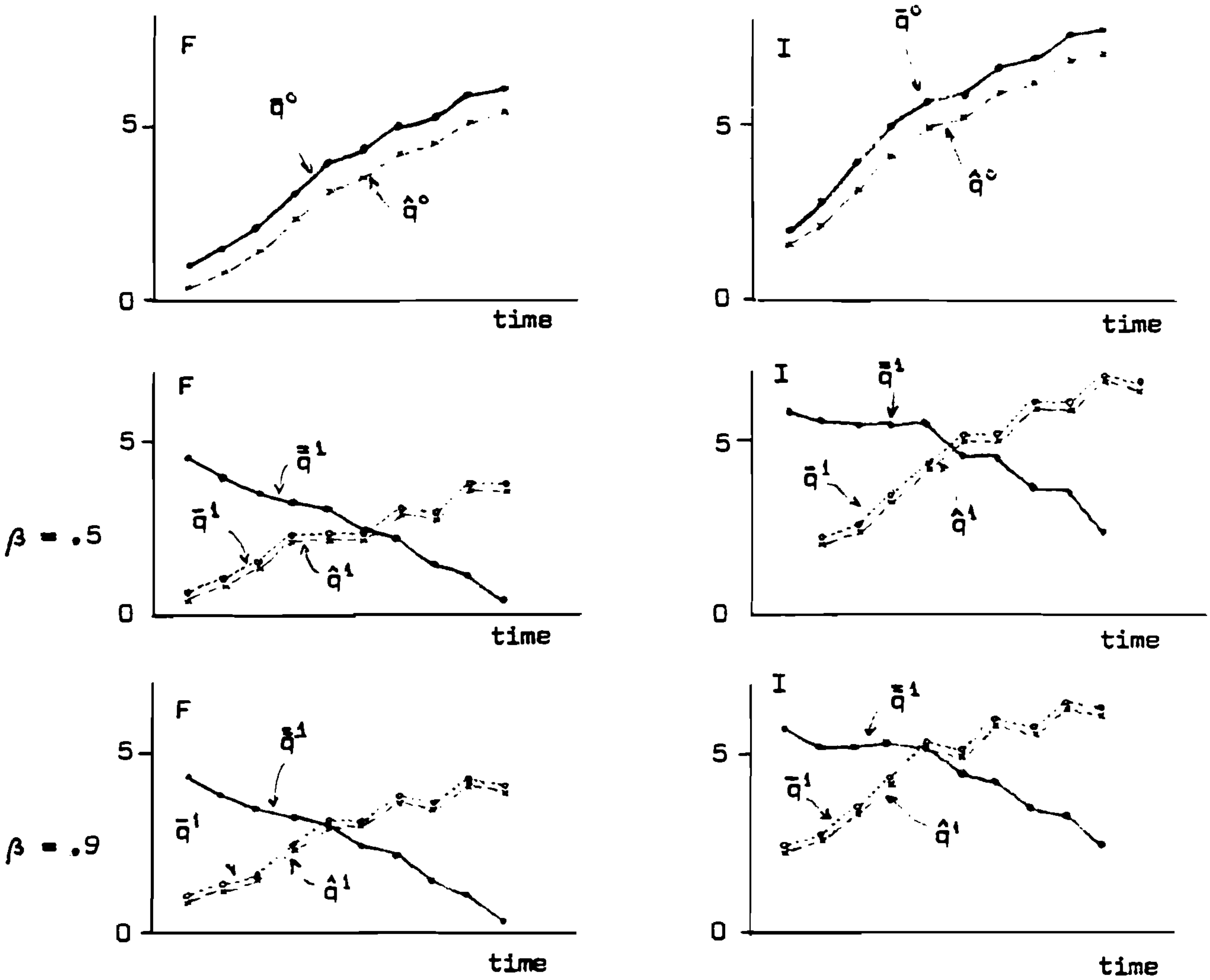

Figure 10. Experiments with CONVERGE: β^k = .5 and β^k = 9.

For the other procedure we chose the bounds y^k as $y^k = 10/2^k$. The results are described in Figure 11. The convergence is now reasonably fast, and therefore, the user has a fair chance to control the sequence of pareto points being generated.

