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THE USE OF RANDOM-UTILITY THEORY IN BUILDING LOCATION-ALLOCATION MODELS*

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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Public Facility Location Task, which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

This paper is both a unifying effort and a contribution to the formulation of such new problems and approaches. It explores the relationships between a recent area of geographic research, random-utility theory, and a recent area of applied mathematic research, the optimization of submodular functions. The fruitfulness of this marriage is shown by some numerical results, which seem to suggest that the approach can yield new, powerful tools for location problems.

Related publications in the Public Facility Location Task are listed at the end of this report.

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ABSTRACT

The most important part of a location-allocation model is the allocation rule, that is, the way clients are assigned to facilities. In the well-known models of the "plant-location" family, the embedded allocation rule is the assignment of the least-travel-cost facility.

This allocation rule depends on the assumption that the cost, or more generally utility, associated with each possible facility choice is deterministically known. The simplest way to generalize a plant-location model is to add a random term to travel costs, with a known probability distribution. Such randomness may be shown to arise in many real-life situations, and the resulting choice models constitute the subject of random-utility theory.

This paper introduces the use of the random-utility modeling philosophy in location-allocation problems. Some relevant properties of the resulting family of models are derived. Among them, of special importance is the submodularity property, which relates the random-utility-based location models to a recent area of research in combinatorial optimization. Submodularity is exploited to develop simple heuristic algorithms, and the effectiveness of the approach is supported with some numerical results.

CONTENTS

THE USE OF RANDOM-UTILITY THEORY IN BUILDING LOCATION-ALLOCATION MODELS

1. INTRODUCTION

Some recent theoretical and computational innovations have revived interest in location problems in the last few years.

On the theoretical side, the need to introduce more realistic and economically sound measures of customer benefits has been felt by many scholars. An outstanding contribution in this field has been given by the use of random-utility theory. This approach, mainly developed for transport demand analysis (Domencich and McFadden, 1975; Williams, 1977), has been rapidly extended to the spatial allocation of economic activities (Coelho, 1977, 1979; Coelho and Williams, 1978; MacGill and Wilson, 1979). Its use in the facility location context first appeared in Coelho and Wilson (1976) and independently in Leonardi (1975), then, shortly after in many other contributions (Williams and Senior, 1977; Leonardi, 1978, 1980a, 1980b; Beaumont, 1979, 1980; Coelho, 1980a).

On the computational side, the major event has been the appearance of the dual ascent method, which amazingly outperforms all previously used methods to solve the classic uncapacitated plant-location problem. The method, first suggested by Bilde and Krarup (1977), has been developed by Erlenkotter (1978) and

Van Roy and Erlenkotter (1980). More interesting than the method itself have been the theoretical investigations it stimulated. Among them, the most fruitful one has been the parallel development of a new framework to analyze combinatorial optimization problems with submodular objective functions. The method has been developed by Cornuejols, Fisher, and Nemhauser (1977); Nemhauser, Wolsey, and Fisher (1978); Fisher, Nemhauser, and Wolsey (1978); and its relationships with the dual-ascent method have been analyzed in Wolsey (1980).

This paper tries a first step in putting the two theories together. Submodularity (the key property leading to the success of the above methods) is shown to hold for all objective functions based on the additive random-utility model. Some simple heuristics based on this property are proposed, and some (surprisingly good) numerical results are shown for a typical random-utility uncapacitated location problem.

As a conclusion, it is argued that further investigation could yield a substantial improvement in the state-of-the-art of location modeling for the near future, and provide superior algorithms for a much wider class of problems than the ones usually considered in the operations research literature.

2. A GENERALIZATION OF THE UNCAPACITATED FACILITY LOCATION PROBLEM

The classic uncapacitated facility location problem (Efroymson and Ray, 1966; Spielberg, 1969) can be formulated in the following way

$$\min_{L,X} \sum_{j \in L} \sum_{i} x_{ij} C_{ij} + \sum_{j \in L} a_j \tag{1}$$

s.t.

$$\sum_{j \in L} x_{ij} = P_i \tag{2}$$

$$x_{ij} \geq 0 \tag{3}$$

where

- i labels customer locations
- j labels facility locations
- P_i is the total number of customers in location i
- x_{ij} is the number of customers in location i served by the facility in location j
- X is the array $\{x_{ij}\}$
- L is the subset of locations of open facilities, to be chosen among all subsets of feasible facility locations
- C_{ij} is the cost paid to serve a customer in location i by the facility in location j; it includes transport costs and possible operating costs
- a_j is the fixed cost paid to establish a facility in location j

The goal of problem (1)-(3) is therefore to find a spatial arrangement of facilities, L, and an assignment of customers to them, X, which minimize total cost.

A well known property of (1)-(3) is that customers are assigned to the least-cost facility. Indeed, for L fixed, problem (1)-(3) is separable for each customer location, and the subproblem associated with a customer location i is

$$\min_X \sum_{j \in L} x_{ij} C_{ij}$$

s.t.

$$\sum_{j \in L} x_{ij} = P_i$$

with the trivial solution

$$x_{ij} = \begin{Bmatrix} P_i, \text{ if } C_{ij} = \min\limits_k C_{ik} \\ 0, \text{ otherwise} \end{Bmatrix} \tag{4}$$

The assignment rule (4) can be given two different interpretations. If the assignment of customers to facilities is controlled by the decision maker, it is the optimal solution to the total cost minimizing problem. If customers are free to choose the facility they want, it is a model for rational choice behavior, stating that customers choose according to the minimum-cost criterion. The last interpretation is the point of departure for the generalization which will be developed.

Let it be assumed that facilities belong to the second class discussed above, so that customers choose facilities, have to travel in order to get served, and pay for transport and possible operating costs. The behavior implied by (4) is not only rational but also deterministic, since no uncertainty, lack of information, or variation in tastes and preferences is accounted for. On the other hand, everyday experience suggests that cost is not the only criterion determining customer choice. If the assumption of rational choice behavior is kept, it can be said that customers maximize their utility, which is basically a function of costs but also includes many other variables. Moreover, not all the components of this utility can be measured easily, since they vary greatly among individuals, and possibly among different points in time for the same individual.

