Control set smoothing method for program package elements calculation

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Problems with incomplete information - an approach by Yu. S. Osipov, A. V. Kryazhimskiy

«The problem of constructing optimal closed-loop control strategies under uncertainty is one of the key problems of the mathematical control theory. Its solution would give a new impetus to the theory's development and create the foundation for its new applications.» Arkady Kryazhimskiy (2013)

- Yu. S. Osipov. *Control Packages: An Approach to Solution of Positional Control Problems with Incomplete Information.* Usp. Mat. Nauk 61:4 (2006), 25–76.
- A. V. Kryazhimskiy, Yu. S. Osipov. *Idealized Program Packages and Problems of Positional Control with Incomplete Information*. Trudy IMM UrO RAN 15:3 (2009), 139–157.
- A. V. Kryazhimskiy, Yu. S. Osipov. *On the solvability of problems of guaranteeing control for partially observable linear dynamical systems*. Proc. Steklov Inst. Math., 277 (2012), 144–159

Guaranteed positional guidance problem at the (pre-defined) time

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t_0 \leq t \leq \vartheta \quad (1)$$

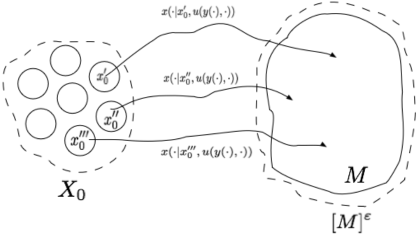
Open-loop control (program) $u(\cdot)$ is measurable.

$u(t) \in P \subset \mathbb{R}^r$, P is a convex compact set

$x(t_0) = x_0 \in X_0 \subset \mathbb{R}^n$, X_0 is a **finite** set

$x(\vartheta) \in M \subset \mathbb{R}^n$, M is a **closed and convex** set

Observed signal $y(t) = Q(t)x(t)$, $Q(\cdot) \in \mathbb{R}^{q \times n}$ is left piecewise continuous



Problem statement

Based on the given arbitrary $\varepsilon > 0$ choose a closed-loop control strategy with memory, **whatever the system's initial state x_0 from the set X_0**, *the system's motion $x(\cdot)$ corresponding to the chosen closed-loop strategy and starting at the time t_0 from the state x_0 reaches the state $x(\vartheta)$ belonging to the ε-neighbourhood of the target set M at the time ϑ.*

Homogeneous signals

Homogeneous system, corresponding to (1)

$$\dot{x}(t) = A(t)x(t)$$

For each $x_0 \in X_0$ its solution is given by the Cauchy formula:

$$x(t) = F(t, t_0)x_0;\ F(t, s)\ (t, s \in [t_0, \vartheta]) \text{ is the fundamental matrix.}$$

Homogeneous signal, corresponding to an admissible initial state $x_0 \in X_0$:

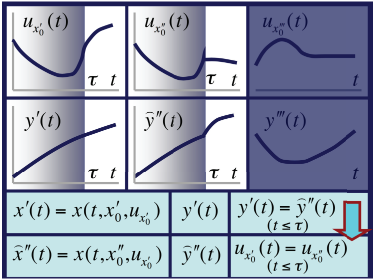
$$g_{x_0}(t) = Q(t)F(t, t_0)x_0\ (t \in [t_0, \vartheta],\ x_0 \in X_0).$$

Let $G = \{g_{x_0}(\cdot) | x_0 \in X_0\}$ be the set of all homogeneous signals and let $X_0(\tau | g(\cdot))$ be the set of all admissible initial states $x_0 \in X_0$, corresponding to the homogeneous signal $g(\cdot) \in G$ *till time point* $\tau \in [t_0, \vartheta]$:

$$X_0(\tau | g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0, \tau]} = g_{x_0}(\cdot)|_{[t_0, \tau]}\}.$$

Package guidance problem

Program package is an open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$, satisfying **non-anticipatory condition**: for any homogeneous signal $g(\cdot)$, any time $\tau \in (t_0, \vartheta]$ and any admissible initial states $x_0', x_0'' \in X_0(\tau | g(\cdot))$ the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for almost all $t \in [t_0, \tau]$.



Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for all $x_0 \in X_0$ holds $x(\vartheta | x_0, u_{x_0}(\cdot)) \in M$.
Package guidance problem is solvable, if a guiding program package exists.

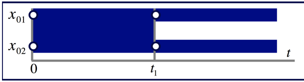
Theorem 1 (Osipov, Kryazhimskiy, 2006)

The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.

Homogeneous signals splitting

For an arbitrary homogeneous signal $g(\cdot)$ let

$$G_0(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G : \lim_{\zeta \to +0} (\tilde{g}(t_0 + \zeta) - g(t_0 + \zeta)) = 0 \right\}$$



be the set of **initially compatible** homogeneous signals and let

$$\tau_1(g(\cdot)) = \max \left\{ \tau \in [t_0, \vartheta] : \max_{\tilde{g}(\cdot) \in G_0(g(\cdot))} \max_{t \in [t_0, \tau]} |\tilde{g}(t) - g(t)| = 0 \right\}$$

be its **first splitting moment**.

