A CONVERGENT REALLOCATION POLICY IN A CONVEX SET

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PREFACE

In a number of economic situations a decision maker is confronted with the problem of modifying a given and unsatisfactory resource allocation in order to improve it. This requires a control strategy to be implemented sequentially over time. Typical constraints are, loosely speaking, the control effort he is willing (or able) to exert, the information requirements on the system "state", the feasibility of intermediate allocations, and the total time in which the process is to be completed. This paper deals with some of these aspects: an "equal reallocation policy" is introduced and appropriate convergence properties are derived. On the basis of income distribution data for the Italian Economy, an example of wealth reallocation over income classes is presented.

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1. INTRODUCTION

A rather primitive concept in economics is the non-negativity of the resources shared by each agent at a given time and in a given social context, as well as the finitness of the resources shared among all agents. A formal translation of this concept leads to an apparently simple set-theoretic property, convexity, from which far-reaching implications and unsuspected results are often derived. In the past and more recently, the somewhat disguising feature of this property has drawn the attention of applied mathematicians and mathematical economists to several aspects of the ensuing "Convex Theory" (Rockafellar, 1969; Nikaido, 1968).

It appears quite natural that a systems theory viewpoint on this matter should be primarily concerned with dynamic systems defined on a convex state space. One such system is considered in this paper. When the matter of concern is convex reallocation dynamics, a fundamental question can be formulated as follows: How would a policy based on taking from the "rich" and giving to the "poor" succeed in equalizing shares over a fixed time horizon? This paper answers that question for the singular but important case of an equal reallocation policy.

2. BASIC ASSUMPTIONS

In addition to the standard assumptions of homogeneity and divisibility of resources, the policy discussed in this paper is based on the following assumptions:

(i) The resource set can be modeled as a Unit Simplex.

(ii) The policy maker operates at discrete time-points and has direct access to one component of the distribution at a time.

iii) The reallocation is evenly redistributed over the remaining components.

A few comments are in order. Regarding (i), it has been argued that reallocation dynamics in a Unit Simplex, to the extent in which it postulates constant-sum resources, contradicts the possibility of growth. It seems more accurate to say, however, that what is omitted from this description is the feedback link between distribution and growth. But this link can be added in an integrated growth-distribution model, if one recognizes that no conceptual difficulty arises in separating a multidimensional growth process into a balanced-growth of all components and a zero-growth redistribution among them.

As for (ii), we should first notice that when access to all components is possible, the reallocation problem becomes mathematically trivial. Convex combinations of unit-sum vectors are unit-sum vectors and such combinations may be chosen at will. Given two vectors, start-end, a trajectory connecting them and containing a desired number of arbitrarily spaced points can easily be constructed.

On the other hand, this case presupposes on the policy maker side a very strict and efficient control on all his resources. This, in practice, may result in costly - if not infeasible - policies. In brief, this case appears both trivial mathematically and of very restricted scope for application.

At the other extreme, we have the case in point. One component is controlled at each step. In order to preserve convexity, it is assumed that the amount by which one component is varied will be evenly redistributed over the remaining components. That this policy should converge to any desired distribution, while intuitively plausible, will require some amount of mathematical reasoning to be rigorously established.

Regarding (iii), some implications of this assumption can be best appreciated in the light of the policy algorithm to be presented in the next paragraph. For this reason, discussion on this point will be deferred until the conclusion.

PROBLEM STATEMENT

As usual, resources are modeled as non-negative unit-sum vectors. A convenient geometric interpretation is suggested by the notion of a Unit Simplex (Nikaido, 1968) in R^N. In two-dimensional space, resource vectors have one end on the line segment through the points (1,0) and (0,1) and this segment is a Unit Simplex in R^2.

