EQUITY, EFFICIENCY, AND ACCESSIBILITY IN URBAN AND REGIONAL HEALTH-CARE SYSTEMS

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FOREWORD

The principal aim of health care research at IIASA has been to develop a family of submodels of national health care systems to be used by health service planners. The modeling work has involved the construction of linked submodels dealing with population, disease prevalence, resource need, resource allocation, and resource supply.

This article considers four resource allocation criteria for assessing the long-term health resource requirements of different areas in a region. The spatial interaction model used here provides a simple method for selecting between different configurations, when population size and structure and resource availability are changing over time and space. The allocation criteria, based on objectives about which there is broad agreement among planners and other actors in the system, are concerned with improving the equity or the efficiency of the system, or the accessibility of the population to the supply of health services.

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Equity, efficiency, and accessibility in urban and regional health-care systems

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Abstract. This paper explores four different criteria of health-care resource allocation at the urban and regional level. The criteria are linked by a common spatial-interaction model. This model is based on the hypothesis that the number of hospital patients generated in a residential zone i is proportional to the relative morbidity of i, and to the availability of resources in treatment zone j, but is in inverse proportion to the accessibility costs of getting from i to j. The resource-allocation criteria are based on objectives on which there is broad agreement among planners and other actors in a health-care system. These objectives are concerned with allocations that conform to notions of equity, efficiency, and two definitions of accessibility. The allocation criteria give mainly aggregate-level information, and are designed with the long-term regional planning of health-care services in mind. The paper starts by defining the criteria, and describes how they are intended to be employed in a planning context. The allocation rules are then formally derived and linked together mathematically. They are then applied to a region, London, England, which is known to have very complex health-care planning problems. As a result of this application, two of the criteria—equity and efficiency—are selected for further analysis. A new model is built and applied that specifically enables the user to trade off one of these criteria against the other.

1 Introduction

This paper describes the theory and application of a set of possible methods to assist in the regional planning of health-care services. These methods are concerned with finding a set of resource allocations in different parts of a region when the morbidity, demographic structure, and resource availability are changing over time and space. They were designed with applications in the strategic planning of health services in mind, where the decisionmakers are concerned mainly with the broad directions and outputs of the system over a period of time. The work presented forms part of a wider research effort being carried out both jointly and independently by the Health Care and Public Facility Location Tasks at IIASA (the former also in conjunction with the Operational Research Service of the Department of Health and Social Security, England). The models that underlie this research are connected by a common spatial-interaction methodology (for example, Wilson, 1974), but each is designed to address a slightly different problem either in the health or in other public sectors. The level of detail in these models varies according to the intended use and the decisionmaking level in the system being studied.

In the present case, the outputs of the model forming the basis for the methods described in this study are highly aggregated, but they are typical of the decision variables used at a regional or supraregional level. After a discussion of the hypothesis underlying the approach employed and the reasons for this choice, the methods are developed in detail. Each is designed to pick a set of allocations according to one of four different criteria on which there is either broad acceptance by actors in the health-care system or considerable precedence in the literature on planning. Particular concern is taken, however, to ensure that the spatial behavior of the patients is correctly embedded in the allocation mechanisms. As a consequence of this concern and of the empirical tests subsequently carried out, two of the criteria are rejected in

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favor of the remaining two. The two accepted criteria address the problems of systems equity and systems efficiency, respectively—two objectives that are shown to pull the spatial pattern of regional resource allocation in different directions. The other two address the problems of accessibility. To conclude the paper, a composite method with an enhanced range of applications is developed that specifically allows users to develop scenarios by trading off the accepted objectives, one against the other.

1.1 *Class of systems*

Not all types or sectors of health-care systems (HCSs) will provide valid applications for the methods to be described. For example, in highly market-oriented HCSs, services are rationed by mechanisms other than these criteria, and so regional disparities in provision may not receive priority or even be considered a problem. The systems for which this work may be appropriate will probably be drawn from the following types:

(a) Payment-free or part-payment systems operating comprehensive health-insurance schemes where there are few market signals to regulate supply and demand.

(b) Systems with national, regional, or local health-care planning machinery and a commitment to the effective territorial planning of health-care services.