For each $i = 1, 2, \ldots$ let

$$G_i(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G_{i-1}(g(\cdot)) : \lim_{\zeta \to +0} (\tilde{g}(\tau_i(g(\cdot)) + \zeta) - g(\tau_i(g(\cdot)) + \zeta)) = 0 \right\}$$

be the set of all homogeneous signals from $G_{i-1}(g(\cdot))$ equal to $g(\cdot)$ in the right-sided neighbourhood of the time-point $\tau_i(g(\cdot))$ and let

$$\tau_{i+1}(g(\cdot)) = \max \left\{ \tau \in (\tau_i(g(\cdot)), \vartheta] : \max_{\tilde{g}(\cdot) \in G_i(g(\cdot))} \max_{t \in [\tau_i(g(\cdot)), \tau]} |\tilde{g}(t) - g(t)| = 0 \right\}$$

be the $(i+1)$**-th splitting moment** of the homogeneous signal $g(\cdot)$.

Initial states set clustering

Let

$$T(g(\cdot)) = \{\tau_j(g(\cdot)) : j = 1, \ldots, k_{g(\cdot)}\}$$

be the set of all splitting moments of the homogeneous signal $g(\cdot)$ and let

$$T = \bigcup_{g(\cdot)\in G} T(g(\cdot))$$

be the set of all splitting moments of all homogeneous signals. T is finite and $|T| \leq |X_0|$. Let us represent this set as $T = \{\tau_1, \ldots, \tau_K\}$, where $t_0 < \tau_1 < \ldots < \tau_K = \vartheta$.
For every $k = 1, \ldots, K$ let the set

$$\mathcal{X}_0(\tau_k) = \{X_0(\tau_k | g(\cdot)) : g(\cdot) \in G\}$$

be the **cluster position** at the time-point τ_k, and let each its element $X_{0j}(\tau_k)$, $j = 1, \ldots, J(\tau_k)$ be a **cluster of initial states** at this time-point; $J(\tau_k)$ is the number of clusters in the cluster position $\mathcal{X}_0(\tau_k)$, $k = 1, \ldots, K$.

Extended space

Let $\mathcal{R}^h$ ($h = 1, 2, \ldots$) be a finite-dimensional Euclidean space of all families $(r_{x_0})_{x_0 \in X_0}$ from $\mathbb{R}^h$ with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{R}^h}$ defined as

$$\langle r', r'' \rangle_{\mathcal{R}^h} = \langle (r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} = \sum_{x_0 \in X_0} \langle r'_{x_0}, r''_{x_0} \rangle_{\mathbb{R}^h} \quad ((r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h).$$

For each non-empty set $\mathcal{E} \subset \mathcal{R}^h$ ($h = 1, 2, \ldots$) let us define its *lower* $\rho^-(\cdot|\mathcal{E}) : \mathcal{R}^h \mapsto \mathbb{R}$ and *upper* support functions $\rho^+(\cdot|\mathcal{E}) : \mathcal{R}^h \mapsto \mathbb{R}$:

$$\rho^-((l_{x_0})_{x_0 \in X_0}|\mathcal{E}) = \inf_{(e_{x_0})_{x_0 \in X_0} \in \mathcal{E}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} \quad ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h),$$

$$\rho^+((l_{x_0})_{x_0 \in X_0}|\mathcal{E}) = \sup_{(e_{x_0})_{x_0 \in X_0} \in \mathcal{E}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} \quad ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h)$$

Extended open-loop control control

Let $\mathcal{P} \subset \mathcal{R}^m$ be the set of all families $(u_{x_0})_{x_0 \in X_0}$ of vectors from P.
Extended open-loop control control is a measurable function $t \mapsto (u_{x_0}(t))_{x_0 \in X_0} : [t_0, \vartheta] \mapsto \mathcal{P}$.
Let us identify arbitrary programs family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ and an extended open-loop control $t \mapsto (u_{x_0}(t))_{x_0 \in X_0}$.

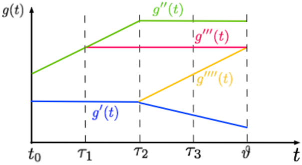
For each $k = 1, \ldots, K$ let $\mathcal{P}_k$ be an **extended admissible control set** on $(\tau_{k-1}, \tau_k]$ in case $k > 1$ and on $[t_0, \tau_1]$ in case $k = 1$ as a set of all vector families $(u_{x_0})_{x_0 \in X_0} \in \mathcal{P}$ such that, for each cluster $X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k), j = 1, \ldots, J(\tau_k)$ and any $x_0', x_0'' \in X_{0j}(\tau_k)$ holds $u_{x_0'} = u_{x_0''}$.

Extended open-loop control control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **admissible**, if for each $k = 1, \ldots, K$ holds $(u_{x_0}(t))_{x_0 \in X_0} \in \mathcal{P}_k$ for almost all $t \in (\tau_{k-1}, \tau_k]$ in case $k > 1$ and for almost all $t \in [t_0, \tau_1]$ in case $k = 1$;

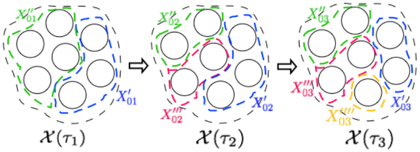
Lemma 2 (Kryazhimskiy (2013))

Extended open-loop control control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a control package if and only if it is admissible.

