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Working Paper **WP-17-016**

Learning from the Past: Supplementary Exercise on Memory, Persistence and Explainable Outreach

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Contents

Abstract

This Working Paper [WP] supports WP-17-015.

> *Toward Handling Uncertainty in Prognostic Scenarios: Advanced Learning from the Past*

by Żebrowski, Jonas & Jarnicka (2017). Their WP (ZJJ WP hereafter) constitutes the main report summarizing the outcome of a one-year project (bearing the same title) under the Earth Systems Sciences [ESS] Research Program of the Austrian Academy of Sciences [OeAW].

The WP focuses on systems with memory, typical in Earth system sciences. Memory allows referring to how strongly a system's past can influence its near-term future (paraphrased *credibility of expectations about a system's future behavior* in the ZJJ WP) by virtue of its persistence. We consider memory an intrinsic property of the system, retrospective in nature; and persistence a consequential (observable) feature of memory, prospective in nature. We delineate the system's near-term future by means of (what we call) its explainable outreach [EO].

This approach to determine the EO of a system complements the approach taken in the ZJJ WP. The WP makes use of a simple synthetic data (time) series example—our control—which we equip, step by step, with realistic physical features such as memory and noise, while exploring the system's persistence and deriving its EO. The prime intention of the WP is to better understand memory and persistence and to consolidate our systems thinking. Therefore, during this explorative state, systemic insight is valued more than mathematical rigor. The example is geared to making the concept of EOs applicable. However, we discuss how consequential it is, where it underperforms, and the questions it provokes.

From our example we conclude that memory allows defining a system's explainable outreach, above and beyond the numerical set up given here. It seems that, even if we know only the temporal extent of memory, a system's EO can be determined. This is promising because it appears possible to determine the temporal extent of memory in the presence of great noise, not exactly but approximately.

However, even with complete knowledge of how memory evolves over time, we are confronted with the challenge of reconstructing best-fit regressions that separate memory and noise—a challenge that we leave for the future.

Acknowledgments

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Matthias Jonas is a senior research scholar with the IIASA Advanced Systems Analysis Program. His interests are in environmental science, and in the development of systems analytical models and tools to address issues of global, universal and regional change, including surprises, and their potential implications for decision and policymakers.

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Learning from the Past:

Supplementary Exercise on Memory, Persistence and Explainable Outreach

Matthias Jonas and Piotr Żebrowski

1 Background

1.1 Motivation

This Working Paper [WP] supports WP-17-015.

> *Toward Handling Uncertainty in Prognostic Scenarios: Advanced Learning from the Past*

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The focus of the WP is on systems with memory, typical in Earth system sciences. Memory allows referring to how strongly a system's past can influence its near-term future. There exist different approaches to capture memory. In our WP we capture memory by way of example with the help of three characteristics: **its temporal extent, and both its weight and quality over time**. The extent of memory quantifies how many historical data directly influence the current one, while the weight of memory describes the strength of this influence. The quality of memory steers how well we know the latter.

The question that attracts our interest in the first place is how well do we need to know these (and/or possibly other) characteristics of memory in order to delineate a system's near-term future, which we seek to do by means of (what we call) the system's explainable outreach [EO]? We have reasons to be optimistic that the system's EO can be derived under both incomplete knowledge of memory and imperfect understanding of how the system is forced.

In our WP the focus is on forced systems. In many cases we do know that a system possesses memory, e.g., because it does not respond instantaneously to the forcing it experiences (what a system with no memory would do). But we find it difficult to quantify memory in a way which is easy to understand, particularly for practitioners and decision-makers.

Figure 1 serves as a prominent example of a forced system. Here the forcing is due to anthropogenic activities, e.g., fossil-fuel burning, cement production, and land use. The figure informs us of the emission reduction paths which we would have to follow at the global scale almost instantaneously if we wanted to keep global warming at or below 2 ºC and prevent the most dangerous impacts of climate change. However, the figure does not inform us on the "degree and extent of persistence" with which greenhouse gas [GHG] emissions will continue on their historical path into the future—knowledge which is crucial for the design, implementation and effectiveness of realistic emission reduction policies and for overcoming path dependences caused by memory.

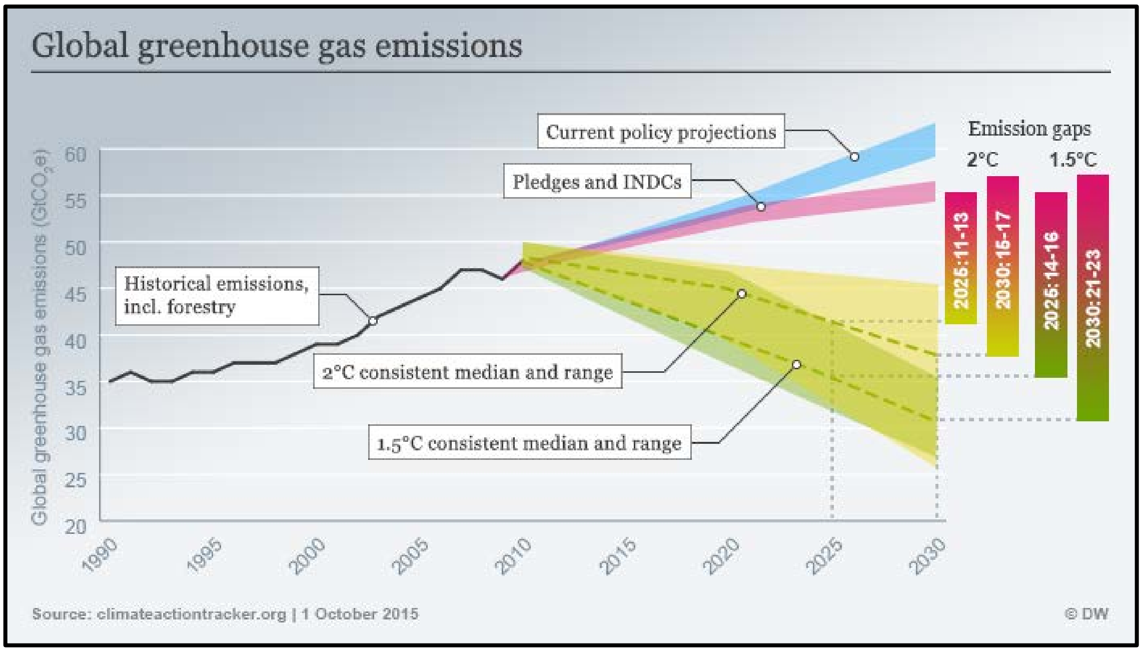

Fig. 1: Illustrating the effect of implementing pledges and so-called intended nationally determined contributions of 146 governments; and comparing these with the expected absolute emissions in 2020, 2025, and 2030 and 1.5 and 2 ºC benchmark emission pathways in accordance with the UNFCCC Paris Agreement (DW, 2015).

The question arises whether we can discriminate and specify the various characteristics of memory (e.g., the ones mentioned above) by way of diagnostic data processing alone? Or, put differently, how much systems understanding do we need to have and inject into the data analysis process in order to enable this discrimination? This question also receives our interest. As yet, we don't see that it can be uniquely answered theoretically. However, we do see a value in exploring this question with the goal of identifying approximate, yet sufficiently robust *modi operandi* to identify EO concepts that are easy to apply in practice.

The main objective of the WP is to pioneer an approach alternative to the one taken by ZJJ to derive a system's EO (paraphrased *credibility of expectations about a system's future behavior* in the ZJJ WP). To these ends, the WP makes use of a simple synthetic data (time) series example—our control—which we equip, step by step, with realistic physical features such as memory and noise, while exploring the system's persistence and deriving its EO (forward mode). The prime intention of the WP is to better understand memory and persistence and to consolidate our systems thinking. Therefore, during this explorative state, systemic insight is valued more than mathematical rigor. The example is geared to making the concept of EOs applicable (Sections 2.1 to 2.3). However, we discuss how consequential it is, where it underperforms, and the questions it provokes (Section 2.4).

The remaining two sections of Chapter 1 help to frame our mindset for Chapter 2. In Section 1.2 we expand on memory, persistence and EO, and in Section 1.3 we explain in greater detail how this WP complements the ZJJ WP. In Chapter 3 we discuss the problems that we envisage in quantifying persistence without having *a-priori* knowledge about memory and its major characteristics (backward mode). Chapter 4 summarizes insights and looks ahead.

1.2 Memory, persistence and explainable outreach

As it will become clear below, there exists some leeway in understanding (and defining) memory, persistence, and explainable outreach.

1. We consider memory an intrinsic property of a system, retrospective in nature; and persistence a consequential (i.e., observable) feature of memory, prospective in nature. Persistence is understood to reflect the tendency of a system to preserve a current value or state and depends on the system's memory which, in turn, reflects how many historical values or states directly influence the current one. The nature of this influence can range from purely deterministic to purely stochastic.

