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THE OPTIMAL LOCATION OF ECONOMIC ACTIVITIES IN A HIERARCHICAL SETTLEMENT SYSTEM IN A RURAL REGION

Ryszard Domanski

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

RD/oc

THE AUTHOR

Professor Ryszard DOMANSKI is the Rector of the Academy of Economics in Poznan, Poland. From January 1979-December 1980, he was a member of the Regional Development Task at IIASA.

PREFACE

This paper is intended as a contribution to the research program being carried out by the Regional Development Task at the International Institute for Applied Systems Analysis. The program is concerned with the development of a model system for application in regional development planning. Although work on the model system is already at an advanced stage, several models have yet to be developed, among which is a settlement-system model for a rural region.

The settlement network is a hierarchical system, hence its activities should be considered within hierarchical framework. So far, such an approach to the location of economic activities is at an initial stage.

In this paper several models of the optimal location of economic activities are reviewed. The models are distinguished by their concern for location problems in hierarchical settlement systems in rural regions. They have been selected from the existing literature on the basis of their potential for further research.

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INTRODUCTION

The development of agriculture and the rural demographic structure brings about many adjustment processes. For example, the general acceptance of technological advances in agriculture increases the demand for machinery, fertilizers, chemicals, construction materials, technical services, etc. New technology contributes to a growth in agricultural production, which in turn stimulates the development of the transportation system and the food-processing industry in rural areas.

Nowadays rural population requires many services: shops, education, health care, repair facilities, entertainment, and public administration. Some of the services that take on a material form can be purchased in urban centers and transported to rural areas for consumption, for example records and television sets. However, most services are consumed in urban areas. It is, therefore, important to ensure that the rural population has access to these centers.

In developed countries, rural-urban migration has led to the depopulation of rural areas. The rate of this process has now slowed down and in some cases the population has remained constant or even increased through in-migration. On the other hand, in less developed countries rural-urban migration is still considerable.

The efficiency of the economic activities in the settlement system depends on the settlement size, its spatial distribution, and its hierarchical pattern. A system with large, spatially concentrated centers can take advantage of economies of scale, thereby reducing unit production costs. At the same time, however, it requires increased transportation. In a system with small, spatially dispersed centers, the enterprises bear higher production costs but lower transportation costs.

The adjustment processes are accompanied by investments in new food-processing plants, services, housing projects, and roads. At this stage, the problem of location arises: How should the new activities be distributed within the settlement system?

In this paper an attempt is made to analyze the distribution problem by drawing upon a selection of the existing literature. However, it is not intended to be a complete record of research on locational theory, rather it is an appraisal of the most significant contributions to the field. The selection was based on three considerations:

-- the adaptability of the methodologies to the problems of rural settlement systems;
-- the importance of the hierarchical framework when dealing with problems of rural settlement systems;
-- the possibilities for further research offered by the various contributions.

ECONOMIES OF SCALE

Production and transportation costs are the most frequently applied criteria for choosing the optimal solution in mathematical programming. Their size depends, among other things, on the hierarchical structure of the settlement system. On a higher level of the hierarchical structure, large enterprises and settlements occur. Lower levels are formed by small enterprises and settlements. The levels in the hierarchy are reflected in the levels of costs.

In the long run, the relationship between average cost (y/x) and level of output (x) is usually described by means of the function:

$$\frac{y}{x} = a - bx + cx^2 \quad , \tag{1}$$

where a, b, c are constants.

When presented in graphic form, it gives the familiar U-shaped average cost curve. The downward bend represents a lowering of the average cost obtained because of the decrease in the unit fixed cost with the increase of output. At higher levels of output, there are also economies from labor specialization. The minimum point of the curve determines the optimal scale of output, while the upward bend represents the increase in average cost occurring when output grows beyond the optimal level. This portion of the curve is, however, disputable. It is argued that the optimal level of output rises with technological and organizational advances, which means that the minimum point shifts to the right and the U-shaped curve is transformed into a curve consisting of a downward bend and flat portion (Figure 1). The latter curve may be described mathematically as:

$$\frac{y}{x} = ax^b \quad . \tag{2}$$

Changes in the average cost of output on various levels of the settlement hierarchy are reflected in the changing values of multiplier a and exponent b.

SETTLEMENT HIERARCHY

Compared to the standard location problem, the problem of locating economic activities within a hierarchical system shows additional complications. The model of a hierarchy of centers developed by Tinbergen (1967) is helpful for gaining an understanding of these complications. It is the work to which other researchers dealing with this subject usually refer.

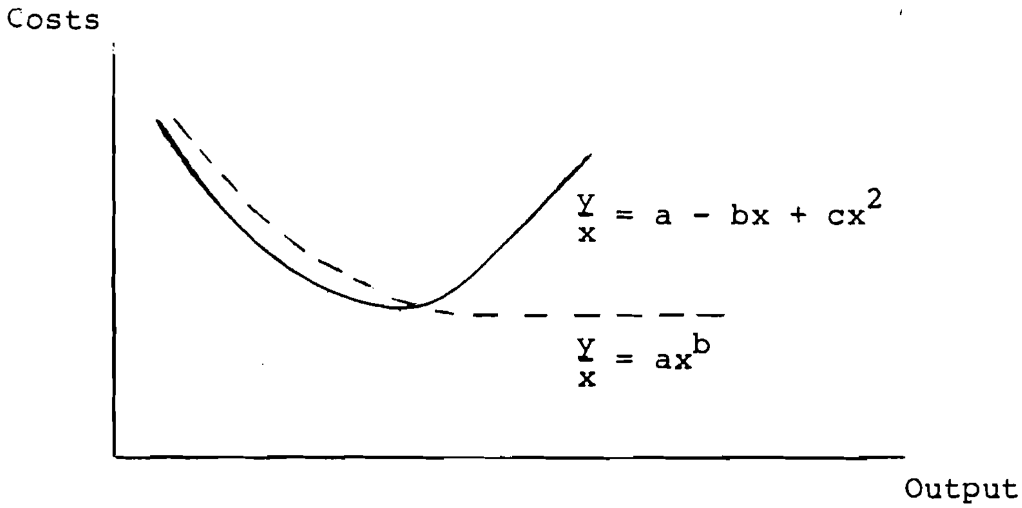

Figure 1. Changes in cost functions according to output levels.

