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Title: Optimal transition to greener production in a pro-environmental society

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Abstract

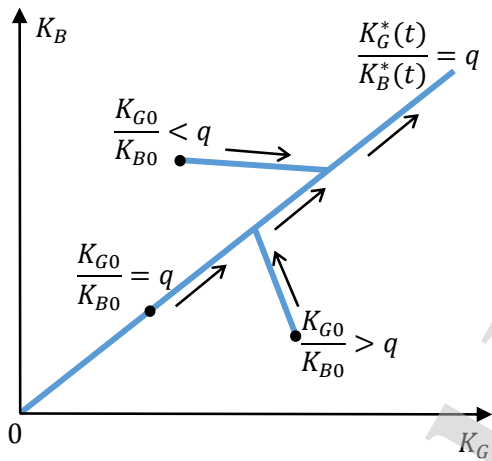
Achieving sustainable development requires a transition from the current production fashion that leads to the environmental degradation to a cleaner production. Such a substitution can be costly if the new technology is less productive. In this paper, we present a two-sector endogenous growth model that analyzes the potential of a transition from a more productive brown sector to a less productive green sector. The representative agent maximizes the weighted sum of the present value of the utility of consumption and the amenity value of green production. We derive a closed-form optimal solution using a suitable version of the Pontryagin Maximum Principle. For an economy, in which the brown sector dominates initially, we obtain that as long as the preference towards green production is positive, the optimal solution always has a single switching point and the following structure. Initially, the representative agent distributes the output between investment in the green sector and consumption, making no investment in the brown sector. This allows attaining a particular critical ratio between green and brown capital stocks in the fastest way. Once this ratio has been reached, the optimal solution switches to that, which allows both capitals to grow at the same rate. The representative agent has to sacrifice his/her consumption to invest in the green sector, especially in the initial period, which is due to the amenity that this sector provides. Under constant productivities, a full substitution of brown production by green production is not possible; rather, they co-exist and evolve proportionally. Three parameters are positively related to the ratio of the green capital stock: the social discount rate, the (augmented) productivity of the green capital, and a representative agent's preference towards the green production amenity.

Keywords: green growth, environmental quality, two-sector economic growth model, AK-model, discount factor

Highlights

- An optimal economic growth model with green and brown sectors is analyzed
- The objective function includes an amenity value of green production
- We find analytically an optimal path for consumption and investments in two sectors
- Full substitution of brown production is not optimal if the productivity gap is big
- Amenity of green production compromises consumption, but not growth

Graphical Abstract



1. Introduction

Achieving the environmentally sustainable development requires a substantial transition from the current production fashion that leads to the degradation of the natural environment to an environmentally friendly production (UNGA, 2015). For an individual country, such a transition can be costly, which raises concerns that it can compromise economic growth and welfare. A comprehensive valuation of both tangible and non-tangible environmental services (Costanza et al., 1997, 2014, 2017) has great potential to be used for their endogenization in economic policy optimization models (Mäler, 1991). This would allow to trade off the preservation of the environment with the required investments.

As a proxy, environmental economists introduce a “damage function” to account for economic losses to production from a lower environmental quality in their models (Manne and Richels, 2005; Maurer et al., 2013; Nordhaus, 2008). Limited available observations make it difficult to establish plausible functional forms for the damage function, let alone their parametrization (Bretschger and Pattakou, 2019; Weitzman, 2012). Researchers adopt conservative assumptions, which leads to model-based levels of the environmental degradation that are often too high from the sustainability perspective. For example, the welfare-optimal temperature rise in the most recent version of the DICE model is ca. 2.5 °C (to be achieved in 2060; Nordhaus, 2017). This is higher than the resolutions of the Paris agreement that states the goal to keep the warming below 2 °C in the 21st century (UNFCCC, 2015).

A number of stylized environmental economics models include a broad notion of the “environmental quality” to represent both tangible and a non-tangible value of the environment and to close the loop and operationalize the negative impact of the environmental degradation on the economy (Acemoglu et al., 2012; Cassou and Hamilton, 2004; Gerlagh and Keyzer, 2004;

Smulders et al., 2014). Again, empirical evidence is insufficient (Galeotti, 2007) and thus modelers examine different assumptions regarding environmental quality aiming to reveal general patterns. For example, Gerlagh and Keyzer (2004) assumed that the environmental quality is positively related to the stock of a natural resource, which is extracted and used for production, and whose depletion creates a disutility for consumers. Smulders et al. (2014) employed a similar but a more general natural resource management model, which additionally assumes that the resource extraction is costly and that the economic output depends also on the natural resource stock; thus, the resource depletion decreases productivity. Acemoglu et al. (2012) considered the environmental quality, which is negatively related to the output of a dirty sector; in one model variation analyzed in this paper, higher values of the environmental quality lowered the productivity of both clean and dirty sectors. Cassou and Hamilton (2004) considered a flow of environmental quality, which is used for production and is inversely related to the stock of the “dirty” capital.

Environmental pollution has two dimensions: domestic and transboundary. In the latter case, harmful emissions are generated in one country, but cause damage in another country by crossing borders with the flows of air or water. Some pollutants can travel over larger distances; greenhouse gases, most notably, carbon dioxide, mix up well in the atmosphere such that its concentration level is fairly uniform globally. The adverse effects of the global warming caused by the carbon dioxide (IPCC, 2014) depend on the vulnerability of a particular locality. Climate change vulnerability varies greatly across the globe (Maplecroft, 2016), but generally the Global South is more vulnerable than the Global North, while the Global North continues to emit the predominant share of the greenhouse gases (GHG) (Pardikar, 2020). A small and heavily polluting country of the Global North does not have enough incentive to transit from dirtier to cleaner

production if a large portion of the pollution that this country generates is transboundary. This argument motivates us to consider in this paper a different way to endogenize environmental concerns in economic models: instead of penalizing production that leads to the environmental degradation (implemented either through a damage function or a disutility from a lower quality of the environment), we consider rewarding production that is environmentally friendly.

Namely, we consider a two-sector economy, in which both brown and green sectors produce an identical final good. We assume a representative agent to be an eco-minded entrepreneur, who maximizes the present value of the future utility that depends on consumption and green production. Thus, the instantaneous utility function is augmented with the amenity value that the agent associates with the green sector output. We assume the mathematically convenient logarithm of the green sector output for the amenity value formula. The environmental impact of brown production is not included in our model. Nevertheless, the representative agent in our model has the intention to transit to a greener production technology, despite its lower productivity. This intention can be motivated by the agent's intrinsic wish to preserve the environment of the planet and, in the model, it is translated into the amenity value that the agent assigns to the green sector.

In the literature that employs stylized environmental economics models, the analysis of positive incentives and inspirations to develop clean production has not received sufficient attention yet. We are aware of only few papers featuring such models. Rauscher (2009) and Moser et al. (2013) assumed that the environmental quality is positively related to the abatement rate and included it in the utility function that is being maximized. In a similar way, a number of papers focusing on the management of natural resources incorporated the resource stock in the utility function (Ayong Le Kama, 2001; Gerlagh and Keyzer, 2004; Lafforgue, 2005; and Wirl, 2004). It is important to emphasize that these models combine positive incentives with penalizing

environmentally-unfriendly production. Our model, on the contrary, focuses on the positive incentives only.

Two approaches to model a transition from brown to green production are prevalent in the literature. In the first approach, it is assumed that the economic output is produced by a single technology, whose adverse impact on the environment can decrease over time due to investment in abatement, resource efficiency or productivity growth (Krautkraemer, 1985; Lans Bovenberg and Smulders, 1995; Rovenskaya, 2010; Smulders, 1995). In the second approach, two competing technologies are considered – a dirtier one and a cleaner one – and over time, the role of the dirtier production is decreasing and the role of a cleaner production is increasing due to appropriate investments in these sectors (Acemoglu et al., 2012, 2016; Boucekkine et al., 2013a; Cassou and Hamilton, 2004; Cunha-e-Sa et al., 2010). Our model is of the latter type, though in the above cited papers, investment aims to enhance sector productivity, while in our model we focus on capital accumulation. The model from (Boucekkine et al., 2013a) is similar to ours as they too employ a two-sector AK model without any productivity growth; however, that model includes pollution dynamics and focuses on an optimal time of the full, binary switch from a dirtier to a cleaner technology, which occurs in order to reduce damages from pollution.

By considering two sectors, green and brown, our model allows us to introduce the amenity of the green production and thus it allows us to model positive motivations for sustainability transformation. Usually, even when stylized, two-sector endogenous growth models are too complex to find their closed-form solutions analytically. Hence, researchers often resort to the steady-state analysis while the trajectories can only be obtained numerically. As we focus on the transition issue, time is a critical dimension, and thus, when building the model, we opted for the simplest possible one that can be solved analytically.

Mathematically, our model is an optimal control problem over an infinite time horizon. We use the logarithmic utility of consumption, which tends to infinity as the consumption rate tends to zero. We solve the model by applying a version of the Pontryagin Maximum Principle from (Besov, 2014), which provides necessary conditions for optimality for infinite-horizon optimal control problems with locally unbounded instantaneous utility functions. We construct a set of candidate, extremal solutions, and, by applying the sufficient conditions for optimality from (Seierstad and Sydsaeter, 1977), select a single extremal solution that satisfies these sufficient conditions and hence is optimal.

The rest of the paper is organized as follows. Section 2 introduces the model and formulates the optimal control problem over an infinite time horizon. Section 3 first presents a set of the necessary conditions for optimality from the Pontryagin Maximum Principle, which includes two regular control regimes and one singular control mode, see subsection 3.1. Subsection 3.2 presents the model assumptions. Next, in subsection 3.3 we consider a special case, in which the initial values of the green and brown sector capital stocks constitute precisely a certain critical ratio that is dependent on the model parameters. We show that in this special case, the optimal control is singular, and the green and brown sectors grow proportionally. In subsection 3.4, we consider a more realistic and general case when the proportion of the initial values of the green and brown capitals is lower than the critical ratio, i.e., the brown sector dominates initially. In this section, we formulate the main result of this paper, namely, in this case, the optimal control consists of two parts. Starting from the initial time moment and as long as the ratio between the stocks of the green and brown capitals remains lower than the critical, it is optimal to invest a maximal (from the necessary conditions) fraction of the total output into the green capital and nothing into the brown capital. This will allow the green capital stock to grow quickly and reach the critical ratio at some

time τ . After this time moment onwards, a singular control is optimal, and the green and brown sectors grow proportionally as in the special case from subsection 3.3. Subsection 3.5 specifies how the switching time τ could be computed numerically and provides necessary mathematical foundations for the suggested approach. Finally, section 4 contains some economic interpretations of the singular optimal path. Section 5 presents a short discussion.

