STRATEGIC RESPONSE WITH PERFECT INFORMATION

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Take a game in extensive form with perfect information. Start with an arbitrary choice of strategies** by the players. Now let each player -- if he can -- deviate unilaterally to a strategy that will improve his payoff, on the assumption that the others stay put with their choices. This gives rise to a new revised choice of strategies. Iterate the process. We will show that the sequence of strategic choices thus generated becomes stationary, and (of course) constitutes a Nash Equilibrium (N.E.) of the game. This result is embarrassingly simple to prove but seems to us to merit being on record. It does somewhat more than re-establish the classical existence of an N.E. for such games. It shows that the N.E.'s are obtained under the simple dynamic of unilateral strategic improvements. Such improvements underlie the very notion of an N.E. (which is defined to occur when they cannot be made), and so it is natural to use them to set up an associated dynamic.

**Throughout this note "strategies" will always mean "pure strategies".

Let us quickly recall the definition of the extensive form game. (For simplicity we will eliminate chance moves, but our result can be shown to hold even then.) There is a tree with a finite number of nodes, one of which is distinguished as the root (the start of the game). The set of nodes that have no followers* are called terminal nodes and correspond to end-points of the game. The non-terminal nodes are the players' moves. Denote by $N = \{1,\ldots,n\}$ the set of players. Each non-terminal node is then labelled by some $i \in N$. Also each $i \in N$ has a (weak) preference relation $\succsim_i$ on the set of terminal nodes, which is assumed to be complete and transitive. (Read "$\alpha \succsim_i \beta$" as "α is liked by i no less than β".) $\succ_i$ is the strict preference induced by $\succsim_i$ ($\alpha \succ_i \beta$ if it is not the case that $\beta \succsim_i \alpha$).

A strategy of $i \in N$ is a choice of an (issuing-out) arc at each node labelled by him. Put

$$S^i = \text{the set of strategies of player } i$$

$$S = S^1 \times \ldots \times S^n$$

Any $s \in S$ gives rise to a path $p(s)$ from the root to some terminal node. Define $\succsim_i$ on S by $s \succsim_i s'$ if $p(s) \succsim_i p(s')$. An n-tuple of strategies $\hat{s} = (\hat{s}^1,\ldots,\hat{s}^n)$ in S is called a <u>Nash Equilibrium</u> (N.E.) if, for each $i \in N$,

$$\hat{s} \succsim_i (\hat{s}|t) \qquad \text{for all } t \in S^i \quad .$$

Here $(\hat{s}|t)$ stands for the n-tuple in S obtained from $\hat{s}$ by replacing $\hat{s}^i$ by t.

Given an $s = (s^1,\ldots,s^n) \in S$, player i may deviate to $t \in S^i$ if it improves him, i.e., $(s|t) \succ_i s$. But such a t may involve eccentric behavior by i on irrelevant parts of the game tree. For example consider the following 2-person game:

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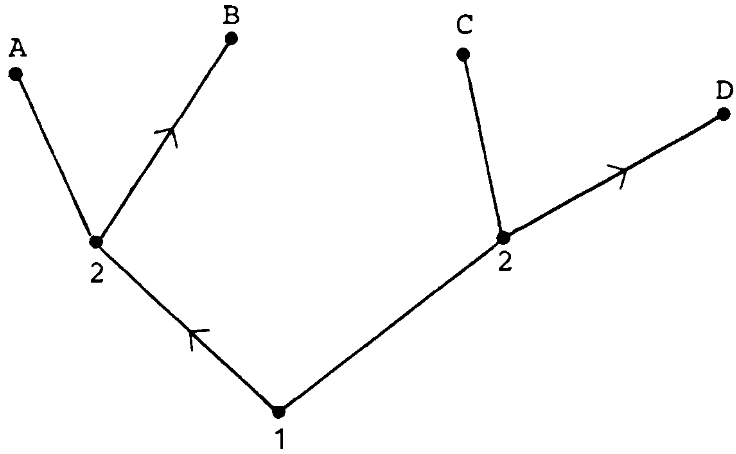
*Node α "follows" node β if β is on the (unique) path connecting the root to α.