2. Deriving an EO should not be confused with prediction (and perfect forecasting).

   In statistics predictability is used in the context of in-sample and out-of-sample predictability, neither of which we are interested in. However, there exists the potential of misunderstanding which is rooted in the way of how an EO is made applicable as a measure of reference. Deriving a time series' EO requires evaluating the series' historical data by applying learning **and** testing (what we also call **learning under controlled prognostic conditions**). Shifting the EO to the end of the series' historical data (= today) has to happen untested and would therefore **not** be permitted. However, the only reason this forward shift is done still is to provide a bridge into the immediate future (see next point), thus **a reference measure for prognostic modelers and decision-makers**. Shifting the EO to today requires a conservative systems view which ensures that the system is not exposed to surprises it had not experienced before. These would cause the system to fall outside the EO.

3. Figure 2 visualizes the idea of using EOs as reference measure for prognostic modelers and decision-makers. An EO is derived from the historical data of a time series only and then shifted to its end (= today). Prognostic scenarios falling outside (above or below) the EO as well as scenarios falling within, but eventually extending beyond the EO are no longer in accordance with the series' past—allowing a decision-maker to inquire about the assumptions made in constructing a forward-looking scenario and to interpret these in terms of how effective planned measures (e.g., emissions reductions) need to be and/or how long the effectiveness of these measures remains uncertain. We consider an EO taking the form of an uncertainty wedge a more appropriate reference measure for the immediate future than a single, model-dependent business-as-usual scenario used as reference by modelers.

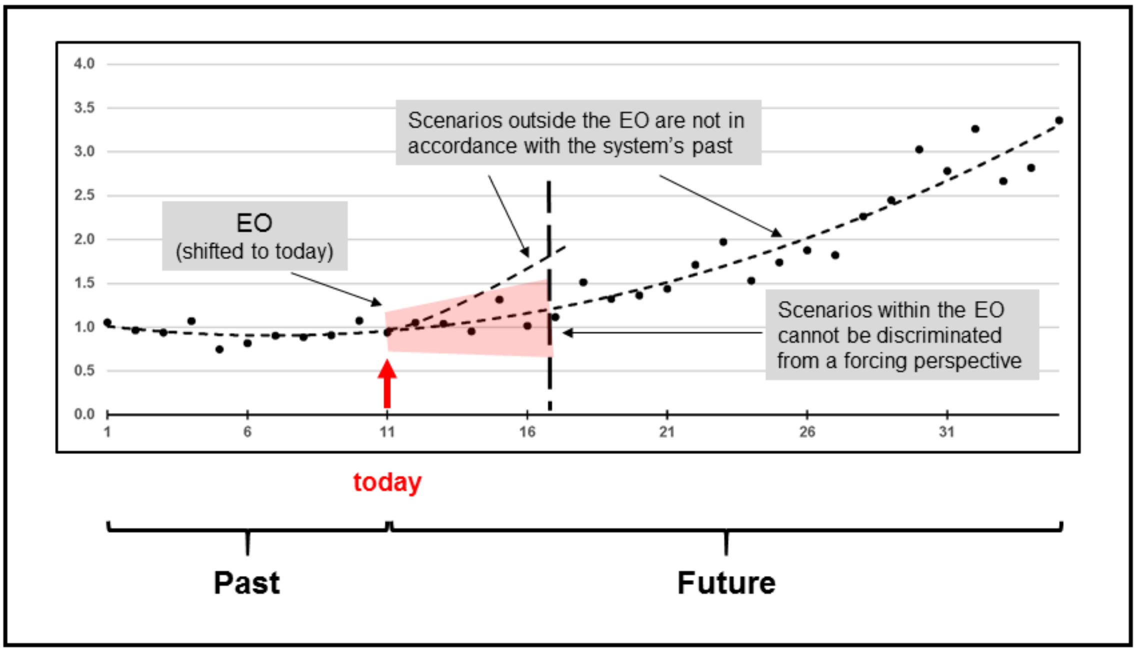

Fig. 2: Illustrating why knowing the EO of a time series is important (see text). For convenience in constructing the figure, we assumed a future being known (see black dots in the future part of the time series).

4. Deriving the EO of a time series must not be confused with signal detection.

"Signal" encourages thinking in terms of deviations from a pre-defined baseline (which can also be the zero line). We practice signal detection elsewhere to evaluate GHG emissions in an emissions change-versus-uncertainty context (cf. Jonas *et al.*, 2014). Figure 2 facilitates understanding why deriving the EO of a data series must not be confused with signal detection. Signal detection requires determining the time at which changes in the data series outstrip uncertainty—which is not done here.

5. Assuming persistence being an **observable** of memory, Figure 2 would allow quantifying (**not** defining) persistence. Given its directional positioning, the red-shaded EO in the figure may be described with the help of two parameters, its extent L and its aperture A at the end. We would then say that a time series with a long and narrow EO (the ratio L/A would be great) exhibits a greater persistence than a time series with a short and wide EO (the ratio L/A would be small).[2]

6. It is in the context of analyzing the structural dependences in the **stochastic component** of a time series where the terms "memory" and "persistence" commonly appear and are widely discussed.[3]

   These terms are not strictly/formally defined since they are regarded as statistical properties resulting from the time series' structural dependences. As a consequence, this leaves room for interpretation and scientific communities do understand these terms differently; and may also apply different methods to analyze them. Table 1 gives an overview of the terminology used by various scientific communities when they refer to memory and persistence; and how they interpret them.

Tab. 1: Memory and persistence as understood and interpreted by various scientific communities. Source: Jonas *et al.* (2017; unpublished document)

Field	Terminology	Interpretation	Literature
Climate Analysis	**Memory**, **dependence** (distinguishing between short-term/short-range and long-term/long range)	Rate of decay of the autocorrelation function (considered geometrically bounded; but also with exponential, power rate, or hyperbolic decay)	Caballero *et al*. (2002); Palma (2007); Franzke (2010); Mudelsee (2010); Lüdecke *et al*. (2013); Barros *et al*. (2016); Belbute & Pereira (2017)
	also **persistence**	Long-range memory (also checked by spectral or fluctuation analysis)	
Economy & Finance	**Serial dependence**, **serial correlation**, **memory**, **dependence**	Statistical dependence in terms of the correlation structure with lags (mostly long memory, i.e., with long lags)	Lo (1991); Chow *et al*. (1995); Barkoulas *et al*.(1996); Dajcman (2012); Hansen & Lunde (2014)
	also **persistence**	positive autocorrelation	
Geophys. & Physics	**Persistence**, **dependence**, also **memory** (mostly long-term)	Correlation structure in terms of Hurst exponent or power spectral density; but also system dynamics expressed by regularities and repeated patterns	Majumdar & Dhar (2001); Kantelhardt *et al*. (2006); Lennartz & Bunde (2009, 2011)

1.3 How is this Working Paper complementary to the Working Paper of Żebrowski et al.?

We restrict our answer to this question to the most important **systemic** and **mathematical** differences in the approaches taken in this WP to quantify the EO and those taken in the ZJJ WP (cf. Tab. 2). In this WP it is the time series' persistence [P]—a characteristic feature of the system—that determines the extent of the EO. Its lower and upper borders are given by the out-of-sample confidence band of a lower-order (linear) polynomial which is used to capture the past. The purpose of computational experiments is to uncover this P–EO linkage and explore its level of applicability.

By way of comparison, in the ZJJ WP both the extent **and** the outside borders of the EO were determined (the latter with the help of the out-of-sample prediction band) by way of computational experiments. That is, computational experiments were needed to determine the extent of the EO in particular.

Tab. 2: The major difference in the approaches followed here to determine the EO and in the ZJJ WP from both a systemic and a mathematical perspective.

	This WP	**ZJJ WP**
Systemically	The system is persistent; its (short-term) memory extends back for a **limited time** only.	The memory of the system may be blurred, but it remembers its **entire past**, limited by the beginning of the time series.
Mathematically	One numerical set-up (allowing multiple experiments): **The extent of the EO is given by the system's persistence** and tested under controlled conditions.	Multi-numerical set-ups (allowing multiple experiments): **The extent of the EO is derived experimentally** and tested under controlled conditions.

2. Example

The focus of the example presented in Chapter 2 is on systemic insight. Its purpose is to illustrate one way (among others) to reflect memory, to see how persistence plays out and to derive an EO. The example has been discussed intensively with respect to how consequential it is, where it underperforms and the questions it provokes—which are listed at the end of Chapter 2. However, the example does **not** exhibit fundamental shortfalls. It does not restrict generalization, while allowing to spot the important research issues which we will be facing in deriving the EO of a data series.

The example is geared to making the concept of EOs applicable. Figure 3 visualizes the different "worlds of knowledge" which we are confronted with in the example. Some of its features are excessively exaggerated to better understand how memory can lead to persistence even under unfavorable conditions, e.g., such as: a forcing which is weak and a memory the extent of which is short in relation to the noise which is superimposed.

The example is dealt with throughout Sections 2.1 to 2.3. Section 2.1 is composed of two steps: In the first step we obliterate the knowledge of our control, a 2^{nd}-order polynomial, by applying a high level of noise; in the second step we limit and steer this obliteration back in time by introducing memory in terms of extent, weight and quality. This allows reconstructing what had been obliterated before. The qualitative and quantitative characteristics of this reconstruction remain to be investigated against a reference. The intention behind this step-wise procedure is to develop an understanding of how memory works and leads to persistence. Section 2.2 visualizes this process graphically; Section 2.3 offers one way of deriving an EO; and Section 2.4 summarizes important insights and questions.