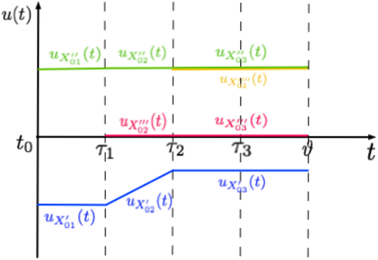
Homogeneous signals, cluster positions and extended open-loop control controls



Homogeneous signals splitting



Initial states set clustering



Extended open-loop control control

Extended problem of program guidance

Extended system (in the space $\mathcal{R}^n$):

$$\begin{cases} \dot{x}_{x_0}(t) = A(t)x_{x_0}(t) + B(t)u_{x_0}(t) + c(t) \\ x_{x_0}(t_0) = x_0 \end{cases}$$

$$(x_0 \in X_0)$$

Extended target set $\mathcal{M}$ is the set of all families $(x_{x_0})_{x_0 \in X_0} \in \mathcal{R}^n$ such, that $x_{x_0} \in M$ for all $x_0 \in X_0$.

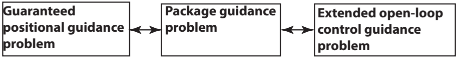
An admissible extended open-loop control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding the extended system**, if $(x(\vartheta | x_0, u_{x_0}(\cdot)))_{x_0 \in X_0} \in \mathcal{M}$.

The **extended problem of open-loop guidance** is solvable, if there exists an admissible extended open-loop control which is guiding the extended system.

Attainability set of the extended system at the time ϑ:
$\mathcal{A} = \{(x(\vartheta | x_0, u_{x_0}(\cdot)))_{x_0 \in X_0} : (u_{x_0}(\cdot))_{x_0 \in X_0} \in \mathcal{U}_{ext}\}$, where $\mathcal{U}_{ext}$ is the set of all admissible extended open-loop control controls.

Theorem 3 (Kryazhimskiy, Strelkovskii (2014))

1) The package guidance problem is solvable if and only if the extended problem of open-loop guidance is solvable. 2) An admissible extended open-loop control is a guiding program package if and only if it is guiding extended system.



Let us denote $D(t) = B^{\mathrm{T}}(t)F^{\mathrm{T}}(\vartheta, t) \quad (t \in [t_0, \vartheta])$ and set the function $p(\cdot,\cdot) : \mathbb{R}^n \times X_0 \mapsto \mathbb{R}$:

$$p(l, x_0) = \langle l, F(\vartheta, t_0)x_0\rangle_{\mathbb{R}^n} + \left\langle l, \int_{t_0}^{\vartheta} F(\vartheta, t)c(t)dt \right\rangle_{\mathbb{R}^n} \qquad (l \in \mathbb{R}^n,\ x_0 \in X_0).$$

Let us set

$$\gamma((l_{x_0})_{x_0\in X_0}) = \rho^-\left((l_{x_0})_{x_0\in X_0}|\mathcal{A}\right) - \rho^+\left((l_{x_0})_{x_0\in X_0}|\mathcal{M}\right) =$$

$$= \sum_{x_0\in X_0} p(l_{x_0}, x_0) - \sum_{x_0\in X_0} \rho^+(l_{x_0}|M) + \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0j}(\tau_k)\in\mathcal{X}_0(\tau_k)} \rho^-\left(\sum_{x_0\in X_{0j}(\tau_k)} D(t)l_{x_0} \Big| P\right) dt.$$

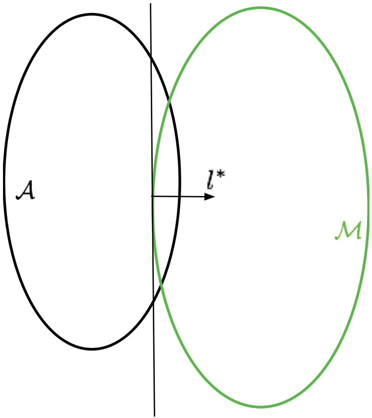
Solvability criterion

Let $\mathcal{L}$ be a compact set in $\mathcal{R}^n$, containing an image of the unit sphere $\mathcal{S}^n$ — for some positive r_1 and $r_2 \geq r_1$ for each $l \in \mathcal{S}^n$ there is $r \in [r_1, r_2]$, for which $rl \in \mathcal{L}$.