A simple model which accounts for the above requirements is as follows. Let it be assumed that the utility of choosing a facility in location j, for a customer in location i, can be split in two parts, a measured (or deterministic) one and a random one

$$\tilde{u}_{ij} = v_{ij} + \tilde{y}_j \tag{5}$$

where

- $\tilde{u}_{ij}$ is the total utility of choosing a facility in location j, for a customer in location i
- v_{ij} is the measured part of utility
- $\tilde{y}_j$ is the random part of utility

The $\tilde{y}_j$ are random variables with a joint distribution function, from which all customers are assumed to draw.

Each customer in i will choose in order to maximize his utility, that is, he will solve the problem

$$\max_{j \in L} \tilde{u}_{ij} \tag{6}$$

But the quantity (6) is a random variable. The expected utility for a customer in i is therefore

$$U_i(L) = E\left(\max_{j \in L} \tilde{u}_{ij}\right) \tag{7}$$

where E denotes the expectation operator. The random-utility counterpart of problem (1)-(3) is therefore

$$\max_{L} \sum_{i} P_i U_i(L) - \sum_{j \in L} a_j \tag{8}$$

Notice that the *max* operator has replaced the *min* operator, since utility maximizing rather than cost minimizing, is used.

Notice also that constraints (2) and (3) have been dropped, and the array X has disappeared, since the assignment subproblem has already been solved by introducing (6) and (7). Problem (8) looks therefore simpler than problem (1)-(3). What price is paid for this "simplicity"? Basically, the linear-integer programming features of (1)-(3) are lost. While problem (1)-(3) can be easily solved by dual ascent and related methods (Bilde and Krarup, 1977; Erlenkotter, 1978; Wolsey, 1980), problem (8) cannot. It is a general combinatorial optimization problem for which satisfactory exact algorithms are not known, except for problems of small size. However, the random-utility assumption will be shown to give problem (8) some special properties, which can be exploited to develop fairly good heuristic approaches.

3. BASIC PROPERTIES OF THE ADDITIVE RANDOM-UTILITY MODEL

In order to operationalize the model loosely introduced in Section 2, some basic notions from random-utility theory are needed. Most results presented in this section can be found in the recent literature (Domencich and McFadden, 1975; Williams, 1977; Daly, 1978; Ben-Akiva and Lerman, 1978).

Some of them are new, and specially tailored to give insight into problem (8).

The following notation will be used. An upper case X,Y, etc., is a vector; a lower case x,y, etc., is either a scalar or a vector with all elements equal to x,y, etc. Therefore

$$x \pm Y = (x \pm y_1, x \pm y_2, \ldots, x \pm y_n)$$

Functions of vectors are defined element by element, e.g.,

$$e^Y = (e^{y_1}, e^{y_2}, \ldots, e^{y_n})$$

$$\log Y = (\log y_1, \log y_2, \ldots, \log y_n)$$

$\tilde{x}_j$ is a *random variable*, usually the jth element of a random vector. $\tilde{X}$ is a *random vector*, whose *distribution function* is

$$F(X) = Pr\{\tilde{X} \leq X\} = Pr\{\tilde{x}_1 \leq x_1, \ldots, \tilde{x}_n \leq x_n\} \tag{9}$$

All the distribution functions considered in this paper are assumed to be continuous and to have derivatives of any order. The *conditional density* of $\tilde{x}_j$ is denoted by $F_j(X)$ and defined by equations

$$F_j(X)dx_j = Pr\{x_j \leq \tilde{x}_j \leq x_j + dx_j,\ \tilde{x}_k \leq x_k,\ \forall k \neq j\} \tag{10}$$

or

$$F_j(X) = \frac{\partial F(X)}{\partial x_j} \tag{11}$$

The *extreme-value distribution* of $\tilde{X}$ is the distribution function of the random variable

$$\max_j \tilde{x}_j \tag{12}$$

It is well known from the theory of extreme order statistics (see Galambos, 1978, for instance) that the following equation holds true

$$Pr\{\max_j \tilde{x}_j \leq x\} = Pr\{\tilde{x}_1 \leq x, \tilde{x}_2 \leq x, \ldots, \tilde{x}_n \leq x\} \tag{13}$$

therefore, from (13) and (9) it follows

$$Pr\{\max_j \tilde{x}_j \leq x\} = F(x) \tag{14}$$

The *extreme-value probability* for element $\tilde{x}_j$ is the probability

$$Pr\{\tilde{x}_j \geq \tilde{x}_k,\ \forall k\} \tag{15}$$

From equation (10) it is easily derived that

$$\Pr\{\tilde{x}_j \geq \tilde{x}_k, \forall k\} = \int_{-\infty}^{\infty} F_j(x)dx \tag{16}$$

Let now the additive random-utility model be introduced. In order to simplify notation, the subscript labeling customer location will be dropped, since it will be kept constant. From (5), the total utility when alternative j is chosen is

$$\tilde{u}_j = v_j + \tilde{y}_j$$

Let F(Y) be the distribution function of $Y = \{\tilde{y}_j\}$. Then the distribution function of $\tilde{U} = \{\tilde{u}_j\}$ is

$$\Pr\{\tilde{U} \leq X\} = \Pr\{V + \tilde{Y} \leq X\} = \Pr\{\tilde{Y} \leq X - V\} = F(X - V) \tag{17}$$

where $V = \{v_j\}$ is the vector of measured utilities.

The extreme value distribution for $\tilde{U}$ is, according to (14)

$$\Pr\{\max_j \tilde{u}_j \leq x\} = F(x - V) \tag{18}$$

and its first moment is

$$\phi(V) = \int_{-\infty}^{\infty} x \; dF(x - V) \tag{19}$$

Equation (19) gives the expected utility for a rational customer.

The extreme value probability for alternative j is the probability that the facility in j is chosen. According to (16) it is given by

$$q_j(V) = \int_{-\infty}^{\infty} F_j(x - V) \; dx \tag{20}$$

where $F_j(X)$ is defined by (11).

The following propositions state some noteworthy properties of the additive random-utility model.