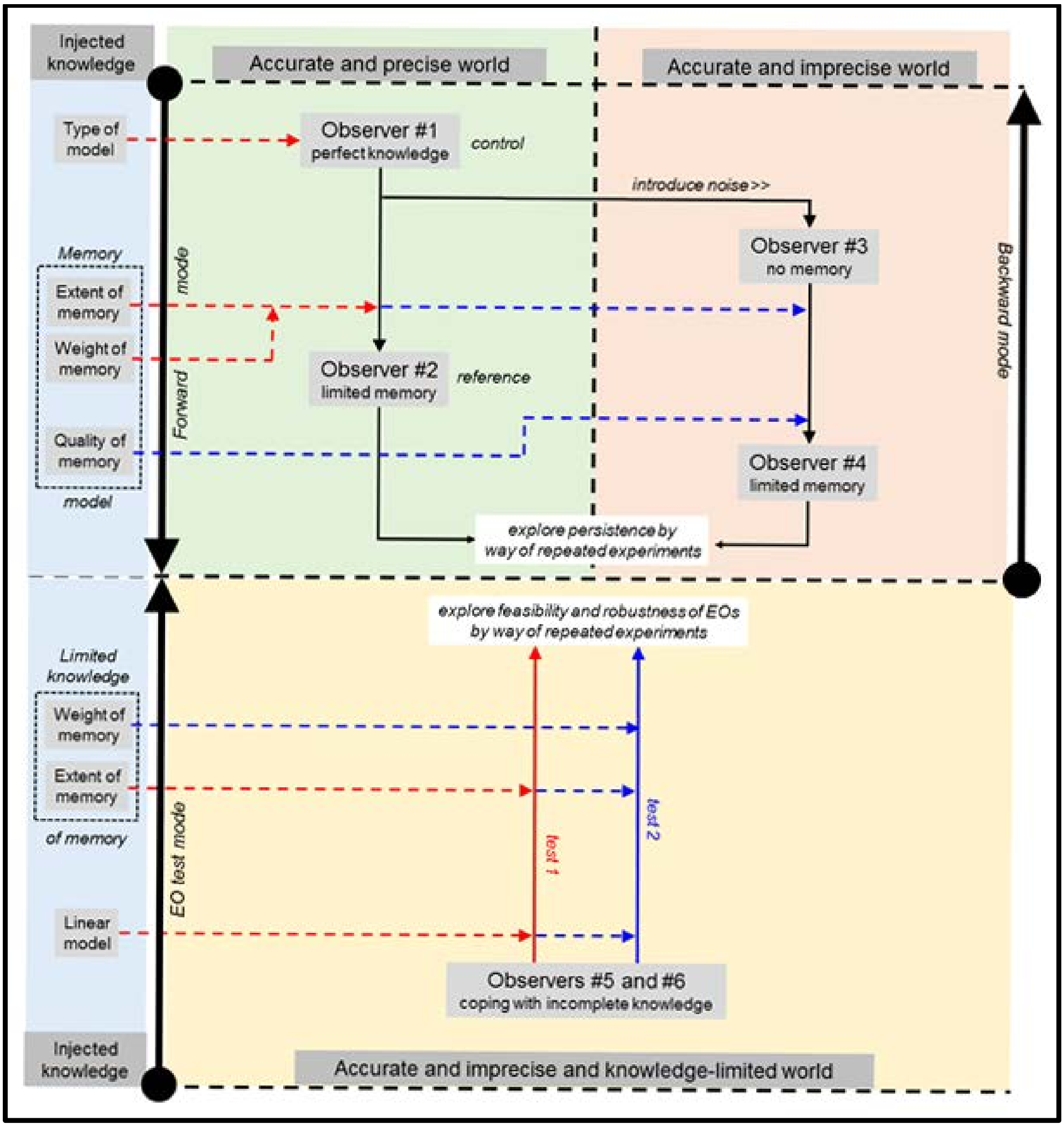

Fig. 3: Graphical visualization of the different "worlds of knowledge" underlying the example discussed in Chapter 2. The figure's main purpose is to distinguish these "worlds" by means of the knowledge that is injected in expanding the example step-by-step.

2.1 Mental and numerical set-up

We work with four functions dependent on x (with $x = 1,...,35$; sufficiently long for illustration purposes) which can, but need not, be interpreted as time series dependent on time t measured in years.[4] The functions can be understood to reflect four observers [O] who perceive an accurate world differently—precisely or imprecisely, and with perfect knowledge, and limited or no memory (cf. upper half of Fig. 3).

To start with, all observers have complete (not necessarily perfect) knowledge of their worlds (i.e., x extending from 1 to 35). We introduce two additional observers later when we split the time series into past (x from 1 to 7) and future (x from 8 to 35). These two observers will have incomplete knowledge because they see the historical part of the time series only (cf. lower half of Fig. 3).

<u>The world of observer O1 is described by</u>

$\boldsymbol{y_{Quad}}$**:** O1's observations are accurate and precise and can be perfectly described by a 2nd-order polynomial,[5] serving as both forcing and control in the following. Its coefficients are chosen such that its initial part exhibits a quasi-linear behavior:

$$y_{Quad}(x) = a_0 + a_1 x + a_2 x^2 ; \tag{2.1}$$

here with $a_0 = 1$, $a_1 = -0.025$, and $a_2 = 0.0025$.[6]

<u>The world of observer O2 is described by</u>

$\boldsymbol{y_{Quad\_wM}}$**:** y_{Quad} with memory [M]. M is chosen by way of assumption (seven years here; justified below) but making sure that it is shorter than the quasi-linear range of y_{Quad}. Each value of y_{Quad_wM} is constructed as a sum over the seven last values of y_{Quad} (including today), the weights of which decrease exponentially back in time:

$$y_{Quad\_wM}(x_k) = \sum_{j=0}^{6} e^{-cj} y_{Quad}(x_{k-j}) \tag{2.2}$$

for $x_k = k \;\; (k = 1,...,35)$ and $y_{Quad} = 0$ for $x_{k-j} = -5,...,0 \;\; (k - j = -5,...,0)$; and with e^{-cj} steering the **weight of memory** (cf. Tab. 3).[7] The exponential weighting is determined such that its value six years back in time (excluding today) is only 0.05, which reflects our cut-off level (**extent of memory**). That is, only 5% of a six-year old y_{Quad} value contributes to constructing the y_{Quad_wM} value of today ($c = ln\,0.05/(-6) = 0.50$). The weighting stays constant during the construction of y_{Quad_wM} and is not yet normalized (which we leave for later).

Tab. 3: The weights of M over seven years back in time (including today).

$-x_j$	$e^{(-c_j)}$
0	1.00
-1	0.61
-2	0.37
-3	0.22
-4	0.14
-5	0.08
-6	0.05
Total	**2.47**

The definition of y_{Quad_wM} demands three important comments: (1) The exponential weighting appears to be a natural choice. With reference to an obvious example (what we term **learning in a diagnostic context**), we see in retrospect that, at the scale of countries, learning (or, conversely, the decrease of uncertainty) in reporting GHG emissions happens exponentially (Hamal, 2010; Halushchak *et al.*, 2017)—leading us to start out here with exponential weighting as well. (2) The notion of memory in connection with y_{Quad_wM} may not appear straightforward, for the following reason: Ideally, y_{Quad} requires the values of only three points (years) to be entirely determined for all times, all the way from the beginning to the end. On the other hand, we use a memory extent of seven years when we construct y_{Quad_wM} with the help of y_{Quad}. Thus, it may be argued that a finite memory becomes meaningless because each individual point of y_{Quad_wM} carries "full memory". However, the situation changes if y_{Quad_wM} is perceived as the extreme outcome of a thought experiment in which the noise surrounding each point of y_{Quad_wM} eventually decreases to zero. (3) It is important to note that the way of how we formalize memory is crucial for how we proceed during the backward mode when we want to quantify persistence without having *a-priori* knowledge about memory and its major characteristics (Chapter 3).

<u>The world of observer O3 is described by</u>

$\boldsymbol{Y_{QwN}}$ **:** y_{Quad} with noise. Y_{QwN} is derived not only by blurring but by obliterating the 2nd-order polynomial character of y_{Quad} by means of great noise, here expressed in relative terms:

$$Y_{QwN}(x) = \left(a_0 + a_1 x + a_2 x^2\right)\left(1 + N u\right) = y_{Quad}(x)\left(1 + N u\right) \tag{2.3}$$

where N is a scaling factor and the values u_k are taken randomly from the u (standard normal) distribution. The equation describes a parabola with a noise component of $N*100\%$ of the "true" values of y_{Quad}.

