# A method for calculation of program package elements for singular clusters

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*"The problem of constructing optimal closed-loop control strategies under uncertainty is one of the key problems of the mathematical control theory. Its solution would give a new impetus to the theory's development and create the foundation for its new applications"* Arkady Kryazhimskiy (2013)

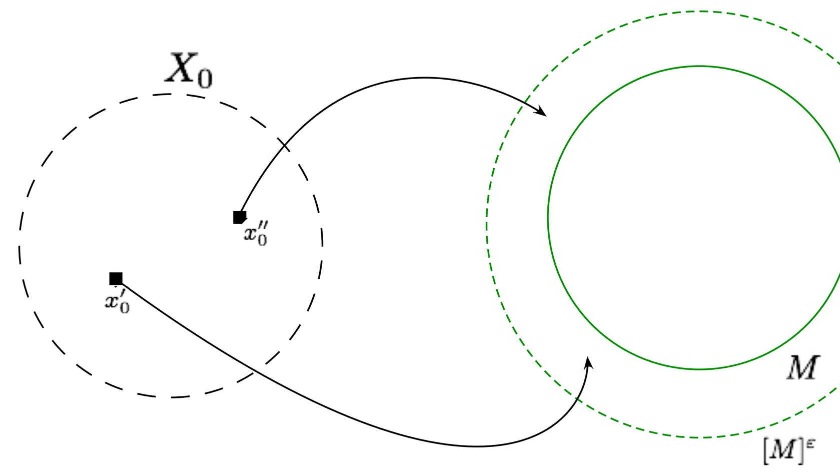
## 1. Problem statement

LET US CONSIDER A LINEAR DYNAMIC CONTROLLED SYSTEM, described by an ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t_0 \le t \le \vartheta \quad (1)$$

Open-loop control (program) $u(\cdot)$ is a measurable function on $[t_0, \vartheta], u(t) \in P \subset \mathbb{R}^r$, $P$ is a convex compact set. The initial state of the system maybe not known a priori: $x(t_0) = x_0 \in X_0 \subset \mathbb{R}^n$, where $X_0$ is a finite known set. The terminal condition $x(\vartheta) \in M \subset \mathbb{R}^n$, where $M$ is a closed and convex set, should hold.

A linear signal $y(t) = Q(t)x(t)$, where $Q(\cdot)$ is left piecewise continuous matrix-function, $Q(t) \in \mathbb{R}^{q \times n}, t \in [t_0, \vartheta]$, is observed by the controlling side.




The problem of positional guidance is formulated as follows - based on the given arbitrary $\varepsilon > 0$ choose a closed-loop control strategy with memory, **whatever the system's initial state** $x_0$ **from the set** $X_0$, the system's motion $x(\cdot)$ corresponding to the chosen closed-loop strategy and starting at the time $t_0$ from the state $x_0$ reaches the state $x(\vartheta)$ belonging to the $\varepsilon$-neighbourhood of the target set M at the time $\vartheta$.

## 2. Program packages method

**Homogeneous system**, corresponding to (1)

$$\dot{x}(t) = A(t)x(t)$$

For each $x_0 \in X_0$ its solution is given by the Cauchy formula:

$x(t) = F(t, t_0)x_0;\ F(t, s)\ (t, s \in [t_0, \vartheta])$ is the fundamental matrix.