(c) Systems in which there is a historical tendency to overallocate resources in some areas and to underallocate them in others, and in which there is a growing desire by statutory authorities to redress these imbalances.

(d) Incipient systems in developing countries, or systems changing from a market approach to a more planned approach in health-care delivery in which considerable reorganization may be required.

In fact, the applications in this paper are based on data from the United Kingdom, which passed the National Health Service Act in 1946. The administrative machinery for regional planning, however, became available only after the National Health Service Reorganisation Act of 1973.

1.2 *Class of model*

The basic model is formed from the following simple hypothesis. It is that the number of patients generated in an origin zone i (place of residence) and treated in a destination zone j (place of treatment) is in proportion to the morbidity or 'patient-generating potential' of i and to the resources available in j, but is in inverse proportion to the accessibility costs of getting from i to j. In its current form, the model assumes that there are not enough resources to satisfy demand and that patients are not restricted by their places of residence to use only certain facilities. The first assumption reflects a view (analyzed in more detail below) that whatever is provided tends to get used. The second is to make it clear that only nonemergency services in the acute sector of the HCS are being discussed, and that some freedom of choice between different facilities is permitted. The type of model that emerges is a gravity model of the attraction-constrained form (Wilson, 1971).

The model is now stated informally; later it will be derived from theoretical grounds. It is

$$T_{ij} = B_j D_j W_i f(\beta, c_{ij}) \,, \tag{1}$$

where

- i, j index the origin and destination zones, respectively, $i = 1, ..., I$; $j = 1, ..., J$;
- T_{ij} is the predicted patient flow from zone i to treatment zone j;
- D_j is a resource measure defined as the case-load capacity in j for treating patients in a specialty or groups of specialties;
- W_i is a patient-generating factor, which is an index of the propensity of the population in i to generate patients in the same group of specialties;

$f(\beta, c_{ij})$ is a spatial discount function such as $\exp(-\beta c_{ij})$ (as used here) or $d_{ij}^{-\beta}$, which is strictly monotonically declining. Later, this function is abbreviated to f_{ij};
β is a spatial discount parameter ($\geqslant 0$) to be determined empirically; and
c_{ij} gives the accessibility costs between i and j;
and where

$$B_j = \left[\sum_i W_i f(\beta, c_{ij})\right]^{-1} . \tag{2}$$

Equation (2) is a constraint that ensures

$$\sum_i T_{ij} = D_j .$$

This is the assumption that all resources in j will be used.

Whereas this model ignores the sometimes complex procedures by which patients are referred between different levels and places of treatment in the system, research has shown that it is possible to describe *and* predict accurately the resulting spatial patterns of patient flows between different i and j (Mayhew and Taket, 1981), which suggests that the model assumptions are sufficient for their intended purposes. The empirical basis for the model, its range of applications, calibration, and various extensions are given elsewhere (Mayhew and Taket, 1980; Mayhew, 1980; 1981).

1.3 *Mode of use*

In conventional usage, the model predicts the impact on patient flows and hospitalization rates that result from changes in patient-generating potential and resource configuration. This permits the evaluation of many alternative allocations, yet it cannot tell the user which is best. For small problems at the local level of decisionmaking, these alternatives will be few, and it is probable that they can be judged for their suitability in only a few computer runs. The strategic level of planning, however, is concerned with the direction of the entire system over a period of time, say ten to fifteen years (DHSS, 1976). If a typical planning region contains one or more cities, several towns, over one hundred hospitals, and a service population in excess of ten million, say, the alternative allocations will be too many to evaluate, and the planner will find it useful to direct his search. The methods described here are designed to assist in this search by narrowing down the possibilities to those that in some sense can be judged best and that can be accomplished during the duration of the plan. To do this, however, the model must be directed to pick resource configurations that satisfy a particular objective or set of objectives. The problem is which objectives to choose and how to express them in a way that can be used by the model.

2 The main objectives of a health-care system

Clearly, an HCS has many objectives, not all of which can be achieved simultaneously. Some objectives, too, will be less important than others, but nevertheless they must be taken into account in some sense (section 2.3). The problem is to understand what the dominant objectives are. It is worth examining the expressed aim of the National Health Service in England and Wales. It is "... to ensure that every man and woman and child can rely on getting all the advice and treatment and care they need in matters of personal health ... [and] ... that their getting these should not depend on whether they can pay for them" (Feldstein, 1963, page 22; quoting from HMSO, 1944).