Tinbergen assumes a closed national economy of regular form evenly distributed with farms except in urban areas. There are H industries, each producing final products indicated by h (h = 0, 1,...,H). The term h denotes the rank of the industry, and the case h = 0 represents agriculture. The demand for product h, which is equal to $a_h Y$ (Y being the country's income, and a_h being a given demand ratio for product h), is satisfied by n_h enterprises, whose size is supposedly optimal.

The industries have been ranked according to the number of firms they include in such a way that:

$$n_1 \geq n_2 \geq n_3 \ldots > n_H \quad . \tag{3}$$

The lowest rank represents the industry with the largest number of firms, and on the highest rank there is only one firm.

Tinbergen's model of the size distribution of centers has the following properties:

1. There are H types of centers (h' = 1,...,H), which means that the number of center types corresponds to the number of sectoral ranks.

2. In any center of type h', only the industries of rank $h \leq h'$ appear.
3. The industries of a rank lower than the center's type satisfy local demand.
4. The industry of a rank corresponding to the center's type ($h = h'$) satisfies both local demand as well as that for the product in the centers of lower types. There is only one enterprise of the highest rank in each center of the given type.

From the assumptions and properties of the model given above, further characteristics can be determined.

1. The size distribution of centers is expressed by cumulative income:

$$Y^0 = a_0 Y \quad , \tag{4}$$

$$Y^0 + Y^1 = \frac{a_0 Y}{1 - a_1} \quad , \tag{5}$$

$$Y^0 + \ldots + Y^{h'} = \frac{a_0 Y}{1 - a_1 - \ldots - a_{h'}} \quad . \tag{6}$$

2. The total income of all centers of type h' amounts to:

$$Y^{h'} = \frac{a_{h'} a_0 Y}{(a_0 + a_{h'+1} + \ldots + a_H)(a_0 + a_{h'} + \ldots + a_H)} \quad . \tag{7}$$

3. The number of centers of type h' is:

$$n^{h'} = n_{h'} \left(\frac{a_0}{1 - a_1 - \ldots a_{h'}}\right) \tag{8}$$

$$= n_{h'} \left(\frac{a_0}{a_0 + a_{h'+1} + \ldots + a_H}\right) \quad .$$

4. The average income per center of type h' is:

$$\bar{Y}^{h'} = \frac{Y^{h'}}{n^{h'}} = \frac{a_{h'}Y}{(a_0 + a_{h'} + \ldots + a_H)n_{h'}} \quad . \qquad (9)$$

Under certain conditions, the model generates an optimal system of centers based on minimizing total transportation costs.

In the model a relationship between the number of enterprises in a sector and its spatial pattern is assumed. If the number of enterprises in a sector is small, then the sector appears only in large urban centers. As the enterprises increase in number, the spatial pattern of the sector changes. They appear in smaller centers. The sectors with the largest number of enterprises are located on the lowest level of the hierarchy of centers.

LOCATION OF AGRICULTURE-RELATED ACTIVITIES

Agriculture-related activities include firms that supply the inputs to agriculture and that process agricultural output, for example the fertilizer, food-processing, and leather industries.

In modeling the spatial distribution of agriculture-related activities, the firms' minimum production and transportation costs are usually accepted as the criteria for the choice of location. Several models may be applied to find the optimal solution. Let us start with the model elaborated by Ulrich (1968).

Ulrich assumes that there are only three possible alternative distribution systems: a central-city distribution system, a service-center distribution system, and a local-center distribution system. In the first case, the firm or firms are located only in the central city of the rural region. The system generates the highest transportation costs and the lowest production costs. In the second case, firms are placed in all services centers (of which there are eight), and in the central city. Such a distribution gives rise to lower transportation costs, but at the same time, however, it may increase production costs. In the third case, firms are dispersed in all local centers (of which there are 72), in the service centers, and in the central center (in total 81 centers).

Each of the three alternative distribution systems bears different production and transportation costs. The average production cost under the individual system is calculated using the following equations:

$$P_s^1 = a_s(X_s^m - X_s^1)^{b_s} + c \quad , \tag{10}$$

$$P_s^2 = a_s(X_s^m - X_s^2)^{b_s} + c \quad , \tag{11}$$

$$P_s^3 = a_s(X_s^m - X_s^3)^{b_s} + c \quad , \tag{12}$$

where

P_s^1, P_s^2, P_s^3 = average production costs for sector s under the first, second, and third distribution systems, respectively;

a_s = multiplier in the production cost function for sector s;

b_s = exponent in the production cost function for sector s;

c = unit cost of output (in dollars) at the most efficient level of output;

X_s^m = minimum efficient level of output in sector s;

X_s^1 = total output of sector s in the region;

X_3^2 = one-ninth of total regional output of sector s;

X_s^3 = one-eighty-first of total regional output of sector s.

The total production cost is a product of the average production cost in each alternative distribution system and the total output of sector s.