PROPOSITION 1. (Translation of expected utility)

$$\phi(V+\alpha) = \phi(V) + \alpha \tag{21}$$

Proof

$$\phi(V+\alpha) = \int_{-\infty}^{\infty} x dF(x-V-\alpha) =$$

$$= \int_{-\infty}^{\infty} y dF(y-V) + \alpha \int_{-\infty}^{\infty} dF(y-V) = \phi(V) + \alpha$$

(The transformation $y = x - \alpha$ has been used.)

PROPOSITION 2. (Translational invariance of choice probabilities)

$$q_j(V+\alpha) = q_j(V) \tag{22}$$

Proof

$$q_j(V+\alpha) = \int_{-\infty}^{\infty} F_j(x-V-\alpha)\, dx = \int_{-\infty}^{\infty} F_j(y-V)\, dy = q_j(V)$$

(The transformation $y = x - \alpha$ has been used.)

PROPOSITION 3. (Hotelling consistency of expected utility)

$$\frac{\partial\phi(V)}{\partial v_j} = q_j(V) \tag{23}$$

Proof

$$\frac{\partial\phi(V)}{\partial v_j} = \frac{\partial}{\partial v_j}\int_{-\infty}^{\infty} x dF(x - V) = \int_{-\infty}^{\infty} x d\frac{\partial F(x - V)}{\partial v_j} =$$

$$= - \int_{-\infty}^{\infty} x dF_j(x - V) =$$

$$= - xF_j(x - V)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} F_j(x - V)\ dx = q_j(V)$$

[Equations (11) and (20) have been used.]

PROPOSITION 4. (Logit-like representation of choice probabilities)

The Function

$$\psi(W) = \exp\phi(\log W) \tag{24}$$

is linear homogeneous, and

$$q_j(V) = \frac{e^{v_j}\ \psi_j(e^V)}{\psi(e^V)} \tag{25}$$

where

$$\psi_j(W) = \frac{\partial\psi(W)}{\partial w_j} \tag{26}$$

Proof:

$\psi(\alpha W) = \exp\phi(\log W + \log \alpha) = \exp[\phi(\log W) + \log \alpha] = \alpha\,\psi(W)$

hence linear homogeneity follows

[Property (21) has been used.]

To prove (25), let (24) be used to write the expected utility in the form

$\phi(V) = \log \psi(e^V)$

then

$$q_j(V) = \frac{\partial \phi(V)}{\partial v_j} = \frac{e^{v_j}\psi_j(e^V)}{\psi(e^V)}$$

[Property (23) has been used.]

PROPOSITION 5. (Equivalence with an entropy maximizing problem)

$$\phi(V) = \max_{Q} \left\{ -\sum_j q_j \log \frac{q_j}{f_j} : \sum_j q_j = 1 \right\} \tag{27}$$

where

$Q = \{q_j\}$

$$f_j = e^{v_j}\,\psi_j(e^V) \tag{28}$$

and $\psi_j(\cdot)$ *is defined by* (26)

Proof:

Define

$$H(Q) = -\sum_j q_j \log \frac{q_j}{f_j}$$

$$H_j(Q) = \frac{\partial H(Q)}{\partial q_j} = -\log \frac{q_j}{f_j} - 1$$

$$H_{jk}(Q) = \frac{\partial^2 H(Q)}{\partial q_j \, \partial q_k} = \begin{cases} -\frac{1}{q_j}, & k = j \\ 0, & k \neq j \end{cases}$$

Then for any $Y = \{y_j\}$

$$\sum_{jk} y_j y_k H_{jk}(Q) = -\sum_j \frac{y_j^2}{q_j} \leq 0$$

hence $H(Q)$ is concave.

The solution to the mathematical program

$$\max_Q \{H(Q);\ \sum_j q_j = 1\}$$

satisfies the conditions

$$H_j(Q) - \nu = 0$$

or

$$q_j = \alpha f_j$$

where ν is a Lagrange multiplier and $\alpha = e^{-(1+\nu)}$.

Elimination of α by the constraint yields

$$q_j = \frac{f_j}{\sum_j f_j}$$

and the concavity of H(Q) ensures that this solution globally maximizes H(Q). But from (28)

$$q_j = \frac{e^{v_j}\,\psi_j(e^V)}{\sum_j e^{v_j}\,\psi_j(e^V)} = \frac{e^{v_j}\,\psi_j(e^V)}{\psi(e^V)}$$

[The linear homogeneity of $\psi(\cdot)$ has been used.] Hence the q_j obtained by solving the above mathematical program are the same as the ones given by (25). Substitution into H(Q) yields

$$H(Q) = -\sum_j q_j \log \frac{1}{\psi(e^V)} = \log \psi(e^V) = \phi(V)$$

[Equation (24) has been used.]

PROPOSITION 6. (Nondecreasing submodularity of expected utility)

The set function

$$U(L) = E(\max_{j \in L} \tilde{u}_j) \tag{29}$$

is submodular nondecreasing.

Proof

The function

$$\tilde{U}(L) = \max_{j \in L} \tilde{u}_j$$

is submodular nondecreasing (Nemhauser, Wolsey, and Fisher, 1978)

This property can be stated as

$$\tilde{U}(S \cup \{j\}) - \tilde{U}(S) \geq \tilde{U}(T \cup \{j\}) - \tilde{U}(T) \geq 0$$

for all S, T, j , $S \subseteq T$, $j \notin T$

Applying the expectation operator to both sides of the above inequality one gets

$$U(S \cup \{j\}) - U(S) \geq U(T \cup \{j\}) - U(T)$$

hence $U(L)$ is submodular nondecreasing.

Besides the above propositions another property will be useful, stating the relationship between (19) and (29). The relationship is

$$U(L) = \lim_{\substack{v_j \to -\infty \\ j \notin L}} \phi(V) \tag{30}$$

Most propositions above are self-explanatory and need only a few comments.

Propositions 1 and 2 state that choice behavior is unaffected by shifts in utilities. In other words, the "zero" of the utility scale may be set arbitrarily.

Proposition 3 is perhaps the most important one, since it states the integrability condition for choice probabilities. The importance of this property is well known, and it has been discussed by many authors (for instance, Williams, 1977; and Daly, 1978).

Proposition 4 gives a useful representation of choice probabilities, and shows how the general additive random-utility model is related to the well-known logit model (included as a special case). Equation (25) generalizes a result first obtained by McFadden (1978) under much more restrictive assumptions.