2. Model

We consider a two-sector competitive market economy consisting of a continuum of identical, infinitely lived agents who act both as producers and as consumers (Cassou and Hamilton, 2004) of a homogenous final good (similar to capital-owning entrepreneurs from (Moser et al., 2013; Rauscher, 2009)). The population is assumed constant and normalized to 1. Agents use the stocks of green capital, $K_G(t)$, and brown capital, $K_B(t)$ to produce the corresponding outputs $Y_G(t)$ and $Y_B(t)$ according to the AK production function (McGrattan, 1998), that is $Y_G(t) = A_G K_G(t)$ and $Y_B(t) = A_B K_B(t)$. The total output then becomes

$$Y(t) = A_G K_G(t) + A_B K_B(t). \quad (1)$$

At each moment of time $t \geq 0$, the total output $Y(t)$ is distributed between consumption $C(t)$, investment $I_G(t)$ in the green sector, and investment in the brown sector $I_B(t)$:

$$Y(t) = C(t) + I_G(t) + I_B(t). \quad (2)$$

Capital stocks $K_G(t)$ and $K_B(t)$ accumulate thanks to investments and depreciate as follows

$$\dot{K}_G(t) = I_G(t) - \delta_G K_G(t), \quad K_G(0) = K_{G0} \quad (3a)$$

and

$$\dot{K}_B(t) = I_B(t) - \delta_B K_B(t), \quad K_B(0) = K_{B0}, \quad (3b)$$

where $K_{G0} > 0$ and $K_{B0} > 0$ determine the initial capital stocks respectively; parameters $\delta_G > 0$ and $\delta_B > 0$ are the respective depreciation rates.

The representative agent's instantaneous utility $U(t)$ consists of two components

$$U(t) = \ln C(t) + \omega \ln Y_G(t), \quad (4)$$

where the first term $\ln C(t)$ is the utility of consumption that is standard in the neoclassical economic growth theory taken in the logarithmic form for simplicity; $\ln Y_G(t)$ in the second term realizes the representative agent's determination to develop green production and $\omega \geq 0$ is the weight he/she attributes to this determination in his/her total utility – we assume diminishing marginal returns and also adopt the logarithmic form for the amenity value of green production for simplicity.

We consider the model over an infinite time horizon and introduce the total utility as

$$J = \int_0^{\infty} e^{-rt} U(t) dt, \quad (5)$$

where $r > 0$ is a social discount rate.

We introduce fractions $c(t) = \frac{C(t)}{Y(t)}$, $u_G(t) = \frac{I_G(t)}{Y(t)}$, and $u_B(t) = \frac{I_B(t)}{Y(t)}$, and treat quantities $u_G(\cdot)$ and $u_B(\cdot)$ as controls. By setting these, the representative agent regulates the capital accumulation processes and defines his/her consumption. We assume that $u_G(\cdot)$ and $u_B(\cdot)$ are piecewise continuous functions over $[0, \infty)$, such that

$$(u_G(t), u_B(t)) \in \mathcal{U} = \{(u_G, u_B) \in \mathbb{R}^2: u_G, u_B \in [0, 1], u_G + u_B < 1\}, t \geq 0. \quad (6)$$

We exclude the boundary $u_G + u_B = 1$ from \mathcal{U} since in this case the instantaneous utility function

(4) $U(t) = \ln[(1 - u_G(t) - u_B(t))(A_G K_G(t) + A_B K_B(t))] + \omega \ln(A_G K_G(t))$ is locally unbounded, as $\ln(1 - u_G - u_B) \rightarrow -\infty$ as $u_G + u_B \rightarrow 1 - 0$. Thus, the set \mathcal{U} of admissible control values is not closed.

Combining (1)-(6), we obtain the following optimal control problem:

$$\max_{u_B(\cdot), u_G(\cdot)} \int_0^{\infty} e^{-rt} (\ln[(1 - u_G(t) - u_B(t))(A_G K_G(t) + A_B K_B(t))] + \omega \ln(A_G K_G(t))) dt$$

(7a)

s.t.:

$$\dot{K}_G(t) = u_G(t)(A_G K_G(t) + A_B K_B(t)) - \delta_G K_G(t), \quad K_G(0) = K_{G0}, \quad (7b)$$

$$\dot{K}_B(t) = u_B(t)(A_G K_G(t) + A_B K_B(t)) - \delta_B K_B(t), \quad K_B(0) = K_{B0}, \quad (7c)$$

$$u_G(t) \in [0,1), \quad u_B(t) \in [0,1), \quad u_G(t) + u_B(t) < 1. \quad (7d)$$

Note that for any admissible controls, the integral in (7a) converges as solutions to the linear equations (7b), (7c) are bounded by exponential functions. Thus, we can apply the standard notion of optimality here, i.e., we seek for controls $u_G(\cdot)$, $u_B(\cdot)$ that would satisfy (7d) and would maximize the utility function (7a) together with the corresponding phase variables $K_G(\cdot)$, $K_B(\cdot)$ satisfying (7b-c). We denote the optimal controls as $u_G^*(\cdot)$ and $u_B^*(\cdot)$, and the optimal trajectories as $K_G^*(\cdot)$ and $K_B^*(\cdot)$.

3. Optimal solution

3.1. Necessary conditions for optimality

The optimal control problem (7) has two important features, which complicate its analysis from the control-theoretic viewpoint. First, it is formulated over an infinite time horizon. This type of optimal control problems may exhibit a certain “degeneracy” in the sense that in the optimum, the Lagrangian multiplier associated with the utility function in the Hamiltonian (λ_0) may equal zero (Aseev and Veliov, 2017). The possibility of such a case requires a careful investigation. Another complication of the infinite-horizon problems is that the “natural” generalizations of the transversality condition, which would allow to narrow down the set of solutions of the adjoint system, may not hold true (Aseev and Veliov, 2017; Aseev et al., 2012).

Second, due to the local unboundedness of the instantaneous utility function, the set of admissible control values \mathcal{U} (6) is not closed. Most of the existence theorems available in the literature are formulated only for compact (e.g., Seierstad and Sydsæter, 1987, Th 3.15) or, at least, closed admissible control sets (Aseev, 2018). Therefore, the existence question requires a special attention. To overcome this complication, one could try to prove that an optimal control problem with a non-closed control set can be replaced by the same optimal control problem, where the control set is a compact subset of the original control set. This replacement would work if one could prove that the sets of optimal solutions in these two problems coincide. In case of optimal control problems over infinite time horizons such a proof may be rather laborious; e.g., Aseev and Kryazhinskii (2007) consider a series of finite approximations of the infinite horizon optimal control problem for this purpose and then prove convergence of their solutions to the solution of the original problem (see Theorem 18.1 therein).

Instead of following this route, in order to solve problem (7) in the presence of the above-mentioned features, we apply a version of the Pontryagin Maximum Principle (PMP) that was specially developed to deal with problems with locally unbounded instantaneous utility functions (Besov, 2014, Th. 3). This version of the PMP therefore does not require the closedness of the set of admissible control values. Thanks to its special assumptions, this theorem excludes the degenerated case of $\lambda_0 = 0$ and furthermore establishes that the optimal adjoint variables take only positive values. Thus, Theorem 3 from (Besov, 2014) allows to significantly narrow down the set of extremal trajectories in comparison to more general PMP theorems for infinite horizon optimal control problems, e.g., that from (Seierstad and Sydsæter, 1987, Th. 3.12). In Appendix A.1, we show that all the conditions of Theorem 3 from (Besov, 2014) hold true for problem (7), and hence this theorem can be applied for problem (7). Note that under these conditions, a solution to problem

(7) exists thanks to Theorem 1 in (Besov, 2014); actually, only some of these conditions are necessary for the existence.

Next, let $\lambda(\cdot) = (\lambda_G(\cdot), \lambda_B(\cdot))^T$ be the vector of present-valued adjoint variables, $K(\cdot) = (K_G(\cdot), K_B(\cdot))^T$ and $u(\cdot) = (u_G(\cdot), u_B(\cdot))^T$ be the vectors of capital stocks and controls respectively. Due to Theorem 3 from (Besov, 2014), we put $\lambda_0 = 1$ and consider only positive-valued adjoint variables $\lambda(\cdot)$. Then the Hamiltonian is as follows

$$H(K, \lambda, u) = \ln[(1 - u_G - u_B)(A_G K_G + A_B K_B)] + \omega \ln(A_G K_G) + \lambda_G (u_G (A_G K_G + A_B K_B) - \delta_G K_G) + \lambda_B (u_B (A_G K_G + A_B K_B) - \delta_B K_B). \quad (8)$$

The following maximizes the Hamiltonian over $u \in \mathcal{U}$ (see Appendix A.2 for details)

$$u^*(K, \lambda) = \begin{cases} \left(\max\left\{0, 1 - \frac{1}{\lambda_G Y}\right\}, \right. & \lambda_G > \lambda_B > 0, \\ 0 & \\ \left. \begin{matrix} 0 \\ \max\left\{0, 1 - \frac{1}{\lambda_B Y}\right\} \end{matrix} \right), & \lambda_B > \lambda_G > 0, \\ \left(\begin{matrix} u_G^* \\ u_B^* \end{matrix} \right) : u_G^* + u_B^* = \max\left\{0, 1 - \frac{1}{\mu Y}\right\}, & \lambda_G = \lambda_B = \mu > 0, \end{cases} \quad (9)$$

with the third case being a singular control. Our analysis in section 3.4 will reveal that starting from some time moment τ , this singular control is optimal in problem (7), and along the corresponding part of the optimal path, the green and brown sectors grow at the same rate.

The adjoint variables evolve according to:

$$\dot{\lambda}_G(t) = (r + \delta_G)\lambda_G(t) - \frac{A_G}{Y(t)} - (\lambda_G(t)u_G(t) + \lambda_B(t)u_B(t))A_G - \frac{\omega}{K_G(t)}, \quad (10a)$$

$$\dot{\lambda}_B(t) = (r + \delta_B)\lambda_B(t) - \frac{A_B}{Y(t)} - (\lambda_G(t)u_G(t) + \lambda_B(t)u_B(t))A_B. \quad (10b)$$

The stationarity² condition for the Hamiltonian takes the following form

² Here “stationarity” means that the Hamiltonian is asymptotically stationary, i.e. it vanishes as time goes to infinity; in this considerations we follow the terminology from (Besov, 2014).

$$e^{-rt}H(K(t), \lambda(t), u^*(K(t), \lambda(t))) = r \int_t^\infty e^{-rs} \left(\ln \left[\left(1 - u_G^*(K(s), \lambda(s)) - u_B^*(K(s), \lambda(s)) \right) Y(s) \right] + \omega \ln(A_G K_G(s)) \right) ds.$$

Tending time t to infinity (see (8), (9)), we obtain the following standard transversality conditions (see Appendix A.3 for details):

$$\lim_{t \rightarrow \infty} e^{-rt} \lambda_G(t) K_G(t) = \lim_{t \rightarrow \infty} e^{-rt} \lambda_B(t) K_B(t) = 0. \quad (10c)$$

As follows from (Besov, 2014, Th.3), the adjoint equations with the transversality conditions (10a-c), the state equations (7b-c) with their initial conditions, and the maximum condition (9) constitute the full set of the necessary conditions of the Pontryagin Maximum Principle for problem (7). It turns out that any trajectory $(K(\cdot), \lambda(\cdot), u(\cdot))$ satisfying (7b-c), (9), and (10a-c) is optimal in problem (7). We establish this fact using the sufficient conditions of optimality from (Seierstad and Sydsaeter, 1977, Ths. 3 and 10), see Theorems 1 and 2 below.