The total transportation costs for the three alternative distribution systems are calculated as follows:

$$T_s^1 = \sum_{d=1}^{54} (U_s + k_s d) N_s F_d^1 \quad , \qquad (13)$$

$$T_s^2 = \sum_{d=1}^{18} (U_s + k_s d) N_s F_d^2 \quad , \qquad (14)$$

$$T_s^3 = \sum_{d=1}^{6} (U_s + k_s d) N_s F_d^3 \quad , \qquad (15)$$

where

T_s^1 = total transportation cost for sector s under the first distribution system;

T_s^2 = total transportation cost for one of the nine centers under the second distribution system;

T_s^3 = total transportation cost for one of the 81 centers under the third distribution system;

U_s = intercept value of the transportation cost function for sector s, representing loading and unloading costs;

k_s = cost per mile of transporting the output of sector s;

d = number of miles from the center to the farm;

N_s = number of two-way trips between the center and the farm for sector s, per year;

F_d^1 = number of farms at distance d from the central city (it is assumed that there are no farms more remote than 18 miles);

F_d^3 = number of farms at distance d from a local center (it is assumed that there are no farms more remote than 6 miles).

The total costs for the alternative distribution systems are obtained by adding transportation costs to production costs:

$$S_s^1 = T_s^1 + (P_s^1 X_s^1) \quad , \tag{16}$$

$$S_s^2 = T_s^2 + (P_s^2 X_s^1) \quad , \tag{17}$$

$$S_s^3 = T_s^3 + (P_s^3 X_s^1) \quad . \tag{18}$$

Using these totals, one can determine the distribution system that minimizes production and transportation costs, i.e. the optimal system.

This procedure is repeated for each sector related to agriculture. When all the sectoral distribution systems are defined, one can calculate the level of output in each center, superimposing one sectoral system upon another. The outcome of this calculation can be presented in the form of a four-element vector. The first element refers to the central-city output, the second to the service center output, the third to the local-center output and the fourth to total regional output.

Agriculture-related activities tend to be placed in two different types of location. Some of its sectors are closely linked to the location of agricultural resources, others are not restricted in this way and may be considered as 'foot-loose' industries. Sugar refineries represent the first type of sector and agricultural machinery plants represent the second type.

The different locational tendencies are included in Gunnarsson's model (1977), in which the idea of a hierarchy of centers presented earlier by Tinbergen (1967) and Bos (1965) has been extended. In particular, the idea behind one of the necessary conditions for the hierarchy of centers, namely indivisibilities leading to economies of scale, is developed. This condition is introduced into the model as a constraint on the output in a center of a foot-loose sector; i.e., output should not fall below the minimum level at which production is feasible, as determined by the plant capacity. The model consists of the following elements:

Data:

$\bar{x}_k$ = total output of a locationally restricted sector k $(k = 1,\dots,n')$;

$\bar{x}_p$ = total output of a foot-loose sector p $(p = n'+1,\dots,n)$;

$\bar{c}_p$ = capacity of a plant in sector p, corresponding to the minimum level at which production is feasible $(p = n'+1,\dots,n)$;

$d^{hh'}$ = distance between centers h and h';

$\bar{G}_p$ = constant satisfying the condition $\bar{G}_p > \bar{x}_p$ $(p = n'+1,\dots,n)$.

Unknowns:

$x_i^{hh'}$ = output of sector i delivered from center h to h' $(h,h' = 1,\dots,H;\ i = 1,\dots,n)$;

x_p^h = output of a foot-loose sector p in center h $(h = 1,\dots,H;\ p = n'+1,\dots,n)$.

Coefficients:

q_k^h = share of center h in total output of a locationally restricted sector k $(\sum_{h=1}^{H} q_k^h = 1)$;

t_i = unit cost of transporting product i per unit of distance;

a_{ij} = input of product i in the production of one unit of product j;

α_i = ratio between final demand for sector i and income (final demand includes the consumption of workers $(\sum_{i=1}^{n} \alpha_i = 1)$);

$a_{ij} + \alpha_i w_j = \nu_{ij}$ = generalized input-output coefficient (w is the ratio between value added and output for sector j);

$1 - (a_{ii} + \alpha_i w_i) = \beta_i$ = surplus per unit of output in sector i after subtraction of internally used output, including employees' consumption.

The model includes the following equations:

$$\beta_i x_i^h + \sum_{h'} x_i^{hh'} - \sum_{h'} x^{hh'} - \sum_{j \neq i} \nu_{ij} x_j^h \geq 0 \quad , \qquad (19)$$

$$i = 1, \ldots, n \quad .$$

This equation states that output plus imports must at least meet local demand and exports in every sector. A distinction is made between locationally restricted sectors (1,...,n') and foot-loose sectors (n'+1,...,n). The location of the restricted sectors' output is assumed to be given and is defined by the equation:

$$x_k^h = q_k^h \bar{x}_k \quad , \qquad k = 1, \ldots, n' \quad . \qquad (20)$$

It has been mentioned earlier that the model accounts for economies of scale. The following example shows how this is done. Assume that the output of sector i in center h amounts to $x_i^h \geq 1000$ or $x_i^h = 0$. This assumption can be presented as:

$$x_i^h - 1000 \geq 0 \quad , \qquad (21)$$

and

$$-x_i^h \geq 0 \quad . \qquad (22)$$

The presence or absence of the sector p in center h can be marked by an integer variable, whose value is either 0 or 1.

Thus, it can be concluded that the assumed level of output should never fall below the capacity of the plant.