Let S^{N-1} be the simplex in R^N defined by

$$S^{N-1} \triangleq \{\underline{x}:\underline{x}=\sum_{i=1}^{N} \lambda_i \underline{e}_i ; \lambda_i \geq 0 \forall_i ; \sum_{i=1}^{N} \lambda_i = 1\} \quad , \qquad (1)$$

where λ's are scalars and $\underline{e}$'s unit vectors. To avoid trivial cases, we will assume $N > 2$. As we are interested in motions within the simplex, we define a trajectory in S^{N-1}.

<u>def. 1</u> *A Trajectory* T *is a collection of vectors in* S^{N-1} *indexed by the integer* t, *i.e.*

$$T \triangleq \{\underline{x}(t); t=1,2,\ldots\} \quad . \qquad (2)$$

For each t, we also introduce

<u>def. 2</u> *An* α*-Neighborhood of* $\underline{x}(t)$ *is the subset of* R^N

$$X(t) \triangleq \{x : x=\underline{x}(t)+\alpha \underline{b}_k ; \alpha \neq 0; k=1,2,\ldots,N\} \quad , \qquad (3)$$

where

$$\underline{b}_k = \{\frac{-1}{N-1} \cdots\cdots 1 \cdots\cdots \frac{-1}{N-1}\}' \quad , \qquad (4)$$

with unit component in position k. The vector αb_k can be regarded as the control vector for the reallocation policy. For fixed α, X(t) contains exactly N vectors. Notice that $\underline{x}(t)$ is not contained in any of its α-Neighborhoods.

def. 3 *A Feasible Neighborhood* $X_f(t)$ *is the intersection*

$$X_f(t) \triangleq X(t) \cap S^{N-1} \tag{5}$$

For a non-void Feasible Neighborhood, the following conditions for α result:

$$\alpha \leq \min \{[1-x_k(t)],\ (N-1) \min_{i\neq k} x_i(t)\} \triangleq \alpha_k^+(t) \text{ if } \alpha > 0. \tag{6}$$

$$|\alpha| \leq \min\{x_k(t), (N-1)[1-\max_{i\neq k} x_i(t)] \triangleq -\alpha_k^-(t) \text{ if } \alpha < 0. \tag{7}$$

These inequalities can be simplified by noting that for any $\underline{x} \in S^{N-1}$

$$\left.\begin{aligned} 1 &= x_k + \sum_{i\neq k} x_i \geq x_k + (N-1) \min_{i\neq k} x_i \\ &\text{and also} \\ 1 &= x_k + \sum_{i\neq k} x_i \geq x_k + \max_{i\neq k} x_i \geq \frac{x_k}{N-1} + \max_{i\neq k} x_i \end{aligned}\right\} \tag{8}$$

Thus no harm is done by choosing

$$\alpha_k^+(t) = (N-1) \min_{i\neq k} x_i(t);\ \alpha_k^- = -x_k^{(t)} \tag{9}$$

and

$$\alpha \in \{[\alpha_k^-(t),0) \cup (0,\alpha_k^+(t)]\} \tag{10}$$

Note that $\alpha_k^-(t) \geq -1$ and $\alpha_k^+(t) \leq 1$ thus $|\alpha| \in (0,1]$, as should be expected.

From an economic standpoint, a Feasible Neighborhood may be regarded as a set of one-step reallocations starting from a given resource vector or initial endowments. Next we introduce:

def. 4 *A Feasible Trajectory* T_f *is a Trajectory in* S^{N-1} *subject to the constraint*

$$\underline{x}(t+1) \in X_f(t)\ \forall t\ . \tag{11}$$

Thus, a Feasible Trajectory is completely specified by (2)-(9). We now turn to the main question. Given two distinct points in S^{N-1} does there exist a Feasible Trajectory starting at one point and ending at the other?

An answer is given, in some sense, by the following reallocation policy.

