On one ill-posed problem of package guidance

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Abstract

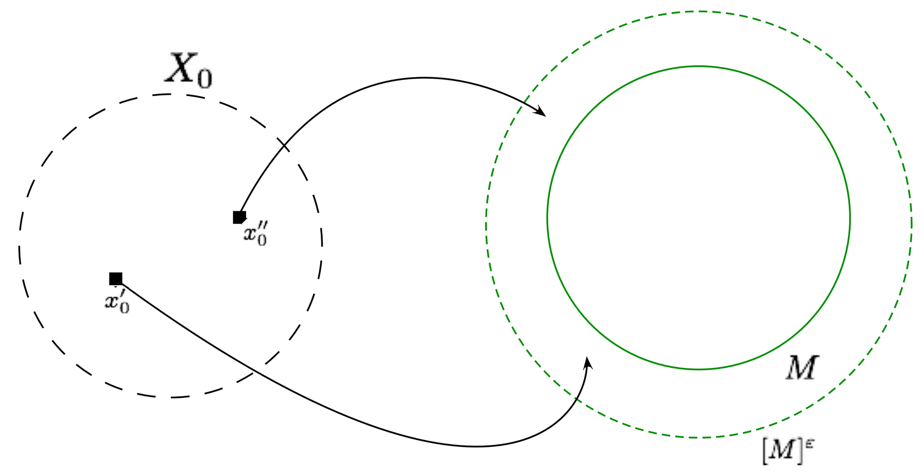
The method of program packages is a tool for solution of the guaranteed positional control problems with incomplete information on the initial state of the system. In this work the application of the method to a linear dynamical system with a linear observed system is considered. An example, for which the corresponding problem turns out to be ill-posed, is presented[1].

1. General problem

Let us consider a linear dynamic controlled system, described by an ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t_0 \le t \le \vartheta \quad (1)$$

Open-loop control (program) $u(\cdot)$ is a measurable function on $[t_0, \vartheta], u(t) \in P \subset \mathbb{R}^r$, P is a convex compact set. The initial state of the system maybe not known a priori: $x(t_0) = x_0 \in X_0 \subset \mathbb{R}^n$, where X_0 is a finite known set. The terminal condition $x(\vartheta) \in M \subset \mathbb{R}^n$, where M is a closed and convex set, should hold.
A linear signal $y(t) = Q(t)x(t)$, where $Q(\cdot)$ is left piecewise continuous matrix-function, $Q(t) \in \mathbb{R}^{q \times n}, t \in [t_0, \vartheta]$, is observed by the controlling side.



The problem of positional guidance is formulated as follows - based on the given arbitrary $\varepsilon > 0$ choose a closed-loop control strategy with memory, **whatever the system's initial state** x_0 **from the set** X_0, the system's motion $x(\cdot)$ corresponding to the chosen closed-loop strategy and starting at the time t_0 from the state x_0 reaches the state $x(\vartheta)$ belonging to the ε-neighbourhood of the target set M at the time ϑ.

2. Program packages method

Homogeneous system, corresponding to (1)

$$\dot{x}(t) = A(t)x(t)$$

For each $x_0 \in X_0$ its solution is given by the Cauchy formula:

$x(t) = F(t, t_0)x_0;\ F(t, s)\ (t, s \in [t_0, \vartheta])$ is the fundamental matrix.

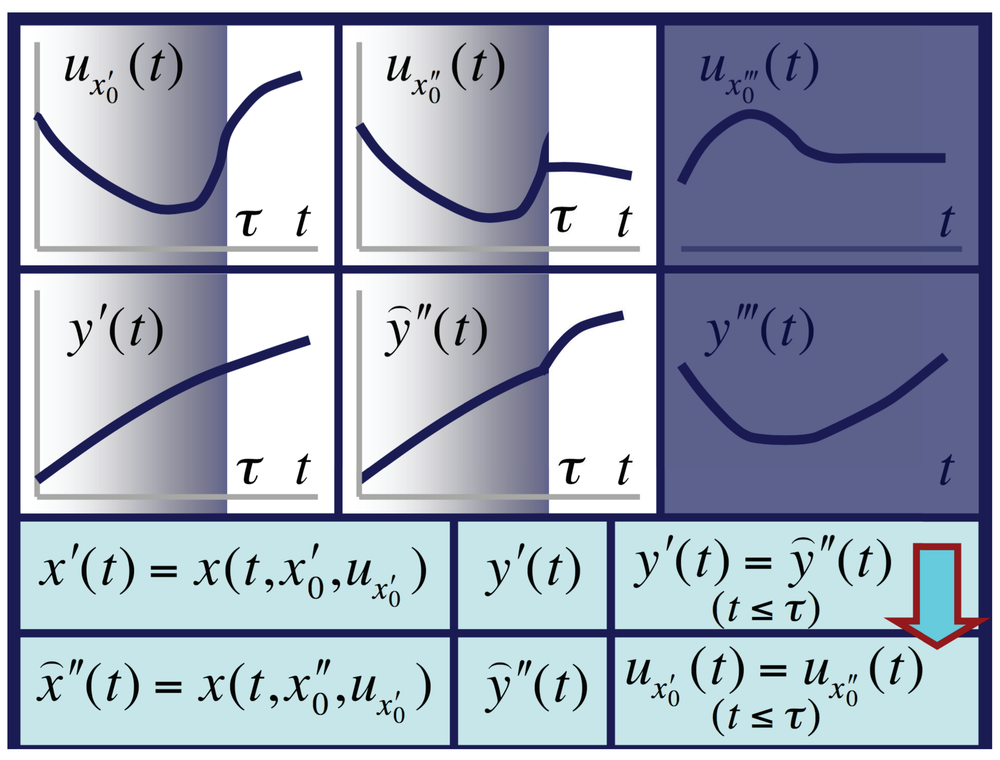
Homogeneous signal, corresponding to an admissible initial state $x_0 \in X_0$:

$$g_{x_0}(t) = Q(t)F(t, t_0)x_0 \ (t \in [t_0, \vartheta],\ x_0 \in X_0).$$

Let $G = \{g_{x_0}(\cdot) | x_0 \in X_0\}$ be the set of all homogeneous signals and let $X_0(\tau | g(\cdot))$ be the set of all admissible initial states $x_0 \in X_0$, corresponding to the homogeneous signal $g(\cdot) \in G$ *till time point* $\tau \in [t_0, \vartheta]$:

$$X_0(\tau | g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0,\tau]} = g_{x_0}(\cdot)|_{[t_0,\tau]}\}.$$

Program package is an open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$, satisfying **non-anticipatory condition**: for any homogeneous signal $g(\cdot)$, any time $\tau \in (t_0, \vartheta]$ and any admissible initial states $x_0', x_0'' \in X_0(\tau | g(\cdot))$ the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for almost all $t \in [t_0, \tau]$.



Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for all $x_0 \in X_0$ holds $x(\vartheta | x_0, u_{x_0}(\cdot)) \in M$. **Package guidance problem** is solvable, if a guiding program package exists.

Theorem 1 ([1]) *The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.*

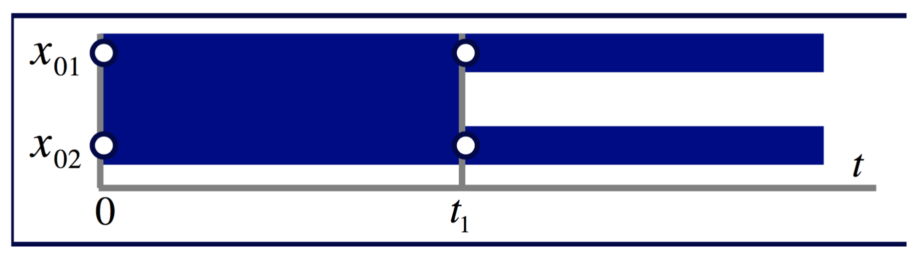
For an arbitrary homogeneous signal $g(\cdot)$ let

$$G_0(g(\cdot)) = \left\{\tilde{g}(\cdot) \in G : \lim_{\zeta \to +0} (\tilde{g}(t_0 + \zeta) - g(t_0 + \zeta)) = 0\right\}$$

be the set of **initially compatible** homogeneous signals and let

$$\tau_1(g(\cdot)) = \max\left\{\tau \in [t_0, \vartheta] : \max_{\tilde{g}(\cdot) \in G_0(g(\cdot))} \max_{t \in [t_0, \tau]} |\tilde{g}(t) - g(t)| = 0\right\}$$

be its **first splitting moment**.



For each $i = 1, 2, \ldots$ let

$$G_i(g(\cdot)) = \left\{\tilde{g}(\cdot) \in G_{i-1}(g(\cdot)) : \lim_{\zeta \to +0} (\tilde{g}(\tau_i(g(\cdot)) + \zeta) - g(\tau_i(g(\cdot)) + \zeta)) = 0\right\}$$

be the set of all homogeneous signals from $G_{i-1}(g(\cdot))$ equal to $g(\cdot)$ in the right-sided neighbourhood of the time-point $\tau_i(g(\cdot))$ and let

$$\tau_{i+1}(g(\cdot)) = \max\left\{\tau \in (\tau_i(g(\cdot)), \vartheta] : \max_{\tilde{g}(\cdot) \in G_i(g(\cdot))} \max_{t \in [\tau_i(g(\cdot)), \tau]} |\tilde{g}(t) - g(t)| = 0\right\}$$