Propositions 5 and 6 state some useful properties to analyze the optimal location problem. Equation (27) relates the additive random-utility model to the wide and well-known class of entropy maximizing models (Wilson, 1970; Wilson and Senior, 1974; Willekens, Por, and Raquillet, 1979). The submodularity property relates the location problem discussed in this paper with the problems analyzed by Nemhauser, Wolsey, and Fisher (1978), for which some approximate optimization results have been produced.

4. BASIC PROPERTIES OF SUBMODULAR SET FUNCTIONS

Most results given in the preceding section were aimed at providing theoretical insight and economic justifications for the behavioral models based on the additive random-utility assumption. At least one of them (namely proposition 6), however, is specially tailored on the more operational goal of providing techniques to solve problems of type (8). Since the submodularity property is the key premise to all subsequent results (the non-decreasing property is not essential), it will be shown to hold true for functions built like (8). First, since the functions $U_i(L)$ are submodular because of proposition 6 and since any positive linear combination of submodular functions is submodular (Nemhauser, Wolsey, and Fisher, 1978), the function

$$g(L) = \sum_i P_i U_i(L) \tag{31}$$

is submodular. Let the set function

$$G(L) = g(L) - \sum_{j \in L} a_j \tag{32}$$

be defined. Then

$$G(L \cup \{j\}) - G(L) = g(L \cup \{j\}) - g(L) - a_j \tag{33}$$

Because of submodularity of g(L), the inequality

$$g(L \cup \{j\}) - g(L) \geq g(T \cup \{j\}) - g(T) \tag{34}$$

holds true for all L,T,j, $L \subseteq T$, $j \notin T$. Subtracting a_j on both sides of (34) and using (33), the inequality

$$G(L \cup \{j\}) - G(L) \geq G(T \cup \{j\}) - G(T) \tag{35}$$

for all L,T,j, $L \subseteq T$, $j \notin T$ follows, thus stating submodularity of G(L). The general optimization problem is therefore to find the unconstrained maximum of a submodular function

$$\max_L = G(L) \tag{36}$$

If G(L) were also nondecreasing the solution to problem (36) would be trivial, i.e., a facility should be established in *all* possible locations. However, the presence of (usually nonzero) fixed charges a_j prevents the nondecreasing property from being true. By means of a_j, economies of scale are introduced, and the higher their value the smaller will be the number of facilities to be established.

The properties of submodular set functions, in the light of combinatorial optimization, have been thoroughly analyzed recently by Cornuejols, Fisher, and Nemhauser (1977); Nemhauser, Wolsey, and Fisher (1978); Fisher, Nemhauser, and Wolsey (1978); and Wolsey (1980). The reader is referred to their work for further results. Here only a few statements will be needed, which do not go far beyond alternative definitions of the submodularity property itself.

Let the following quantities be defined as

$$\rho_j(S) = G(S \cup \{j\}) - G(S) \qquad \text{the incremental value of adding element } j \text{ to the set } S, \text{ for } j \notin S \tag{37}$$

The $\rho_j(S)$ give information on how the value of the objective function changes when the solution set S is augmented by one location j.

Often use will be made of the change in the value of the objective function due to the deletion of one location $j \in S$.

From equation (37) it follows that this change is given by

$$G(S - \{j\}) - G(S) = -[G(S) - G(S - \{j\})] = -\rho_j(S - \{j\})$$

The statements on submodular set functions which will be used in the rest of the paper are summarized in the following proposition.

PROPOSITION 7.

Each of the following inequalities defines a submodular set function

$$\rho_j(S) \geq \rho_j(T) \;, \qquad \textit{for all} \quad S \subseteq T \quad \textit{and} \quad j \notin T \tag{38}$$

$$G(T) \leq G(S) + \sum_{j \in T-S} \rho_j(S) - \sum_{j \in S-T} \rho_j(S \cup T - \{j\}) \quad \text{for all } S \text{ and } T \tag{39}$$

$$G(T) \leq G(S) + \sum_{j \in T-S} \rho_j(S) \qquad \text{for all } S \subseteq T \tag{40}$$

$$G(T) \leq G(S) - \sum_{j \in S-T} \rho_j(S - \{j\}) \qquad \text{for all } T \subseteq S \tag{41}$$

The proof of the above proposition is found in the four references listed above.

For an intuitive feeling of how a submodular set function looks, compared to the usual functions of sets of real numbers, one could note that the submodularity property is a kind of generalization of concavity. Indeed, if the quantities $\rho_j(S)$ are considered as analogous to derivatives, property (38) suggests the notion of decreasing return to scale; i.e., adding one more element to the solution set is always more beneficial for smaller sets than for bigger ones.

In order to devise techniques to solve problem (36), a first step is to impose some restrictive requirements on possible solution sets. One is thus led to look for some analog of *local maxima*, i.e., sets which are optimal within some suitably defined neighborhood.

The simplest local properties for maxima are stated in the following definitions

DEFINITION 1.

A weak (or first order) local maximum of a set function $G(S)$ *is a set* S *such that*

$$\begin{cases} \rho_j(S) \leq 0 \,, & \text{for all } j \in S \\ \rho_j(S - \{j\}) \geq 0 \,, & \text{for all } j \in S \end{cases} \tag{42}$$

DEFINITION 2.

A strong (or second order) local maximum of a set function G(S) *is a set* S *such that*

$$\begin{cases} \rho_j(S) = \min_{k \in S \cup \{j\}} \rho_k(S \cup \{j\} - \{k\}) \leq 0 \quad , \quad \textit{for all } j \notin S \\ \textit{or, equivalently} \\ \rho_j(S - \{j\}) = \max_{k \notin S - \{j\}} \rho_k(S - \{j\}) \geq 0 \quad , \quad \textit{for all } j \in S \end{cases} \tag{43}$$

Stated in words, a weak local maximum is a set which cannot be improved by the addition or deletion of a single element. A strong local maximum is a weak local maximum too; moreover, it cannot be improved by any paired interchange between two elements.

Of course generalizations to higher order local maxima can be devised, requiring stability under interchange of n elements, for $n > 2$. But the numerical results shown later seem to suggest that going beyond the second order is seldom needed.