Policy 1 *Given* $\underline{x}(0)$ *and* $\underline{x} \neq \underline{x}(0)$ *in* S^{N-1}, *construct the sequence* $\{\underline{x}^*(t)\}$ *according to*

$$\begin{cases} \underline{x}^*(1) = \underline{x}(0) \\ \underline{x}^*(t) = \underline{x}^*(t-1) + \alpha \underline{b}_k \ , \end{cases} \tag{12}$$

$t = 2,3,4,\ldots,$

with $\underline{b}_k$ *specified in (3) and at each* t,k *and* α *chosen according to the "Robin Hood Rule":*

Evaluate:

$$\beta \triangleq \ |x_j - x_j^*(t)| \geq |x_i - x_i^*(t)|; \ i = 1,2,\ldots,N \ , \tag{13}$$

$$\gamma \triangleq \ (x_h^*(t)-x_h) \geq (x_i^*(t) - x_i); \quad i = 1,2,\ldots,N \ , \tag{14}$$

$$\delta \triangleq \ \min(\alpha_j^{+}(t), x_j - x_j^{*}(t)) \tag{15}$$

Then

$$\text{if } x_j - x_j^*(t) > 0 \text{ and } \delta \neq 0 \text{ choose } \alpha=\delta \text{ and } k=j \ , \tag{16}$$

$$\text{if } x_j - x_j^*(t) > 0 \text{ and } \delta = 0 \text{ choose } \alpha=-\gamma \text{ and } k=h \ , \tag{17}$$

$$\text{if } x_j - x_j^*(t) < 0 \qquad \text{choose } \alpha=-\gamma \text{ and } k=j \ , \tag{18}$$

if none of the above applies stop the algorithm. (19)

Remark We will try to comment briefly on this algorithm. At each step, the current state $\underline{x}^*(t)$ is compared to the desired state $\underline{x}$. The difference vector $\underline{x}^*(t) - \underline{x}$ is analyzed component-wise to find:
- the highest absolute value (β; component j), and
- the highest value (γ; component h).

Now the idea is to modify either component of the current state so as to obtain a new state vector "closer" to the desired state. If j=h, we decrease the j-th component of the current state. No problems arise with constraints in this case (18). If $j \neq h$, we increase the j-th component to the extent permitted by the α_j^+ constraint (16). But there might be cases where no increase is permitted, i.e., $\alpha_j^+=0$. In this case, we decrease the h-th component (17).

Convergence results are summarized in the following:

Theorem

The sequence (12) in Policy 1 is a Feasible Trajectory. If this sequence is finite, its last element is $\underline{x}$*, otherwise* $\{\underline{x}^*(t);\ t=1,2,\ldots\}$ *monotonically converges to* $\underline{x}$ *in the Euclidean Norm.*

Proof

(13)-(18) imply (10), (11) thus $\{\underline{x}^*(t);\ t=1,2,\ldots\}$ is a Feasible Trajectory. If (12) contains a finite number of points, then at some $t = T < \infty$ step (19) of Policy 1 has been reached. This means $0 = x_j - x_j^*(T) = |x_j - x_j^*(T)|$ and, by (13) $\underline{x} = \underline{x}^*(T)$.

To prove monotonic convergence, equip R^N with the Euclidean Norm $||\cdot||^2$. Then

$$||\underline{x}-\underline{x}^*(t)||^2 = ||\underline{x}-\underline{x}^*(t-1)-\alpha\underline{b}_k||^2 = ||\underline{x}-\underline{x}^*(t-1)||^2 +$$

$$\alpha^2||\underline{b}_k||^2 - 2\alpha <\underline{x}-\underline{x}^*(t-1),\underline{b}_k> =$$

$$||\underline{x}-\underline{x}^*(t-1)||^2 + \alpha^2(1 + \frac{N-1}{(N-1)^2}) - 2\alpha(\Delta x_k(t-1) + \frac{\Delta x_k(t-1)}{N-1}) =$$

$$||\underline{x}-\underline{x}^*(t-1)||^2 + \frac{N}{N-1}(\alpha^2 - 2\alpha\Delta x_k(t-1)) \quad , \qquad (20)$$

where we put

$$\Delta x_k(t) \triangleq x_k - x_k^*(t) \quad .$$

Observe that (16)-(18) imply

$$|\alpha| \in (0,1];\ \alpha\Delta x_k(t) \geq 0;\ |\alpha| \leq |\Delta x_k(t)| \quad , \qquad (21)$$