be the $(i+1)$**-th splitting moment** of the homogeneous signal $g(\cdot)$.
Let

$$T(g(\cdot)) = \{\tau_j(g(\cdot)) : j = 1, \ldots, k_{g(\cdot)}\}$$

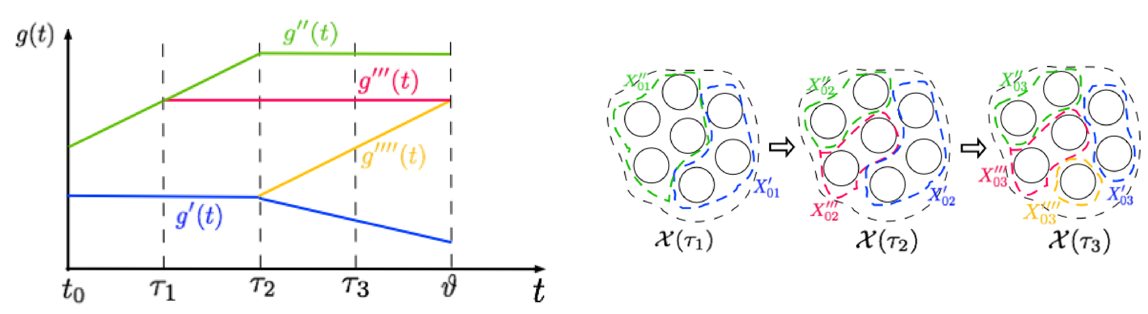
be the set of all splitting moments of the homogeneous signal $g(\cdot)$ and let

$$T = \bigcup_{g(\cdot) \in G} T(g(\cdot))$$

be the set of all splitting moments of all homogeneous signals. T is finite and $|T| \le |X_0|$. Let us represent this set as $T = \{\tau_1, \ldots, \tau_K\}$, where $t_0 < \tau_1 < \ldots < \tau_K = \vartheta$.
For every $k = 1, \ldots, K$ let the set

$$\mathcal{X}_0(\tau_k) = \{X_0(\tau_k | g(\cdot)) : g(\cdot) \in G\}$$

be the **cluster position** at the time-point τ_k, and let each its element $X_{0j}(\tau_k)$, $j = 1, \ldots, J(\tau_k)$ be a **cluster of initial states** at this time-point; $J(\tau_k)$ is the number of clusters in the cluster position $\mathcal{X}_0(\tau_k)$, $k = 1, \ldots, K$.



Let us denote $D(t) = B^{\mathrm{T}}(t)F^{\mathrm{T}}(\vartheta, t) \quad (t \in [t_0, \vartheta])$ and set the function $p(\cdot, \cdot) : \mathbb{R}^n \times X_0 \mapsto \mathbb{R}$:

$$p(l, x_0) = \langle l, F(\vartheta, t_0)x_0\rangle_{\mathbb{R}^n} + \left\langle l, \int_{t_0}^{\vartheta} F(\vartheta, t)c(t)dt \right\rangle_{\mathbb{R}^n} \qquad (l \in \mathbb{R}^n,\ x_0 \in X_0).$$

Let us set

$$\gamma((l_{x_0})_{x_0 \in X_0}) = \rho^-\left((l_{x_0})_{x_0 \in X_0} | \mathcal{A}\right) - \rho^+\left((l_{x_0})_{x_0 \in X_0} | \mathcal{M}\right) =$$
$$= \sum_{x_0 \in X_0} p(l_{x_0}, x_0) - \sum_{x_0 \in X_0} \rho^+(l_{x_0} | M) +$$
$$+ \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k)} \rho^-\left(\sum_{x_0 \in X_{0j}(\tau_k)} D(t)l_{x_0} \Big| P\right) dt.$$

Let $\mathcal{L}$ be a compact set in $\mathcal{R}^n$, containing an image of the unit sphere $\mathcal{S}^n$ — for some positive r_1 and $r_2 \ge r_1$ for each $\ell \in \mathcal{S}^n$ there is $r \in [r_1, r_2]$, for which $r\ell \in \mathcal{L}$.