3.2. Useful notations and additional assumptions on model parameters

We denote by $D = D_B - D_G$ the difference between augmented productivities of the brown and green capitals respectively, where $D_B = A_B - \delta_B > 0$ and $D_G = A_G - \delta_G > 0$. Further, in this study, we make the following assumptions.

(A1) The brown capital is more productive than the green capital: $A_B > A_G$.

(A2) The augmented productivity of the brown sector exceeds the augmented productivity of the green sector at least by the discount factor: $D > r$.

(A3) The productivity of the green sector is greater than the discount factor: $A_G > r$.

Let us also denote

$$q = \left(\frac{D}{\omega r} + \frac{D-r}{r} \right)^{-1}. \quad (11)$$

Parameter q represents a critical ratio between the green and brown capital stocks, which delineates three economy types depending on the initial conditions as will be shown in the theorems below.

3.3. Optimal solution for the economy, in which the initial brown and green capital stock values constitute exactly the critical ratio q .

First, we consider the special case of the parameter values, in which the initial capital stock values constitute exactly the critical ratio q (11): $\frac{K_{G0}}{K_{B0}} = q$. Theorem 1 below provides the optimal controls in problem (7) for this special case, which also serves as a useful input to the analysis of the more general and realistic case of an economy, in which the brown capital initially dominates: $\frac{K_{G0}}{K_{B0}} < q$.

This case is presented in section 3.4.

Theorem 1. *If $\frac{K_{G0}}{K_{B0}} = q$, an optimal solution satisfies the third (singular) case in (9) over the entire time interval $[0, \infty)$. That is, constant controls*

$$u_G^*(t) = \frac{q(A_G - r + D)}{qA_G + A_B} = u_G^* \text{ and } u_B^*(t) = \frac{A_B - r}{qA_G + A_B} = u_B^* \text{ for all } t \geq 0 \quad (12)$$

are optimal in problem (7). The corresponding optimal brown and green capital stocks grow at the same rate $(D_B - r)$ as follows

$$K_G^*(t) = K_{G0} e^{(D_B - r)t} \text{ and } K_B^*(t) = K_{B0} e^{(D_B - r)t} \text{ for all } t \geq 0 \quad (13)$$

and the adjoint variables coincide and decay at the same rate as follows

$$\lambda_G(t) = \lambda_B(t) = \frac{\omega}{DK_{G0}} e^{-(D_B - r)t} \text{ for all } t \geq 0.$$

The proof of Theorem 1 is presented in Appendix B. Note that under the conditions of Theorem 1, along the optimal solution

$$\lambda_G(t)K_G^*(t) = \frac{\omega}{D} \text{ and } \frac{K_G^*(t)}{K_B^*(t)} = q \text{ for all } t \geq 0; \quad (14)$$

and optimal controls can be expressed as follows

$$u_G^*(t) = u_G^* = \frac{(A_G+D-r)K_G^*(t)}{Y^*(t)} \text{ and } u_B^*(t) = u_B^* = \frac{(A_B-r)K_B^*(t)}{Y^*(t)}, t \geq 0, \quad (15)$$

where $Y^*(t) = A_G K_G^*(t) + A_B K_B^*(t)$. Relations (14) are important for Theorem 2 in the next subsection and the representation of optimal controls (15) allows for interpretations provided in subsection 4.1. Note that in this special case, optimal controls are constant over time and, hence, there is no switching.

3.4. Optimal solution for the economy, in which the brown sector dominates initially

As this paper aims to investigate the possibility and potential of greening of the economy, we assume that at the beginning of the model's time horizon, the brown sector significantly dominates in the economy, i.e., here we focus on the case of $\frac{K_{G0}}{K_{B0}} < q$. Theorem 2 below provides optimal controls in problem (7) in this case, which is the main result of our paper.

Theorem 2. *If $\frac{K_{G0}}{K_{B0}} < q$, there exists a unique switching time $\tau > 0$, such that for $t \in [0, \tau]$ it is the first case in (9) that delivers an optimal solution, and for $t > \tau$, an optimal solution satisfies the third (singular) case in (9). That is, controls*

$$u_G^*(t) = \begin{cases} 1 - \frac{1}{\bar{\lambda}_G(t)\bar{Y}(t)}, & t \in [0, \tau], \\ u_G^*, & t > \tau, \end{cases} \text{ and } u_B^*(t) = \begin{cases} 0, & t \in [0, \tau], \\ u_B^*, & t > \tau, \end{cases} \quad (16)$$

are optimal in problem (7); here $\bar{Y}(t) = A_G \bar{K}_G(t) + A_B \bar{K}_B(t)$. Optimal state and adjoint variables are

$$K_G^*(t) = \begin{cases} \bar{K}_G(t), & t \in [0, \tau], \\ \bar{K}_G(\tau)e^{(D_B-r)(t-\tau)}, & t > \tau, \end{cases} \text{ and } K_B^*(t) = \begin{cases} \bar{K}_B(t), & t \in [0, \tau], \\ \bar{K}_B(\tau)e^{(D_B-r)(t-\tau)}, & t > \tau, \end{cases} \quad (17)$$

and

$$\lambda_G(t) = \begin{cases} \bar{\lambda}_G(t), & t \in [0, \tau], \\ \bar{\lambda}_G(\tau)e^{-(D_B-r)(t-\tau)}, & t > \tau, \end{cases} \text{ and } \lambda_B(t) = \begin{cases} \bar{\lambda}_B(t), & t \in [0, \tau], \\ \bar{\lambda}_B(\tau)e^{(D_B-r)(t-\tau)}, & t > \tau. \end{cases} \quad (18)$$

The switching time $\tau > 0$ and functions $(\bar{K}_G(\cdot), \bar{K}_B(\cdot), \bar{\lambda}_G(\cdot), \bar{\lambda}_B(\cdot))$ satisfy the following boundary value problem over $[0, \tau]$

$$\dot{\bar{K}}_G(t) = D_G \bar{K}_G(t) + A_B \bar{K}_B(t) - \frac{1}{\bar{\lambda}_G(t)}, \quad \bar{K}_G(0) = K_{G0}, \quad \bar{K}_G(\tau) = q \bar{K}_B(\tau), \quad (19a)$$

$$\dot{\bar{K}}_B(t) = -\delta_B \bar{K}_B(t), \quad \bar{K}_B(0) = K_{B0}, \quad (19b)$$

$$\dot{\bar{\lambda}}_G(t) = -(D_G - r) \bar{\lambda}_G(t) - \frac{\omega}{\bar{K}_G(t)}, \quad \bar{\lambda}_G(\tau) = \frac{\omega}{D \bar{K}_G(\tau)}, \quad (19c)$$

$$\dot{\bar{\lambda}}_B(t) = (r + \delta_B) \bar{\lambda}_B(t) - A_B \bar{\lambda}_G(t), \quad \bar{\lambda}_B(\tau) = \bar{\lambda}_G(\tau) = \frac{\omega}{D \bar{K}_G(\tau)}. \quad (19d)$$

Over $[0, \tau]$, the optimal trajectory and adjoint variables are defined as a solution to equations (19), which are obtained by substituting controls (16) into equations (7b), (7c), (10a), (10b). Conditions at the right-hand-side boundary in (19a) and (19d), $\bar{K}_G(\tau) = q \bar{K}_B(\tau)$ and $\bar{\lambda}_G(\tau) = \bar{\lambda}_B(\tau) = \frac{\omega}{D \bar{K}_G(\tau)}$, ensure the continuity of optimal trajectories and adjoint variables when transiting from the first ($t \in [0, \tau]$) to the third ($t \in (\tau, \infty)$) case of (9) (compare (19a), (19c) with relations (14)). The BVP (19) contains four differential equations over a time interval with an unknown right-hand-side boundary τ and five boundary conditions. Lemma 1 in Appendix C.1 establishes that a solution to (19) exist, while Lemma 2 in Appendix C.2 proves that this solution indeed satisfies the first condition in (9). In this case $\frac{K_G^*(t)}{K_B^*(t)} = \frac{\bar{K}_G(t)}{\bar{K}_B(t)} < q$ for all $t \in [0, \tau]$. Over (τ, ∞) , the optimal control, phase and adjoint trajectories are the same as those in Theorem 1 (up to the shift by τ) and in this case, $\frac{K_G^*(t)}{K_B^*(t)} = q$ for all $t \in [\tau, \infty)$.

In order to prove Theorem 2, first, we will show that relations (16)-(18) satisfy the necessary conditions for optimality of the PMP, then we will prove that solution (16)-(18) is

optimal applying the sufficient conditions for optimality, which we show to hold true for the extremal solution (16)-(18). The proof of Theorem 2 can be found in Appendix C.3.

Optimal solution (16)-(18) has the following interpretation. As long as the brown sector dominates in the economy, i.e., as long as the ratio between the two capital stocks is lower than the critical ratio: $\frac{K_G^*(t)}{K_B^*(t)} < q$, it is optimal to invest the fraction of the total output equal to $u_G^*(t) = 1 - \frac{1}{\lambda_G(t)\bar{Y}(t)}$ into the green capital and nothing into the brown capital. This will allow the green capital stock to grow quickly and reach the critical ratio $\frac{K_G^*(\tau)}{K_B^*(\tau)} = q$ at time $t = \tau$. After $t = \tau$, it is optimal to invest constant fractions of the total output in the development of the green and brown sectors (specified in Theorem 1), which will maintain the same proportion of the green and brown capitals over time, i.e., $\frac{K_G(t)}{K_B(t)} = q$ for all $t \geq \tau$. Note that if the ratio between the initial capital stocks were the opposite, i.e., in the case of an economy, in which the green sector dominates initially $\frac{K_{G0}}{K_{B0}} > q$, the “mirror” strategy would be optimal. This dynamic is schematically illustrated in the phase plane of variables K_G, K_B in Figure 1.

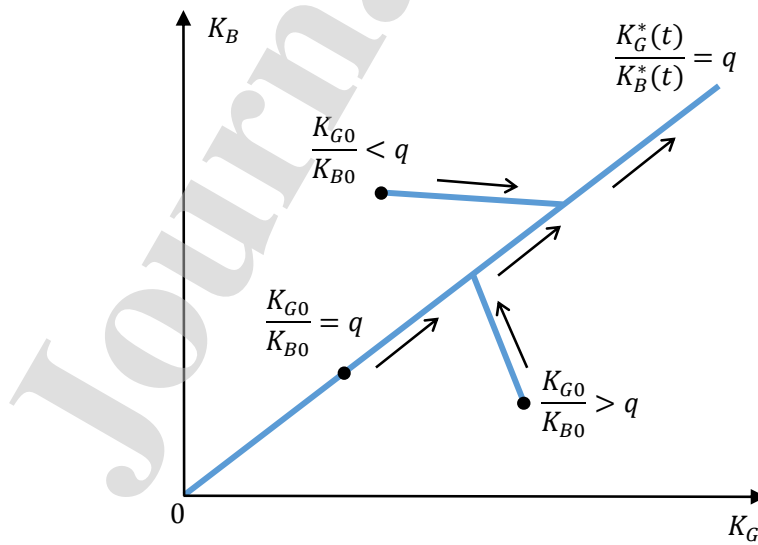


Figure 1. Optimal trajectories in the phase space for three qualitatively different cases of the initial conditions: $\frac{K_{G0}}{K_{B0}} = q$, $\frac{K_{G0}}{K_{B0}} < q$, and $\frac{K_{G0}}{K_{B0}} > q$. In the second/third case, in order to reach the singular ray $\frac{K_G}{K_B} = q$, the investment into the brown/green sectors should be zero. Once the singular ray is reached, investments in the green and brown capitals should be such that $K_B^*(t)$ and $K_G^*(t)$ grow proportionally.