Therefore,

$$\alpha\Delta x_k(t) = |\alpha||\Delta x_k(t)| \geq \alpha^2 \quad , \tag{22}$$

and

$$\alpha^2 - 2\alpha\Delta x_k(t-1) \leq 0 \quad , \tag{23}$$

in (20), which proves monotonic convergence of the sequence $\{||\underline{x}-\underline{x}^*(t)||^2;\ t=1,2,\ldots\}$

To show that the limit of this sequence is zero, consider first the case in which (16) applies. Then, for $\underline{x},\underline{x}^*(t) \in S^{N-1}$

$$||\underline{x}-\underline{x}^*(t)||^2 \leq \sum_{i=1}^{N} |x_i - x_i^*(t)| \leq N|\Delta x_k(t)| \quad . \tag{24}$$

If (17) or (20) apply instead

$$||\underline{x}-\underline{x}^*(t)||^2 \leq \sum_{i\in I^-}(x_i - x_i^*(t)) + \sum_{i\in I^+}(x_i^*(t)-x_i) =$$

$$2\sum_{i\in I^+}(x_i^*(t)-x_i) \leq 2|I^+| \max_i (x_i(t)-x_i) \leq 2(N-1)\Delta x_k(t) \tag{25}$$

where I^+ is the set of indices where $x_i^*(t) > x_i$ and $|I^+|$ its cardinal.

Using (21), (24), in (20), we get when (16) applies

$$||\underline{x}-\underline{x}^*(t-1)||^2 - ||\underline{x}-\underline{x}^*(t)||^2 = \frac{N}{N-1}(2\alpha\Delta x_k(t-1)-\alpha^2) \geq$$

$$\frac{N}{N-1}(2\alpha\Delta x_k(t-1) - |\alpha||\Delta x_k(t-1)|) \geq \frac{|\alpha|}{N-1}\ ||\underline{x}-\underline{x}^*(t-1)||^2 , \tag{26}$$

from which

$$||\underline{x}-\underline{x}^*(t)||^2 \leq \left(1 - \frac{|\alpha|}{N-1}\right)\ ||\underline{x}-\underline{x}^*(t-1)||^2 . \tag{27}$$

Using (21), (25) in (20) and proceeding likewise when (17) or (20) apply

$$||\underline{x}-\underline{x}^*(t)||^2 \leq \left(1 - \frac{N}{2(N-1)^2}|\alpha|\right)\ ||\underline{x}-\underline{x}^*(t-1)||^2 \quad . \tag{28}$$

In both cases $|\alpha| \in (0,1]$ and $\underline{x}^*(t) \to \underline{x}$ in norm.

Remark 1 The Trajectory obtained with (12)-(18) satisfies a local optimality criterion in the following sense. At each step t, $||\underline{x} - \underline{x}^*(t)||^2$ is decreased proportionally to $\Delta x_k(t)$ and by (13)-(15) any different choice of $\Delta x_k(t)$ whould yield a lower value of $\Delta x_k(t)$.
However, local optimality does not ensure global optimality, that is a locally non-optimal choice of k may yield faster convergence than a locally optimal one, as shown by this counter-example. Assume initial state = $\frac{1}{14}\begin{pmatrix}2\\7\\5\end{pmatrix}$ and finale state = $\frac{1}{14}\begin{pmatrix}5\\4\\5\end{pmatrix}$. A globally optimal Feasible Trajectory is the finite sequence $\frac{1}{14}\left\{\begin{pmatrix}2\\7\\5\end{pmatrix}\begin{pmatrix}4\\6\\4\end{pmatrix}\begin{pmatrix}5\\4\\5\end{pmatrix}\right\}$ whereas a locally optimal Feasible Trajectory would generate $\left\{\begin{pmatrix}2\\7\\5\end{pmatrix},\begin{pmatrix}3.5\\4.0\\6.5\end{pmatrix},\begin{pmatrix}5.00\\3.25\\5.75\end{pmatrix},\begin{pmatrix}5.325\\3.575\\5.000\end{pmatrix}\right\}$, ...