If G(S) is submodular, the following propositions can be stated for weak local maxima.

PROPOSITION 8. (Dominance over all supersets)

If S *is a weak local maximum and* $S \subseteq T$*, then*

$$G(S) \geq G(T) \tag{44}$$

<u>*Proof*</u>

from (42) $\rho_j(S) \leq 0$, for $j \in T - S$; substitution into (40) yields

$$G(T) \leq G(S)$$

PROPOSITION 9. (Dominance over all subsets)

If S *is a weak local maximum and* $T \subseteq S$, *then*

$$G(S) \geq G(T) \tag{45}$$

Proof

from (42) $\rho_j(S - \{j\}) \geq 0$, for $j \in S - T$;
substitution into (41) yields

$$G(T) \leq G(S)$$

The above two simple propositions surprisingly widen the neighborhood dominated by a weak local maximum. What they state is that if a weak local maximum is detected, then any of its subsets or supersets can be dropped from further search.

Two further propositions will be useful to build rules of improvement for a possible trial solution.

PROPOSITION 10. (Nondecreasing point detection)

If $\rho_j(T) \geq 0$ *for some* $j \notin T$, *then no* $S \subset T$ *is a weak local maximum*

Proof

from (38) , $\rho_j(S) \geq \rho_j(T) \geq 0$, for some $j \notin S$

which contradicts (42).

PROPOSITION 11. (Nonincreasing point detection)

If $\rho_j(T - \{j\}) \leq 0$ *for some* $j \in T$, *then no* $S \supseteq T$ *is a weak local maximum*

Proof

since $S - \{j\} \supseteq T - \{j\}$, from (38)

$\rho_j(S - \{j\}) \leq \rho_j(T - \{j\}) \leq 0$, for some $j \in S$,

which contradicts (42).

Propositions 10 and 11 are useful for building possible tree-search algorithms. If the search goes downward, that is building small and smaller subsets, then Proposition 10 may be used as a rule to stop searching in a subset. If the search goes upward, that is building bigger and bigger supersets, then Proposition 11 may be used as a rule to stop searching in a superset.

Proposition 8-11 refer to weak local maxima. One would hope to find some more restrictive conditions when strong local maxima are used. Unfortunately, this does not seem to be the case. No further property of strong local maxima has been found as yet, except those stated in the definition.

5. SOME ALGORITHMS

To be safe, one could approach problem (36) via some tree-search scheme. This would surely yield the exact optimal solution, but the computing time required might become prohibitive.

On the other hand, one could try to develop some heuristic approaches exploiting submodularity as far as possible, hoping to find at least a good local maximum. This would usually require negligible computing time, but unfortunately submodularity alone does not provide any sufficient condition for a global maximum.

However, the successful experience with simple heuristics applied to maximizing submodular functions (although of a less general nature than the one considered here, see Wolsey, 1980) suggests that the actual performance of such heuristics could be very good. In an exploratory stage of research, it is therefore worth checking the results obtained with heuristic algorithms against the result obtained with an exact method. If numerical experience shows that heuristics work almost or just as good as the exact algorithms, then good reasons for further theoretical investigation are provided.

A tree search can be easily organized by exploiting propositions 7-11. Suppose, for instance, the search goes downward, i.e., building smaller subsets. Then the search within a given subset S can be stopped when:

a. S is a local maximum;
b. S is a nondecreasing point (see Proposition 10);
c. an upper bound to G(T), $T \subseteq S$, as computed by (41), is less than the highest value of the objective function found so far.

For condition c, an upper bound can be computed from (41) as follows

$$G(T) \leq G(S) - \sum_{j \in S^*} \rho_j(S - \{j\}) \quad , \qquad \text{for all } T \subseteq S$$

where

$$S^* = \left\{ j : j \in S,\ \rho_j(S - \{j\}) < 0 \right\}$$

The procedure outlined above is based on the submodularity property only. Another method should be mentioned, which has been developed and used in Erlenkotter and Leonardi (forthcoming). This method, referred to as INTLOC, does not use any submodularity at all, but rather works with the continuous relaxation of (36) to get approximate integer solutions and bounds in the tree search.

More precisely, let the function

$$\Omega(X) = \sum_i P_i\, \phi_i(V + \log X) - \sum_j x_j\, a_j \tag{46}$$

be defined over all real nonnegative vectors $X = \{x_j\}$, where the $\phi_i(V)$ are the functions defined by equation (19). Then, if

$$x_j = \begin{cases} 1, & j \in L \\ 0, & j \notin L \end{cases}$$

it follows from (30), (31), and (32) that

$$\Omega(X) = G(L) \tag{47}$$

Therefore the solution to the mathematical program

$$\max_X \{\Omega(X) : 0 \leq x_j \leq 1 \, , \forall j\} \tag{48}$$

provides an upper bound to the optimal solution of (36).

If problem (48) is solved by a simple Frank-Wolfe method, feasible integer solutions are also generated as a by-product at each iteration. The best of such solutions can therefore be taken as an approximation to the solution of the problem

$$\max_X \{\Omega(X) : x_j \in \{0,1\} \, , \forall j\} \tag{49}$$

which is of course equivalent to (36).

Such a procedure can be easily embedded into a branch-and-bound search scheme. In its present version INTLOC is not designed to work with a general function of type (46). It assumes a special structure, the same one described in Section 6 of this paper.

The heuristic procedures will be kept as simple as possible. The first proposed heuristic tries to find a weak local maximum, while monotonically increasing the value of the objective function. This can be done as follows. Assume a given iteration starts with a trial solution set S which does not meet conditions (42). This means that the current value of the objective function, G(S), can be increased by adding or dropping one element. Let the change giving the highest increase in the objective function be introduced and repeat the step. The procedure stops when no element can be conveniently added or deleted, that is, when a weak local maximum is detected. This procedure will be stated formally:

HEURISTIC 1. (Ascent towards a weak local maximum)

1. guess a starting S

2. find $\rho_i(S) = \max_{j \notin S} \rho_j(S)$ and

$$\rho_k(S - \{k\}) = \min_{j \in S} \rho_j(S - \{j\})$$

3. if $\rho_i(S) \leq 0$ and $\rho_k(S - \{k\}) \geq 0$ stop

if $\rho_i(S) > -\rho_k(S - \{k\})$ replace S by $S \cup \{i\}$

if $\rho_i(S) \leq -\rho_k(S - \{k\})$ replace S by $S - \{k\}$

4. go to 2

The starting guess is quite arbitrary, and different starts may lead to different local maxima. When no better start is available, a reasonable one is

$$S = \left\{ i:\ G(\{i\}) = \max_k G(\{k\}) \right\} \tag{50}$$

The second heuristic tries to find a strong local maximum, while monotonically increasing the value of the objective function. It works as follows. Suppose a given iteration starts with a weak local maximum. If at least one paired interchange improves the value of the objective function, a new and better weak local maximum is produced by restarting Heuristic 1 with the interchanged solution, and the iteration is repeated. The procedure stops when no paired interchange can improve the current weak local maximum, which will therefore be a strong local maximum too.