Theorem 4 (Kryazhimskiy, Strelkovskii (2014))

Each of the three problems – (i) the extended open-loop control guidance problem, (ii) the package guidance problem and (iii) the guaranteed positional guidance problem – is solvable if and only if

$$\max_{(l_{x_0})_{x_0 \in X_0} \in \mathcal{L}} \gamma((l_{x_0})_{x_0 \in X_0}) \leq 0. \qquad (2)$$



Construction of the guiding program package

Assuming that the solvability criterion (2) is satisfied, let us introduce the function $\hat{\gamma}(\cdot,\cdot) : \mathcal{R}^n \times [0,1] \mapsto \mathbb{R}$:

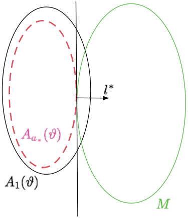
$$\hat{\gamma}((l_{x_0})_{x_0\in X_0}, a) = \sum_{x_0\in X_0} \langle l_{x_0}, F(\vartheta, t_0)x_0\rangle_{\mathbb{R}^n} + \left\langle l_{x_0}, \int_{t_0}^{\vartheta} F(\vartheta, t)c(t)dt \right\rangle_{\mathbb{R}^n} - \sum_{x_0\in X_0} \rho^+(l_{x_0}|M) - \\ - \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0j}(\tau_k)\in\mathcal{X}_0(\tau_k)} \rho^- \left(\sum_{x_0\in X_{0j}(\tau_k)} D(t)l_{x_0} \Big| aP \right) dt. \quad (3)$$

Program package $(u^0_{x_0}(\cdot))_{x_0\in X_0}$ is **zero-valued**, if $u^0_{x_0}(t) = 0$ for almost all $t \in [t_0, \vartheta], x_0 \in X_0$.

Lemma 5 (Kryazhimskiy (2014))

If the solvability criterion (2) holds and zero-valued program package is not guiding the extended system, then exists $\mathbf{a}_* \in (0,1]$ *such, that*

$$\max_{(l_{x_0})_{x_0\in X_0}\in\mathcal{L}} \hat{\gamma}((l_{x_0})_{x_0\in X_0}, \mathbf{a}_*) = 0. \quad (4)$$



Construction of the guiding program package

For each program package $(u_{x_0}(\cdot))_{x_0\in X_0}$, arbitrary cluster $X_{0j}(\tau_k)\in\mathcal{X}(\tau_k)$, $j=1,\ldots,J(\tau_k)$, $k=1,\ldots,K$ and arbitrary $t\in[\tau_{k-1},\tau_k)$ let us denote $u_{X_{0j}(\tau_k)}(t)$ program values $u_{x_0}(t)$, which are equal for all $x_0\in X_{0j}(\tau_k)$.
Let $(\mathbf{l}^*_{\mathbf{x_0}})_{\mathbf{x_0}\in\mathbf{X_0}}$ be the maximizer of the left handside of (4). Cluster $X_{0j}(\tau_k)$ is **regular**, if

$$\sum_{x_0\in X_{0j}(\tau_k)} D(t)\mathbf{l}^*_{\mathbf{x_0}}\neq 0,\ \ t\in[\tau_{k-1},\tau_k).$$

Otherwise the cluster is **singular**.

Theorem 6 (Kryazhimskiy (2014))

Let P be a strcitly convex compact set, containing the zero vector; condition (4) holds and the program package $(u^*_{x_0}(\cdot))_{x_0\in X_0}$ satisfies the condition $u^*_{x_0}(t)\in \mathbf{a}_*P\quad (x_0\in X_0,\ t\in[t_0,\vartheta])$. Let the clusters $X_{0j}(\tau_k)\in\mathcal{X}_0(\tau_k)$, $k=1,\ldots,K$, $j=1,\ldots,J(\tau_k)$ be regular, and for each of them the following equality holds

$$\left\langle D(t)\sum_{x_0\in X_{0j}(\tau_k)}\mathbf{l}^*_{\mathbf{x_0}},\,u^*_{X_{0j}(\tau_k)}(t)\right\rangle_{\mathbb{R}^m}=\rho^-\left(D(t)\sum_{x_0\in X_{0j}(\tau_k)}\mathbf{l}^*_{\mathbf{x_0}}\,\middle|\,\mathbf{a}_*P\right)\ (t\in[\tau_{k-1},\tau_k)).\qquad(5)$$

Then the program package $(u^*_{x_0}(\cdot))_{x_0\in X_0}$ is guiding.

Algorithm for calculating elements of the guiding program package based on the theorem 6 and the method of succesing approximations was proposed in [6].
The algorithm is applicable only if there now singular clusters, i. e. that for the problem (8) for any direction $\ell^i \in \mathbb{R}^{nN}$, $i = 0, \ldots$ corresponding to the i-th step of the algorithm and for ℓ^*, maximizing (4), holds that there exist unique $v^i(\cdot)$, $i = 1, \ldots,$ и $v^*(\cdot)$ such, that

$$\rho^-(\mathcal{V}(t), \ell^i) = \langle \ell^i, v^i(t) \rangle, i = 1, \ldots, n; \ \rho^-(\mathcal{V}(t), \ell^*) = \langle \ell^*, v^*(t) \rangle, \quad t \in [t, \vartheta]. \quad (6)$$

Approximation of the control set with a smooth convex compact set

Let us get back to the extended control problem and consider it in the Euclidean space $\mathbb{R}^{nN}$ instead of the extended space $\mathcal{R}^n$, where $N = |X_0|$, $|X_0|$ is the number on elements in X_0. Let us denote the admissible extended control, i. e. the family $u_{x_0}(\cdot)_{x_0 \in X_0}$ with $\hat{u}(\cdot)$, the corresponding motion of the extended system $(x(\cdot|x_0, u_{x_0}(\cdot)))_{x_0 \in X_0}$ with $\hat{x}(\cdot)$, and the vector $(x_0)_{x_0 \in X_0}$ with $\hat{x}_0$. So we have

$$\begin{aligned} &\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{B}(t)\hat{u}(t) + \hat{c}(t), \quad \hat{x}(t_0) = \hat{x}_0, \\ &\hat{x}(\vartheta) \in \hat{M}, \quad \hat{u}(t) \in \hat{P}(t). \end{aligned} \tag{7}$$