This seems an uncontroversial statement for the HCSs we have in mind. At least, two serious problems, however, are associated with the ideals expressed in it that are preventing its objectives from being attained. The first is that, as long as patients pay in time, money, discomfort, and other costs for access to facilities, there will always be a negative influence in the volume of per capita health-care consumption in different areas no matter which country or what type of HCS is considered. The second is that the assumption in 1944 that all needs could be catered to has proved unrealistic. The budget for health care and the consumption of health-care services in general, continues to rise at an alarming rate in the majority of countries, not only in England and Wales. In all countries, too, it has proved impossible to measure at a general level the marginal benefits of this increased expenditure, to determine the extent to which genuine needs are being satisfied, or to define an objective set of standards on which to base supply.

2.1 *Demand and availability*

Figure 1 illustrates empirically what usually happens in practice when there are uncertainties about outputs, accessibility costs to pay, and excess demands in the system. The discharges and deaths per thousand *catchment* population[(1)] (the population mostly dependent on the facilities in an area) are plotted against the hospital bed availabilities in each catchment area in Southeast England in 1977. The diagram demonstrates

(1) the strength of the supply side, and not relative need, in determining demand in the areas influenced by the facilities, particularly the way demand seems to rise so that it meets supply[(2)], and

(2) the strong dependence of the population on the local availability of facilities.

Figure 2 emphasizes point (2) in another way. It is a histogram showing the relationship between the percentage of patients using facilities in the London area and the distance from the hospital. It is based on a sample of about 2000 patients at fourteen hospitals. It shows clearly the marked preference among patients to use local facilities.

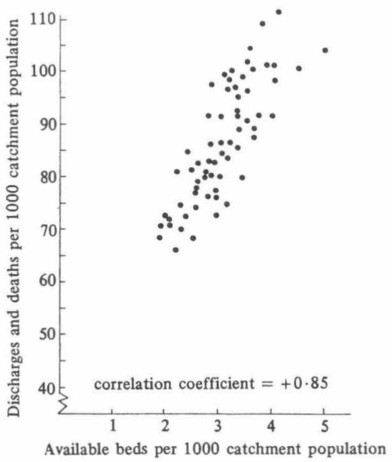


Figure 1. The relationship between hospitalization rates and level of provision for health district catchment populations in Southeast England in 1977 (source: LHPC, 1979a, page 26).

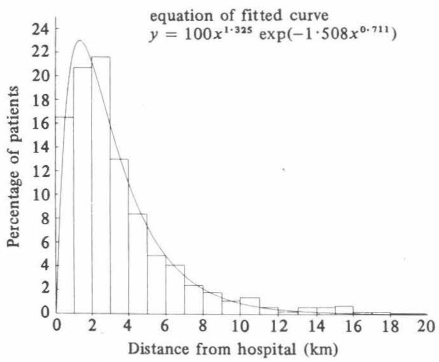


Figure 2. The relationship between the percentage of patients using hospital facilities and the distance from hospitals in London in 1977 for general medical and surgical specialties (source: Mayhew, 1979).

(1) A catchment population is defined by C_j, where $C_j = \sum_i E_{ij}P_i$, $E_{ij} = T_{ij} \Big/ \sum_j T_{ij}$, and P_i is the resident population in i.

(2) The relationship is not strictly linear since lengths of hospital stay are also an increasing function of bed supply, but this consideration is unimportant in the resource range examined here.

2.2 *Equity, efficiency, and accessibility*

Though from the above and other recent evidence, it would appear difficult for an HCS to satisfy all the actual and potential demands for health care, certain criteria stand out as being both sensible and applicable when budget constraints and uncertain outputs are both dominant considerations. These criteria are the improvement of the fairness of the system (equity), the increase in benefits to the patients (efficiency), and the equalization of the friction of distance between demand and supply (accessibility).