Remark 2 In several applications, reallocation policies that are "smoother" than implied by this rule may be desired. This, however, can be obtained by restricting the α-range to a suitable sub-interval of (0,1] with no prejudice on convergence results.

4. AN EXAMPLE: INCOME DISTRIBUTION IN THE IATLIAN ECONOMY

As a possible application of the preceding results, consider the per-capita income distribution in Italy during the period of 1974-76. The dynamics of the distribution is best illustrated by normalized histograms, as in Figure 1. Per capita income has been divided into ten income classes ranging from one fifth to twice the average income. The diagrams show the percentage of the national income allocated to each income class. The area of the histograms is proportional to the gross national income and is normalized to one.

This representation permits the comparison of income distributions at different epochs, irrespective of variations in population size and gross national income. The three-year record shows no substantial change in the distribution, thus it is reasonable to assume a balanced growth of all classes at a common growth rate.

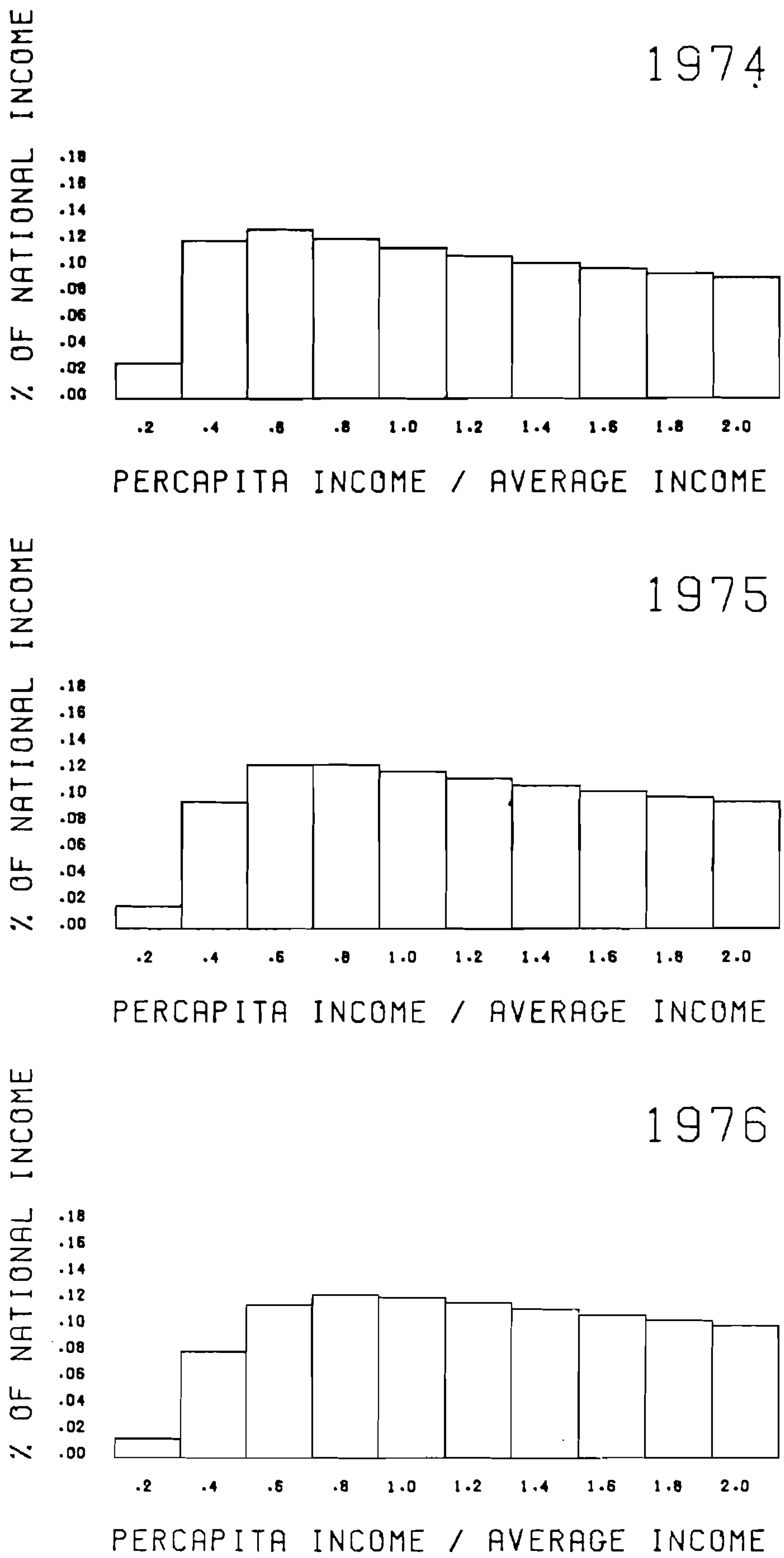

Fig. 1 Income distribution in Italy (Source: Bulletin of Bank of Italy, yrs 1966-78)

A rather controversial issue in Economic Theory is whether or not the observed distribution represents an economic optimum - aside from the question of whether or not such an optimum really exists. I will not enter that dispute here. But, for illustration purposes, I shall assume that some benevolent governmental goal has been set so as to reach the target distribution in Figure 