The single arrows give s. Suppose that A $\succ_2$ B, and D $\succ_2$ C. Then one profitable deviation by 2 is the pair of choices given by $\{(2,A),(2,D)\}$. But there is another, namely $\{(2,A),(2,C)\}$. Since the right branch of the tree is not reached given 1's choice at the root, 2 can behave perversely at this node by choosing C instead of D. We wish to rule this out. Accordingly let us define $t \in S^i$ to be an <u>improvement-response</u> to s if, for every subgame Γ having a node of i as its root,

$$t_\Gamma \neq s^i_\Gamma \Rightarrow (s_\Gamma | t_\Gamma) \succ_i s_\Gamma \quad .$$

The symbol $s^i_\Gamma(t_\Gamma)$ stands for the restriction of $s^i(t)$ to Γ; $s_\Gamma \equiv (s^1_\Gamma,\ldots,s^n_\Gamma)$. Denote the set of improvement-responses* of i to s by $I^i(s)$.

Consider an infinite sequence $\{s(1),\ldots,s(\ell),\ldots\}$, $s(\ell) = (s^1(\ell),\ldots,s^n(\ell)) \in S^1 \times \ldots \times S^n$, of strategic choices by the players. We will say that <u>player</u> i <u>is improving in</u> $\{s(1),\ldots, s(\ell),\ldots\}$ if there exists an increasing sequence of integers $t^i(1),\ldots,t^i(k),\ldots$ such that

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*Note that an improvement-response of i does <u>not</u> require him to make an improvement in every subgame. In particular $s^i \in I^i(\ldots,s^i,\ldots)$ always. Perhaps the name "non-disimprovement responses" would be better.

$$s^i(t^i(k+1)) \in I^i(s(t^i(k))) \quad ,$$

for all k; and

$$s^i(\ell) = s^i(t^i(k))$$

for $t^i(k) \leq \ell < t^i(k+1)$. This simply says that (i) whenever player i changes his strategy it is an improvement to the n-tuple of strategies that he is informed about at that time; (ii) any change in his information is always an up-date. Thus, to give two extreme cases, i is improving in $\{s(1),\ldots,s(\ell),\ldots\}$ if $s^i(\ell) = s^i(1)$ for all ℓ; or if $s^i(\ell+1) \in I^i(s(\ell))$ for all ℓ.

Claim: Let $\{s(1),\ldots,s(\ell),\ldots\}$ be an infinite sequence of strategic choices in which each $i \in N$ is improving. Then there is an integer L such that $s(\ell) = s(L)$ for $\ell \geq L$, i.e., the sequence becomes stationary.

Proof: This will be by induction on the number of arcs k in the game tree. If k = 1 the claim is obvious. Assume it to be true whenever $k \leq n$ and take the case $k = n+1$.

If $\{s(1),\ldots,s(\ell),\ldots\}$ does not become stationary, then there is a node α, labelled by player i, such that i picks at least two arcs at α infinitely often in $\{s^i(1),\ldots,s^i(\ell),\ldots\}$. Consider the subgames $\Gamma_1,\ldots,\Gamma_m$ ($m \geq 2$) which start at the end of the arcs picked infinitely often at α. In each such Γ_j, the sequence $\{s_{\Gamma_j}(1),\ldots,s_{\Gamma_j}(\ell),\ldots\}$ clearly satisfies the hyothesis of the claim, hence -- by the inductive assumption -- becomes stationary. But then i would pick only one arc at α infinitely often, a contradiction.

Q.E.D.

So far we have not insisted in the sequence $\{s(1),\ldots,s(\ell),\ldots\}$ of the claim that if a player can strictly improve himself, then he should do so. Suppose improvements take place as follows. In each "round" the players are ranked in some order and called upon in turn to make unilateral deviations. Each player i is informed of some (possibly empty) subset of the changes made prior to his turn.* Moreover, knowing $s \in S$ at his turn, if there is a $t \in I^i(s)$ with the property $(s|t) \succ_i s$, then he deviates to some such t. The claim applies, and after a finite number of rounds we arrive at a stationary n-tuple of strategies which is clearly an N.E. There is no guarantee that it will be a "perfect" N.E. (i.e. one in which, in each subgame, the restricted strategies form an N.E.). To ensure convergence to that, we would have to require that the players strictly improve (if possible) in each subgame rooted at their nodes when their turn comes.

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*Indeed some players may make several changes before others. All we need is that each player gets at least one chance to change his strategy in every round.