2.2.1 *Equity* The equity criterion is defined as choosing a resource configuration such that the relative needs (not the absolute needs as above) in each part of a region are satisfied. Relative needs can be expressed as the expected number of hospital admissions in one or more acute clinical specialties that would be generated by an area of residence if national utilization rates by age, sex, and specialty were applied to the local demographic structure. [This is analogous to the method of calculating the patient-generating factor in equation (1); it is simply an indicator of *expected* demand.]

2.2.2 *Efficiency* The efficiency criterion is defined as choosing a resource configuration that maximizes the benefits to consumers (patients) by satisfying their preferences for treatment in different locations. This criterion is rooted in notions of consumer surplus favored by transport planners, economists, and others, and is presented formally below in section 4.3.

2.2.3 *Accessibility* The accessibility criterion is defined in two ways for reasons that will become apparent:

Accessibility 1—The first way is to choose a resource configuration that equalizes the average costs of travel from places of residence to places of treatment. Somewhat related accessibility criteria have precedents particularly in the operations research literature (for example, Toregas et al, 1971), though very strong assumptions are typically made concerning the nature of demand and the allocation of this demand to particular facilities (for example, the 'nearest facility rule'). Here, these assumptions are relaxed to preserve the observed spatial choice behavior of patients.

Accessibility 2—Equalizing the average accessibility costs will be inefficient if the variance in the observed costs between different places of residence is large. Thus a second criterion is defined: it is to choose a resource configuration that minimizes the variance in the accessibility costs from places of residence to places of treatment. In this way, those patients with very high or very low accessibility costs may be taken into account.

2.3 *Systems constraints*

It is inevitable that in the use of one or more of these objectives others will conflict in the process. For example, in addition to treating patients, an HCS carries out medical research and trains physicians, nurses, and other personnel. The consequent resource requirements for these activities can conflict with the service requirements of the population (LHPC, 1979a). Also, the possibilities for allocating resources among different areas will be constrained by the existing stock of facilities, the availability of land, manpower, economies of scale, finance capital, political, and many other considerations.

These constraints could, if they were sufficiently strong, dominate completely, and allow no room in the strategic plan for any maneuver. In practice, although few new facilities will ever be added to well-established systems and although all the factors described are important to differing degrees, surprisingly large reallocations (for example, −30% to +16% in zones in Southeast England between 1975 and 1977) take place through mechanisms such as the updating or enlargement of existing facilities, the closure or reduction in size of old facilities, or a redistribution of more mobile resources such as manpower. The problem, hence, is to include these constraints in a way that will direct the system towards its prime objectives, but with due regard to the operating environment.

Such constraints are clearly important, and it is taken for granted that they would be specified only after detailed discussions with all the actors in the system, including patient representatives, medical staff, and other experts. Even then, it is anticipated that more than one scenario with a variation of the constraints will need to be tested, with the model used in a 'what if' manner.

3 The input variables

There are three input variables in the model—resources, patient-generating factors, and accessibility costs—whose estimation is now discussed in more detail before the formal derivation of the model and its application is given.

3.1 *Patient-generating factor*

A patient-generating factor is calculated as

$$W_i(t) = \sum_{k,m} P_{ik}(t) u_{mk}(t) , \tag{3}$$

where $P_{ik}(t)$ is the forecasted population in time t, zone i, and age-sex category k, and $u_{mk}(t)$ is the projected national hospital utilization rate in clinical specialty m in category k. Although P and u are the dominant considerations in the consumption of health care, the definition of the patient-generating factor is incomplete in the sense that it ignores certain socioeconomic differences among areas that are also believed to influence the use of the services (LHPC, 1979a). Some research on identifying these factors has been done and more work is in progress. The projected populations in each area can be determined by means of conventional demographic methods; a method for forecasting utilization rates is described in LHPC (1979a), LHPC (1979b) and is summarized in Mayhew (1980, appendix B). The latter assumes a saturation effect, arguing that utilization rates in each clinical category, though generally increasing, will gradually level out in the future.