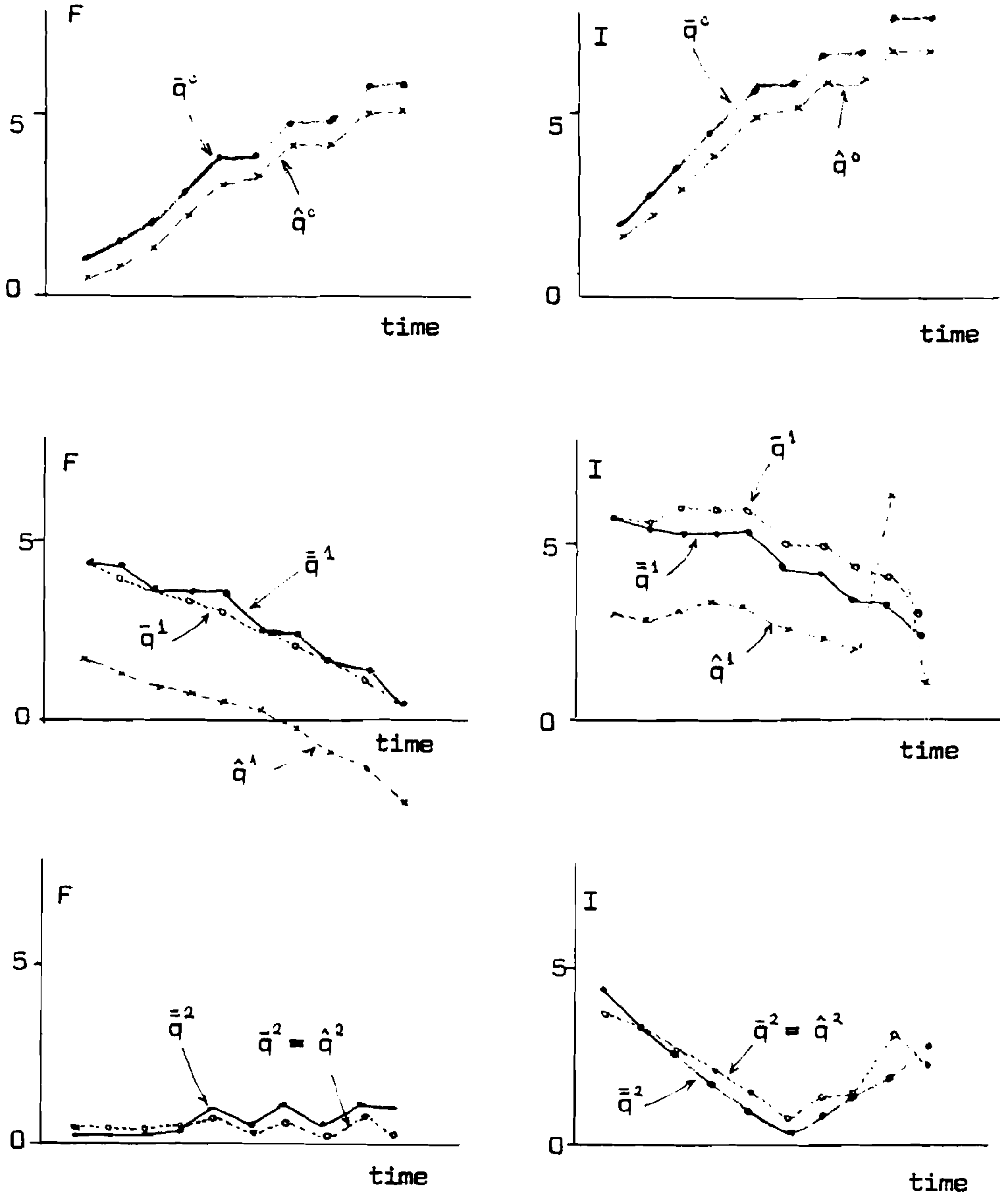

Figure 11. Experiments with CONVERGE: $y^k = 10/2^k$.

6. SUMMARY AND CONCLUSIONS

In this paper we have investigated the reference point approach for linear multiobjective optimization (Wierzbicki 1979a, b). In our opinion, the basic concept proves to be very useful, in particular, because of its simplicity. The method does not necessarily aim at finding an optimum under any utility function, but rather it is used to generate a sequence of interesting pareto points. In order to guarantee usefulness of the information being generated, we let the decision maker interfere with the model system. In the course of such an interactive process he suggests reference objectives which normally reflect his desired levels of various objectives. The optimization system is used to find, in some sense, the nearest pareto point to each reference objective.

As a measure of "distance" between the reference points and the pareto set we use the penalty scalarizing function (1) which in our experience has very favorable properties: first, the problem of finding the nearest pareto point to a reference point amounts to a linear programming problem, and second, it allows the user a reasonable control over the sequence of pareto points generated (given that the penalty coefficient ρ is close to the number of objectives). To clarify the latter point we have observed that some scalarizing functions have an undesirable property of favoring arbitrarily one or a few components of the objective vector. In such a case, the objective levels at the pareto point and at the reference objective may be close to each other in all except one component where the pareto point is far superior to the reference objective. In dynamic cases this phenomenon usually causes spikes in trajectories of the objectives (see Figure 5 for large values of the penalty coefficient ρ).