2 (light line) starting from the 1976 situation. Assume that income transfers are controlled by some government action (fiscal policy, social benefits, interest rates, etc.) on a trimester basis. Assume further that the policy employed is selective, i.e. at each trimester t the fraction of the national income allocated to one particular income class is varied by a value not exceeding a pre-set transfer rate g(t). If this class is chosen according to Rule (13)-(19), the reallocation process becomes a Feasible Trajectory in Income Space with

$$\begin{aligned} \alpha &= \min(\delta, g(t)) \quad \text{if (16) applies, otherwise} \\ \alpha &= \max(-\gamma, -g(t)) \ . \end{aligned} \tag{29}$$

Assuming a constant 5% transfer rate, the hypothesized policy results in the graphs shown in figure 2 (dark line).

Table 1 shows the number of steps required to get to the target distribution within a 1% norm error, in function of the maximum transfer rate which is assumed constant at each step. It may be of (academic?) interest to note that no improvement in the convergence (number of steps) can be achieved by transfer rates higher than 5.86%.

5. CONCLUSIONS

There are cases where equal readjustment occurs as a spontaneous property of an economic system. In the Theory of Demand, for instance (Hildebrand, 1974), there is room for the case in which a price increase in one commodity affects an agent's consumption plan by a decrease in the budget share of that commodity and a common increase of the remaining ones. The equal readjustment process is endogenously performed by the system, i.e., the commodity market, whose behavior is - so to speak - inherently convex.