All these considerations relating to economies of scale can be written as:

$$x_p^h + (z_p^h - 1)\bar{c}_p \geq 0 \quad , \qquad p = n'+1, \ldots, n \quad , \qquad (23)$$

$$-x_p^h - (z_p^h - 1)\bar{G}_p \geq 0 \quad , \qquad p = n'+1,\dots,n \quad , \tag{24}$$

$$z_p^h \quad \text{either 0 or 1} \quad . \tag{25}$$

The nonnegativity requirements assume the form:

$$x_i^{hh'} \geq 0 \quad , \tag{26}$$

$$x_p^h \geq 0 \quad . \tag{27}$$

The model can be used to determine the output of foot-loose sectors in centers and the deliveries between centers such that the total transportation costs are minimized. Its objective function is written as:

$$\min \sum_h \sum_{h'} \sum_i x_i^{hh'} d^{hh'} t_i \quad , \qquad h \neq h' \quad . \tag{28}$$

The model has two weaknesses, one of which is that it allows the concentration of foot-loose sectors in one large center. This deficiency does not apply to restricted sectors, whose output is located in a given place. The numerical examples given above suggest, however, that a concentration in one large center occurs only if there is an even spatial distribution of agriculture. In the case where the spatial distribution of agriculture is uneven, this tendency is not evident.

The other weakness is that the number and location of centers must be specified. This does not allow the locational pattern of output to be changed by the formation of new centers. The model also assumes a closed economy. The weakness of this assumption, however, may be eliminated by the introduction of trade into the model.

The hierarchical structure is more explicitly embodied in the model of a nodal hierarchy in a network system, developed by Scott (1971). The network system consists of a given set of nodes, from which a subset of nodes is selected as locations for some central facilities. These facilities produce or transmit the commodities consumed at each node. The set of central facilities forms a hierarchical system with m levels and each level incurs different fixed capital costs as well as transportation costs. The system is organized such that these costs are minimized by selecting the number and locations of facilities and organizing commodity flows from facilities to nodes in an optimal way.

The system modeled by Scott has some limitations resulting from the following characteristics: the levels of the hierarchy of facilities are functionally distinct; any facility on a given level may receive (despatch) goods only from (to) the closest higher level and despatch (receive) goods only to (from) the closest lower level; the goods transferred from producers to consumers do not themselves undergo material changes at intermediate levels.

The notations used in the model are given below:

a^k = amortized capital cost of any facility at level k (it is usually expected that $a^1 \geq a^2 \geq \ldots \geq a^m$);

d^k_{ij} = minimum cost of transportation in the given network between i and j per unit of commodity transported from the facility on level k (it is usually expected that $d^1_{ij} \leq d^2_{ij} \ldots \leq d^m_{ij}$);

B^k = capacity of any facility on level k;

D_j = final demand for the commodity at node j;

$\lambda^k_i = \begin{cases} 1 & \text{if a facility of order k is built at node i,} \\ 0 & \text{otherwise;} \end{cases}$

x^k_{ij} = number of commodities delivered to node j from the facility on level k located at i (solution variable);

M = arbitrarily large number.

The model described above can be stated as follows:

$$\text{Minimize } Z = \sum_{k=1}^{m} \sum_{i=1}^{n} (a^k \lambda_i^k + \sum_{j=1}^{n} d_{ij}^k x_{ij}^k) \quad , \tag{29}$$

subject to

$$\sum_{j=1}^{n} x_{ij}^k \leq B^k \quad , \tag{30}$$

$$\sum_{i=1}^{n} x_{ij}^m = D_j \quad , \tag{31}$$

$$\sum_{i=1}^{n} x_{ij}^{k-1} - \sum_{i=1}^{n} x_{ji}^k = 0 \quad , \tag{32}$$

$$M\lambda_i^k - \sum_{j=1}^{n} x_{ij}^k \geq 0 \quad , \tag{33}$$

$$\lambda_i^k = \begin{cases} 1 \\ 0 \end{cases} \quad , \tag{34}$$

$$x_{ij}^k \geq 0 \quad . \tag{35}$$

The constraints denote that: maximum capacities are not exceeded; all demands are met; all inputs from higher-level facilities to any k-level facility located at node j are equal to all outputs from the latter facility; whenever any demand, however small, is made upon a k-level facility at node i, then a facility is located at that node. The last two constraints are largely self-explanatory.

The solution of the model identifies the optimal commodity flows as well as the optimal number and location of the facilities. It is worthwhile noting that the model can be reduced to a standard transshipment problem. In the latter form its solution is simpler.

LOCATION OF SERVICES

The service sector also develops in rural settlements, although more slowly than in urban areas. It is necessary both for stimulating further growth of agricultural production as well as for increasing the standard of living of the rural population. Without a substantial improvement in services, it would be difficult to keep the population in rural areas at a size adequate to maintain dynamic demographic and economic development.

In most countries the introduction of service facilities in rural areas has been seriously neglected. The need for such facilities is now urgent, requiring large investments. As the availability of investment increases, so the problems associated with the location of services also increase. Investment decisions not only concern the location of individual facilities, but also the configuration of whole systems. For example: How should the optimal combination of central-facility locations be selected from among a number of potential centrally located villages? How should a given number of central facilities be distributed so that the maximum distance between all of the users and the closest facility is minimized? Where should the minimum number of central facilities be located such that the distance between the users and the closest facility does not exceed a given maximum admissible distance? What configuration of given central facilities maximizes the accessibility of the locations to users?

Optimal Selection of Key Villages

An attempt was made in Domanski (1980) to simulate the development process of key villages, conceived as service centers. The simulation technique applied was not used to evaluate the means or outcome of development.

We now extend our inquiry to this evaluation problem by asking the question: What combination of key villages selected from among a number of villages will be optimal? There is no optimization criterion that can be considered as fully satisfactory. As usual in such a situation, we apply some surrogate measure. The total distance traveled or time spent on traveling by the population from dispersed villages to key villages may constitute such a measure.

A problem of this type belongs to the class of problems referred to as central-facility location. Hence, to solve the problem, it may be possible to adapt earlier models of central-facility location.

The following symbols will be used:

a_i = population of village i (i = 1,...,n);

d_{ij} = shortest distance in the network from village i to key village j (j = 1,...,n; $i \neq j$);

$$x_{ij} = \begin{cases} 0 \text{ if village } i \text{ is not assigned to key village } j, \\ 1 \text{ if village } i \text{ is assigned to key village } j; \end{cases}$$

m = number of central facilities (key villages).