Note that for optimal consumption share the following formula is valid:

$$c^*(t) = 1 - u_G^*(t) - u_B^*(t) = \frac{1}{\lambda_G(t)Y^*(t)},$$

and as

$$c^*(t)Y^*(t) = C^*(t) = \left(\left(\frac{d}{dc} \ln C \right)_{c=C^*(t)} \right)^{-1},$$

we obtain that

$$\left(\frac{d}{dc} \ln C \right)_{c=C^*(t)} = \lambda_G(t).$$

Thus – see formula (4) – we obtain that the marginal utility of consumption is equal to the shadow price of the green capital. Note that such equality appears also as a necessary condition for optimality in the classical one-sector Ramsey model, see e.g. (Acemoglu, 2009, p. 299, Eq. (8.34)).

The next section is devoted to how one can find a solution to (19) numerically.

3.5. Finding the switching time τ numerically

To find τ from (19), it is sufficient to consider equations (19a), (19b), and (19c) as these equations are independent from $\bar{\lambda}_B(\cdot)$. Using an explicit solution to (19b), $\bar{K}_B(t) = K_{B0}e^{-\delta_B t}$ for all $t \in [0, \tau]$, the BVP (19) can be reduced to

$$\begin{aligned} \dot{\bar{K}}_G(t) &= D_G \bar{K}_G(t) + A_B \bar{K}_B(t) - \frac{1}{\bar{\lambda}_G(t)}, & \bar{K}_G(0) &= K_{G0}, \bar{K}_G(\tau) = qK_{B0}e^{-\delta_B \tau}, \\ \dot{\bar{\lambda}}_G(t) &= -(D_G - r)\bar{\lambda}_G(t) - \frac{\omega}{\bar{K}_G(t)}, & \bar{\lambda}_G(\tau) &= \frac{\omega}{qDK_{B0}e^{-\delta_B \tau}}. \end{aligned} \quad (20)$$

Let us introduce $z(t) = \frac{\bar{K}_G(t)}{\bar{K}_B(t)}$ and $s(t) = \bar{\lambda}_G(t)\bar{K}_B(t)$. Also, let $z_0 = \frac{K_{G0}}{K_{B0}}$. Then (20) is equivalent to the following:

$$\begin{aligned} \dot{z}(t) &= (D_G + \delta_B)z(t) + A_B - \frac{1}{s(t)}, & z(0) &= z_0, & z(\tau) &= q, \\ \dot{s}(t) &= -(D_G + \delta_B - r)s(t) - \frac{\omega}{z(t)}, & s(\tau) &= \frac{\omega}{qD}. \end{aligned} \quad (21)$$

The convenience of this transformation is due to the fact that in (21), the boundary conditions do not contain the time moment τ in the right-hand side. If we find a solution of the BVP (21), then the solution of the BVP (20) can be found by the inverse transformation of variables: $\bar{K}_G(t) = \bar{K}_B(t)z(t) = K_{B0}e^{-\delta_B t}z(t)$ and $\bar{\lambda}_G(t) = \frac{s(t)}{K_{B0}e^{-\delta_B t}}$. We have the following

Proposition 1. Solution $(s(\cdot), z(\cdot))$ to (21) exists and satisfies the following inequalities

$$z_0 \leq z(t) \leq q \text{ and } s(t) \geq \frac{\omega}{qD} \text{ for all } t \in [0, \tau].$$

The proof of Proposition 1 can be found in Appendix D. Proposition 1 allows to make the following estimation of the right-hand side of the first equation in (21) for all $t \in [0, \tau]$:

$$0 < (D_G + \delta_B)z_0 + A_B - \frac{qD}{\omega} \leq (D_G + \delta_B)z(t) + A_B - \frac{1}{s(t)} \leq (D_G + \delta_B)q + A_B. \quad (22)$$

Now using solution $t \mapsto (s(t), z(t))$ to (21) over $[0, \tau]$, we exclude t and introduce function $z \mapsto s(z)$ over $[z_0, q]$. This function is subject to the following initial value problem (IVP) over $[z_0, q]$:

$$s'(z) = \frac{-(D_G + \delta_B - r)s(z) - \frac{\omega}{z}}{(D_G + \delta_B)z + A_B - \frac{1}{s(z)}}, \quad s(q) = \frac{\omega}{qD}. \quad (23)$$

Estimates (22) guarantee the existence of a solution to (23) on the entire segment $[z_0, q]$.

In order to find τ numerically, one should first solve the IVP (23) and obtain $s_0 = s(z_0)$. Then, one should solve the IVP for the differential equations from (21) with initial conditions $z(0) = z_0$ and $s(0) = s_0$. Time moment $\tau > 0$ such that $z(\tau) = q$ will be the sought-for switching time. Note that in this case, equality $s(\tau) = \frac{\omega}{qD}$ holds automatically because of the initial condition in (23). The following proposition provides a lower estimate and an upper estimate for τ .

Proposition 2. The following estimate for switching time τ holds:

$$\frac{1}{D_G + \delta_B} \ln \frac{q(D_G + \delta_B) + A_B}{z_0(D_G + \delta_B) + A_B} \leq \tau \leq \frac{1}{D_G + \delta_B} \ln \frac{q(D_G + \delta_B) + A_B - qD/\omega}{z_0(D_G + \delta_B) + A_B - qD/\omega}.$$

This estimate is derived as a part of the proof of Lemma 1 in Appendix C.1. The difference between the upper and lower bounds decreases with decrease of qD/ω and τ is greater for bigger proportion q/z_0 . Thus, the greater is the initial gap between green and brown sector's endowments ($z_0 = K_{G0}/K_{B0} < 1$), the longer it takes to achieve the singular mode of optimal proportional growth.

4. Some properties of the proportional optimal growth path

In this section, we will focus on the singular control (ray) described in section 3.3, so we assume that $\frac{K_{G0}}{K_{B0}} = q$. We make four major observations.

4.1. The brown sector is the sole engine of growth, whose rate is determined by the augmented productivity of the brown capital and the social discount factor

As can be seen from formulas (13), the optimal growth rate of both capitals in the singular control mode is $(D_B - r)$; due to the AK-model assumption, the output grows with the same rate too. Notably, this optimal growth rate depends neither on the parameters of the green sector nor on

weight ω . In fact, in this model, the growth rate of the economy would be the same even in the absence of green capital (compare e.g. with (Acemoglu, 2009, p. 391, Eq. (12.15)). This is because thanks to its higher productivity, it is the brown capital that solely generates growth in the model. Indeed, from (15) one can see that the optimal investment in the accumulation of the green capital, $I_G^*(t) = u_G^* Y^*(t) = (A_G + (D - r))K_G^*(t)$, exceeds the entire output of the green sector, $Y_G^*(t) = A_G K_G^*(t)$, while the optimal investment in accumulation of the brown capital, $I_B^*(t) = u_B^* Y^*(t) = (A_B - r)K_B^*(t)$, is smaller than the output of the brown sector, $Y_B^*(t) = A_B K_B^*(t)$. The part of the brown sector's output $rK_B^*(t)$, which is not used for investment in this sector, is distributed between consumption and investment in the green sector. Such a donor-recipient relation between brown and green production happens because there is no technological change in our model. The green sector has a lower productivity than the brown sector over the entire time horizon, and, since it also starts from a lower capital stock, it can never overtake the brown sector. As it does not contribute to consumption, the only incentive to develop the green sector is due to its amenity value defined by weight ω .

4.2. A full replacement of brown production by green production is not possible if the productivity gap between the brown and green capitals is greater than the social discount rate

The fraction of the green capital in the total capital stock becomes

$$\frac{K_G(t)}{K_B(t) + K_G(t)} = \frac{r}{D} \frac{\omega}{1 + \omega}. \quad (24)$$

We make three observations. First, obviously, if the representative agent places zero weight on the amenity of green production (i.e., if $\omega = 0$), the optimal fraction of the green capital in the total capital stock is zero, which means that due to $A_B > A_G$, it is optimal to develop the brown sector only.

Second, higher values of ω promote green production, however, even under an infinitely large ω , the optimal fraction of the green capital in the total capital stock only reaches as high as $\frac{r}{D} < 1$ (see Assumption (A2)). This is because in this model, the brown sector is needed for growth due to its higher productivity and even a very high amenity of green production alone is not able to lead to a full transition from brown to green production.

This critical role of the brown sector for growth can be seen from the following reasoning. The linear production function implies that the time gradients of the brown and green capital stocks are proportional to the total output. The maximization of the discounted weighted sum of the logarithmic utility of consumption and the logarithmic amenity value of green production is equivalent to the maximization of the discounted weighted sum of the logarithm of the consumption rate, the output growth rate, and the growth rate of the green capital. If the weight of the amenity value of green production were zero, the optimal output growth rate would be $(D_B - r)$. Due to the productivity gap between brown and green production, the optimal output growth rate in the model with a positive amenity weight cannot be higher than $(D_B - r)$. For example, if over the entire time horizon, the investment into the brown sector were zero and even if consumption were zero too, the green production would grow with rate D_G , which, according to our assumptions, is lower than $(D_B - r)$. Despite the positive amenity weight shifts the redistribution of the available resources to allow for the green sector to grow, the output rate $(D_B - r)$ is still attainable and hence is optimal in this case too.

Third, the share of the green capital in the total capital stock is negatively related to the difference between the augmented productivities of the brown and green sectors D . Hence, the relative role of the brown capital is lower for a higher augmented productivity of the green capital D_G (as $D = D_B - D_G$) and/or a higher discount rate r . While the latter would lead to lower growth

rate, the former would not affect the growth rate. In the limit case, when the difference between the augmented productivities of the brown and green sectors D tends to the discount factor r , then $\frac{\kappa_G^*(t)}{\kappa_B^*(t)} = \omega$. Thus, only in this limit case, a full transition to green production is possible, which requires an infinitely high weight ω .