3.2 *Resources*

Resources are defined in terms of case load, the number of patients treated by the system in a particular time period (usually one year). The regional case load is a function of the availability of hospital beds, the efficiency with which patients can be treated, finance, and other factors. All have to be taken into account. The fundamental relationship in a clinical specialty between cases, beds, and throughput, for example, is

$$B_m(t) = \frac{d_m(t)[l_m(t) + t_m(t)]}{365} , \tag{4}$$

where $B_m(t)$ is the number of beds in specialty m in time t, $d_m(t)$ is the number of cases, $l_m(t)$ is the average length of stay between admission and discharge, and $t_m(t)$ is the average length of time between the discharge and admission of a new patient. Lengths of stay depend on clinical practice, the pressure on beds, and other considerations. In some specialties, lengths of stay are declining because of improved methods of treatment, and so it is desirable to introduce these trends into the case-load estimates. Turnover intervals are not constant either, and they must also be carefully considered. Suitable methods for dealing with these measures were used by the LHPC (LHPC, 1979a) and are also briefly described in Mayhew (1980).

It is simplest to build the resource measures at a regional level, but if local conditions are quite varied, it may be argued that an aggregation of the separate trends in each place of treatment would be more accurate. In the simpler case only, however,

$$Q(t) = 365 \sum_m \frac{B_m(t)}{l_m(t) + t_m(t)} , \tag{5}$$

where $Q(t)$ is the forecasted case load to be allocated among the places of treatment. Constraints on each place of treatment may now be introduced. Suppose that after much analysis, a proportionate increase/decrease of more than $\pm p$ in resource levels is regarded as undesirable or unmanageable in a planning period. The constraints are then set as

$$D_j(t)(1+p) \geqslant D_j(t) \geqslant D_j(t)(1-p) , \tag{6}$$

where D_j is the case load in j and t is the planning horizon. Between these constraints the system is presumed indifferent to the outcome of the allocative methods.

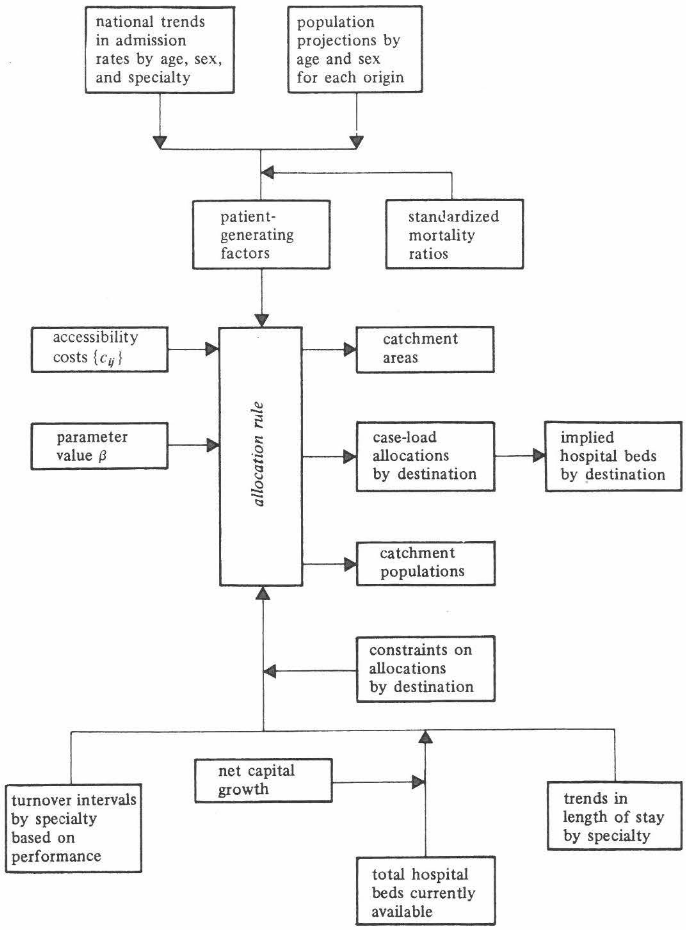


Figure 3. Planning acute inpatient hospital services using the allocation model: the inputs and outputs.