The problem of selecting key villages consists in distinguishing m of n villages ($m < n$) in such a way that the total distance traveled by the population between i and j is minimized.

The model of this problem (ReVelle and Swain 1970) can be written as:

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i d_{ij} x_{ij} \quad , \tag{36}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad , \qquad i = 1,\ldots,n \quad , \tag{37}$$

$$x_{jj} \geq x_{ij} \quad , \qquad i = 1,\ldots,n \ , \quad j = 1,\ldots,n \ , \quad i \neq j \ , \tag{38}$$

$$\sum_{i=1}^{n} x_{ii} = m \quad , \tag{39}$$

$$x_{ij} \geq 0 \quad , \quad i = 1,\ldots,n \quad , \quad j = 1,\ldots,n \quad . \tag{40}$$

The first constraint states that each village is assigned to one and only one key village. The villages assigned to individual key villages must form nonoverlapping districts that cover the whole region. This requirement is particularly important when key villages are assigned some administrative functions.

The second constraint states that key villages to which other villages are assigned must serve their own area.

The third constraint limits the number of central facilities and thus the number of key villages, i.e. villages that serve themselves.

In the above model, it was assumed that the number of key villages is given. Such an assumption is justified, since the number might frequently be defined by a political body rather than by a scientific method. However, since the creation of key villages is costly, the assumed number should be checked from the economic point of view. First, the program for creating key villages should be checked against the size of feasible investments. If we introduce the investment constraint, the model for selecting key villages, conceived as service centers, will take the form (Rojeski and ReVelle 1970):

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i d_{ij} x_{ij} \quad , \tag{41}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad , \quad i = 1,\ldots,n \quad , \tag{42}$$

$$x_{jj} \geq x_{ij} \quad , \quad i = 1,\ldots,n \quad , \quad j = 1,\ldots,n \quad , \quad i \neq j \quad , \tag{43}$$

$$\sum_{j=1}^{n} f_j x_{jj} + \sum_{j=1}^{n} b_j \sum_{i=1}^{n} a_i x_{ij} \leq C \quad , \tag{44}$$

$$x_{ij} \geq x_{jj} - x_{ii} \quad \text{for adjacent i-j pairs} \quad , \tag{45}$$

$$x_{ij} \geq 0 \quad , \qquad i = 1,\ldots,n \; , \; j = 1,\ldots,n \; , \tag{46}$$

where

f_j = fixed cost of establishing facility j;
b_j = coefficient of increase in variable cost of facility j;
C = investment limit.

Other notations are the same as in the previous model.

The first constraint requires that the population of each village be fully assigned. It will be assigned to one key village if the solution consists only of zero-one variables (integer programming), and to n key villages in the event of a non-integer solution (linear programming).

By the second constraint, the assignment is restricted to those key villages that serve themselves. This condition, however, will be enforced only if the solution consists exclusively of zero-one variables.

The third constraint demands that the total funds expended do not exceed the investment limit.

The fourth constraint requires that each village be assigned to the closest key village. If, for example, village A has no central facility and village B does, the constraint assumes the form:

$$X_{AB} \geq X_{BB} - X_{AA} \quad , \tag{47}$$

or

$$X_{AB} \geq 1 - 0 = 1 \quad . \tag{48}$$

Location Set-Covering and Maximal-Covering Location Problems

When searching for an optimal network of key villages as well as for the optimal location of service facilities in an already established network, it should be determined how well a particular location configuration fulfills the objectives that it is supposed to serve. Two measures of fulfillment have received attention in location models (Church and ReVelle 1974): total weighted distance or time for travel to the facilities and maximal service distance; i.e. the distance (or time) that the user most distant from the facility would have to travel to reach that facility.

The models presented so far in this section have minimized the total weighted distance. Now, models applying the concept of maximal service distance will be presented. Such models can be proposed particularly for the location of emergency facilities such as fire stations and ambulance depots. The concept of maximal service distance reflects well both the behavior of a country dweller and the decision process of those responsible for the location of public services. The country dweller is interested in obtaining this type of service within a critical time. This requirement is also of prime importance in the preparation of location schemes.

The concept of maximal service distance appears in two location problems: (a) the location set-covering problem, and (b) the maximal-covering location problem.

The location set-covering problem is concerned with finding the minimum number of facilities ensuring that the users at each point of demand will find services within a desired maximal service distance. In order to formulate a mathematical model (Toregas and ReVelle 1972), the following definitions will be needed:

I = set of demand points;

J = set of possible facility sites;

d_{ji} = shortest distance from site j to point i;

s = maximal service distance that may separate any demand point from its nearest facility;

$N_i = [j \in J | d_{ji} \leq s]$ for all i in I = set of facility sites eligible to provide coverage to demand point i.

The model assumes the form:

$$\text{Minimize } Z = \sum_{j \in J} x_j \quad , \tag{49}$$

subject to

$$\sum_{j \in N_i} x_j \geq 1 \quad , \qquad i \in I \quad , \tag{50}$$

$$x_j = (0,1) \quad , \qquad j \in J \quad . \tag{51}$$

By the first constraint, each demand point i must be covered by at least one facility. The objective function minimizes the number of facilities that satisfy demand. The solution also specifies the location of these facilities.