This procedure will be stated formally:

HEURISTIC 2. (Ascent towards a strong local maximum)

1. use Heuristic 1 with any start to generate a weak local maximum S

2. set $S_o = S$ and $T = S$

3. if $T = \emptyset$ stop

4. choose some $j \in T$ and replace T by $T - \{j\}$

5. use Heuristic 1 with start $S_o - \{j\}$ to generate a weak local maximum S

6. if $S \neq S_o$ go to 2

7. go to 3

Just as for Heursitic 1, no uniqueness of the final solution is assured in principle. However, a later section on numerical experiments will show that the performance of Heuristic 2 is surprisingly better than might be foreseen from theory.

6. A SPECIAL CASE

The random-utility model considered so far is quite general, since no restrictive assumption has been introduced for the distribution functions F(Y).

A simplifying assumption often found in the literature (Domencich and McFadden, 1975; Daly, 1978; and Van Lierop and Nijkamp, 1979) is that Y is a sequence of independent identically distributed random variables with a common distribution function

$$D(x) = \exp\left[-e^{-(\alpha x - \beta)}\right] , \qquad \alpha > 0 \tag{51}$$

where α and β are called the shape and the location parameters, respectively. This distribution is known as the Gumbel distribution, and it plays an important role in extreme order statistics (Gumbel, 1958; and Galambos, 1978).

The mean of (51) is known to be

$$\mu = \int_{-\infty}^{\infty} x dD(x) = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \tag{52}$$

where $\gamma = 0.5772157...$ is Euler's constant.

Because of independency, F(Y) takes the form

$$F(Y) = \prod_j D(y_j) = \exp\left[-e^{\beta} \sum_j e^{-\alpha y_j}\right]$$

and the extreme value distribution is, according to (18)

$$F(x - V) = \exp\left[-e^{\beta} h(V) e^{-\alpha x}\right] = \exp\left\{-e^{-[\alpha x - \beta - \log h(V)]}\right\} \tag{53}$$

where

$$h(V) = \sum_j e^{\alpha v_j} \tag{54}$$

Comparison of (53) with (51) shows that F(x - V) is still a Gumbel distribution, with shape parameter α and location parameter $\beta + \log h(V)$. Therefore, according to (19) and (52), the expected utility for a customer in a given origin is

$$\phi(V) = \int_{-\infty}^{\infty} x dF(x - V) = \frac{1}{\alpha} \log h(V) + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \tag{55}$$

Since, because of Propositions 1 and 2, a shift in the origin of the utility scale does not affect customer choice behavior, additive constants can be dropped from (55) and the expected utility can be redefined as

$$\phi(V) = \frac{1}{\alpha} \log h(V) \tag{56}$$

The choice probabilities can be found using (23) and (54). They are

$$q_j(V) = \frac{\partial \phi(V)}{\partial v_j} = \frac{e^{\alpha v_j}}{\sum\limits_j e^{\alpha v_j}} \tag{57}$$

This is the well known multinominal logit model, extensively used in transport demand analysis (Domencich and McFadden, 1975; Williams, 1977).

Let all these results be introduced in problem (36). From (30) one gets

$$U_i(L) = \frac{1}{\alpha} \log \sum_{j \in L} e^{\alpha v_{ij}}$$

Substitution in (31) yields

$$g(L) = \frac{1}{\alpha} \sum_i P_i \log \sum_{j \in L} e^{\alpha v_{ij}} \tag{58}$$

and substitution of (58) in (32) gives the objective function

$$G(L) = \frac{1}{\alpha} \sum_i P_i \log \sum_{j \in L} e^{\alpha v_{ij}} - \sum_{j \in L} a_j$$

Since no generality is lost if the fixed charges are rescaled by α and G(L) is multiplied by α, the objective function can be redefined as

$$G(L) = \sum_i P_i \log \sum_{j \in L} f_{ij} - \sum_{j \in L} a_j \tag{59}$$

where

$$f_{ij} = e^{\alpha v_{ij}}$$

Under the above assumptions, problem (36) takes the form

$$\max_L \sum_i P_i \log \sum_{j \in L} f_{ij} - \sum_{j \in L} a_j \tag{60}$$

Problem (60) may be given an alternative formulation, in the spirit of Proposition 5. Since this formulation is closely related to the one extensively used by Coelho (1977, 1979, 1980a, and 1980b) the following will be called

PROPOSITION 12. (Coelho representation)

$$\max_L \left\{ \sum_i P_i \log \sum_{j \in L} f_{ij} - \sum_{j \in L} a_j \right\} =$$

$$= \max_{L,Q} \left\{ -\sum_i \sum_{j \in L} Q_{ij} \log \frac{Q_{ij}}{P_i f_{ij}} - \sum_{j \in L} a_j : \sum_{j \in L} Q_{ij} = P_i \,, \forall_i \right\} \tag{61}$$

Proof

the proof parallels the one of Proposition 5.