In general, we deal with noise in the order of $N \approx 0.10$ (that is, $N*100\% \approx 10\%$).[8] Here, however, we increase N by one order, namely to $N = 3.0$ (that is, $N*100\% = 300\%$), which may result in perceiving Y_{QwN} as a whole as random noise with some directional drift, if at all, rather than a signal that is clearly visible albeit superimposed by noise. It is the almost complete obliteration of Y_{QwN} why we argue that we can freely choose the extent of memory in constructing y_{Quad_wM} (observer O2 above) and Y_{QwN_wM} (observer O4 below).

<u>The world of observer O4 is described by</u>

$\boldsymbol{Y_{QwN\_wM}}$**:** Y_{QwN} with M (seven years). Y_{QwN_wM} is given by:

$$Y_{QwN\_wM}\left(x_k\right) = \sum_{j=0}^{6} e^{-c\,j}\, y_{Quad}\left(x_{k-j}\right)\left[1+\left(1-De^{-d\,j}\right)N\,u_{k-j}\right] \tag{2.4}$$

with $1-De^{-d\,j}$ steering the **quality of memory** (cf. Tab. 4). This term is determined such that it allows only 0.05 parts (5%) of random noise for today, meaning that our memory is fairly precise; while it allows 0.95 parts (95%) of random noise when our memory gets as old as six years (excluding today), meaning that our memory is highly imprecise ($D = 0.95$ and $d = ln\left(0.05/0.95\right)/\left(-6\right) = 0.49$). Or, if interpreted systemically in a GHG emissions-concentration context,[4] the contribution of old emissions to today's concentration in the atmosphere is not only smaller than that of more recent emissions; but their contribution is also less well known. The quality stays constant during the construction of Y_{QwN_wM} and can be easily refined.[9]

To summarize, in introducing memory we make use of three characteristics: its temporal extent (here dealt with by way of "insightful decision"), and both its weight and quality over time. We show in Section 2.2 that memory can, but need not, allow partial reconstruction of what had been obliterated before.

Tab. 4: Quality of M over seven years back in time (including today). The last column shows the interaction of both weights (Tab. 3) and quality of M (second last column in this table) over time in the case that $a_0 = 1$, $a_1 = a_2 = 0$, and $N u_{k-j} = 1$ for all *k* and *j* as specified in the text.

$-x_j$	$De^{(-d_j)}$	$1-De^{(-d_j)}$	$e^{(-c_j)}\left[1+\left(1-De^{(-d_j)}\right)\right]$
0	0.95	0.05	1.05
-1	0.58	0.42	0.86
-2	0.36	0.64	0.61
-3	0.22	0.78	0.40
-4	0.13	0.87	0.25
-5	0.08	0.92	0.16
-6	0.05	0.95	0.10
Total	**2.37**		**3.43**

2.2 An experimental realization

Our mental-numerical set-up allows multiple experiments. A new experiment is launched with a new set of u_k taken randomly from the standard normal distribution, while all other parameters are kept constant.[10] Each experiment consists of two parts: I) Construction and graphical visualization of y_{Quad}, y_{Quad_wM}, Y_{QwN}, and Y_{QwN_wM}; and II) linear regression of the first seven points of Y_{QwN_wM}. The deeper understanding of Part II is (1) that we now split the world with respect to time into two parts, past ($x = 1,...,7$) and future ($x = 8,...,35$); making, in particular, the step from observer O4 who has complete knowledge of his/her world—**the world which we ultimately experience and have to deal with**—to observers (O5 and O6; cf. also lower half of Fig. 3) who have incomplete knowledge of that world, namely of its historical part only (seven years; in accordance with the extent of memory); and (2) that these observers can perceive the historical part of the "O4 world" only by way of linear regression, at the best.

Part I: Construction and graphical visualization of y_{Quad}, y_{Quad_wM}, Y_{QwN}, and Y_{QwN_wM}

Figures 4a and 4b show the graphical visualization of an experiment. Figure 4a shows y_{Quad} (orange), y_{Quad_wM} (black), Y_{QwN} (blue) and Y_{QwN_wM} (red); while Figure 4b shows only Y_{QwN} (blue) and Y_{QwN_wM} (red). Dashed lines indicate 2nd-order regressions and their coefficients of determination (R^2) which were determined using Excel.[11] The purpose of showing the 2nd-order regressions *of* y_{Quad}, y_{Quad_wM} and Y_{QwN} in Figure 4a and Y_{QwN_wM} in Figure 4b, along with their R^2-values, is to facilitate understanding. Knowing that our control is a 2nd-order polynomial, these regressions and their R^2-values allow following the obliteration of y_{Quad}, and its incomplete reconstruction thereafter.

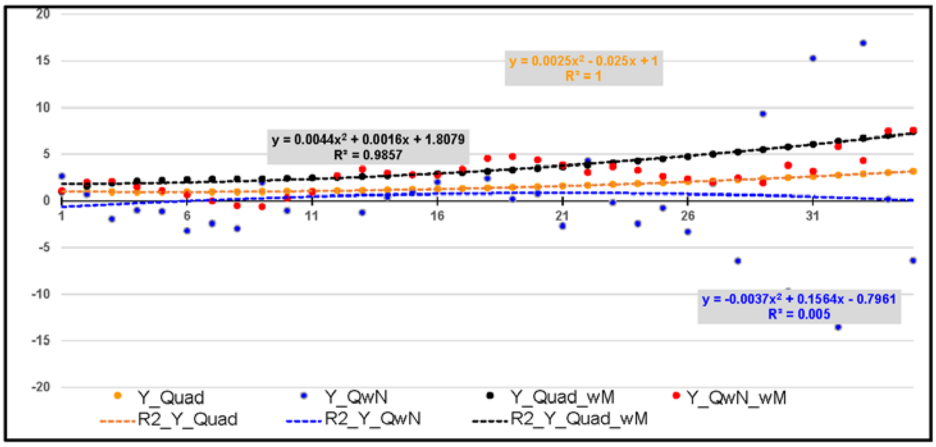

Fig. 4a: An experimental realization: y_{Quad} (orange; invariant), y_{Quad_wM} (black; invariant), Y_{QwN} (blue; variable) and Y_{QwN_wM} (red; variable). Dashed lines indicate the 2nd-order regressions and their coefficients of determination (R^2). Here, the regression of Y_{Quad_wM} falls above the regression of y_{Quad} because we have not yet normalized the coefficients of Y_{Quad_wM}, which steer the weight of memory over time.

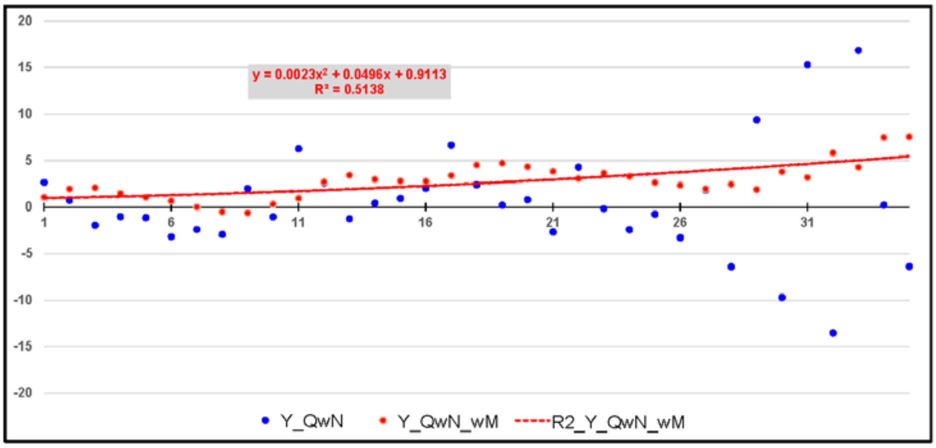

Fig. 4b: Like Figure 4a, but showing for a better overview only Y_{QwN} (blue; variable) and Y_{QwN_wM} (red; variable) with its 2nd-order regression (red solid line).

The experiment is very insightful because it is not (yet) as successful as we wish it to be. As expected, the application of great noise obliterates y_{Quad}. The blue points (Y_{QwN}) do not seem to follow a clear trend. Still, if one wanted to assign a 2nd-order regression to these points just for the sake of it, the regression would exhibit (here) a concave curvature—which would be opposite to the convex curvature of y_{Quad}—and a low R^2-value of 0.005 (cf. also Tab. 5), confirming the complete obliteration of y_{Quad}.[12]

Y_{QwN_wM} overcomes much of that obliteration, bringing the curvature back to convex and increasing the R^2-value substantially, here to greater than 0.5 (cf. also Tab. 5).

Part II: Linear regression of the first seven points of y_{QwN_wM}

Figures 5 expands Figure 4 (cf. also lower half of Fig. 3). Figure 5a shows a linear regression called *R1_Y_QwN_wM_hist_uw* (in the figure) and $Y_{Lin,7yr}$ (in Tab. 5) for the first seven points of Y_{QwN_wM} where we assume that it is only these seven points of Y_{QwN_wM} that an observer (observer O5 hereafter) knows. **It is this assumption—knowing the extent of memory—that requires discussion**. In deriving the linear regression, the seven points are weighted equally (unit weighting [uw] back in time), resulting in a low R^2-value of about 0.51 but, more importantly, in the wrong direction (downward).[13] Note that the overall direction of Y_{QwN_wM} is upward (cf. also Tab. 5).