Theorem 2 (Solvability criterion [2]) *Both problems – the package guidance problem and he guaranteed positional guidance problem – are solvable if and only if*

$$\max_{(l_{x_0})_{x_0 \in X_0} \in \mathcal{L}} \gamma((l_{x_0})_{x_0 \in X_0}) \le 0. \quad (2)$$

3. Ill-posed problem example

Let us consider an arbitrary linear dynamic system (1) and the observed signal

$$y(t) = Q(t)x(t), t \in [t_0, \vartheta], \text{ where } Q(t) = \begin{pmatrix} 0 & 1 \end{pmatrix}, t \in [t_0, \vartheta].$$

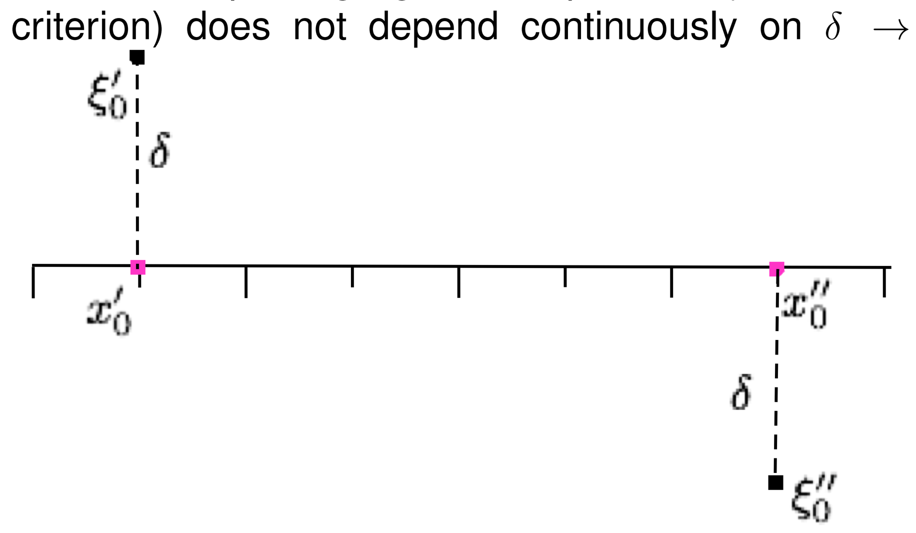
Let us assume that the program package guidance problem is solvable for the set

$$X_0 = \{x_0', x_0''\}, \text{ where } x_0' = \begin{pmatrix} x_{01}' \\ \delta \end{pmatrix} \text{ and } x_0'' = \begin{pmatrix} x_{01}'' \\ -\delta \end{pmatrix}, \delta > 0.$$

It is clear that uniform signals

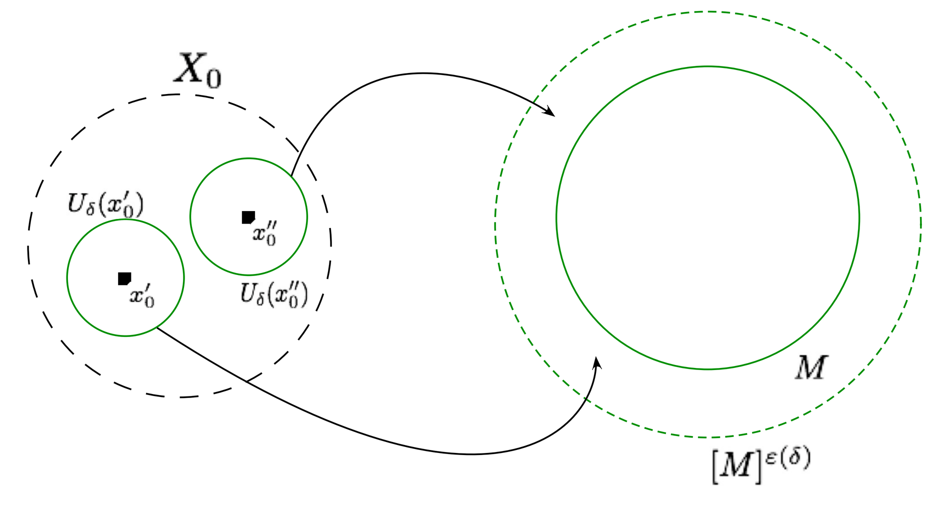
$$g_{x_0'}(t) = \delta, t \in [t_0, \vartheta] \text{ and } g_{x_0''} = -\delta, t \in [t_0, \vartheta]$$

are not initially compatible for any $\delta > 0$, thus, the initial states x_0' and x_0'' belong to the different clusters of the cluster position $\mathcal{X}_0(t_0)$. But if $\delta = 0$, then $g_{x_0'} = g_{x_0''} = 0, t \in [t_0, \vartheta]$ and it is impossible to distinguish the initial states x_0' and x_0'', so the solution of the package guidance problem (i.e. the solvability criterion) does not depend continuously on $\delta \to 0$.



4. Possible solution

One of the possible regularisation methods is foreseen a guidance of the whole δ-vicinity of any initial state $x_0 \in X_0$ for a relatively small $\delta > 0$. The obtained program package is to depend on δ, thus, not corresponding to the original program package, but, taking into account the approximate nature of the initial (positional guidance) problem statement, the resulting positional strategy can be constructed using the methods suggested in [4], satisfying the requirement of ε-guidance.



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