We expand the reference point approach for the adaptation of information which accumulates on the decision maker's preferences in the course of the interactive process. In this case we exclude from consideration every pareto point which is not optimal under any linear utility function consistent with the information obtained so far. Thus the pareto point being generated is the nearest one among the rest of the pareto points.

We have implemented the reference point approach using the interactive mathematical programming system, called SESAME (Orchard-Hays 1978). The package of programs consists of essentially two parts: first, a DATAMAT program which reformulates a linear programming model in the form (P) of reference point optimization, and second, a routine to carry out an interactive iteration (i.e., to insert a reference objective, and to compute and display the pareto point). The current implementation employs the scalarizing function (1) with the parameter vector ε being equal to zero. The system is now capable of handling large practical multicriteria linear programs with up to 99 objectives and one or two thousand constraints.

For computational experimentation we used a dynamic LP model of a forest sector with about 700 rows and 900 columns. There are two objectives defined for each of the ten time periods of the model, i.e., there are twenty objectives in total. We experimented first with different values of the penalty coefficient ρ. The results suggest that for ρ one should use a value which is equal to or slightly larger than p, the number of objectives. Based on this observation, we set $\rho = p = 20$ for further numerical test runs. A sample of reference points is tried out and the overall performance of the method is found satisfactory. In a few cases, however, we observed some undesirable unsmoothness in the computed trajectroeis of the two objectives (see Figures 8b and 8c). This may be due to the fact that only weak pareto optimality is guaranteed, for $\varepsilon = 0$ (see Lemma 1). Thus, we expect the problem to disappear when the scalarizing function is implemented for $\varepsilon > 0$ in the next stage.

A general observation is that the pareto trajectories tend to agree with the reference objectives shifted up or down. This property was found not to be valid when experimenting with the extension of employing cumulative information on preferences. However, after this information began to influence the solution the pareto trajectories generally seemed likely to be better than those obtained disregarding this information(see Figures 9f and 9g).

APPENDIX

Derivation of Problem (P)

Denote by $W \equiv \{w | -w+Cx=\bar{q},\ Ax=b,\ x\geq 0\}$ the feasible set for vector w. Then the reference point optimization problem, when the scalarizing function (1) is applied, is as follows:

$$\min_{w \in W}\{-\min\{\rho \min_i w_i, \sum_i w_i\} - \varepsilon w\}$$

$$= \min_{w \in W}\{\max\{\max_i(-\rho w_i)\ ,\ -\sum_i w_i\} - \varepsilon w\}$$

$$= \min_{w \in W}\{\max\{\max_i(-\rho w_i - \varepsilon w)\ ,\ -\sum_i w_i - \varepsilon w\}\}$$

$$= \min_{\substack{w \in W \\ z \in R}}\{z | z \geq -\rho w_i - \varepsilon w, \text{ for all } i,\ z \geq -\sum_i w_i - \varepsilon w\}$$

$$= \min_{\substack{w \in W \\ y \in R}}\{y - \varepsilon w | -y - \rho w_i \leq 0, \text{ for all } i,\ -y - \sum_i w_i \leq 0\} \quad ,$$

where we have substituted $y = z + \varepsilon w$.

REFERENCES

Bell, D., R. Keeney, and H. Raiffa (eds.) (1977) Conflicting Objectives in Decisions. IIASA International Series on Applied Systems Analysis. New York: Wiley.

Kallio, M., A. Propoi, and R. Seppälä. A Model for the Forest Sector. Laxenburg, Austria: International Institute for Applied Systems Analysis. Forthcoming.

Orchard-Hays, W. (1977) A Simplified Introduction to VM/CMS and SESAME/DATAMAT Software. Technical Report (unpublished). Laxenburg, Austria: International Institute for Applied Systems Analysis.

Orchard-Hays, W. (1978) Anatomy of a mathematical programming system. In: Design and Implementation of Optimization Software, edited by H. Greenberg. Netherlands: Sijthoff and Noordhoff.

Orchard-Hays, W. (1979) Multi-objective Criterion Optimization. User's Guide (unpublished). Laxenburg, Austria: International Institute for Applied Systems Analysis.

Starr, M., and M. Zeleny (eds.) (1977) Multiple criteria decision making. TIMS Studies in the Management Sciences, 6.

Wierzbicki, A. (1979a) The Use of Reference Objectives in Multiobjective Optimization. Theoretical Implications and Practical Experience. WP-79-66. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Wierzbicki, A. (1979b) A Methodological Guide to Multiobjective Optimization. WP-79-122. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Zionts, S., and J. Wallenius (1976) An interactive programming method for solving the multiple criteria problem. Management Science 22:652-663.