By way of contrast, in deriving the linear regression in Figure 5b the first seven points are weighted exponentially [ew] over time. Here we assume that an observer (observer O6 hereafter) knows, like observer O5, only the first seven points (i.e., the extent of memory) of Y_{QwN_wM} but, in addition, **also the weight of memory over time—an assumption that requires discussion as well**. The exponential weighting (the same which underlies Y_{QwN_wM}) results in a more confident linear regression called *R1_Y_QwN_wM_hist_ew* (in the figure) and $Y_{Lin\_exp,7yr}$ (in Tab. 5) with an R^2-value of about 0.90 and an even greater downward trend (- 0.40 versus - 0.24; cf. Tab. 5). Figure 5b also shows the confidence bands belonging to $Y_{Lin\_exp,7yr}$ for the first seven years [inConf] and beyond; the latter by means of the out-of-sample [outConf] continuation of the seven-year confidence band. As can be seen, Y_{QwN_wM} crosses the seven-year confidence band from below to above and falls above the out-of-sample confidence band.

The purpose of selecting this (unsuccessful), and not another (successful) experimental realization is to prepare for the next section where we ask the question of whether we can make use of repeated regression analyses to capture the immediate future of Y_{QwN_wM}? This will cause the experimental outlook to change from unsuccessful to promising.

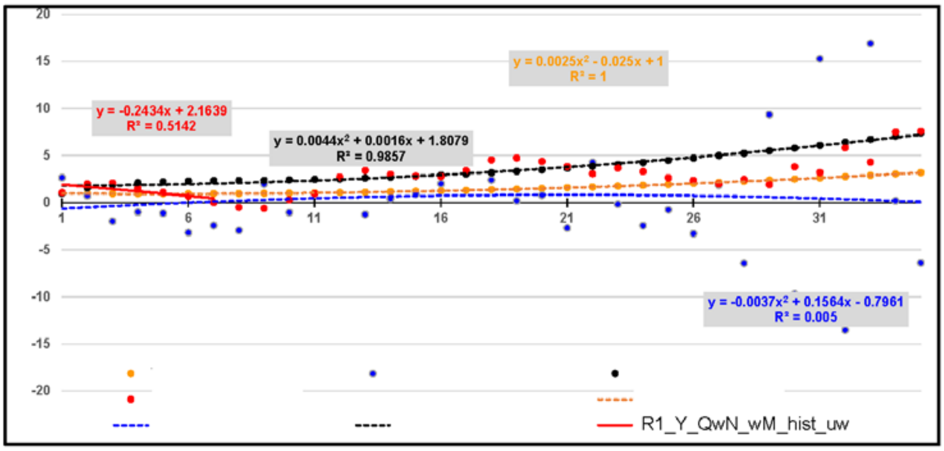

Fig. 5a: Like Figure 4a but additionally showing *R1_Y_QwN_wM_hist_uw*, a linear regression applying unit weighting [uw] back in time for the first seven points of Y_{QwN_wM} (red; variable). The assumption here is that it is only these points (i.e., the extent of memory) of Y_{QwN_wM} that observer O5 knows.

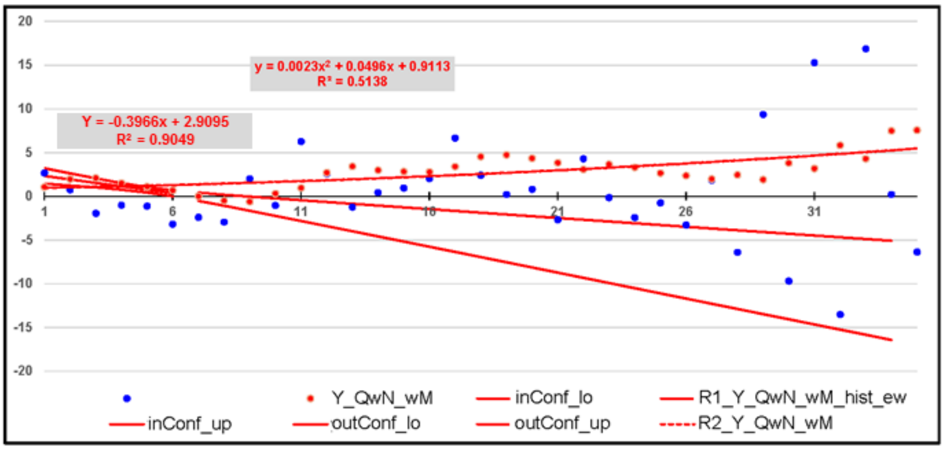

Fig. 5b: Like Figure 4b but additionally showing *R1_Y_QwN_wM_hist_ew*, a linear regression applying exponential weighting [ew] back in time for the first seven points of Y_{QwN_wM}, together with its in-sample [inConf] and out-of-sample [outConf] confidence bands. The borders of the confidence bands are indicated by upper [up] and lower [lo]. The assumption here is that observer O6 knows, like observer O5 only the first seven points (i.e., the extent of memory) of Y_{QwN_wM} but, in addition, also the weight of memory over time.

Tab. 5: Supplementing Figures 4 and 5 (while recalling Endnote 11): Compilation of regression parameters and coefficients of correlation, the latter between: (1) y_{Quad} and y_{Quad_wM} (invariant); (2) y_{Quad} and Y_{QwN} (variable); (3) Y_{QwN_wM} and $Y_{QwN\_wM-7\,yr}$ (variable) with $Y_{QwN\_wM-7\,yr}$ being identical to Y_{QwN_wM} but shifted backward in time (year 8 becomes year 1, year 9 year 2, and so on; while dropping the first seven years of Y_{QwN_wM}); and (4) y_{Quad} and Y_{QwN_wM} (variable). The first correlation coefficient indicates that limiting only the extent of memory back in time is not sufficient to overcome the "full memory" of y_{Quad}.[5] Correlation coefficients 2 and 3 seem to confirm that applying a high level of noise completely obliterates the 2nd-order polynomial character of y_{Quad}; and memory does not extend beyond seven years. Finally, correlation coefficient 4 seems to confirm that memory (that is, Y_{QwN_wM}) nullifies much of the obliteration brought about by Y_{QwN}.

Polynomial / Regression for	a_2	a_1	a_0	R^2
y_{Quad}	0.0025	- 0.0250	1.0000	1.000
y_{Quad_wM}	0.0044	0.0016	1.8079	0.9857
Y_{QwN}	- 0.0037	0.1564	- 0.7961	0.005
Y_{QwN_wM}	0.0023	0.0496	0.9113	0.5138
$Y_{Lin,7\,yr}$	----	- 0.2434	2.1639	0.5142
$Y_{Lin\_exp,7\,yr}$	----	- 0.3966	2.9095	0.9049
Coefficient of Correlation between				
1) y_{Quad} & y_{Quad_wM}	Influence of memory (w/o noise)			0.99
2) y_{Quad} & Y_{QwN}	Influence of noise (obliteration)			0.02
3) Y_{QwN_wM} & $Y_{QwN\_wM-7\,yr}$	Influence of memory after 7 yr (w noise)			0.06
4) y_{Quad} & Y_{QwN_wM}	Influence of memory in the presence of noise (reconstruction)			0.71

2.3 Toward a robust EO

We now repeat the experiment described in Section 2.2 multiple times (cf. also lower half of Fig. 3). Table 6 summarizes the results of 100 consecutive experiments where Y_{QwN_wM} falls within the (in-sample and out-of-sample) confidence band of $Y_{Lin\_exp,7\,yr}$ for a time that corresponds to two times the extent of memory (= 14.5 yr in the numerical set-up). These experimental realizations are denoted by "1: Y_{QwN_wM} in". All other experiments without exception by "0: Y_{QwN_wM} out". This repetition indicates how often shifting an EO with an extent of seven years to today (here: year 7) is justified; using "one times the extent of memory" as reference for both the shift and the extent of the EO. Table 6 indicates that this is the case in 42% of all experiments.

But we can learn more from the statistics than just success and failure. Table 6 also suggests that the R^2-value of $Y_{Lin\_exp,7yr}$, as well as that of $Y_{Lin,7yr}$, seems to be the right leverage point to discriminate “0-experiments” from “1-experiments”. In the numerical set-up given here, a grouping of experiments depending on whether the R^2-value of $Y_{Lin\_exp,7yr}$ is greater or smaller than 0.50 seems to be a success. This is shown in the fact that the R^2-values of $Y_{Lin\_exp,7yr}$ and those of $Y_{Lin,7yr}$ do not overlap:

$$Y_{Lin\_exp,7yr}: \quad R^2 > 0.50: \quad 0.82 \pm 0.13 = [0.69, 0.95]$$

$$R^2 < 0.50: \quad 0.18 \pm 0.11 = [0.07, 0.29]$$

$$Y_{Lin,7yr}: \quad R^2: \quad 0.68 \pm 0.26 = [0.42, 0.94]$$

$$R^2: \quad 0.19 \pm 0.15 = [0.04, 0.34].$$

Tab. 6: Summary of results of 100 consecutive experiments where Y_{QwN_wP} falls within the (in-sample and out-of-sample) confidence bands of $Y_{Lin\_exp,7yr}$ for a time that corresponds to two times the extent of memory (= 14.5 yr in the numerical set-up.). These experimental realizations are denoted by “1” (Y_{QwN_wM} in); all others by “0” (Y_{QwN_wM} out); indicating how often it is justified to shift the EO to today (here: year 7).