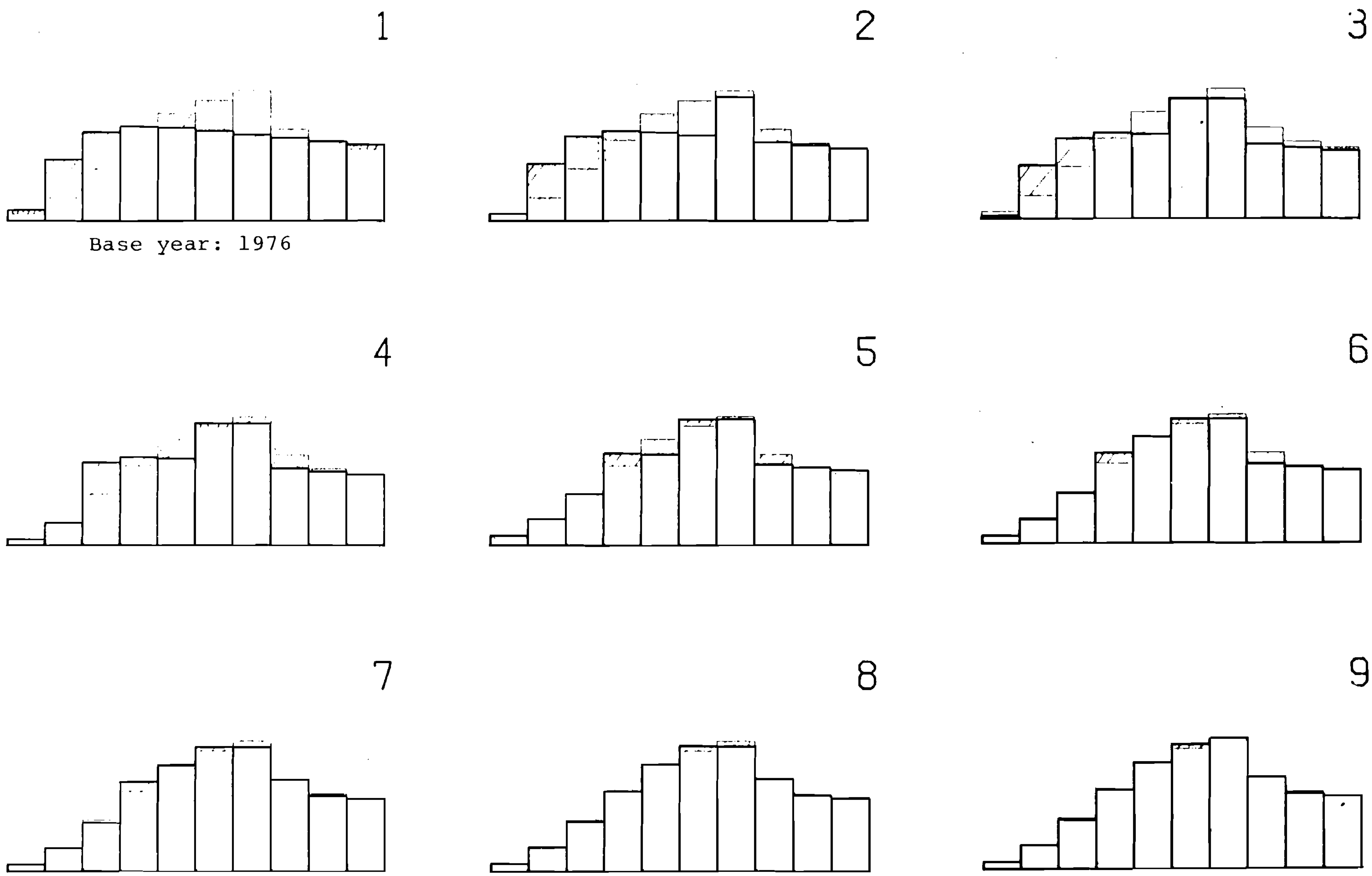

Fig. 2 Convergence of equal reallocation policy in 9 periods up to 1% norm error.
Shaded area shows difference between desired (light) and current (dark) allocation.

MAX TRANSFER RATE (%)	100	5.86	5.00	4.00	3.00	2.00	1.00
NUMBER OF STEPS	8	8	9	9	12	14	22

Table 1. Convergence up to 1% norm error as a function of maximum transfer rate.

In other economic situations a system may not possess that property, in which case, of course, it becomes the responsibility of the policy maker to enforce equal readjustment over the remaining components. The advantage of concentrating the control effort on a single component is lost in this case.