Let us substitute the variables $v(t) = \hat{B}(t)\hat{u}(t)$, so that $v(t) \in \mathcal{V}(t) = \{v(t) = \hat{B}(t)\hat{u}(t) : \hat{u}(t) \in \hat{P}(t)\}$. Then the problem (7) will take the form

$$\begin{aligned} &\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + v(t) + \hat{c}(t), \quad \hat{x}(t_0) = \hat{x}_0, \\ &\hat{x}(\vartheta) \in \hat{M}, \quad v(t) \in \mathcal{V}(t). \end{aligned} \tag{8}$$

Smoothing of the control set I

Let us introduce the set $\mathcal{V}_\varepsilon(t)$ such that for any $t \in [t_0, \vartheta]$ holds $\mathcal{V}(t) \subset \mathcal{V}_\varepsilon(t)$ and $h(\mathcal{V}_\varepsilon(t), \mathcal{V}(t)) \leq \varepsilon$, where $h(A, B)$ is the Hausdorf distance between the sets A и B.
Let us consider the augmented problem

$$\begin{aligned} &\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + v(t) + \hat{c}(t), \quad \hat{x}(t_0) = \hat{x}_0, \\ &\hat{x}(\vartheta) \in \hat{M}, \quad v(t) \in \mathcal{V}_\varepsilon(t). \end{aligned} \tag{9}$$

Theorem 7

Let $\mathcal{V}_\varepsilon(t)$, $\varepsilon > 0$ be a smooth convex compact set and $(\hat{x}_\varepsilon(\cdot), v_\varepsilon(\cdot))$ is a solution of the problem (9). Then for any control $v(\cdot) : v(t) \in \mathcal{V}(t), t \in [0, \theta]$, such that $\|v_\varepsilon(t) - v(t)\| \leq \varepsilon$, the corresponding motion $x(\cdot)$ value of the system (8) at the time θ differs from $\hat{x}_\varepsilon(\theta)$ by not more than $\underline{O}(\varepsilon)$.

If we use instead of the original control set $\mathcal{V}(\cdot)$ which could be non-strictly convex and have an empty interior which leads to singular clusters in the original package guidance problem (8), a smooth convex compact set $\mathcal{V}_\varepsilon(\cdot)$, $\varepsilon > 0$, then in a small vicinity of the augmented problem (9) there is always a trajectory of the original

problem (8), whose end is close to the end of the trajectory of the solution of the augmented problem (9), and for constructing a solution of the augmented problem (9) it is possible to use the algorithm [6], which converges to the exact solution since the are no singular clusters.
So for constructing an approximate solution of the original problem (8), we need to consider the augmented problem (9) with a small $\varepsilon > 0$ and solve it approximately with the algorithm [6]. This approach has two obstacles:
1) The trajectory of the exact solution of the augmented problem would not always guide on the target set
2) The iterative process leads to a new approximate solution on each step which guides to a (smaller after each step) vicinity of the target set, i .e. there is no exact guidance onto the target set.
To solve these problems we can guide the object in the augmented problem (9) inside the target set to satisfy the estimate of the theorem 7 and inaccuracy of the iterative process of the algorithm [6]. Then any trajectory from the ε-vicinity (with regard to the control) of the obtained approximate solution of the augmented problem (9) will guide precisely onto the target set in the original problem (8).

Target set compression

Let us fix a $\delta > 0$ and demote $\hat{M}_\delta$ as an arbitrary convex closed set satisfying the condition $\hat{M}_\delta + S_\delta(0) \subset \hat{M}$ (here $S_\delta(0)$ is a sphere with the radius δ in the space $\mathbb{R}^{nN}$ with its center in the coordinate center). Let us consider the following guidance problem:

$$\begin{aligned} &\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + v(t) + \hat{c}(t), \quad \hat{x}(t_0) = \hat{x}_0, \\ &\hat{x}(\vartheta) \in \hat{M}_\delta, \quad v(t) \in \mathcal{V}_\varepsilon(t). \end{aligned} \tag{10}$$

Theorem 8

Let us assume that $\operatorname{int} \hat{M} \neq \varnothing$ *and the problem* (8) *has a solution with the trajectory end guiding inside the set* $\hat{M}$*. Then exist such* $\delta > 0$ *and a set* $\hat{M}_\delta : \hat{M}_\delta + S_\delta(0) \subset \hat{M}$*, that the following statements hold:*

1. *For any* $\varepsilon > 0$ *the problem* (10) *is solvable.*
2. *There exists such a* $\varepsilon > 0$*, that for any solution* $(\hat{x}_{\varepsilon\delta}(\cdot), v_{\varepsilon\delta}(\cdot))$ *of the problem* (10) *any control* $v(\cdot)$ *such, that* $v(t) \in \mathcal{V}(t)$ *and* $\|v(t) - v_{\varepsilon\delta}(t)\| \leq \varepsilon$*, guides the motion of the system* (8) *to the target set* $\hat{M}$*.*