Grouping of Experiments	**Coefficient of Determination for**			**Coefficient of Correlation for**			**No. of Exp**
	$Y_{Lin\_exp,7yr}$	$Y_{Lin,7yr}$	Y_{QwN_wM}	y_{Quad} & Y_{QwN}	Y_{QwN_wM} & Y_{QwN_wM-7yr}	y_{Quad} & Y_{QwN_wM}	
No grouping	0.58 ± 0.32	0.50 ± 0.31	0.53 ± 0.22	0.10 ± 0.23	0.19 ± 0.35	0.62 ± 0.26	100
0: Y_{QwN_wM} **out**	0.72 ± 0.26	0.60 ± 0.30	0.55 ± 0.22	0.15 ± 0.22	0.20 ± 0.37	0.65 ± 0.27	58
1: Y_{QwN_wM} **in**	0.38 ± 0.30	0.35 ± 0.27	0.50 ± 0.21	0.03 ± 0.22	0.17 ± 0.32	0.59 ± 0.25	42
0: Y_{QwN_wM} **out** and **R²** of $Y_{Lin\_exp,7yr}$ **> 0.30**	0.77 ± 0.19	0.64 ± 0.28	0.56 ± 0.21	0.15 ± 0.21	0.19 ± 0.37	0.67 ± 0.24	53
1: Y_{QwN_wM} **in** and **R²** of $Y_{Lin\_exp,7yr}$ **< 0.70**	0.27 ± 0.22	0.25 ± 0.19	0.50 ± 0.23	0.04 ± 0.23	0.19 ± 0.34	0.61 ± 0.25	34
0: Y_{QwN_wM} **out** and **R²** of $Y_{Lin\_exp,7yr}$ **> 0.50**	0.82 ± 0.13	0.68 ± 0.26	0.54 ± 0.21	0.13 ± 0.20	0.17 ± 0.36	0.66 ± 0.25	48
1: Y_{QwN_wM} **in** and **R²** of $Y_{Lin\_exp,7yr}$ **< 0.50**	0.18 ± 0.11	0.19 ± 0.15	0.50 ± 0.22	0.03 ± 0.23	0.18 ± 0.33	0.61 ± 0.26	27

In addition, Table 6 indicates (while recalling Endnote 11) that the obliteration of y_{Quad} appears to be slightly greater on average for “1-experiments” than for “0-experiments” (cf. coefficients of correlation between y_{Quad} and Y_{QwN}: 0.03 ± 0.23 versus 0.13 ± 0.20). However, it seems that “1-experiments” perform, on average, slightly better in terms of reconstruction than “0-experiments.” In fact, they almost catch up (cf. coefficients of correlation between y_{Quad} and Y_{QwN_wM}: 0.61 ± 0.26 versus 0.66 ± 0.25).

In a nutshell, Table 6 confirms what common sense tells us: **A world perceived too precisely is difficult to "project" even into the immediate future.** Conversely, this is much easier to achieve if we are confronted with a highly imprecise world (forcing us to acknowledge our ignorance). It is exactly this insight which tells us (1) that we should avoid following the footsteps of "perfect forecasting" to derive the EO of a data series (cf. Section 1.2); and (2) that we can even derive a robust EO if we resist attempting to describe the world we perceive too precisely.

2.4 Pertinent insights and questions

Part I: Insights and questions of systemic nature

We recall that our WP reflects only a small step toward making the derivation of EOs an integral part of model building—what we aim at in the long-term (Jonas *et al.*, 2015). With this in mind:

1. Is the approach robust of deriving EOs which deliberately perceives the historical part of a data series imprecisely (by way of linear regression in our example)? How imprecisely shall we perceive the data series' historical part? This needs to be researched [Tbr].
2. We are confident that we can reduce the problem of studying memory and persistence systemically to studying single time series initially, if we allow flexible approaches to capture memory ranging from purely deterministic to purely stochastic; while keeping the issue of data availability in mind. In our example, we capture memory (by way of approximation) in terms of extent, weight and quality, with the latter interacting with the data series' stochastic component. However, different approaches to capture memory may require deriving EOs differently. Tbr
3. Even if our understanding is imperfect of how a system is forced, we still need to know one (or more?) characteristics of memory—in our example we need to know at least the extent of memory—in order to quantify a system's EO. How well do we need to know/can we know these characteristics in the presence of great noise? How much systems understanding do we need to inject in order to specify all characteristics of memory? Tbr
4. Shall we consider an upper ceiling for noise? We are aware of concerns that require pre-selecting/ conditioning observations (estimates) of systems so that its noise is ≤ 100% ($N \leq 1$); in particular, for system variables which balance at/around zero under (near-) equilibrium conditions. Tbr
5. In our example we have taken advantage of being able to repeat experiments multiple times—which we may not be able to do in reality. We would have to apply an alternative, e.g., a moving-window technique, where the length of the window coincides with the extent of memory. To start with, can we determine the extent of

memory with sufficient precision under great noise? How long must a data series be to allow achieving as robust findings as by way of repetition? Tbr

Part II: Insights and questions of mathematical nature

6. Our example underperforms mathematically in various ways, e.g.: What are the consequences of applying weights of memory that are not (yet) normalized and thus come with a "phase-in" effect? Was it justified to choose the extent of memory freely in the example's forward mode? Is the R^2-value a good measure to discriminate EOs robustly considering that a data series' historical part can also be perceived by way of nonlinear regression? Under which conditions is the use of confidence bands more appropriate than the use of prediction bands, or vice versa, to determine the shape of EOs? Tbr

7. In our example we need to know at least the extent of memory in order to derive a system's EO. Which technique(s) can be applied to determine the extent of memory in the presence of great noise? Can we think of an iterative trial-and-error procedure (including stacking) which would result in "de-noising" and, as a consequence, in determining the extent of memory? Tbr

8. Can time series analysis be applied in a flexible way so to allow testing approaches to capture memory, ranging from purely deterministic to purely stochastic? In this context it is noted that de-trending a time series, as our example shows, is not readily possible without knowing how memory plays out. (We are **not** able to make the step from Y_{QwN} to Y_{QwN_wM} if we do not inject the knowledge of how memory works.) Do other de-trending approaches exist that can be used? Tbr

3 Inverse problem: A glimpse into extracting persistence

The purpose of this chapter is to give a brief overview of the problems which we anticipate in determining memory (cf. Section 2.4: Point 3). To this end we proceed in two steps: the first referring to the deterministic case, and the second to the stochastic case.

Case I: *From* y_{Quad_wM} *to* y_{Quad}

Here we assume that we know y_{Quad_wM} and are interested in resolving the pertinent characteristics of memory (extent and weight) and, if possible, in reconstructing y_{Quad}. To start with, it is worth noting that it is not uncommon that we have some, if not a fairly good *a-priori* understanding of the system under investigation, including the temporal extent of its memory. Figure 6 seems to suggest that the coefficient of correlation between y_{Quad_wM} and y_{Quad_wM} shifted backward in time, designated $Y_{Quad\_wM-i\,yr}$ $(i=1,\ldots,19$

in the figure), allows detecting the temporal extent of memory in the vicinity around our/a insightful *a-priori* assumption (here: seven years). The figure shows that the correlation coefficient decreases slowly during the first seven years, that is, as long as memory provides a bond between y_{Quad_wM} and $Y_{Quad\_wM-i\,yr}$ (a consequence of Eq. 2.2); and decreases more strongly thereafter.

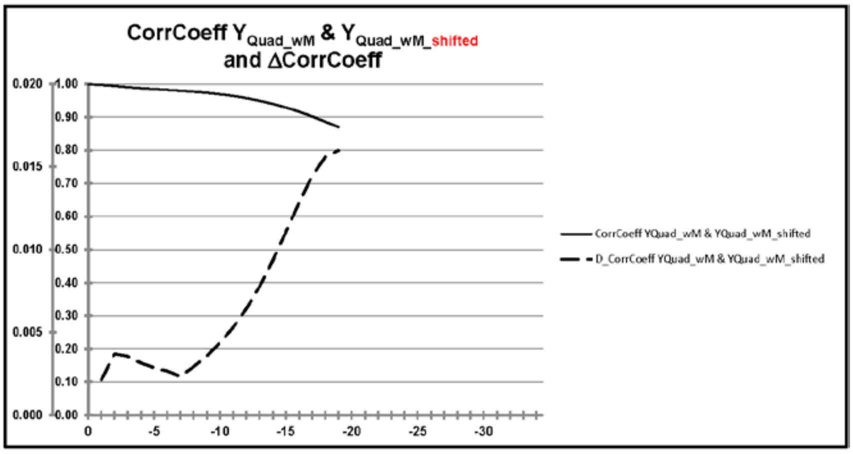

Fig. 6: **Solid black line and inner (right) vertical axis:** Coefficient of correlation between Y_{Quad_wM} and Y_{Quad_wM} shifted by $i = 1,\ldots,19$ years back in time, designated $Y_{Quad\_wM-i\,yr}$. For instance, $Y_{Quad\_wM-1\,yr}$ is identical to Y_{Quad_wM} but shifted backward by one year (year 2 becomes year 1, year 3 year 2, and so on; while dropping the first year of Y_{Quad_wM}). The correlation coefficient decreases over the range of shifted years shown here. **Dashed black line and outer (left) vertical axis:** The year-to-year difference in the correlation coefficient indicates that this decrease exhibits a local minimum between years -7 and -6 (disregarding the minimum between years -1 and 0 which is an artifact resulting from how the phase-in of Y_{Quad_wM} is currently realized; see text).