Total coverage of the demand area within a desired maximal service distance may be impossible because of budget constraints. This will result in a limitation on the resources available, which may not be sufficient for constructing the number of facilities specified in the model solution. In such a situation, the decision maker will have to reformulate his objective function. He may abandon his goal of total coverage and instead attempt to minimize the number of people that will not be served within a desired maximal service distance. In other words, he may try to find such a distribution of facilities that as many people as possible are included within the desired area. This problem is termed the maximal-covering location problem. Its mathematical formulation can be stated as (Church and ReVelle 1974):

$$\text{Maximize } Z = \sum_{i \in I} a_i y_i \quad , \tag{52}$$

subject to

$$\sum_{j \in N_i} x_j \geq y_i \quad , \qquad \text{for all } i \in I \quad , \tag{53}$$

$$\sum_{j \in J} x_j = P \quad , \tag{54}$$

$$x_i = (0,1) \quad , \qquad \text{for all } j \in J \quad , \tag{55}$$

$$y_i = (0,1) \quad , \qquad \text{for all } i \in I \quad , \tag{56}$$

where

$N_i = \{j \in J | d_{ij} \leq s\}$ for all i in I = set of facility sites eligible to provide coverage to demand point i;

s = distance beyond which a demand point is considered to be not covered;

a_i = population to be served at demand point i;

$$y_i = \begin{cases} 1 \text{ if demand node i is covered by a facility with distance s,} \\ 0 \text{ otherwise;} \end{cases}$$

P = number of facilities to be located.

All other notations are the same as in the previous model.

The objective function maximizes the number of people lying within the desired service distance. The first constraint allows y_i to be 1 only when the facility closest to the demand point i is at a distance not further than s. The second constraint restricts the number of facilities.

Another formulation of the maximal-covering location problem is possible. Given the restricted number of facilities, we may seek their distribution such that the maximum distance between the facilities and the users is minimized. This problem is structured mathematically in the following way (Bach 1980):

$$\text{Minimize } Z = \max_{j} \min_{i} d_{ij}^{b} a_{ij} \quad , \tag{57}$$

subject to

$$a_{ij} = 0 \text{ or } 1 \quad , \qquad \begin{array}{l} i = 1,\ldots,m \quad , \\ j = 1,\ldots,n \quad , \end{array} \tag{58}$$

$$\sum_{i=1}^{m} a_{ij} \geq 1 \quad , \qquad j = 1,\ldots,n \quad , \tag{59}$$

$$d_{ij} \geq 0 \quad , \qquad \begin{array}{l} i = 1,\ldots,m \quad , \\ j = 1,\ldots,n \quad . \end{array} \tag{60}$$

Constraints (58) and (59) allow each user to be assigned to to at least one facility. Constraint (60) expresses the non-negativity condition. The problem stated in this way has more than one optimal solution, i.e. one may obtain alternative distributions of facilities fulfilling the conditions of minimizing the maximum service distance.*

Social Welfare Maximization

Most models of facilities' location exhibit the following two characteristics: they assume a fixed demand function and an objective function minimizing travel costs. Both assumptions, although reasonable and commonly accepted, ignore essential features of the real world. The demand for services usually varies with the cost of obtaining them; it decreases with the increase in cost. The models describe real situations if they assume declining demand functions. Minimization of travel costs as an efficiency criterion is rather narrow. Social welfare would constitute a more adequate criterion.

The models assuming fixed demand functions and travel-cost-minimization objective functions yield, in fact, only suboptimal solutions. The assumption of fixed demand requires demand to be satisfied even if customers attach to the service a value that is lower than the marginal costs of providing it. The level of services obtained from such models tends to be higher than the Pareto optimal level.

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*Readers interested in the unifying framework for public-facility-location problems are referred to Leonardi (1980a).

Wagner and Falkson (1975) elaborated models of public-facility location that eliminate these drawbacks. The models include a declining demand function and maximize social welfare explicitly. The surplus of consumers plus producers is accepted as a measure of social welfare. The notion of surplus is defined as follows: the consumers' surplus (CS) is equal to the sum over all consumers of the difference between the highest willingness-to-pay for a product and the amount actually paid; the producers' surplus (PS) is the sum over all producers of the difference between the revenue actually received for a product and the lowest willingness-to-sell. The notion is structured mathematically in the following way:

$$CS = \sum_{j=1}^{n} \sum_{i=1}^{n} (V_i - t_{ij}d_{ij} - P_j^S)a_iX_{ij} \quad , \qquad (61)$$

$$PS = \sum_{i=1}^{n} \sum_{j=1}^{n} (P_j^S - b_j)a_iX_{ij} - \sum_{j=1}^{n} f_jY_j \quad , \qquad (62)$$

where

V_i = willingness-to-pay of each member of community i for a unit of product per unit time period;

$t_{ij}d_{ij}$ = transportation cost per unit of product between demand node i and supply node j;

P_j^S = factory price at supply node j;

a_i = amount of service demanded at community i;

b_j = marginal cost of service at j;

f_j = fixed cost of service at j;

X_{ij} = assignment variable representing the fraction of demand of community i satisfied by supply node j;

Y_j = zero-one location variable.

In summing the two surpluses, the P_j^Ss cancel out and the sum is reduced to:

$$\sum_{j=1}^{n} \sum_{i=1}^{n} (V_i - b_j - t_{ij}d_{ij})a_iX_{ij} - \sum_{j=1}^{n} f_jY_j \quad . \qquad (63)$$

The models of Wagner and Falkson (1975) distinguish between two institutional environments that differ with respect to the consumers' freedom of choice of facilities. These are: public fiat environment and serve-all-comers environment. In the public fiat environment, the consumers do not have the freedom of choice but may be assigned arbitrarily to facilities and may be denied service. In the serve-all-comers environment, the consumers can choose the facilities freely and must be served by the facility of their choice. Below, the models for both environments are specified.