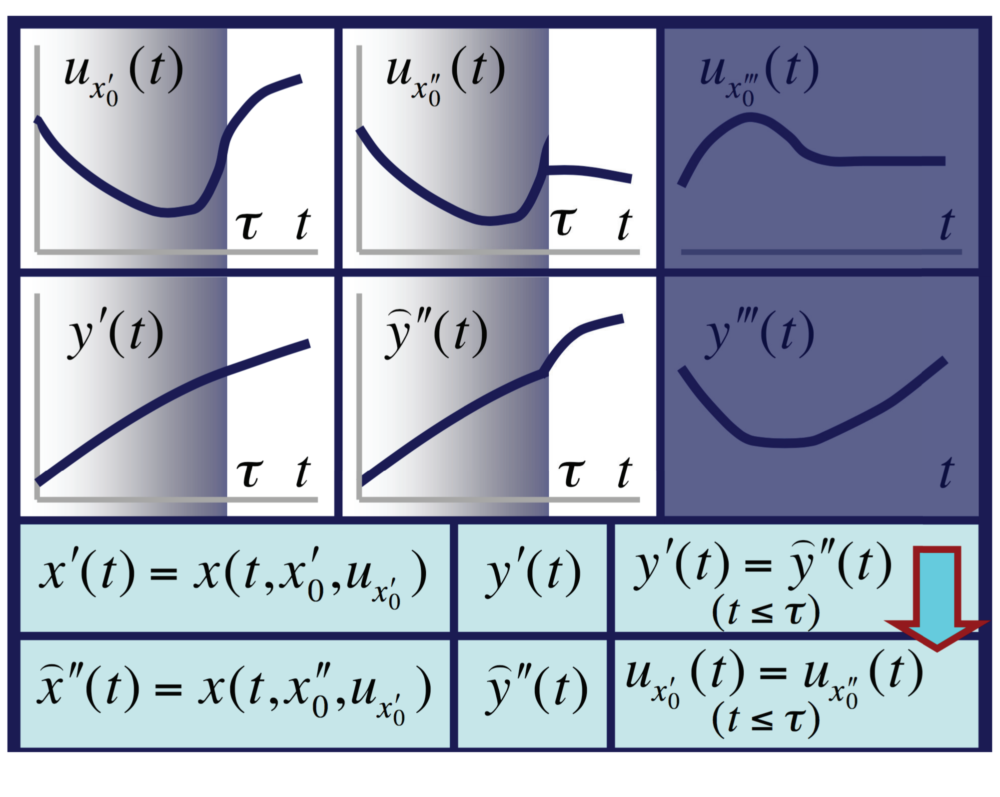
**Homogeneous signal**, corresponding to an admissible initial state $x_0 \in X_0$:

$$g_{x_0}(t) = Q(t)F(t, t_0)x_0\ (t \in [t_0, \vartheta],\ x_0 \in X_0).$$

Let $G = \{g_{x_0}(\cdot) | x_0 \in X_0\}$ be the set of all homogeneous signals and let $X_0(\tau|g(\cdot))$ be the set of all admissible initial states $x_0 \in X_0$, corresponding to the homogeneous signal $g(\cdot) \in G$ *till time point* $\tau \in [t_0, \vartheta]$:

$$X_0(\tau|g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0,\tau]} = g_{x_0}(\cdot)|_{[t_0,\tau]}\}.$$

**Program package** is an open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$, satisfying **non-anticipatory condition**: for any homogeneous signal $g(\cdot)$, any time $\tau \in (t_0, \vartheta]$ and any admissible initial states $x_0', x_0'' \in X_0(\tau|g(\cdot))$ the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for almost all $t \in [t_0, \tau]$.




Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for all $x_0 \in X_0$ holds $x(\vartheta|x_0, u_{x_0}(\cdot)) \in M$. **Package guidance problem** is solvable, if a guiding program package exists.

**Theorem 1** ([1]). *The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.*

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## 3. Algorithm for singular clusters

An algorithm for the package guidance problem solution is proposed in paper [4], however, in some cases (so called *singular clusters* of the initial states set) it may not be applied, because the minimum condition degenerates. In this work an additional algorithm for tackling such cases is presented.

At a step $i = 0, 1, \ldots$ of the algorithm for arbitrary cluster $X_{0r}(\tau_k) \in \mathcal{X}_0(\tau_k), r = 1, \ldots, R(\tau_k), k = 1, \ldots, K$ a non-empty set $w_0$ of indices is estimated $j, 1 \le j \le m$ such that to $j \in w_0$ corresponds non-empty set $\sigma_j \subset [\tau_{k-1}, \tau_k]$, such that

$$\sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l^{*(i)}_{x_{0q}} \equiv 0 \quad (t \in \sigma_j). \quad (2)$$