4.3. A higher social discount leads to a higher contribution of the green capital stock to the total capital stock in the economy

A large body of literature within the environmental economics discusses the role of the social discount rate (SDR) pointing out that lower SDRs are likely to be more favorable to pro-environmental policies. For example, in the DICE model, lower SDRs lead to higher social costs of carbon (Nordhaus, 2017) and more stringent abatement policies (Nordhaus, 2008, Chapter 9). Similar results were derived in a number of other models ((Horowitz, 1996; Huang and Cai, 1994), to name just a few). This positive effect of the lower SDRs on the environmental quality is thanks to the fact that in these models, due to the cumulative effects, the current decisions on mitigation translate into economic damages from the environmental degradation in the future. Higher SDRs discount these future damages more strongly and this disincentivizes current mitigation.

Our model does not include this mechanism. Rather, higher SDRs discount more strongly both the future utility of consumption and the amenity of the green production, equally. Due to this fact, we observe that a higher r implies a higher share of the green capital in the total capital stock (see formula (24), subsection 4.2) and thus higher SDRs are favorable for the transition to a cleaner technology. At the same time, higher SDRs lower economic growth rate ($D_B - r$) as they lead to lower investment rates u_B^* in the brown sector (see (12)) and higher consumption rates (see subsection 4.4). Hence, since it is the brown capital that is the driver of growth in the model, a

higher discount factor means less growth and a smaller share of the brown capital in the total capital stock.

4.4. Optimal consumption rate is lowered by a higher amenity of green production and a lower social discount rate

Using (12), we obtain

$$c^*(t) = 1 - u_G^*(t) - u_B^*(t) = \frac{Dr}{\omega(A_G r + A_B(D-r)) + A_B D}.$$

Thus, in our model, a higher weight ω leads to a lower optimal consumption rate. Although increasing consumption and, hence, decreasing investment in capital accumulation can limit economic growth thus reducing industrial pollution, in our model, as well as in some others (Cunha-e-Sa et al., 2010; Gradus and Smulders, 1993; Rauscher, 2009), where a cleaner production is an available option, it is optimal to decrease consumption and use the released funds to finance the transition to a cleaner production.

However, only a part of the burden associated with the development of the green sector falls on consumption. From formulas (11), (12), we can see that the optimal investment rate into the brown sector u_B^* is lower for higher values of ω and the optimal investment rate into the green sector u_G^* is higher. In other words, a greater preference of the representative agent towards green production drives the share of the brown capital down and the share of the green capital up and hence the green sector development happens also at the expense of less investment into brown sector while keeping the same growth rate.

Furthermore, due to assumption (A1), higher social discount rates r increase the optimal consumption rate, which is a rather standard outcome for such type of models (Cunha-e-Sa et al.,

2010; Rauscher, 2009). At the same time, the difference between the consumption rate at $\omega = 0$ and at a given positive weight ω becomes

$$c(t|0) - c(t|\omega) = \frac{\omega r}{A_B} \frac{A_G r + A_B(D-r)}{\omega(A_G r + A_B(D-r)) + A_B D}.$$

Here one can see that a higher discount rate leads to higher consumption losses and in case of ω tending to infinity, the loss in the consumption rate tends to $\frac{r}{A_B}$.

5. Discussion

This paper contributes to the literature, which analyzes the relation between the long-term economic growth and the environmental quality seeking to find solutions, which reduce the trade-offs and take advantage of synergies. We presented and analyzed a stylized two-sector growth model that includes the brown and green sectors. The production function is linear, and the total production output is a sum of the outputs of the two sectors. The representative agent, who is a capital-owning eco-minded entrepreneur, maximizes the weighted sum of the present value of the utility of consumption and an amenity value of green production.

From the mathematical point of view, this problem is interesting because the logarithmic form of the utility components and the linearity of the production function allows for deriving a two-dimensional optimal control analytically. This turns to be possible despite the constraints on the admissible values of the two controls are interrelated, the set of admissible values in \mathbb{R}^2 is not closed and instantaneous utility function is locally unbounded. Consequently, we were able to derive and analyze the optimal trajectories in the model – contrary to similar models in the economic literature, which usually at their most allow for the explicit derivation of a steady state, to which the optimal path converges in the long run. For instance, a similar-to-ours, two-sector AK model from (Boucekkine et al., 2013a), which we mentioned in the Introduction, requires a

special form of the first-order conditions (Boucekkine et al., 2013b) to derive the optimal solution along with the optimal switching time; and, consequently, it is not possible to obtain an analytical solution in that model. With the optimal trajectories being available analytically, our model provides input to the discussion of the transition to a cleaner production over time.

The analysis of the model presented here is done with the full mathematical rigor, namely, we carefully address the problem of existence of an optimal solution and verify the validity of a convenient form of the transversality conditions. For these purposes, we used an existence theorem and a version of the Pontryagin Maximum Principle that are formulated for optimal control problems over infinite time horizons with locally unbounded instantaneous utility functions. Under some conditions, which hold true for our problem, this version of the PMP provides a set of necessary conditions for optimality. These conditions include a stationarity condition, from which we derived a transversality condition.

We considered our two-sector growth model under assumptions (A1)-(A3), which seem to be plausible. (A1) assumes that the brown capital is more productive than the green capital, which is a standard assumption in models analyzing a transition to a cleaner technology (Acemoglu et al., 2012, 2016; Boucekkine et al., 2013a). (A2) is a stronger version of (A1) as it assumes that the productivity of the brown sector exceeds the productivity of the green sector at least by the discount factor plus the difference of the depreciation rates of the brown and green capitals. While it is very difficult to obtain reliable empirical estimates of productivities A_B and A_G and thus validate assumption (A2), we find this case of a significantly lower productivity of the green capital to be most interesting and relevant for the challenge of the sustainability transition. A higher productivity of the green capital would facilitate this transition; thus, one can expect that if A_G were higher than A_B , the representative agent would prefer to invest in the green capital

accumulation only. However, a rigorous analysis of cases $A_B \leq A_G$ and especially $A_G < A_B \leq A_G + r + \delta_B - \delta_G$ would require a significant effort and space for presentation. Hence, we postpone it to future studies. Finally, (A3) assumes that the productivity of the green sector is greater than the discount factor. A sufficiently high productivity is necessary to ensure positive growth; for example, in one-sector AK model, the productivity of capital is considered to be higher than the discount factor plus the depreciation rate (see (Acemoglu, 2009; p. 390, Eq. (12.12))), which is an even stronger assumption than ours.

From the economic point of view, our model yields a few interesting insights. Recall that the only reason to develop the green sector in our model is its amenity value for the representative agent. The main result of this paper is that in this model, a full substitution of brown production by green production is not possible; rather, in an optimal solution, they co-exist and evolve proportionally. The representative agent has to sacrifice his/her consumption to invest in green production, especially in the initial period. Three parameters are positively related to the ratio of the green capital stock: the social discount rate, the (augmented) productivity of the green capital, and a representative agent's preference towards the green production amenity. However, higher social discount rates lead to lower economic growth rates, while the green capital productivity under assumption (A1) as well as the preference towards the green production amenity do not affect growth.

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Appendix A. Analysis of necessary conditions for optimality

A.1. Checking conditions and applicability of Theorem 3 from (Besov, 2014)

Denote the right-hand side of (7b), (7c) by $f(K, u) = (f_G(K, u), f_B(K, u))^T$ and the discounted function in the integrand in (7a) by $g(K, u)$. We check conditions (A1)-(A3), (A5), (A6), (A_g) of Theorem 3 in (Besov, 2014) as follows.

Condition (A1):

The following inequalities hold for any $K > 0$ and $u \in \mathcal{U}$:

$$\begin{aligned} \langle K, f(K, u) \rangle &= K_G u_G (A_G K_G + A_B K_B) - \delta_G K_G^2 + K_B u_B (A_G K_G + A_B K_B) - \delta_B K_B^2 \leq (A_G - \\ &\delta_G) K_G^2 + (A_B - \delta_B) K_B^2 + (A_G + A_B) K_G K_B \leq (A_G - \delta_G) K_G^2 + (A_B - \delta_B) K_B^2 + (A_G + A_B) (K_G^2 + \\ &K_B^2) / 2 \leq C \|K\|^2, \end{aligned}$$

where $C = \max\{(A_G - \delta_G + (A_G + A_B)/2), A_B - \delta_B + (A_G + A_B)/2\}$.

Condition (A2):

Set $Q(K) = \{(x, x^0) \in \mathbb{R}^3: x^0 \leq g(K, u), x = f(K, u), u \in \mathcal{U}\}$ is convex for any $K > 0$ thanks to the linearity of $f(K, \cdot)$ in u , the convexity of \mathcal{U} , and the concavity of $g(K, \cdot)$ in u for any $K > 0$.

Condition (A3):

It is obvious that for any admissible control $u(\cdot): u(t) \in \mathcal{U}, t \geq 0$, trajectories $K_G(\cdot)$ and $K_B(\cdot)$ are bounded, i.e., there exist some positive $C_1 > 0$ and $C_2 > 0$, such that $K_G(t) \leq C_1 e^{C_2 t}$ and $K_B(t) \leq C_1 e^{C_2 t}$ for all $t \geq 0$. Then

$$\begin{aligned} g(K(t), u(t)) &= \ln[(1 - u_G(t) - u_B(t))(A_G K_G(t) + A_B K_B(t))] + \omega \ln(A_G K_G(t)) \leq \\ &\ln[2A_B C_1 e^{C_2 t}] + \omega \ln(A_G C_1 e^{C_2 t}) = C_3 + \bar{C}_2 t, \end{aligned} \quad (\text{A1-1})$$

where $C_3 = \ln[2A_B C_1] + \omega \ln(A_G C_1)$ and $\bar{C}_2 = C_2(1 + \omega)$. Then

$$\int_T^{T'} e^{-rt} g(K(t), u(t)) dt \leq \int_T^\infty e^{-rt} [C_3 + \bar{C}_2 t] dt = \frac{e^{-rT} [C_3 r + \bar{C}_2 T r + \bar{C}_2]}{r^2} = \square(T),$$

where $h(T) > 0$, $h(T) \rightarrow +0$, while $T \rightarrow +\infty$.

Condition (A5):

We need to prove that for any $K > 0$, there exists $u \in \mathcal{U}$ such that $f(K, u) > 0$. Note that both components of $f(K, u)$ are linear and strictly increasing in u_G and u_B . Also, $f(K, \bar{u}) = 0$ for

$$\bar{u}_G = \frac{\delta_G K_G}{A_G K_G + A_B K_B} \quad \text{and} \quad \bar{u}_B = \frac{\delta_B K_B}{A_G K_G + A_B K_B}.$$

Next, as $\delta_G < A_G$ and $\delta_B < A_B$, it holds that $\bar{u}_G + \bar{u}_B < 1$.

Consider controls $u_G = \bar{u}_G + \varepsilon$ and $u_B = \bar{u}_B + \varepsilon$ with such (sufficiently small) $\varepsilon > 0$ that these controls are admissible. Then $f(K, u) > 0$ for these controls.