In order to implement the algorithms of Section 5, computational formulas for the incremental values $\rho_j(S)$ and $\rho_j(S - \{j\})$

are needed. From (37) and (59) these are

$$\rho_j(S) = \sum_i P_i \log\left(1 + \frac{f_{ij}}{\sum_{k \in S} f_{ik}}\right) - a_j \quad , \qquad j \notin S \tag{62}$$

$$\left.\begin{aligned} \rho_j(S - \{j\}) = \sum_i P_i \log\left(1 + \frac{f_{ij}}{\sum_{k \in S} f_{ik} - f_{ij}}\right) - a_j \quad , \\ j \in S \end{aligned}\right\} \tag{63}$$

7. SOME NUMERICAL RESULTS

The exact and heuristic algorithms of Section 5 have been applied to a test problem. The problem data refer to the location of high schools in Turin, Italy. A detailed description of the data and the geographical setting can be found in Leonardi and Bertuglia (1981) and in Ermoliev, Leonardi, and Vira (1981), where they have been used to test somewhat different optimal location algorithms (namely, a problem with constraints on facility size and a stochastic programming approach). The input data used in the tests are reported in the Appendix. From the point of view of numerical testing, salient features of the problem are:

a. Turin is divided into 23 districts, each district being both a place of residence of high school demand and a possible location for a high school facility. No limitation is placed on the choice of districts of destination, so that in principle a customer living in a given district can use a facility in any other district.
b. The utility has been simply set equal to travel time changed in sign, i.e.:

$$f_{ij} = e^{-\alpha c_{ij}} \tag{64}$$

where

c_{ij} is the travel time from district i to district j, in minutes

Travel times are measured on the public transport network. The within-district travel time has been given a standard value of five minutes, in accordance with empirical findings.

The parameter α has been given a value

$$\alpha = 0.194 \tag{65}$$

According to more recent origin-destination surveys on home-to-school trips, this parameter should be set equal to values around 0.15.

However, the value (65) has been kept, in order to make results comparable with previous studies (Erlenkotter and Leonardi, forthcoming).

c. The quantities P_i appearing in (60) are the number of high school students living in each district, as of 1977 (Provincia di Torino, 1978). The values of these quantities range from 500 to 2,500, approximately.

d. The fixed charges a_j have been set equal for all districts

$$a_j = a$$

and the algorithms have been tested for values ranging from 500 to 5,000. This range has been used for testing purposes only [the difficulty of (60) usually increases with a], and there is no claim of realism in it.

The results of the numerical testing are summarized in Table 1. The results for Heuristics 1 and 2 have been produced with the standard start (50).

Table 1 is surprising in many ways. First of all, it shows the unexpected power of Heuristic 1. If the 2nd and 4th column are compared, it is seen that Heuristic 1 failed to find the exact solution only for a fixed charge of 2,500. Even in that case, the value of the objective function (boxed in Table 1) is very close to the optimal one. Comparison of the 3rd and 4th column is also revealing. The 3rd column shows the results obtained in the first stage of the INTLOC algorithm, that is the best lower bounds produced before entering the branch-and-bound routine. Except for a fixed charge value of 500 (for which all methods give the optimal solutions), these values are *always* nonoptimal. Even for a fixed charge of 2,500 the value found with Heuristic 1, although nonoptimal, is closer to the optimal one than the one produced with the first stage of INTLOC. Heuristic 1 seems therefore to definitely outperform the first stage of INTLOC.

Heuristic 1 has also been tried with starts different from (50), and the results (not reported here) have not always been so good. The procedure often terminated on nonoptimal local maxima. However, even in the worst cases, the comparison with the first stage of INTLOC has always been in favor of Heuristic 1.

The second important fact shown by Table 1 is the effectiveness of Heuristic 2, which seems to work much better than exact algorithms. This effectiveness is actually much higher than shown in the table. Heuristic 2 has been systematically tested with many different starts, and it *never* failed to reach the optimum for every fixed charge value.

The comparison of computing times is also revealing. For the exact search methods, only the CPU times for the method based on submodularity bounds have been recorded, but they are of the same order of magnitude as for INTLOC. The asterisks in the 6th column indicate that the tree search was too slow to reasonably wait for a solution, and the already available solution

Table 1. Comparison of the performance of exact and heuristic algorithms for the Turin high school test problem.

	Objective function (changed in sign)				CPU time (seconds)[a]		
Fixed charge	Tree search[b]	First stage of INTLOC	Heuristic 1	Heuristic 2	Tree search[c]	Heuristic 1	Heuristic 2
500	25899	25899	25899	25899	1.2	4.1	15.7
1000	37399	39692	37399	37399	1.3	4.2	15.7
1500	48685	48783	48685	48685	1.2	4.0	11.5
2000	59000	60663	59000	59000	1.9	3.9	13.3
2500	68097	69817	68324	68097	10.0	3.4	16.6
3000	76430	78214	76430	76430	50.3	3.1	11.5
3500	82725	84644	82725	82725	***	2.4	15.3
4000	87921	95237	87921	87921	***	2.4	9.9
4500	92730	99201	92730	92730	***	1.9	18.8
5000	96730	104167	96730	96730	***	2.2	19.8

[a] On the IIASA VAX computer.

[b] Both with the search based on submodularity bounds and with INTLOC [the Frank-Wolfe based branch-and-bound algorithm used in Erlenkotter and Leonardi (forthcoming)]; the results are the same, and correspond to the exact solution.

[c] Recorded CPU times refer to the search based on submodularity bounds, which provided an answer in reasonable time for fixed charges up to 3000. For higher fixed charge values the computing time was too long to wait for an answer. Although exact times for INTLOC were not recorded, it had to be run one night to provide the results for high fixed charge values.

previously produced with INTLOC (which was just as slow, although its users were less impatient) was kept.

The CPU time of tree search methods is very low for small fixed charge values, but it starts a fast increase after a fixed charge of 2,500 and becomes infeasible for fixed charge values higher than 3,000.

The CPU times for Heuristic 1 are all around 2 ~ 4 seconds, no matter what the value of the fixed charge. (In Table 1, they seem to decrease with the fixed charge, but this cannot be generalized, since it depends on the starting solution used.) Heuristic 1 seems therefore a bit slower than the tree search for small fixed charge values, but this is more than counterbalanced by its performance for high fixed charge values.

The CPU times for Heuristic 2 vary roughly between 10 and 20 seconds, again independently of the fixed charge value. They are not negligible, but they are still much lower than the ones for the tree-search algorithms in the hard cases. This must be coupled with the fact that apparently Heuristic 2 *never* fails to reach the optimum.