Despite this obvious drawback, the equal readjustment policy still appears to merit special attention over the other, possibly more flexible, policies when considered from the viewpoint of information requirements. As information is minimal in a vector of equal components, it is plausible to expect that information costs would be minimal with a control vector containing as many common values as possible, as is the case with $\alpha\underline{b}_k$ type of controls. Although it would be desirable to support this statement on the basis of a firmer formal theory, an attempt to clarify this point on heuristic grounds can be made along the line of reasoning contained in the Appendix.

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APPENDIX

We will try to justify the minimality of the information requirement associated to an equal readjustment policy on the basis of the block-diagram shown in figure A1:

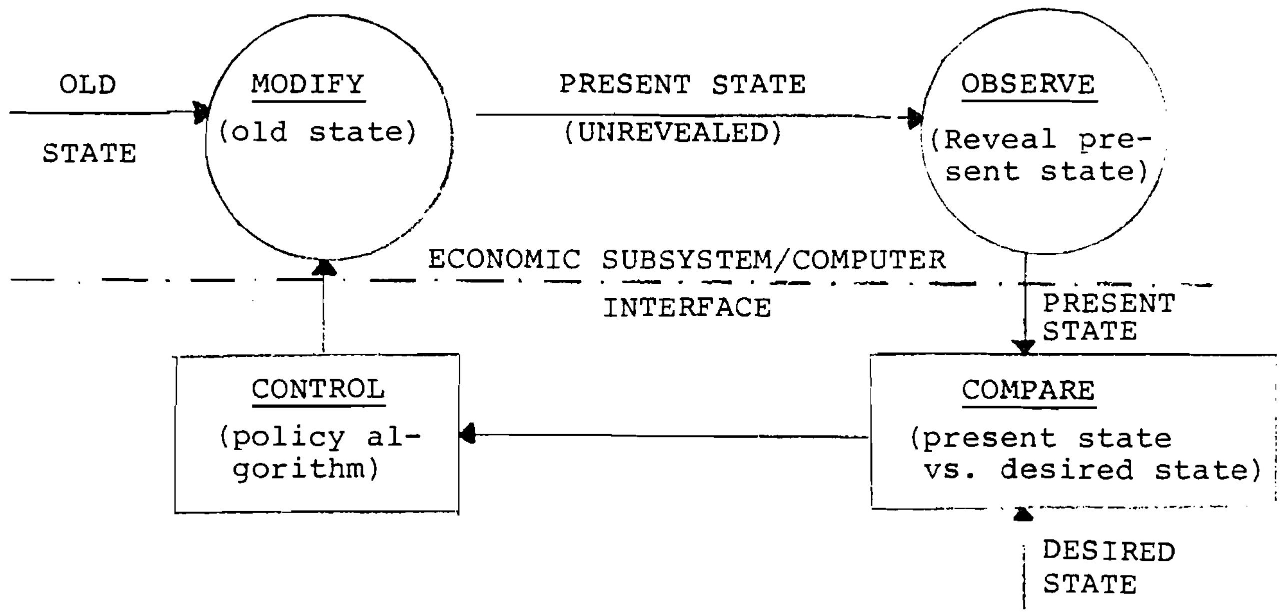

Figure A1. Block Diagram of a Reallocation Policy

Each block in the diagram is representative of a finite set of elementary operations: storage and retrieval of information. Some of these are to be performed by computer (square blocks), others involve interaction between the economic subsystem and the appropriate governing agency (round blocks). Assuming that major costs are concentrated on the latter, the equal re-adjustment policy involves:

- a sorting of an N-component vector (eqs. (13),(14),(15)) in the OBSERVE block: no storage is necessary and
- an assignment of two values in the MODIFY block: the amount by which k-th component is to be changed and the amount by which the remaining components are to be changed. However, remaining components need not be identified. In this case, storage requirement is independent of the size N of the problem.

A more flexible policy would require instead:

- an appraisal of N different values in the OBSERVE block - these are to be stored for subsequent processing and
- an assignment of N different values and additional information on the identity of each component in the MODIFY block. Information is dependent on the size N of the problem in this case.