Condition (A6):

The following inequalities hold for any $K > 0$ and $u \in \mathcal{U}$:

$$\frac{\partial g(K, u)}{\partial K_G} = \frac{A_G}{A_G K_G + A_B K_B} + \frac{\omega}{K_G} > 0 \quad \text{and} \quad \frac{\partial g(K, u)}{\partial K_B} = \frac{A_B}{A_G K_G + A_B K_B} > 0,$$

$$\frac{\partial f_G(K, u)}{\partial K_B} = A_B u_G \geq 0 \quad \text{and} \quad \frac{\partial f_B(K, u)}{\partial K_G} = A_G u_B \geq 0.$$

Condition (A_g):

First, it is obvious that $g(K, u) \rightarrow -\infty$ as $(u_G + u_B) \rightarrow 1 - 0$. Let us put $a(K) = \left\| \frac{\partial g(K, u)}{\partial K} \right\|$; $\frac{\partial g(K, u)}{\partial K}$ does not depend on u – see the proof of condition (A6) above. Then

$$\left\| \frac{\partial g(K, u)}{\partial K} \right\| \leq a(K)(|g(K, u)| + 1), \text{ for all } K > 0 \text{ and } u \in \mathcal{U}.$$

Thus, all the conditions of Theorem 3 from (Besov, 2014) hold true for problem (7) and hence its statements can be used to solve this problem. In particular, we can put $\lambda_0 = 1$ and consider only positive-valued adjoint variables $\lambda_G(\cdot) > 0$ and $\lambda_B(\cdot) > 0$.

A.2. Finding a maximizer of the Hamiltonian

First, let us rewrite the Hamiltonian (8) as follows

$$H(K, \lambda, u) = h(u) + \ln Y + \omega \ln(A_G K_G) - \lambda_G \delta_G K_G - \lambda_B \delta_B K_B,$$

where

$h(u) = \ln(1 - u_G - u_B) + (\lambda_G u_G + \lambda_B u_B)Y$ and $Y = A_G K_G + A_B K_B$. From this representation of the Hamiltonian, it is obvious that its maximizer $u^* = (u_G^*, u_B^*)$ with respect to $u \in \mathcal{U}$ coincides with the maximizer of function $h(u)$. Depending on the relation between (positive) λ_G and λ_B , there may be three different cases.

Case 1: $\lambda_G > \lambda_B > 0$. Introducing a new variable $v = u_G + u_B$, we rewrite function $h(u)$ as a sum of two functions: $h(u) = h_1(v) + h_2(u_B)$, where $h_1(v) = \ln(1 - v) + \lambda_G Y v$, $v \in [0, 1)$ and $h_2(u_B) = -(\lambda_G - \lambda_B)Y u_B$, $u_B \in [0, 1)$. Maximizers v^* and u_B^* of functions $h_1(v)$ and $h_2(u_B)$ are $v^* = \max\left\{0, 1 - \frac{1}{\lambda_G Y}\right\}$ and $u_B^* = 0$ correspondingly. Returning to the original variables, we have $u_G^* = \max\left\{0, 1 - \frac{1}{\lambda_G Y}\right\}$.

Case 2: $\lambda_B > \lambda_G > 0$. Similarly to case 1, using representation $h(u) = h_3(v) + h_4(u_G)$, where $h_3(v) = \ln(1 - v) + \lambda_B Y v$, $v \in [0,1)$, and $h_4(u_G) = -(\lambda_B - \lambda_G) Y u_G$, $u_G \in [0,1)$, we get that $u_B^* = \max\left\{0, 1 - \frac{1}{\lambda_B Y}\right\}$ and $u_G^* = 0$.

Case 3: $\lambda_G = \lambda_B = \mu > 0$. In this case, $h(u) = \ln(1 - v) + \mu Y v$, $v \in [0,1)$, from which we obtain that $v^* = \max\left\{0, 1 - \frac{1}{\mu Y}\right\}$ and, in the original variables, any vector $u^* = (u_G^*, u_B^*) \in \mathcal{U}$ such that $u_G^* + u_B^* = v^* = \max\left\{0, 1 - \frac{1}{\mu Y}\right\}$ maximizes $h(u)$.

A.3. Deriving transversality conditions

Tending time t to infinity in the stationarity condition

$$e^{-rt} H(K(t), \lambda(t), u^*(K(t), \lambda(t))) = r \int_t^\infty e^{-rs} \left(\ln \left[\left(1 - u_G^*(K(s), \lambda(s)) - u_B^*(K(s), \lambda(s)) \right) Y(s) \right] + \omega \ln(A_G K_G(s)) \right) ds,$$

the right-hand side vanishes, and we obtain the following equality:

$$\lim_{t \rightarrow \infty} e^{-rt} H(K(t), \lambda(t), u^*(K(t), \lambda(t))) = 0. \quad (\text{A3-1})$$

Due to formula (9), we have the four following cases for $u^*(K(t), \lambda(t))$, $t \geq 0$:

1. $u_G^*(t) = u_B^*(t) = 0$,
2. $u_G^*(t) = 1 - \frac{1}{\lambda_G(t) Y(t)}$, $u_B^*(t) = 0$,
3. $u_G^*(t) = 0$, $u_B^*(t) = 1 - \frac{1}{\lambda_B(t) Y(t)}$,
4. $u_G^*(t) = u_B^*(t) = 1 - \frac{1}{\mu(t) Y(t)}$.

Substituting each of these cases into formulas (8) and (A3-1) and taking into consideration estimate (A1-1), we obtain that in all these four cases conditions (10c) must be satisfied for the formula (A3-1) to hold true.

Appendix B. Proof of Theorem 1

Here and below we omit to indicate the dependence of state, adjoint variables and control on t to make formulas more succinct and transparent. All the formulas with omitted t in this section are true for all $t \geq 0$.

Let us consider the third case in formula (9), that is, assume that $\lambda_G = \lambda_B = \mu > 0$ and $u_G^* + u_B^* = \max\left\{0, 1 - \frac{1}{\mu Y}\right\}$. By differentiating the former equality over time, we obtain $\dot{\lambda}_G = \dot{\lambda}_B$, or, after applying formulas (10),

$$(r + \delta_G)\lambda_G - \frac{A_G}{Y} - (\lambda_G u_G + \lambda_B u_B)A_G - \frac{\omega}{K_G} = (r + \delta_B)\lambda_B - \frac{A_B}{Y} - (\lambda_G u_G + \lambda_B u_B)A_B. \quad (\text{B-1})$$

Let us assume that $u_G^* + u_B^* = 1 - \frac{1}{\mu Y} > 0$. Substituting this relation into (B-1), using that $\lambda_G = \lambda_B = \mu$ and simplifying, we rewrite (B-1) as follows:

$$\mu K_G = \frac{\omega}{D}. \quad (\text{B-2})$$

By differentiating equation (B-2) over time, we obtain

$$\dot{\mu} K_G + \mu \dot{K}_G = 0,$$

or, by using differential equations (7b) and (10a),

$$\left((r + \delta_G)\mu - \mu A_G - \frac{\omega}{K_G}\right) K_G + \mu(u_G^* Y - \delta_G K_G) = 0.$$

Then, resolving for u_G^* , we get

$$u_G^* = \frac{(A_G - r + D)K_G}{Y} \quad (\text{B-3})$$

and

$$u_B^* = 1 - \frac{1}{\mu Y} - u_G^* = \frac{A_B \mu K_B + r \mu K_G - \omega - 1}{\mu Y}. \quad (\text{B-4})$$

Then, we can write a differential equation for the adjoint variable μ using equation (10b):

$$\dot{\mu} = (r + \delta_B)\mu - \frac{A_B}{Y} - \mu \left(1 - \frac{1}{\mu Y}\right) A_B = -(D_B - r)\mu.$$

With the initial condition $\mu(0) = \frac{\omega}{DK_G(0)} = \frac{\omega}{DK_{G0}}$, which we obtain from (B-2) at $t = 0$, we get the

solution to the Cauchy problem as follows

$$\mu(t) = \frac{\omega}{DK_{G0}} e^{-(D_B-r)t} = \lambda_G(t) = \lambda_B(t), \quad t \geq 0. \quad (\text{B-5})$$

From condition (B-2) further we obtain

$$K_G(t) = K_{G0} e^{(D_B-r)t}, \quad t \geq 0. \quad (\text{B-6})$$

Let us substitute relations (B-2) and (B-4) into differential equation (7c):

$$\dot{K}_B = u_B^* Y - \delta_B K_B = D_B K_B - \left(\frac{D}{\omega} + D - r \right) K_G. \quad (\text{B-7})$$

Substituting (B-6) in (B-7) with the initial condition $K_B(0) = K_{B0}$, we solve the linear initial value problem and obtain

$$K_B(t) = \left(-K_{G0} \left(\frac{D}{\omega r} + \frac{D-r}{r} \right) (1 - e^{-rt}) + K_{B0} \right) e^{D_B t}, \quad t \geq 0.$$

As $\frac{K_{G0}}{K_{B0}} = q$, we get

$$K_B(t) = K_{B0} e^{(D_B-r)t}, \quad t \geq 0. \quad (\text{B-8})$$

Hence,

$$\frac{K_G(t)}{K_B(t)} = q, \quad t \geq 0. \quad (\text{B-9})$$

Let us derive the formulas for u_G^* and u_B^* by substituting relations (B-2), (B-5), (B-6), (B-8) and (B-9) into relations (B-3) and (B-4):

$$u_G^* = \frac{q(A_G - r + D)}{qA_G + A_B},$$

$$u_B^* = \frac{A_B - r}{qA_G + A_B}.$$

Note that assumptions (A1)-(A3) ensure that $q > 0$, $u_G^* > 0$, $u_B^* > 0$, as well as that $D_B > r$, which means that capitals (B-6), (B-8) grow at the same rate and adjoint variables (B-5) decay.

Thus, we proved that controls (12) and corresponding trajectories (13) with adjoint variables (B-5) satisfy the conditions of the Pontryagin Maximum Principle.

To prove their optimality, we check the conditions of Theorems 3 and 10 from (Seierstad and Sydsaeter, 1977), which supply the sufficient conditions for optimality. Let $\bar{\mu} = \max(\lambda_G, \lambda_B)$ and consider

$$H^*(K, \lambda) = \max_{u \in \mathcal{U}} H(K, \lambda, u) = \begin{cases} \ln(A_G K_G + A_B K_B) + \omega \ln(A_G K_G) - \lambda_G \delta_G K_G - \lambda_B \delta_B K_B, & \text{if } \lambda_G \leq 0 \text{ and } \lambda_B \leq 0, \\ -\ln \bar{\mu} + \omega \ln A_G K_G + \bar{\mu}((A_G - \delta_G)K_G + (A_B - \delta_B)K_B) - 1, & \text{otherwise.} \end{cases}$$

Obviously, function $H^*(\cdot, \lambda)$ is concave in K for any $\lambda \in \mathbb{R}^2$.