Being able to determine the temporal extent of memory is already an important first step. However, determining by how much past values contribute to today's value is more difficult. It requires knowing **how** this happens. Recall that we had applied an exponential function to weight memory over time (cf. Eq. 2.2). If, and only if, the exponential weighting approach holds—indeed, it would be good to know if this approach even holds in general—we would be able to deduce y_{Quad} value by value, starting at its beginning. The smallest weighting (we had chosen 0.05 as cut-off, leading to $c = 0.50$, the function's exponent) could be dealt with by way of agreement; while the phase-in could be overcome, for example, by recourse to the system's equilibrium state. The latter statement requires further explanation: We constructed y_{Quad_wM} according to

$$
\begin{aligned}
& y_{Quad\_wM}(1) = 1\, y_{Quad}(1) \\
& y_{Quad\_wM}(2) = 1\, y_{Quad}(2) + 0.61\, y_{Quad}(1) \\
& y_{Quad\_wM}(3) = 1\, y_{Quad}(3) + 0.61\, y_{Quad}(2) + 0.37\, y_{Quad}(1) \\
& \ldots
\end{aligned}
\tag{5.1}
$$

(cf. Tab. 3 for the coefficients), assuming that y_{Quad} does **not** exist before year 1, which may not be in accordance with reality.

However, we would still be able to deduce y_{Quad} if we were justified in assuming that y_{Quad} starts out from equilibrium (while still disregarding normalization):

$$
\begin{aligned}
& y_{Quad\_wM}(1) = 1\, y_{Quad}(1) + 0.61\, y_{Quad}(0) + \ldots + 0.05\, y_{Quad}(-6) \\
& y_{Quad\_wM}(2) = 1\, y_{Quad}(2) + 0.61\, y_{Quad}(1) + \ldots + 0.05\, y_{Quad}(-5) \\
& \ldots
\end{aligned}
\tag{5.2}
$$

with $y_{Quad}(1) = y_{Quad}(0) = \ldots = y_{Quad}(-6)$.

To sum up, it is important to keep in mind that the deduction of y_{Quad} will only be possible if the exponential approach holds of weighting memory back in time.

Case II: *From* Y_{QwN_wM} *to* y_{Quad}

Here we assume that we know Y_{QwN_wM} and are interested in resolving the pertinent characteristics of memory (extent, weight and quality) and, if possible, in reconstructing y_{Quad}. Recall that we are now confronted with random experimental realizations (depending on the u_k which are taken randomly from the standard normal distribution). Figure 7 refers to two such random realizations. Table 7 provides additional information. Figures 7a and 7b are similar to Figure 6 but show the coefficient of correlation between Y_{QwN_wM} and $Y_{QwN\_wM-i\,yr}$ $(i = 1,\ldots,19)$, and the year-to-year change in this coefficient. The figures indicate that: (1) these two quantities, the correlation coefficient and its year-to-year change, become quite variable; and (2) a temporal extent of memory of seven years cannot be so easily identified as in Figure 6. This does not come as a surprise—it is the result of allowing a high level of random noise.

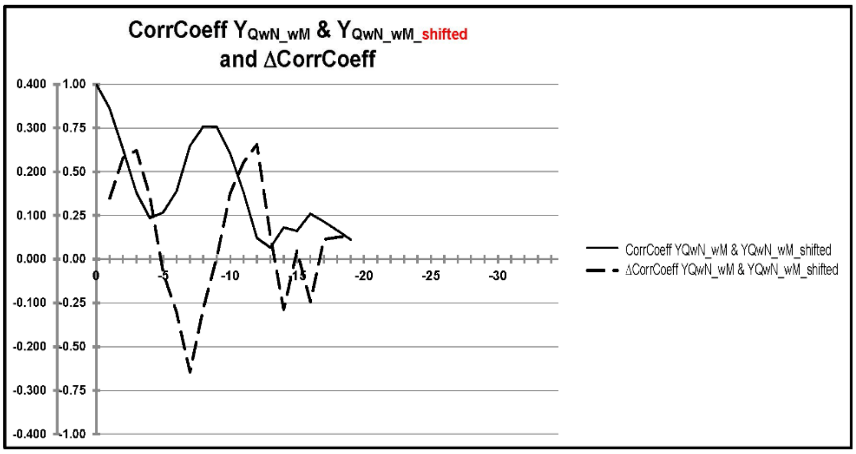

Fig. 7a: Like Figure 6 but for the coefficient of correlation between Y_{QwN_wM} and $Y_{QwN\_wM-i\,yr}$ $(i=1,\ldots,19)$, and the change in this coefficient. From the perspective of Figure 5b, Y_{QwN_wM} can be described to fall within both the in-sample and the out-of-sample confidence band belonging to $Y_{Lin\_exp,7\,yr}$. For further information see Table 7.

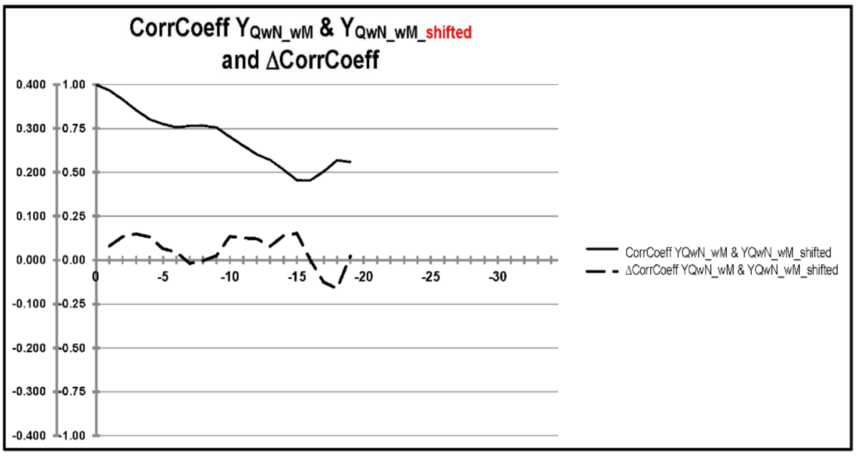

Fig. 7b: Like Figure 7a. From the perspective of Figure 5b, Y_{QwN_wM} can be described to fall within and to leave the in-sample confidence band belonging to $Y_{Lin\_exp,7\,yr}$ above and to stay above its out-of-sample confidence band. For further information see Table 7.

Tab. 7: Additional information on the experiments underlying Figures 7a and 7b.

Additional Information to	**Coefficient of Determination for**			**Coefficient of Correlation for**		
	$Y_{Lin\_exp,7\,yr}$	$Y_{Lin,7\,yr}$	Y_{QwN_wM}	y_{Quad} & Y_{QwN}	Y_{QwN_wM} & $Y_{QwN\_wM-7\,yr}$	y_{Quad} & Y_{QwN_wM}
Fig. 7a	0.0254	0.0645	0.6818[a)]	0.29	0.65	0.82
Fig. 7b	0.9194	0.8827	0.9051[b)]	0.50	0.77	0.95

a) $Y_{QwN\_wM} = 0.0126\,x^2 - 0.2238\,x + 2.7973$; b) $Y_{QwN\_wM} = 0.014\,x^2 - 0.1934\,x + 0.6469$

Figures 7a and 7b reflect special experimental realizations: (1) the obliteration of y_{Quad} is less severe than on average—the coefficient of correlation between y_{Quad} and Y_{QwN} ranges between 0.29 and 0.50 (cf. Tab. 7 and compare with Tab. 6); and (2) Y_{QwN_wM} nullifies more of that obliteration than on average—the coefficient of correlation between y_{Quad} and Y_{QwN_wM} ranges between 0.82 and 0.95 (cf. Tab. 7 and compare with Tab. 6).

The reason behind this experimental selection is to prepare for a potential way forward. We know that, **in the case of zero noise**, Figures 7a and 7b coincide with Figure 6 (Y_{QwN_wM} coincides with y_{Quad_wM}). By way of contrast, it appears that **in the case of non-zero noise**, the random, Figure-7-like experimental realizations exhibit a behavior similar to that in Figure 6, on average, especially in the beginning during the first seven years when memory still provides a bond between Y_{QwN_wM} and $Y_{QwN\_wM-i\,yr}$; and becomes arbitrarily variable thereafter. That is, it should be possible to overlay many Figure-7-like realizations to identify a behavior like that in Figure 6 and to determine the temporal extent of memory, **not exactly but approximately**. The option of stacking Figure-7-like realizations, however, would require a sufficiently long time series to allow applying a moving-window technique.