The problem of estimation of the components $(u^*_{X_{0r}(\tau_k)}(t))_j$ of the guiding program package elements on the sets $\sigma_j, j \in w_0$ reduces to the following system of equations

$$\begin{cases} \sum\limits_{j \in w_0} \int\limits_{\sigma_j} e_j(t) u_j(t) dt = v_{x_0}, \quad |u_j(t)| \le a_* p, \\ x_0 \in X_{0r}(\tau_k) \end{cases} \quad (3)$$

where

$$v_{x_0} = z_{x_0} - F(\vartheta, t_0)x_0 + a_* p \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{j=1}^{r} e_j(t) \operatorname{sign} \left( \sum_{\substack{\bar{x}_0 \in X_{0r}(\tau_k) \\ r: x_0 \in X_{0r}(\tau_k)}} \sum_{q=1}^{n} D_{jq}(t) l^*_{x_{0q}} \right) d\tau,$$

$$z_{x_0} \in \operatorname{Arg}\max_{z \in M} \langle z, l^*_{x_0} \rangle_{\mathbb{R}^n},$$

$$e(t) = \begin{pmatrix} D(t)_{j1} \\ \ldots \\ D(t)_{jn} \end{pmatrix}, j = 1, \ldots, m.$$

Let us introduce functions $\lambda(\cdot,\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \times [\tau_{k-1}, \tau_k] \times \mathbb{N} \mapsto \mathbb{R}$ $\Lambda(\cdot,\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^n \times [0,1] \times [\tau_{k-1}, \tau_k] \times \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$:

$$\lambda(z_{x_0}, l_{x_0}, \mu, \sigma, w) = \sum_{x_0 \in X_{0r}(\tau_k)} \langle l_{x_0}, z_{x_0} \rangle_{\mathbb{R}^n} - \mu \sum_{j \in w} \int_{\sigma^j} \left| \sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}} \right| dt,$$

$$\Lambda(z_{x_0}, \mu, \sigma, w, h) = \max_{\substack{\sum\limits_{x_0 \in X_{0r}(\tau_k)} \|l_{x_0}\|_{\mathbb{R}^n} = 1 \\ l_{x_0} \in \mathbb{R}^h}} \lambda(z_{x_0}, l_{x_0}, \mu, \sigma, w).$$

The input of the algorithm is

$$\mu^{(0)} = 1, \sigma_j^{(0)} = \sigma_j, w^{(0)} = w_0, z_{x_0}^{(0)} = v_{x_0} \quad (j \in w_0).$$

After the step $h - 1, h = 1, 2, \ldots)$ of the algorithm the following is known

$$\mu^{(h-1)}, w^{(h-1)}, \sigma_j^{(h-1)}, z_{x_0}^{(h-1)}, l^{*(1)}_{x_0}, \ldots, l^{*(h-1)}_{x_0}, (u^*_{X_{0r}(\tau_k)}(t))_j\ (t \in \sigma_j / \sigma_j^{(h-1)}, j \in w_0).$$

Let us describe the step $h, h = 1, 2, \ldots$ of the algorithm:

1. $\mu^{(h)}$ $(0 \le \mu^{(h)} \le \mu^{(h-1)})$ is calculated as the smallest root of the equation
$$\Lambda\left(z_{x_0}^{(h-1)}, \mu, \sigma^{(h-1)}, w^{(h-1)}, n - h + 1\right) = 0.$$
2. If $\mu^{(h)} = 0$, then $(u^*_{X_{0r}(\tau_k)}(t))_j = 0, t \in \sigma_j^{(h-1)}, j \in w^{(h-1)}$, and the required element of the guiding program package is found. Otherwise the algorithm proceeds to the next step immediately.
3. The vector family $(l^{*(h)}_{x_0})_{x_0 \in X_0}$ is estimated such, that
$$\sum_{x_0 \in X_{0r}(\tau_k)} \left\langle l^{*(h)}_{x_0}, l^{*(d)}_{x_0} \right\rangle = 0$$
for all $d = 1, \ldots, h - 1$ and
$$\begin{aligned} &\Lambda\left(z_{x_0}^{(h-1)}, \mu^{(h)}, \sigma^{(h-1)}, w^{(h-1)}, n - h + 1\right) = \\ &= \max_{\substack{\sum\limits_{x_0 \in X_{0r}(\tau_k)} \|l_{x_0}\| = 1 \\ l_{x_0} \in \mathbb{R}^{n-h+1}}} \lambda\left(z^{(h-1)}, l^{(h)}_{x_0}, \mu^{(h)}, \sigma_j^{(h-1)}, w^{(h-1)}\right) = \\ &= \sum_{x_0 \in X_{0r}(\tau_k)} \left\langle l^{*(h)}_{x_0}, z^{(h-1)}_{x_0} \right\rangle - \mu^{(h)} \sum_{j \in w^{(h-1)}} \int_{\sigma_j^{(h-1)}} \left| \sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l^{*(h)}_{x_{0q}} \right| = \\ &= 0. \end{aligned} \quad (4)$$
4. The set of indices $w^{(h)} \subset w^{(h-1)}$ is estimated and for each $j \in w^{(h)}$ a set $\sigma_j^{(h)} \subset \sigma_k^{(h-1)}, k \in w^{(h-1)}$ is found such that
$$\sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l^{*(h)}_{x_{0_{q1}}} \equiv 0 \quad (j \in w^{(h)}, t \in \sigma_j^{(h)}).$$
5. On the sets $\sigma_k^{(h-1)} / \sigma_j^{(h)}, k \in w^{(h-1)}, j \in w^{(h)}$ the components $(u^*_{X_{0r}(\tau_k)}(t))_j$ of the guiding program package elements are
$$(u^*_{X_{0r}(\tau_k)}(t))_j = a_* p \operatorname{sign}\left( \sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l^{*(h)}_{x_{0q}} \right).$$
6. If $\operatorname{mes}\sigma^{(h)} = \operatorname{mes} \bigcup\limits_{j \in w^{(h)}} \sigma_j^{(h)} = 0$, then $(u^*_{X_{0r}(\tau_k)}(t))_j$ is known on the whole set $\sigma = \bigcup \sigma_j$ and the problem is solved. Otherwise the algorithm proceeds to the next step
7. The following approximation is calculated
$$z^{(h)} = z^{(h-1)} - a_* p \mu^{(h)} \sum_{j \in w^{(h-1)}} \int_{\sigma_j^{(h-1)}} e_j(t) \operatorname{sign} \left( \sum_{\substack{\bar{x}_0 \in X_{0r}(\tau_k) \\ r: x_0 \in X_{0r}(\tau_k)}} \sum_{q=1}^{n} D_{jq}(t) l^{*(h)}_{x_{0q}} \right) dt;$$
and the algorithm proceeds to the step $h + 1$.

After $n \times m$ steps, in a generic case, $m$-dimensional control in $n$-dimensional problem is calculated on the whole half-interval $[\tau_{k-1}, \tau_k)$. Convergence of the algorithm follows from convergence of the corresponding algorithm in Euclidean space [3].

## 4. Example

Let us consider a linear control system

$$\begin{cases} \dot{x}_1 = x_2, x_1(0) = x_{01} \\ \dot{x}_2 = u, x_2(0) = x_{02} \end{cases} \quad (5)$$

on the time segment $[0, 2]$.

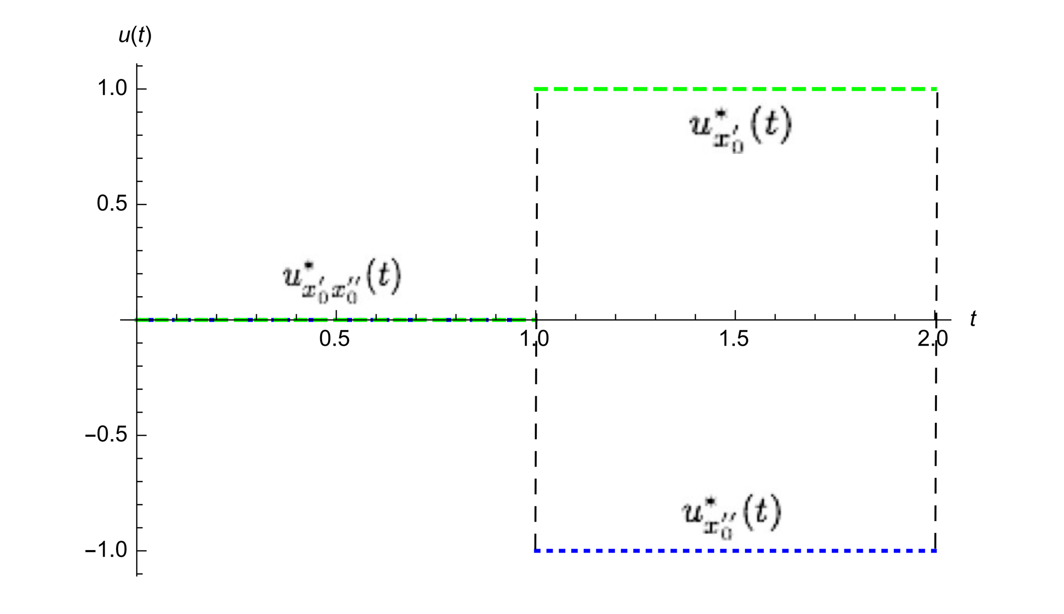
$$M = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 : |x_1| \le 1, x_2 \in \mathbb{R} \right\}; X_0 = \left\{ \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} \right\}$$