Let us now prove that $\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) - K^*(t) \rangle \geq 0$ for any admissible $K(\cdot)$ and adjoint variable (B-5). The following equalities are true:

$$e^{-rt} \mu(t) K_G^*(t) = e^{-rt} \frac{\omega}{DK_{G0}} e^{-(D_B-r)t} K_{G0} e^{(D_B-r)t} = \frac{\omega}{D} e^{-rt}$$

and

$$e^{-rt} \mu(t) K_B^*(t) = e^{-rt} \frac{\omega}{DK_{B0}} e^{-(D_B-r)t} K_{B0} e^{(D_B-r)t} = \frac{\omega}{qD} e^{-rt}.$$

Therefore, transversality condition (10c) holds, and $\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K^*(t) \rangle = 0$. Then

$$\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) - K^*(t) \rangle = \lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) \rangle \geq 0 \text{ thanks to the positivity of } \lambda(t) \text{ and}$$

$K(t), t \geq 0$. Hence, solution (12), (13), (B-5) is optimal in problem (7). ■

Appendix C. Proof of Theorem 2

Subsections C.1 and C.2 present auxiliary lemmas useful to prove Theorem 2. Subsection C.3 presents the proof of Theorem 2.

C.1. Proof of Lemma 1

Lemma 1. $(\bar{K}_G(\cdot), \bar{K}_B(\cdot), \bar{\lambda}_G(\cdot), \bar{\lambda}_B(\cdot))$ and $\tau > 0$ as a solution to (19) exists.

Proof.

The BVP (19) can be reduced to

$$\begin{aligned} \dot{\bar{K}}_G &= D_G \bar{K}_G + A_B \bar{K}_B - \frac{1}{\bar{\lambda}_G}, & \bar{K}_G(0) &= K_{G0}, \bar{K}_G(\tau) = qK_{B0}e^{-\delta_B \tau}, \\ \dot{\bar{\lambda}}_G &= -(D_G - r)\bar{\lambda}_G - \frac{\omega}{\bar{K}_G}, & \bar{\lambda}_G(\tau) &= \frac{\omega}{qDK_{B0}e^{-\delta_B \tau}}. \end{aligned} \quad (C1-1)$$

Let us introduce $z = \frac{\bar{K}_G}{\bar{K}_B}$ and $s = \bar{\lambda}_G \bar{K}_B$, where $\bar{K}_B(t) = K_{B0}e^{-\delta_B t}$ is the solution of (19b).

Also, let $z_0 = \frac{K_{G0}}{K_{B0}}$. Then (C1-1) is equivalent to the following:

$$\begin{aligned} \dot{z} &= (D_G + \delta_B)z + A_B - \frac{1}{s}, & z(0) &= z_0, z(\tau) = q, \\ \dot{s} &= -(D_G + \delta_B - r)s - \frac{\omega}{z}, & s(\tau) &= \frac{\omega}{qD}. \end{aligned} \quad (C1-2)$$

Let us get rid of the time dimension by dividing the second equation in (C1-2) by the first one and consider the following initial value problem (IVP):

$$\frac{ds}{dz} = \frac{-(D_G + \delta_B - r)s - \frac{\omega}{z}}{(D_G + \delta_B)z + A_B - \frac{1}{s}}, \quad s(q) = \frac{\omega}{qD}. \quad (C1-3)$$

over $z \in [z_0, q]$.

Let us first prove the existence of a solution to (C1-3) over $z \in [z_0, q]$. Consider a compact set $\Omega = \left\{ (z, s) : z \in [z_0, q], s \in \left[\frac{\omega}{qD}, \bar{S} \right] \right\}$ where $\bar{S} > \frac{\omega}{qD}$ is some fixed number. Due to the positivity of the denominator in Ω , the right-hand side (RHS) of equation (C1-3) and its derivative w.r.t. variable s are continuous in Ω . Then RHS of (C1-3) is Lipschitz continuous in Ω and hence there exists a unique solution to (C1-3) defined in some neighborhood of $z = q$.

We have the following estimates of the RHS of (C1-3) in Ω :

$$f_-(s) = \frac{-(D_G + \delta_B - r)s - \frac{\omega}{z_0}}{(D_G + \delta_B)q + A_B} \leq \frac{-(D_G + \delta_B - r)s - \frac{\omega}{z}}{(D_G + \delta_B)z + A_B - \frac{1}{s}} \leq \frac{-(D_G + \delta_B - r)s - \frac{\omega}{q}}{(D_G + \delta_B)z_0 + A_B - \frac{qD}{\omega}} = f_+(s).$$

Let us calculate the lower and upper solutions $s_-(\cdot)$, $s_+(\cdot)$ of the IVPs with $f_-(s)$ and $f_+(s)$ as RHSs, and the initial condition from (C1-3):

$$s_-(z) = \begin{cases} -\frac{\omega}{z_0(D_G + \delta_B - r)} + \left[\frac{\omega}{qD} + \frac{\omega}{z_0(D_G + \delta_B - r)} \right] \exp\left(-\frac{D_G + \delta_B - r}{(D_G + \delta_B)q + A_B} (z - q) \right), & \text{if } D_G + \delta_B - r \neq 0, \\ -\frac{\omega}{z_0((D_G + \delta_B)q + A_B)} (z - q) + \frac{\omega}{qD}, & \text{if } D_G + \delta_B - r = 0 \end{cases}$$

and

$$s_+(z) = \begin{cases} -\frac{\omega}{q(D_G + \delta_B - r)} + \left[\frac{\omega}{qD} + \frac{\omega}{q(D_G + \delta_B - r)} \right] \exp\left(-\frac{D_G + \delta_B - r}{(D_G + \delta_B)z_0 + A_B - \frac{qD}{\omega}} (z - q) \right), & \text{if } D_G + \delta_B - r \neq 0, \\ -\frac{\omega}{q((D_G + \delta_B)z_0 + A_B - \frac{qD}{\omega})} (z - q) + \frac{\omega}{qD}, & \text{if } D_G + \delta_B - r = 0. \end{cases}$$

It is easy to prove that in all three cases $-D_G + \delta_B - r > 0$, $D_G + \delta_B - r < 0$, and $D_G + \delta_B - r = 0$ – functions $s_-(\cdot)$ and $s_+(\cdot)$ are decreasing. It means that $s_-(z) > \frac{\omega}{qD}$ and $s_+(z) > \frac{\omega}{qD}$ for all $z \in [z_0, q]$. Then by applying the Chaplygin Existence Theorem (Szarski, 1965, Prop. 31.1) one

obtains the existence and uniqueness of a solution to (C1-3) for $z \in [z_0, q]$ as well as estimates $s_-(z) < \bar{s}(z) < s_+(z)$ where $\bar{s}(z)$ is a solution to (C1-3) and $z \in [z_0, q]$.

Let $s_0 = \bar{s}(z_0)$. Then a solution to the IVP with equations from (B2) and initial conditions $z(0) = z_0$ and $s(0) = s_0$ exists. Then one can find such τ that $z(\tau) = q$, while $s(\tau) = \frac{\omega}{qD}$. Such τ exists because of the following estimates of $z(\cdot)$:

$$\dot{z} = (D_G + \delta_B)z + A_B - \frac{1}{s} \geq (D_G + \delta_B)z + \left[A_B - \frac{qD}{\omega}\right], t \in [0, \tau],$$

$$\dot{z} = (D_G + \delta_B)z + A_B - \frac{1}{s} \leq (D_G + \delta_B)z + A_B, t \in [0, \tau],$$

or

$$z(t) \geq -\frac{A_B - qD/\omega}{D_G + \delta_B} + \left(z_0 + \frac{A_B - qD/\omega}{D_G + \delta_B}\right) \exp((D_G + \delta_B)t), t \in [0, \tau],$$

$$z(t) \leq -\frac{A_B}{D_G + \delta_B} + \left(z_0 + \frac{A_B}{D_G + \delta_B}\right) \exp((D_G + \delta_B)t), t \in [0, \tau],$$

which leads to estimate

$$\frac{1}{D_G + \delta_B} \ln \frac{q(D_G + \delta_B) + A_B}{z_0(D_G + \delta_B) + A_B} \leq \tau \leq \frac{1}{D_G + \delta_B} \ln \frac{q(D_G + \delta_B) + A_B - qD/\omega}{z_0(D_G + \delta_B) + A_B - qD/\omega},$$

and to the fact that the corresponding curves $(z(t), s(t), t \in [0, \tau])$ and $(\bar{s}(z), z \in [z_0, q])$ coincide in the space of variables (z, s) . The constructed solution will be a solution to (C1-2) and therefore we finally obtain a solution to (C1-1). ■

C.2. Proof of Lemma 2

Lemma 2. $(\bar{K}_G(\cdot), \bar{K}_B(\cdot), \bar{\lambda}_G(\cdot), \bar{\lambda}_B(\cdot))$ and $\tau > 0$ as a solution to (19) has the following properties:

(i) $1 - \frac{1}{\bar{\lambda}_G(t)\bar{Y}(t)} > 0$ for all $t \in [0, \tau]$, and

(ii) $\bar{\lambda}_G(t) > \bar{\lambda}_B(t) > 0$ for all $t \in [0, \tau]$.

Proof.

Let us first introduce new variables, which will be used in the proof of Lemma 2 below. We denote

$$m = \frac{\bar{\lambda}_G}{\bar{\lambda}_B}, \quad n = \bar{\lambda}_B \bar{K}_G, \quad h = \bar{\lambda}_G \bar{Y}, \quad k = \bar{\lambda}_G \bar{K}_G, \quad l = \bar{\lambda}_B \bar{K}_B$$

and derive differential equations for all new variables as follows.

$$\dot{m} = \frac{\dot{\bar{\lambda}}_G}{\bar{\lambda}_B} - \frac{\bar{\lambda}_G \dot{\bar{\lambda}}_B}{\bar{\lambda}_B^2} = -(D_G + \delta_B)m - \frac{\omega}{n} + A_B m^2, \quad (\text{C2-1})$$

$$\dot{n} = \dot{\bar{\lambda}}_B \bar{K}_G + \bar{\lambda}_B \dot{\bar{K}}_G = (D_G + \delta_B + r)n + A_B l - A_B k - \frac{1}{m}, \quad (\text{C2-2})$$

$$\dot{h} = \dot{\bar{\lambda}}_G \bar{Y} + \bar{\lambda}_G \dot{\bar{Y}} = rh - (\omega + 1)A_G - \frac{\omega A_B}{z} + A_B(\delta_G - \delta_B)s, \quad (\text{C2-3})$$

$$\dot{k} = \dot{\bar{\lambda}}_G \bar{K}_G + \bar{\lambda}_G \dot{\bar{K}}_G = rk + A_B s - \omega - 1, \quad (\text{C2-4})$$

$$\dot{l} = \dot{\bar{\lambda}}_B \bar{K}_B + \bar{\lambda}_B \dot{\bar{K}}_B = rl - A_B s. \quad (\text{C2-5})$$

Let us prove the following

Proposition 3. Let $(\bar{K}_G(\cdot), \bar{K}_B(\cdot), \bar{\lambda}_G(\cdot), \bar{\lambda}_B(\cdot))$ and $\tau > 0$ be a solution to (19). Then the following inequalities hold:

$k(t) \leq \frac{\omega}{D}$ and $l(t) \geq \frac{\omega}{qD}$ for all $t \in [0, \tau]$.