As in Case I, knowing the temporal extent of memory is an important step, if not the most important—it allows constructing $\boldsymbol{Y_{Lin,7\,yr}}$ **the $\boldsymbol{R^2}$-value of which appears to be an appropriate means to successfully identify robust EOs** (cf. Tab. 6). But we are interested in more, namely in how memory evolves back in time in terms of both weight and quality. Recall that knowing how the weight of memory evolves back in time allows constructing $Y_{Lin\_exp,7\,yr}$ the R^2-value of which appears to be an even better means to identify the robustness of EOs (cf. Tab. 6).

In our example the two exponentials that we applied to describe weight and quality back in time are not independent—they share the same j_{max} (cf. Eq. 2.4 and Tab. 4), which allow us to treat them in combination and proceed as in Case I (meaning that initial and cut-off values determining c, D and d could be dealt with by way of agreement). Of course, **even with the knowledge of the two exponential functions at hand, it will not be possible to reconstruct $\boldsymbol{y_{Quad}}$**. This is because we do not know the noise component individually at each point in time. Nonetheless, in the case that the two exponential functions can be deduced by systemic insight, it should be possible—while proceeding as in Case I—to "knowledge-correct" Y_{QwN} point by point, the best-fit regression of which would exhibit a behavior close to that of y_{Quad_wM} (ideally also a 2nd-order one). Since we will know c, D and d only imprecisely at best, we will find a set of (ideally) 2nd-order best-fit regressions. It remains to be seen whether Y_{QwN_wM} will turn out as the mean of that range—a challenge which we leave for later.

To sum up, it seems possible to determine the temporal extent of memory. However, the deduction of y_{Quad} will not be possible; only best-fit regressions centering around y_{Quad_wM} at the best if the exponential approach holds of describing weights and quality of memory back in time.

4 Summary and outlook

The WP focuses on systems with memory, typical in Earth system sciences. Memory allows referring to how strongly a system's past can influence its near-term future by virtue of its persistence. We consider memory an intrinsic property of the system, retrospective in nature; and persistence a consequential (observable) feature of memory, prospective in nature.

The WP's main objective is to delineate a system's near-term future by means of its EO and to pioneer an approach that is complementary to the one taken by ZJJ. The WP makes use of a simple synthetic data (time) series example—our control—which we equip, step by step, with realistic physical features such as memory and noise, while exploring the system's persistence and deriving its EO (forward mode). The prime intention of the WP is to better understand memory and persistence and to consolidate our systems thinking. Therefore, during this explorative state, systemic insight is valued more than mathematical rigor. The example is geared to making the concept of EOs applicable.

There exist different approaches to capture memory. In our WP we capture memory by way of example with the help of three characteristics: its temporal extent, and both its weight and quality over time. The extent of memory quantifies how many historical data directly influence the current one, while the weight of memory describes the strength of this influence. The quality of memory steers how well we know the latter.

The question that attracts our interest in the first place is how well do we need to know these (and/or possibly other) characteristics of memory in order to delineate a system's EO (backward mode)? We have reasons to be optimistic that the system's EO can be derived under both incomplete knowledge of memory and imperfect understanding of how the system is forced.

We speculate that memory allows defining a system's EO conveniently; that is, in general, above and beyond the numerical example given here. It appears that, if we only know the temporal extent of memory, the system's EO can be determined (to this end we make us of the R^2-value of $Y_{Lin,7yr}$). This is promising because it is not uncommon that we have some, if not a fairly good *a-priori* understanding of the system under investigation, including the temporal extent of its memory. If we also know how the weight of memory evolves back in time, it even appears possible to reinforce the robustness with which a system's EO can be determined (to this end we make us of the R^2-value of $Y_{Lin\_exp,7yr}$).

The issue of partial versus complete knowledge of memory (here: extent, weight and quality) guides our discussion of the backward mode, which becomes intricate in the presence of noise (in this case, at a high level). It appears to be possible to determine the temporal extent of memory, not exactly but approximately. While this has yet to be investigated thoroughly it would allow us, as mentioned above, to determine a system's EO.

In contrast, achieving complete knowledge of memory appears to be possible but requires additional *a-priori* insight of the system; such as the insight that weight and quality of memory back in time can be approximated sufficiently well by exponential functions. It is this insight of the exponential functions in general, not specifically, that we would need to have available.

However, even with complete knowledge of memory, we are confronted with the challenge of reconstructing best-fit regressions that separate memory and noise. We leave this challenge for the future.

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Acronyms and Nomenclature

A	EO aperture
ASA	Advanced Systems Analysis Program (IIASA)
C	concentration
E	emissions
EO	explainable outreach
ESS	Earth Systems Sciences
ew	exponential weighting
GHG	greenhouse gas
IIASA	International Institute for Applied Systems Analysis
IID	independent and identically distributed
inConf	in-sample confidence band
L	EO length
lo	lower
O	observer
OeAW	Austrian Academy of Sciences
outConf	out-of-sample confidence band
P	persistence
R	resolution
t	time
Tbr	to be researched
up	upper
uw	unit weighting
WP	Working Paper
ZJJ	Żebrowski, Jonas & Jarnicka

Endnotes

[1] OeAW's 2013 ESS call: http://www.oeaw.ac.at/fileadmin/NEWS/2013/pdf/ESS_Calltext_Englisch.pdf

[2] *L* and *A* are characteristics of the model which is selected to analyze the given time series. We expect that increasing the resolution ($R\uparrow$) with which the system is observed will result in a decreasing EO extent ($L\downarrow$). The EO of the selected model is likely to prove less suitable to capture the more fluctuating time series, which discloses more detail (surprises) than before (resulting in $A\uparrow$). In order to increase persistence [P] again to its prior value, our model would have to be modified so it smooths the time series more than before.

[3] In modeling the **deterministic component** of a time series, memory effects are typically believed **not** to be of major concern and that this component can be captured by means of suitable regression methods (e.g., linear/polynomial or non-parametric)—what is easy to refute by means of a (our) time series example in Chapter 2.

[4] This choice of interpretation allows simplifying our example but it is systemically important. Consider, e.g., the logical link between GHG emissions – atmospheric concentration – global mean surface warming, and the lag (memory) effect between any two of them; say concentration [C] and emissions [E]. It is this two-data-series perspective, here the $C = C(E)$ perspective in the t-E-C space, which practitioners are interested in. However, reducing the two-data-series perspective to the perspective of a single time series, here $E = E(t)$ or $C = C(t)$, comes useful. It allows, if done cleverly, describing memory deterministically **and/or** stochastically. Here we prefer the single-time-series perspective. However, to acknowledge the wider perspective, we continue using x as variable, **not** t.

[5] We see it as a consequence of fitting a model to the data why the term "full memory" is also used. Knowing the model's coefficients (three in our case) perfectly well leads us to believe that we know the time series all the way from its beginning (past) to its end (which we may even extrapolate into the future). That is, the model's coefficients are interpreted to embody the "full memory" of the time series.

[6] To facilitate following the fate of our control, we use descriptor type of indices (such as "Quad"). Otherwise, our mathematical terminology is standard (cf., e.g., Wolberg, 2006): small letters are used for model related variables, while capital letters are used for values that are observed (or estimated).

[7] The way y_{Quad_wM} operates falls under smoothing, referring to techniques for smoothing time series data; here by assigning weights to past observations or estimates which decrease exponentially over time. Of particular interest is that smoothing introduces a phase shift into the data (cf. https://en.wikipedia.org/wiki/Exponential_smoothing).

[8] As a real-world pendant with noise in the order of $N \approx 1$ one may think of, e.g., the net biome production of the terrestrial biosphere.

[9] For example, the quality can be made to follow an S-shape of a normalized cumulated distribution function more closely if memory accumulates over time (which we leave for later).

[10] These are: $a_0 = 1$; $a_1 = -0.025$; $a_2 = 0.0025$; $c = 0.50$; $D = 0.95$; $d = 0.49$; $N = 3.0$; $k_{min} = 1$; $k_{max} = 35$; $j_{min} = 0$; $j_{max} = 6$.

[11] We are aware that the coefficient of determination for a nonlinear regression exhibits limitations (http://blog.minitab.com/blog/adventures-in-statistics/why-is-there-no-r-squared-for-nonlinear-regression). This is why this coefficient should be understood as a qualitative indicator only, as done here and explained in the text. An appropriate alternative would be the use of the standard error of the regression.

[12] Concave = concave downward (or convex upward); convex = convex downward (or concave upward).

[13] We are aware (indeed, tolerate) that the assumption of independent and identically distributed [IID] random variables underlying linear regression theory may be violated (cf. https://en.wikipedia.org/wiki/Independent_and_identically_distributed_random_variables).