Corollary 9

Let $(\hat{x}^{\Delta}_{\varepsilon\delta}(\cdot), v^{\Delta}_{\varepsilon\delta}(\cdot))$ *be an approximate solution of the problem* (10)*, calculated using the algorithm [6], and* $\Delta > 0$ *is guiding inaccuracy onto the target set* $\hat{M}_\delta$*, i. e.* $\min\limits_{\hat{m} \in \hat{M}_\delta} \|\hat{x}^{\Delta}_{\varepsilon\delta}(\vartheta) - \hat{m}\| = \Delta$*.*

Then $\hat{x}^{\Delta}_{\varepsilon\delta}(\vartheta) \in \hat{M}_\delta + S_\Delta(0)$ *and for satisfying the statement 2 of the theorem 8 for the approximate solution* $(\hat{x}^{\Delta}_{\varepsilon\delta}(\cdot), v^{\Delta}_{\varepsilon\delta}(\cdot))$*, is necessary that for* ε*,* δ*,* Δ *holds the condition* $\varepsilon C \leq \delta + \Delta$*.*

Concrete examples

The proposed theorems do not suggest concrete ways of constructing the sets $\mathcal{V}_\varepsilon(\cdot)$ и $\hat{M}_\delta$. For applications it is important to have constructive formulas for smoothing the set $\hat{M}$ and for approximation of the control set $\mathcal{V}(\cdot)$ with a smooth convex compact set.
The following methods for constructing the set M_δ could be used: 1) based on the set compression relative to the compression center (see [7]) and 2) based on the geometrical difference of the sets (see definition in [9]) of the set M_δ and the sphere $S_\delta(0)$. For constructing a smooth convex compact set $\mathcal{V}_\varepsilon(\cdot)$ the following methods could be used ([10], [11]). For the sets motivated by applications there are suggested convenient formulas in [10], [11], [12]
Let us consider a typical example there the control set is a segment symmetrical with respect to the coordinate center and an algebraic sum of two segments. Let the set

$$U_0 = I_1 + \ldots + I_L,\ I_j = [-b_j, b_j], \quad j = 1, \ldots, L, \tag{11}$$

be an algebraic sum of the segments in the space $\mathbb{R}^n$ ($b_j \in \mathbb{R}^n$, $j = 1, \ldots, L$). Then the upper support function of the smoothed set (with a parameter $\mu > 0$) is

$$\rho^+(U_\mu, l) = \sum_{j=1}^{L} \sqrt{\langle l, b_j\rangle^2 + \mu\left[\|l\|^2 - \frac{\langle l, b_j\rangle^2}{\|b_j\|^2}\right]}. \tag{12}$$

Lemma 10

Let the set U_0 be of the form (11), and the set U_μ зdefined by the support function (12). Then, for any $\mu > 0$ the inclusion $U_0 \subset U_\mu$ holds and for any given $\varepsilon > 0$ it is possible to choose $\mu > 0$, such that $h(U_0, U_\mu) \leq \varepsilon$.

Model example I

Let us consider a dynamical controlled system on the time segment $[0,2]$:

$$\begin{cases} \dot{x}_1 = x_2, x_1(0) = x_{01} \\ \dot{x}_2 = u, x_2(0) = x_{02}. \end{cases}$$

$$X_0 = \left\{ \begin{pmatrix} -7/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 7/5 \\ 0 \end{pmatrix} \right\}; \qquad M = \left\{ (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1 \right\}$$

$$u(t) \in P = \{u : |u| \leq 1\}, \ t \in [0,2]; \qquad Q(t) = \begin{cases} (0,0), & t \in [0,1/2], \\ (1,0), & t \in (1/2,2]. \end{cases}$$

The package guidance problem is solvable and the cluster positions are $\mathcal{X}_0(1/2) = \{X_0\}$ and $\mathcal{X}_0(2) = \{\{x_0'\}, \{x_0''\}\}$. To calculate the guiding program package let us try to apply the algorithm [6]. On the zero step let us estimate the motion value of the extended system under the zero-valued extended control, so we get the vector $d = (-7/5 \ 0 \ 7/5 \ 0)^{\mathrm{T}}$. It is obvious that $d \notin \mathcal{M}$, where $\mathcal{M} = M \times M$. Let us find the closest to d point of the set $\mathcal{M}$: $\bar{z} = (-1 \ 0 \ 1 \ 0)^{\mathrm{T}}$. Let us calculate the zeroth approximation of the support vector:

$$l^{*(0)} = \frac{d - z}{\|d - z\|} = \left(-\frac{\sqrt{2}}{2} \ 0 \ \frac{\sqrt{2}}{2} \ 0 \right)^{\mathrm{T}}.$$