Although the numerical tests discussed above cannot be claimed to be exhaustive, they seem to be enough to state that

a. Heuristics 1 and 2 provide a uniformly superior start than any other method for a further tree-search refinement.
b. The high performance of Heuristic 2 deserves further theoretical investigation, along the lines of looking for possible sufficiency conditions.
c. For real sensitivity analysis problems, Heuristic 2 can be safely recommended; for a preliminary rough analysis, Heurisitc 1 can be enough.

Of course the validity of the above statements is provisionally limited to objective functions of the form (60), although it might be argued that the performance would be just as good with many other submodular set functions.

Statement a is perhaps the most intriguing one. Indeed the tree search starting with the results of Heuristic 2 has been attempted, but this has led to no significant improvement in its performance. In other words, recognizing the starting solution as an optimal one seems to take just as long as building an optimal solution from a bad start. Improving the tree search and tightening its bounds is therefore another subject for further investigation.

Statement b is related to the possible development of an effective duality theory for problems of type (60), or more generally of type (8). The effectiveness of dual relationships is the main reason for the successful algorithms recently developed to solve problems of type (1)-(3) (Bilde and Krarup, 1977; Erlenkotter, 1978), and this encourages the search for similar results for the more difficult problems (8) and (60).

Statement c reminds us that the mathematical interest in finding an exact solution is not necessarily realistic. Assuming an impatient decision maker would have used only Heuristic 1, according to Table 1 his maximum loss (in terms of relative difference between obtained and optimal objective value) would have been (for the fixed charge value of 2,500)

$$\frac{68324 - 68097}{68097} = 0.0033$$

that is about 0.3%. How many input data have measuring errors less than this figure?

8. CONLCUDING COMMENTS AND ISSUES FOR FUTURE RESEARCH

The main aim of the last sections of this paper has been to show the effectiveness of simple ascent heuristics when applied to nonconventional facility location problems (i.e., whose objective function is based on random-utility theory). The key property leading to these results seems to be the submodularity of the objective function. The problem formulation itself may lack some realistic features, since actual location problems often have constraints not considered here, like budget and size constraints. However, the successful solution of the simple problem is the main step towards solving more complex ones. Related work has shown (Leonardi and Bertuglia, 1981) that effective ascent heuristics can be built for problems with capacity constraints as well. Future progress can be expected in solving problems of the type recently explored by Coelho (1980b), including fixed charges, capacity constraints and p-median type constraints (that is, constraints on the number of facilities to be established).

Another strand of future research is the exploration of different objective functions. The numerical results reported in this paper are based on the special assumptions introduced in Section 6. Other forms of random-utility distributions could be tried, leading to other objective functions. However, the submodularity property would still hold because of Proposition 6.

Further mathematical investigation is also required, as already stated in Section 7. The reason why the proposed heuristics outperform other methods is far from being fully understood, although some theoretical results for related simpler problems (Wolsey, 1980) seem to suggest that such a good performance is not too surprising. This paper is only an exploratory one, showing a way which might be worth following.

APPENDIX: INPUT DATA USED FOR NUMERICAL TESTING (Turin High Schools)

Travel times (minutes) on public transport

District	High school demand (No. students)	to district / from district	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	1402	1	5	17	13	21	19	19	17	23	29	30	22	32	26	26	27	30	28	13	32	30	50	28	34
2	1332	2	14	5	14	25	21	26	24	29	18	21	24	32	31	32	36	41	41	28	48	36	50	13	28
3	1540	3	8	12	5	13	9	17	17	25	18	23	12	20	18	21	28	29	31	22	39	30	51	20	27
4	1153	4	21	25	14	5	7	22	31	39	21	22	13	14	9	23	30	34	40	31	44	44	61	26	27
5	1437	5	17	22	10	8	5	16	25	33	30	32	22	22	15	22	34	35	42	30	46	38	57	32	37
6	1428	6	17	27	18	20	15	5	26	32	35	40	30	31	23	9	28	30	33	27	39	39	56	37	43
7	1199	7	16	23	19	32	26	25	5	21	31	32	33	39	38	32	36	39	34	18	34	15	49	33	35
8	1224	8	22	29	26	38	32	30	20	5	37	39	39	46	41	37	38	40	33	17	34	24	37	34	42
9	1268	9	27	19	20	21	30	34	33	40	5	15	19	29	31	41	45	46	51	38	57	46	62	12	19
10	2356	10	28	19	24	25	34	39	34	41	15	5	9	20	32	43	48	51	52	39	58	46	58	16	18
11	2615	11	20	25	14	14	23	30	33	40	21	9	5	7	20	34	41	39	47	34	53	46	64	23	19
12	2301	12	32	35	23	16	22	35	43	50	32	20	8	5	20	40	48	42	53	44	57	55	73	34	27
13	2234	13	27	29	17	10	14	25	36	40	31	33	20	20	5	29	38	32	48	38	52	49	67	37	38
14	1834	14	24	33	21	25	22	12	35	41	41	43	33	29	28	5	29	28	42	33	45	48	64	43	46
15	1496	15	28	38	32	34	37	31	38	40	46	47	44	45	41	31	5	18	29	24	44	39	70	48	51
16	1572	16	32	42	31	39	37	33	37	40	48	48	36	30	31	30	15	5	28	24	37	39	73	52	52
17	1441	17	31	43	34	42	43	32	34	34	51	52	47	52	48	39	29	31	5	18	36	33	71	51	55
18	1422	18	14	29	24	32	31	28	20	18	38	39	34	44	37	35	23	26	18	5	17	17	57	39	42
19	1050	19	35	49	40	46	47	40	37	35	58	58	54	57	52	47	44	40	35	18	5	34	74	60	63
20	1045	20	32	37	36	48	42	40	18	26	46	47	50	55	54	47	39	41	34	18	35	5	58	48	51
21	516	21	53	53	54	66	63	60	52	40	63	60	68	73	71	67	71	75	73	59	75	56	5	59	68
22	822	22	23	13	22	26	31	35	33	34	11	17	23	33	36	42	45	50	49	37	57	45	54	5	30
23	2171	23	30	27	26	27	35	41	35	42	20	18	18	28	37	44	50	53	52	39	59	48	64	32	5

Travel time discount rate: $\alpha = 0.194$.

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