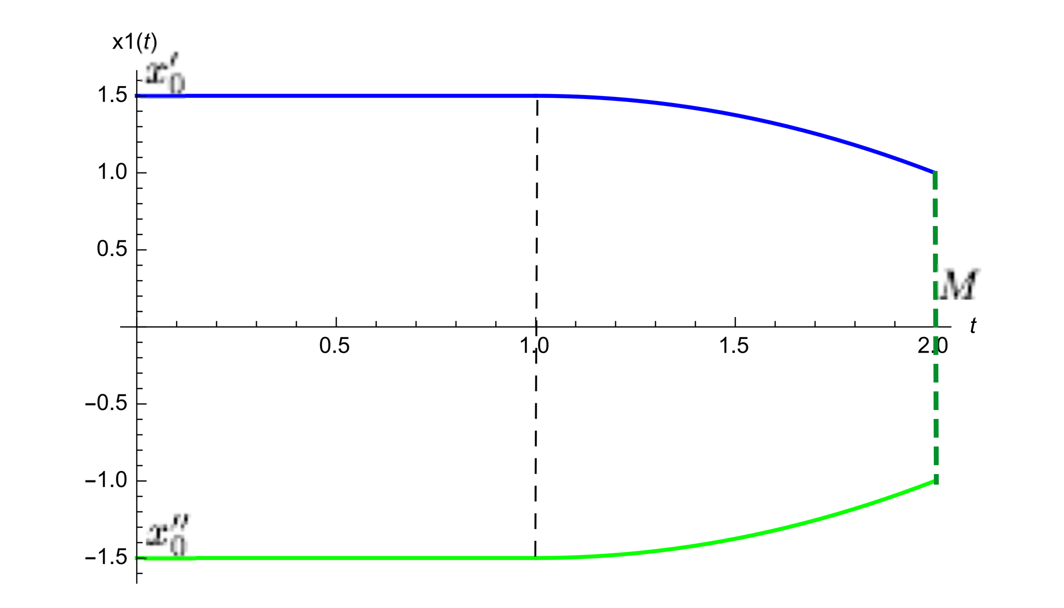
$$Q(t) = \begin{cases} (0, 0), t \in [0, 1] \\ (1, 0), t \in (1, 2] \end{cases} \quad u(t) \in P = \{u : |u| \le 1\},\ t \in [0, 2].$$

Since the solvability criterion holds, the package guidance problem is solvable. Using the algorithm from [4] the guiding program package elements for $x_0'$ and $x_0''$ can be calculated. However, for the cluster $X_0 = \{x_0', x_0''\}$ on the time interval $[0, 1]$ there is an ambiguity in the minimum condition and the standard algorithm is not applicable. The presented method allows to reduce the problem to the system of two integral equations:

$$\begin{cases} x_1(2|x_0') = \frac{3}{2} + \frac{1}{2} u_{X_0}(t) + \int\limits_1^2 \left(u_{X_0}(t) - t + 1\right) dt = 1 \\ x_1(2|x_0'') = -\frac{3}{2} + \frac{1}{2} u_{X_0}(t) + \int\limits_1^2 \left(u_{X_0}(t) + t - 1\right) dt = -1 \end{cases}$$

Its solution gives that $u_{X_0}(t) \equiv 0,\ t \in [0, 1]$.

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