Since the first two elements of the vector $l^{*(0)}$ correspond to the initial state x_0' and the next two elements to the initial state x_0'', for the cluster $X_0 \in \mathcal{X}_0\,(1/2)$ the support vector in the right hand-side of the (5) is

$$\begin{pmatrix} 2-t & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 2-t & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \equiv 0,\ t \in [0, 1/2),$$

i. e. the cluster X_0 is singular on the segment $[0, 1/2)$, and the algorithm is not applicable. Let us write the extended system in the space $\mathbb{R}^4$ in the form (7), where

$$\hat{x}(t) = \begin{pmatrix} x_{x_0'1}(t) \\ x_{x_0'2}(t) \\ x_{x_0''1}(t) \\ x_{x_0''2}(t) \end{pmatrix},\ \hat{u}(t) = \begin{pmatrix} 0 \\ u_{x_0'}(t) \\ 0 \\ u_{x_0''}(t) \end{pmatrix},\ \hat{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \hat{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

So we have

$$\begin{cases} \dot{x}_{x_0'1}(t) = x_{x_0'2}(t), & x_{x_0'1}(0) = x_{01}', \\ \dot{x}_{x_0'2}(t) = u_{x_0'}(t), & x_{x_0'2}(0) = x_{02}', \\ \dot{x}_{x_0''1}(t) = x_{x_0''2}(t), & x_{x_0''1}(0) = x_{01}'', \\ \dot{x}_{x_0''2}(t) = u_{x_0''}(t), & x_{x_0''2}(0) = x_{02}''. \end{cases} \tag{13}$$

Model example III

The extended control resource takes the form

$$\hat{P}(t) = \begin{cases} P_1, & t \in [0, 1/2), \\ P_2, & t \in [1/2, 2], \end{cases}$$

where

$$\begin{aligned} P_1 &= \left\{ (u_{x_0})_{x_0 \in X_0} \in [-1,1] \times [-1,1] : u_{x_0'} = u_{x_0''} \right\}, \\ P_2 &= \left\{ (u_{x_0})_{x_0 \in X_0} \in [-1,1] \times [-1,1] \right\}. \end{aligned}$$

Let us substitute variables

$$v(t) = \hat{B}\hat{u}(t) = \begin{pmatrix} v_{x_0' 1}(t) \\ v_{x_0' 2}(t) \\ v_{x_0'' 1}(t) \\ v_{x_0'' 2}(t) \end{pmatrix} \in \mathcal{V}(t) = \hat{B}\hat{P}(t) = \begin{cases} V_1, & t \in [0, 1/2), \\ V_2, & t \in [1/2, 2], \end{cases}$$

where

$$\begin{aligned} V_1 &= \left\{ v \in \mathbb{R}^4 : v_{x_0' 1} = v_{x_0'' 1} = 0,\, v_{x_0' 2} = v_{x_0'' 2} \in [-1,1] \right\}, \\ V_2 &= \left\{ v \in \mathbb{R}^4 : v_{x_0' 1} = v_{x_0'' 1} = 0,\, v_{x_0' 2}, v_{x_0'' 2} \in [-1,1] \right\}. \end{aligned}$$

Model example IV

The system (13) takes the form

$$\begin{cases} \dot{x}_{x_0'1}(t) = x_{x_0'2}(t) + v_{x_0'1}(t), & x_{x_0'1}(0) = x_{01}', \\ \dot{x}_{x_0'2}(t) = v_{x_0'2}(t), & x_{x_0'2}(0) = x_{02}', \\ \dot{x}_{x_0''1}(t) = x_{x_0''2}(t) + v_{x_0''1}(t), & x_{x_0''1}(0) = x_{01}'', \\ \dot{x}_{x_0''2}(t) = v_{x_0''2}(t), & x_{x_0''2}(0) = x_{02}''. \end{cases} \tag{14}$$

For smoothing the set $\mathcal{V}(t)$ let us apply the formula (12), and for $t \in [0, 1/2]$ let it be $L_1 = 1$ and $b_1 = (0\ 1\ 0\ 1)^{\mathrm{T}}$ while for $t \in [1/2, 2]$ let it be $L_2 = 2$ and $b_1 = (0\ 1\ 0\ 0)^{\mathrm{T}}$, $b_2 = (0\ 0\ 0\ 1)^{\mathrm{T}}$. For illustrative reasons let us take $\mu = 0.01$.

Let us apply the algorithm [6] to the system (14) with a smoothed control set defined by the support function (12) with the mentioned parameters. After 14 iterations of the algorithm the guidance inaccuracy (i .e., the distance from the motion end to the target set) is $\Delta \approx 0.001$. We will have the trajectories $(x^{\mu}_{x_0'}(\cdot), x^{\mu}_{x_0''}(\cdot))$ and the corresponding controls $(v^{\mu}_{x_0'}(\cdot), v^{\mu}_{x_0''}(\cdot))$. Using the theorem 7 let us find in the μ-vicinity of the obtained controls original system (14) controls, namely, let us use these controls:

$$\begin{aligned} \bar{v}_{x_0'1}(t) &= \bar{v}_{x_0''1}(t) \equiv 0, && t \in [0, 2], \\ \bar{v}_{x_0'2}(t) &= \bar{v}_{x_0''2}(t) = v^{\mu}_{x_0'2}(t), && t \in [0, 1/2], \\ \bar{v}_{x_0'2}(t) &= v^{\mu}_{x_0'2}(t), && t \in [1/2, 2], \\ \bar{v}_{x_0''2}(t) &= v^{\mu}_{x_0''2}(t), && t \in [1/2, 2]. \end{aligned} \tag{15}$$