Proof. Consider differential equations (C2-4) and (C2-5). For $t \in [0, \tau)$, we obtain the following estimates for derivatives \dot{k} and \dot{l} (here we use inequality $s(t) > \frac{\omega}{qD}$, $t \in [0, \tau)$, proven in the proof of Lemma 1):

$$\dot{k} = rk + A_B s - \omega - 1 > rk + A_B \frac{\omega}{D} \left(\frac{D}{\omega r} + \frac{D-r}{r} \right) - \omega - 1 > r \left(k - \frac{\omega}{D} \right) \quad (\text{C2-6})$$

and

$$\dot{l} = rl - A_B s < rl - A_B \frac{\omega}{qD} < r \left(l - \frac{\omega}{qD} \right). \quad (\text{C2-7})$$

The last transitions in (C2-6) and (C2-7) are due to assumptions (A1), (A3).

Consider differential equation (C2-4) and a differential equation with the RHS equal to the last expression in (C2-6), both with the initial condition $k(\tau) = \bar{\lambda}_G(\tau) \bar{K}_G(\tau) = \frac{\omega}{D}$. The solution to the Cauchy problem for the latter differential equation is obviously $k(t) = \frac{\omega}{D}$ for all $t \in [0, \tau]$. Then from the Comparison Theorem (Budincevic, 2010) and estimate (C2-6), we obtain that $k(t) < \frac{\omega}{D}$, $t \in [0, \tau)$.

Using the same argumentation for (C2-5) and (C2-7) with the initial condition $l(\tau) = \bar{\lambda}_B(\tau) \bar{K}_B(\tau) = \frac{\omega}{qD}$, we prove that $l(t) > \frac{\omega}{qD}$, $t \in [0, \tau)$. ■

We return to the proof of Lemma 2 and below we prove condition (i). Let us prove that $h(t) > 1$ while $t \in [0, \tau]$, which will prove condition (i). We have

$$h(\tau) = \bar{\lambda}_G(\tau)\bar{Y}(\tau) = \frac{\omega}{qD}(qA_G + A_B) > \frac{\omega}{qD}A_B > 1;$$

the last inequality follows from assumptions (A1)-(A3). Then there exists an interval $(\bar{\tau}, \tau]$ such that $h(t) > 1$ while $t \in (\bar{\tau}, \tau]$.

We are going to prove that $h(t) > 1$ also for all $t \in [0, \tau]$. Let us suppose the contrary, namely, $h(\bar{\tau}) = 1$. Using formula (C2-3) and assumptions (A1), (A3), we have then

$$\dot{h}(\bar{\tau}) = r - (\omega + 1)A_G - \frac{\omega A_B}{z(\bar{\tau})} + A_B(\delta_G - \delta_B)s(\bar{\tau}) < -\frac{\omega A_B}{z(\bar{\tau})} + A_B(\delta_G - \delta_B)s(\bar{\tau}).$$

If $\delta_B \geq \delta_G$, due to the positivity of $z(\cdot)$ and $s(\cdot)$ we have

$$-\frac{\omega A_B}{z(\bar{\tau})} + A_B(\delta_G - \delta_B)s(\bar{\tau}) < 0.$$

If $\delta_G > \delta_B$ then

$$-\frac{\omega A_B}{z(\bar{\tau})} + A_B(\delta_G - \delta_B)s(\bar{\tau}) = \frac{A_B(\delta_G - \delta_B)}{z(\bar{\tau})} \left(k(\bar{\tau}) - \frac{\omega}{\delta_G - \delta_B} \right) < \frac{A_B(\delta_G - \delta_B)}{z(\bar{\tau})} \left(\frac{\omega}{D} - \frac{\omega}{\delta_G - \delta_B} \right) < 0$$

because $D > \delta_G - \delta_B$ (as $A_B > A_G$ thanks to assumption (A1)). Hence, $\dot{h}(\bar{\tau}) < 0$ and therefore $h(t) < h(\bar{\tau}) = 1$ in a right neighborhood of $\bar{\tau}$, which contradicts the assumption that $h(t) > 1$ in $(\bar{\tau}, \tau]$. Thus, we proved that $h(t) > 1$, while $t \in [0, \tau]$, which proves condition (i).

Let us now proceed to the proof of condition (ii) of Lemma 2. From the Proposition 3, we obtain $\lambda_B(t) = \frac{l(t)}{\bar{K}_B(t)} \geq \frac{\omega}{qDK_{B0}} > 0$, $t \in [0, \tau]$. The next step is to prove that $m(t) > 1$, $t \in [0, \tau]$, which will imply condition (ii) of Lemma 2.

We know that $m(\tau) = \frac{\bar{\lambda}_G(\tau)}{\bar{\lambda}_B(\tau)} = 1$. From formula (C2-1) and because $n(\tau) = \bar{\lambda}_B(\tau)\bar{K}_G(\tau) =$

$\frac{\omega}{D}$, we have

$$\dot{m}(\tau) = -(D_G + \delta_B) - D + A_B = 0.$$

By differentiating (C2-1) and substituting (C2-2), we further obtain

$$\ddot{m} = -(D_G + \delta_B + 2A_B m)\dot{m} + \frac{\omega}{n^2} \left((D_G + \delta_B + r)n + A_B l - A_B k - \frac{1}{m} \right).$$

Recalling that $\dot{m}(\tau) = 0$ and $m(\tau) = 1$, we have

$$\ddot{m}(\tau) = \frac{\omega}{\left(\frac{\omega}{D}\right)^2} \left((D_G + \delta_B + r) \frac{\omega}{D} + A_B \frac{\omega}{qD} - A_B \frac{\omega}{D} - 1 \right) = \frac{D(A_B - r)}{q} > 0.$$

Hence, $\ddot{m}(\tau) > 0$ and then there exists such $\hat{t} \in [0, \tau)$ that $\dot{m}(t) < 0$, $t \in (\hat{t}, \tau)$ and thus $m(\cdot)$ decreases while $t \in (\hat{t}, \tau)$ and, in turn, $m(t) > 1$, $t \in (\hat{t}, \tau)$.

We want to prove that $m(t) > 1$ for all $t \in [0, \tau)$ and not only in the interval (\hat{t}, τ) . Let us suppose the contrary, namely $m(\hat{t}) = 1$. We have

$$\dot{m}(\hat{t}) = -(D_G + \delta_B)m(\hat{t}) - \frac{\omega}{n(\hat{t})} + A_B m^2(\hat{t}) = D - \frac{\omega}{n(\hat{t})}. \quad (\text{C2-8})$$

Let us consider the differential equation (C2-2) in the interval $[\hat{t}, \tau)$. Using the Proposition 3, assumptions (A1)-(A3), and the fact that $m(t) \geq 1$, $t \in [\hat{t}, \tau)$, we obtain the following estimate of RHS of (C2-2) for $t \in [\hat{t}, \tau)$:

$$\begin{aligned} \dot{n} &= (D_G + \delta_B + r)n + A_B l - A_B k - \frac{1}{m} \geq (A_B - D + r)n + A_B \frac{\omega}{D} \left(\frac{D}{\omega r} + \frac{D-r}{r} \right) - A_B \frac{\omega}{D} - 1 > \\ &(A_B - D + r)n - (A_B - D + r) \frac{\omega}{D} = (A_B - D + r) \left(n - \frac{\omega}{D} \right). \end{aligned} \quad (\text{C2-9})$$

Consider differential equation (C2-2) and a differential equation with the RHS equal to the last expression in (C2-9), both with the initial condition $n(\tau) = \bar{\lambda}_B(\tau)\bar{K}_G(\tau) = \frac{\omega}{D}$. The solution to the Cauchy problem for the latter differential equation is obviously $n(t) = \frac{\omega}{D}$ for all $t \in [\hat{t}, \tau]$. Then

from the Comparison Theorem (Budincevic, 2010) and estimate (C2-9), we obtain that $n(t) < \frac{\omega}{D}$, $t \in [\hat{t}, \tau)$.

Hence, $n(\hat{t}) < \frac{\omega}{D}$ and from the relation (C2-8) we obtain that $\dot{m}(\hat{t}) < 0$. This means that in a sufficiently small right neighborhood of \hat{t} inequality $m(t) < 1$ holds. This is a contradiction with the supposition that $m(t) > 1$ while $t \in (\hat{t}, \tau)$. Hence, $m(t) > 1$ when $t \in [0, \tau)$ and condition (ii) of Lemma 2 is proved. ■

C.3. Proof of Theorem 2

In Lemmas 1 and 2 above, it is proven that controls (16) and corresponding trajectories (17) with adjoint variables (18) exist and satisfy the conditions of the Pontryagin Maximum Principle. To prove their optimality, as in the proof of Theorem 1, we check the conditions of Theorems 3 and 10 from the paper (Seierstad and Sydsaeter, 1977) as follows.

In Appendix B we proved that maximized Hamiltonian $H^*(\cdot, \lambda)$ is concave in K for any $\lambda \in \mathbb{R}^2$. Let us now prove that $\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) - K^*(t) \rangle \geq 0$ for any admissible $K(\cdot)$ and adjoint variable (18). The following equalities are true for $t \geq \tau$:

$$e^{-rt} \lambda_G(t) K_G^*(t) = e^{-rt} \frac{\omega}{qDK_{B0}e^{-\delta_B \tau}} e^{-(D_B-r)(t-\tau)} qK_{B0} e^{-\delta_B \tau} e^{(D_B-r)(t-\tau)} = \frac{\omega}{D} e^{-rt}$$

and

$$e^{-rt} \lambda_B(t) K_B^*(t) = e^{-rt} \frac{\omega}{qDK_{B0}e^{-\delta_B \tau}} e^{-(D_B-r)(t-\tau)} K_{B0} e^{-\delta_B \tau} e^{(D_B-r)(t-\tau)} = \frac{\omega}{qD} e^{-rt}.$$

Therefore, $\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K^*(t) \rangle = 0$. Then $\lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) - K^*(t) \rangle = \lim_{t \rightarrow \infty} \langle e^{-rt} \lambda(t), K(t) \rangle \geq 0$ thanks to positivity of $\lambda(t)$ and $K(t)$, $t \geq 0$. Hence, found solution (16)-(18) is optimal for problem (7). Theorem 2 is proved. ■

Appendix D. Proof of Proposition 1

In the proof of Lemma 1, we show that solution $(z(t), s(t), t \in [0, \tau])$ to problem (C1-2) and solution $(s(z), z \in [z_0, q])$ to problem (C1-3) coincide in the phase space. Then we automatically obtain the estimate $z_0 \leq z(t) \leq q$ for all $t \in [0, \tau]$. Inequality $s(t) \geq \frac{\omega}{qD}$, $t \in [0, \tau]$, follows from the following, established in the proof of Lemma 1, estimates of the solution $(s(z), z \in [z_0, q])$ to problem (C1-3):

$s_-(z) < s(z) < s_+(z)$, where $s_-(z) > \frac{\omega}{qD}$ and $s_+(z) > \frac{\omega}{qD}$ for all $z \in [z_0, q]$. ■

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