The model for the public fiat environment can be stated as:

$$\text{Max} \sum_{j=1}^{n} \sum_{i=1}^{n} (V_i - b_j - t_{ij}d_{ij})a_iX_{ij} - \sum_{j=1}^{n} f_jY_j \quad , \tag{64}$$

subject to

$$\sum_{j=1}^{n} X_{ij} \leq 1 \quad , \qquad i = 1,\ldots,n \quad , \tag{65}$$

$$Y_j \geq X_{ij} \quad , \qquad i,j = 1,\ldots,n \quad , \tag{66}$$

$$Y_j = (0,1) \quad , \qquad j = 1,\ldots,n \quad , \tag{67}$$

$$X_{ij} \geq 0 \quad , \qquad i,j = 1,\ldots,n \quad . \tag{68}$$

Constraint (65)-(66) expresses the demand requirement. It has the form of an inequality since the environment does not require that maximum potential demand at any site be met. Constraint (67)-(68) states that production at node j necessarily bears fixed costs.

The model for serve-all-comers can be stated as:

$$\text{Max} \sum_{j=1}^{n} \sum_{i=1}^{n} (V_i - b_j - t_{ij}d_{ij})a_iX_{ij} - \sum_{j=1}^{n} f_jY_j \quad , \tag{69}$$

subject to

$$\sum_{j=1}^{n} X_{ij} \leq 1 \quad , \qquad i = 1,\ldots,n \quad , \qquad (70)$$

$$Y_j \geq X_{ij} \quad , \qquad i,j = 1,\ldots,n \quad , \qquad (71)$$

$$Y_k + \sum_{i \in N_{ik}} X_{ij} \leq 1 \quad , \qquad i,k = 1,\ldots,n \quad , \qquad (72)$$

$$\sum_{j=1}^{n} X_{ij} \geq Y_{ij} \quad , \qquad \text{for all } i,j \text{ such that } V_i - t_{ij}d_{ij} - P_j^s > 0 \quad , \qquad (73)$$

$$Y_j = (0,1) \quad , \qquad j = 1,\ldots,n \quad , \qquad (74)$$

$$X_{ij} \geq 0 \quad . \qquad (75)$$

The latter model contains two additional sets of constraints: (71) and (73). It is assumed that in a serve-all-comers environment the customers will choose the closest operating facility. Constraint (72) ensures the assignment of customers to their closest operating facility. It is also necessary to ensure that all customers who require services are assigned to some facility. This is stated in constraint (73).

The simple welfare-maximization model (64)-(68) can be adjusted to the case of declining community demand functions (of delivered price). In this case only the coefficients of each X_{ij} in the objective function must be changed. In the simple model they assume the form $a_i(V_i - t_{ij}d_{ij} - b_j)$. In the case of continuous declining demand functions, they are replaced by the form:

$$a_i \int_{t_{ij}d_{ij} + b_j}^{\infty} (V_i - b_i - t_{ij}d_{ij}) f_i(V_i) dV_i \quad , \qquad (76)$$

where $f_i(V_i)$ is the relative frequency function for values V_i in community i.

Replacement is possible because of the relationship established between the demand function and the frequency distribution. An aggregate demand function, being in fact a cumulative willingness-to-pay function, can be transformed into a cumulative frequency distribution. This is done by measuring the fraction of potential consumers who are willing to pay an amount of money greater than, or equal to, any specified amount.

Hierarchical Location-Allocation Problem

The majority of research on the spatial distribution of services assumes that the facilities form single-level systems. Multilevel systems are rarely considered. However, Dokmeci (1973) and Banerji and Fisher (1974) consider such a case. Dokmeci's model, which is presented in this section, determines the optimal distribution of hierarchically coordinated facilities over a bounded space.

Demand points, the trip requirements of each demand point, as well as facility and transportation costs for each level of the hierarchy are given. The fixed part of facility costs varies according to the size of the facility and reflects economies of scale. The unknown variables to be defined in an optimal fashion are: number, size, and location of facilities over a bounded space. The model is used to find the minimum total cost of the system, i.e. facility and transportation costs.

Demand is nonuniformly distributed at m points P_j. The set of these points forms a zero-th level or demand level, L_0. There are N facility levels, L_β ($\beta = 1,\ldots,N$), each demand point being serviced by one facility on each facility level. The hierarchical structure is given by:

$$L_0 = \{P_j | j = 1,\ldots,m\} \quad , \tag{77}$$

$$L_1 = \{P_j | j = m+1,\ldots,n\} \quad , \tag{78}$$

$$L_2 = \{P_j | j = n+1,\ldots,p\} \quad , \tag{79}$$

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$$L_N = \{P_j | j = q+1,\ldots,r\} \quad . \tag{80}$$

The model's structure allows for interactions between facilities of different levels but not of the same level.

The total transportation cost is given by:

$$T = \sum_{i=1}^{r} \sum_{j=1}^{r} u_{ij} a_{ij} d_{ij} t \quad . \tag{81}$$

The facility cost (F) varies according to the level of facilities. The total facility cost is given by:

$$F = \sum_{i=1}^{r} (b_i + c_i K_i) \quad , \tag{82}$$

or by:

$$F = \sum_{\beta=1}^{N} (g_\beta n_\beta + h_\beta A_\beta) \quad , \tag{83}$$

where

u_{ij} = number of trips made from point j to facility i;

$a_{ij} = \begin{cases} 1 \text{ if facility i supplies point j,} \\ 0 \text{ otherwise;} \end{cases}$

d_{ij} = distance between i and j;

t = unit transportation cost;

b_i, c_i = specified constants for each level of facilities;

K_i = unknown capacity of facility i;

A_β = specified total capacity requirement;

n_β = number of facilities on level β;

$g_\beta h_\beta$ = specified constants for level β of facilities.