Model example V

It is possible to choose different controls $(\bar{v}_{x_0'}(\cdot), \bar{v}_{x_0''}(\cdot))$ from the μ-vicinity of the controls $(v^{\mu}_{x_0'}(\cdot), v^{\mu}_{x_0''}(\cdot))$, for some of them a precise guidance onto the target set is possible, for some of them not. Let us take such controls $(\bar{v}_{x_0'}(\cdot), \bar{v}_{x_0''}(\cdot))$ that there is no precise guidance onto the target set. On the Fig. 1 the trajectories $(x^{\mu}_{x_0'}(\cdot), x^{\mu}_{x_0''}(\cdot))$ are shown, corresponding to the calculated by the algorithm controls $(v^{\mu}_{x_0'}(\cdot), v^{\mu}_{x_0''}(\cdot))$, and the trajectories $(\bar{x}_{x_0'}(\cdot), \bar{x}_{x_0''}(\cdot))$, corresponding to the controls (15), in the state space x_1, x_2. It is clear that the obtained trajectories $(\bar{x}_{x_0'}(\cdot), \bar{x}_{x_0''}(\cdot))$ are not guiding precisely onto the target set and the guidance inaccuracy does not exceed $\sqrt{\mu} L_1 C_1 + \sqrt{\mu} L_2 C_2 + \Delta$, where

$$C_1 = \int\limits_0^{1/2} \|\hat{F}(2,t)\|_F \, dt \leq 1.6, \quad C_2 = \int\limits_{1/2}^{2} \|\hat{F}(2,t)\|_F \, dt \leq 3.5$$

according to the theorems 7 and 8.

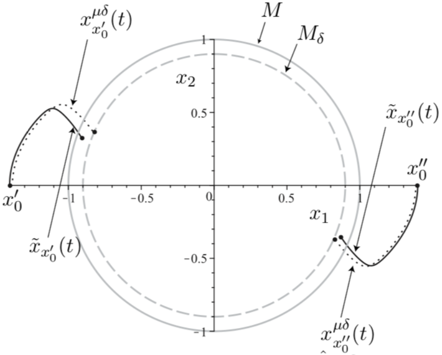


Fig 1. Guidance onto the set $\hat{M}$ (projection on $\mathbb{R}^2$)

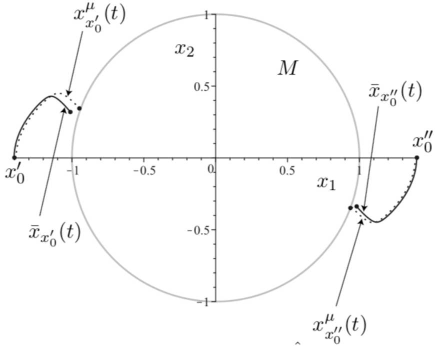


Fig 2. Guidance onto the set $\hat{M}_\delta, \delta = 0.1$ (projection on $\mathbb{R}^2$)

For precise guidance of the trajectories of the original system (14), let us use the results of the theorem 8. Let is apply the algorithm [6] to the extended guidance problem for the system (14) with a coefficient for smoothing the control set of $\mu = 0.0001$ onto the compressed target set $\hat{M}_\delta = M_\delta \times M_\delta$, where $M_\delta = \{(x_0', x_0'')^{\mathrm{T}} : (x_0')^2 + (x_0'')^2 \leq (1-\delta)^2\}$, $\delta = 0.1$. After 17 iterations of

the algorithm the computation inaccuracy is $\Delta \approx 0.001$. We have obtained the trajectories $(x^{\mu\delta}_{x_0'}(\cdot), x^{\mu\delta}_{x_0''}(\cdot))$ and the corresponding controls $(v^{\mu\delta}_{x_0'}(\cdot), v^{\mu\delta}_{x_0''}(\cdot))$. So, the trajectories corresponding to any control from the μ-vicinity
$(v^{\mu\delta}_{x_0'}(\cdot), v^{\mu\delta}_{x_0''}(\cdot))$, are guiding precise onto the target set. As an example the following controls were chosen:

$$\begin{aligned}
\tilde{v}_{x_0'1}(t) &= \tilde{v}_{x_0''1}(t) \equiv 0, & t &\in [0,2],\\
\tilde{v}_{x_0'2}(t) &= \tilde{v}_{x_0''2}(t) = v^{\mu\delta}_{x_0'2}(t), & t &\in [0,1/2],\\
\tilde{v}_{x_0'2}(t) &= v^{\mu\delta}_{x_0'2}(t), & t &\in [1/2,2],\\
\tilde{v}_{x_0''2}(t) &= v^{\mu\delta}_{x_0''2}(t), & t &\in [1/2,2].
\end{aligned} \tag{16}$$

Trajectories $(\tilde{x}_{x_0'}(\cdot), \tilde{x}_{x_0''}(\cdot))$, corresponding to the controls (16) are shown on the Fig. 2. It is clear that they are guiding inside the target set $\hat{M}$.

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Thanks for your attention!