The hierarchical location-allocation problem can now be formulated.

$$\text{Minimize } Z = \sum_{i=1}^{r} \sum_{j=1}^{r} u_{ij} a_{ij} d_{ij} + \sum_{i=1}^{r} (b_i + c_i K_i) \quad , \tag{84}$$

subject to

$$\sum_{i \in \beta} K_i \geq A_\beta \quad . \tag{85}$$

The hierarchical problem described above is difficult to resolve numerically because of the nonlinear objective function. An easier way to find a solution is to apply a heuristic algorithm.

ACCESSIBILITY

One of the major characteristics of a settlement system of any region is that it provides the population with jobs, housing, and services. It is important that these opportunities be accessible to the inhabitants of the region. Spatial accessibility of jobs, housing, and services can be considered as one of the components of quality of life. Regional planning aims to improve accessibility through the extension and proper location of facilities of various kinds (Domanski 1979).

When planning the construction and location of new facilities, decision makers may take maximization of spatial accessibility as their objective function. The equalization of the accessibility of facilities from users' locations can be considered as an alternative objective function. By equalizing accessibility, spatial equity may be increased.

Let us first consider the location of service facilities maximizing spatial accessibility. Before presenting the model of this problem, let it be understood that the concept of accessibility is equivalent to the concept of potential in spatial analysis. There are several formulae to calculate the potential. According to one (Bach 1980), the potential created at location i by a central facility located at j amounts to:

$$\pi_{ij} = \frac{a_j}{1 + d_{ij}^{\beta}} \quad , \tag{86}$$

where

a_j = attractiveness of the central facility at j (it can be measured by the facility size weighted with some quality index;

d_{ij} = distance between users at i and the central facility at j;

β = exponent representing the influence of distance (transportation cost) on the interaction between i and j.

The potential caused by all facilities interacting with the given location of users can be written as:

$$\pi_i = \sum_j \frac{a_j}{1 + d_{ij}^{\beta}} \quad . \tag{87}$$

We will now present the model elaborated by Bach (1980).

Data:

U_i = locations of users (i = 1,...,m);

n = number of central facilities ($n \geq 2$);

a_j = attractiveness of central facilities (j = 1,...,n).

Unknowns:

C_j = locations of central facilities (j = 1,...,n).

Constraints:

Each user must be assigned to exactly one central facility in such a way that nonoverlapping districts covering the whole region are formed.

The model aims to find those locations of central facilities and configurations of districts that would maximize the sum of potentials (accessibilities) created by central facilities at the users' locations.

This problem may now be written in programmatic form as follows:

$$\text{Maximize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \pi_{ij}\alpha_{ij} \quad , \tag{88}$$

subject to

$$\alpha_{ij} = \begin{cases} 0 \\ 1 \end{cases} , \quad \begin{array}{l} i = 1,\ldots,m \quad , \\ j = 1,\ldots,n \quad , \end{array} \tag{89}$$

$$\sum_{j=1}^{n} a_{ij} = 1 \quad , \quad i = 1,\ldots,m \quad , \tag{90}$$

$$d_{ij} \geq 0 \quad , \quad \begin{array}{l} i = 1,\ldots,m \quad , \\ j = 1,\ldots,n \quad . \end{array} \tag{91}$$

The first two constraints ensure that each user is assigned to one, and only one, central facility.

The given settlement system may ensure accessibility to services, but not to jobs or housing, and vice versa. Such a system is spatially unbalanced. It does not ensure a satisfactory level of social welfare, which would require all components of welfare to be accessible.

Klaassen, Paelinck, and Wagenaar (1979) developed a method that allows individual accessibilities to be integrated into one total accessibility. It simulates the characteristics of the living conditions of individual villages and towns.*

*A multiactivity location problem with accessibility and congestion-sensitive demand is being studied at IIASA (Leonardi 1980b).

The use of a special type of social-welfare function is the main element of this method. The function is characteristic in that the individual accessibilities (potentials) are its arguments. The social-welfare function integrating all accessibilities can be written as:

$$w_i = 1(\pi_i^h, \pi_i^s, \pi_i^v) \quad , \qquad \text{for all } i = 1,\ldots,m \quad . \qquad (92)$$

where

π_i^h = the accessibility to housing from location i;

π_i^s = the accessibility to services;

π_i^v = the accessibility to jobs.

It can be specified, for example, as a Cobb-Douglas-type function:

$$w_i = (\pi_i^h)^{a^h} \ (\pi_i^s)^{a^s} \ (\pi_i^v)^{a^v} \quad . \qquad (93)$$

The function presented above shows some typical characteristics. First, different combinations of arguments (potentials, accessibilities) can lead to the same value of the function. This is the well-known principle of substitution. Second, if the arguments are interdependent, then the change in one argument will influence the others. Third, the total accessibility of location i (village, town) depends not only on the availability of facilities in location i itself, but also on the availability of those in neighboring locations within reach of region i.

This form of social-welfare function may be used to reflect the effects of governmental investment policy. The government can allocate its budget among different facilities and locations. Each allocation may bring different results. It is then necessary to determine what allocation will bring the maximum increase in the welfare of all locations together. The answer can be obtained by finding the maximum of the social-welfare function.

CONCLUSION

As a result of the assessment of the various models discussed above, we conclude that three features are important when adapting or developing models to solve the problems of rural settlement systems. These features are associated with the hierarchical framework of such systems, their spatial accessibility, and their unique characteristics, and they should all be considered in the model. Some comments on these features are given below.

The hierarchical framework allows for a more accurate representation and analysis of economic activities in settlement systems. While there are few such models currently in operation, a number of promising directions for further research have been identified.

Spatial accessibility should be one of the major criteria in evaluating the development plans of settlement systems. Its improvement would contribute to an increase in the quality of life and social equity.

Settlement systems in rural regions have several unique characteristics that the hierarchical optimization models should reflect. Some of the existing models can be adapted to include these characteristics and therefore may be suitable for solving rural problems.

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