3.3 *Accessibility costs*

Accessibility costs $\{c_{ij}\}$ express the difficulty of someone in zone i being admitted as a patient in treatment zone j. In an HCS the factors determining the way a patient chooses (or is referred to) a particular destination may be complex. In some cases, the decision may be based on convenience; in others it may be the result of a series of referrals from a general practitioner or specialists lower in the HCS hierarchy. In still other cases, the patient may be taken in an emergency to a destination unrelated to his place of residence. In spite of these complexities, a number of measures, including distance, modified distance, and journey time, have proved reliable indicators of this process, which underlines that access is still the dominant consideration in most cases. These measures are further described in Mayhew and Taket (1980).

3.4 *Flow chart*

These input variables and the way they are related to the allocation rules are shown in the flowchart given in figure 3. This provides one example of how the model may be constructed and linked together; it has already been tried in practice but in another context (LHPC, 1979a). The outputs are the resources in each place of treatment (right-hand box) and other information of value. These outputs will depend on the total resources available, the configuration of demand, the specification of the constraints, the accessibilities, the model parameter, and the allocation rule. Attention is now turned to the formal derivation of the model and the methods for solving it in the case of each allocative criterion.

4 The model: a formal derivation

It has become customary in recent years to embed gravity models, such as the one described in section 3, in types of benefit functions that are derived from concepts of consumer surplus (Wilson and Kirwan, 1969; Neuburger, 1971; Cochrane, 1975; Williams, 1977; Coelho and Williams, 1978; Leonardi, 1978; 1980a; Coelho, 1980), entropy (Cohen, 1961; Wilson, 1967; Dacey and Norcliffe, 1977; Jefferson and Scott, 1979), random utility (Domencich and McFadden, 1975; Ben-Akiva and Lerman, 1978; Leonardi, 1981), or simple utility theory (Mayhew, 1981). These provide the models with a consistent theoretical basis, linked to welfare or other considerations. They enable the consideration of a wider range of systems characteristics, and enrich the variety of eventual models and the uses to which they may be put.

The embedding functions may be built by means of only minimal assumptions about the spatial behavior of people, and this is one of their main attractions. In the present case, a function is built for an activity (health care) in which there are excess demands and accessibility costs to pay. The function maximized is subject to the known, and presumed constraints acting in the HCS in order to determine the most likely spatial behavior of the patients.

4.1 *Benefit embedding functions*

This embedding function F is written in a form that incorporates the conclusions of the empirical examples in section 2.1; it is related to well-known types of entropy functions and can be shown to be equivalent to a suitably defined consumer-surplus function. It also takes into explicit consideration the elastic demand mechanism introduced in Leonardi (1980b)

$$F = -\sum_{ij} T_{ij}\left[\ln\left(\frac{T_{ij}}{f_{ij}}\right) - 1\right] - \sum_{i} U_i\left[\ln\left(\frac{U_i}{h_i}\right) - 1\right] , \qquad i = 1, ..., I , \quad j = 1, ..., J , \tag{7}$$

where

T_{ij} is the predicted patient flow between i and j,

U_i is the unsatisfied demand in i,

f_{ij} is a space discount function $\exp(-\beta c_{ij})$, where c_{ij} are the accessibility costs between i and j,
β is a spatial discount parameter, and
h_i is a parameter related to the disutility of not receiving treatment.
In equation (7), U_i may be thought of as consisting of reported demand in the form of waiting lists, queues, or as unreported demand in the form of sick people who have not presented themselves to a doctor.

Satisfied and unsatisfied demand are related by the identity

$$\sum_j T_{ij} + U_i = V_i \,, \tag{8}$$

where V_i measures the total demand in i.
The problem is to maximize F subject to constraint (8), the total demand in the system, and to a resource constraint D_j in each place of treatment j:

$$\sum_i T_{ij} = D_j \,. \tag{9}$$

That is,

$$\underset{T,\,U}{\text{maximize}}\; F \,. \tag{10}$$

This is equivalent to finding the saddle point of the Lagrangian function C, where

$$C = F + \sum_i \lambda_i \left(V_i - \sum_j T_{ij} - U_i \right) + \sum_j \nu_j \left(D_j - \sum_i T_{ij} \right) \,, \tag{11}$$

and where λ_i and ν_i are the Lagrangian multipliers associated with the constraints (8) and (9). The solution is found by equating the first derivatives of C with respect to T_{ij}, U_i, λ_i, and ν_j to zero and then solving the $J + I(J+2)$ equations:

$$\frac{\partial C}{\partial T_{ij}} = 0 \,, \tag{12}$$

$$\frac{\partial C}{\partial U_i} = 0 \,, \tag{13}$$

$$\frac{\partial C}{\partial \lambda_i} = 0 \,, \tag{14}$$

$$\frac{\partial C}{\partial \nu_j} = 0 \,. \tag{15}$$

From equations (11) and (12), and with the rearranging of terms,

$$T_{ij} = f_{ij} \exp -(\lambda_i + \nu_j) \,. \tag{16}$$

Similarly, from equations (11) and (13),

$$U_i = \exp(-\lambda_i) h_i \,. \tag{17}$$

Also, from equations (9), (11), and (16),

$$D_j = \sum_i T_{ij} = \exp(-\nu_j) \sum_i \exp(-\lambda_i) f_{ij} \,. \tag{18}$$

Therefore

$$\exp(-\nu_j) = D_j \left[\sum_i \exp(-\lambda_i) f_{ij} \right]^{-1} \,, \tag{19}$$

which in equation (16) gives

$$T_{ij} = D_j \frac{\exp(-\lambda_i) f_{ij}}{\sum\limits_i \exp(-\lambda_i) f_{ij}} . \tag{20}$$

But, this is

$$T_{ij} = D_j \frac{U_i h_i^{-1} f_{ij}}{\sum\limits_i U_i h_i^{-1} f_{ij}} , \tag{21}$$

where $U_i h_i^{-1}$ is the ratio of unsatisfied demand to the disutility of not receiving treatment. If it is assumed that U_i is sufficiently large so that $\sum\limits_i T_{ij}$ can be considered negligible, then U_i from constraint (8) equals V_i. If W_i, the morbidity factor, is defined as $V_i h_i^{-1}$, then we obtain the attraction-constrained model in equation (1):

$$T_{ij} = \frac{D_j W_i f_{ij}}{\psi_j} , \tag{1}$$

where B_j has now been replaced by ψ_j^{-1},

$$\psi_j = \sum_i W_i f_{ij} = B_j^{-1} . \tag{22}$$

The path to equation (1) thus underlines the nature of the assumptions, which hitherto have only been stated informally. We now develop the four criteria (equity, efficiency, accessibility 1 and 2) with which to allocate resources among places of treatment.

4.2 *Equity*

The objective of the equity criterion is to choose a resource configuration such that the patients generated in each i are in proportion to the relative needs of i.

From equation (1) and with summation over j, the predicted number of patients generated by i is given by

$$\sum_j T_{ij} = W_i \sum_j \frac{D_j f_{ij}}{\psi_j} . \tag{23}$$

Since W_i, an index of patient-generating potential, is also the expected number of patients, the expression

$$\sum_j \frac{T_{ij}}{W_i} = \sum_j \frac{D_j f_{ij}}{\psi_j} \tag{24}$$

gives therefore for origin i the ratio of the predicted to the expected number of patients. More importantly, it is also the ratio of the predicted service levels to the relative needs, and, as we have defined it, the objective is to ensure that this ratio is constant in all origins i by choosing the appropriate values for D_j. However, this quantity cannot be calculated directly without a priori knowledge of the service prediction, $\sum\limits_j T_{ij}$. Fortunately, it is completely analogous to base the estimation of this ratio on the total resources available in the system, Q, and W_i. Thus, a new term α is defined which is given by

$$\alpha = \frac{Q}{\sum\limits_i W_i} . \tag{25}$$

This is simply the total resources divided by the total relative needs in the region of interest. If Q reflects resource availability over the whole country, and if the generating factors are based on the expected number of patients, then α will be 1. If W_i is calculated in another way this result will not follow automatically.

If the constraints on change permitted at each destination are taken into account, the reformulated problem can be written as

$$\underset{D_j}{\text{minimize}} \sum_i \left(\sum_j \frac{D_j f_{ij}}{\psi_j} - \alpha \right)^2 = Z \,, \tag{26}$$

subject to

$$D_j^{\max} \geqslant D_j \geqslant D_j^{\min} \,, \qquad \forall j \,, \tag{27}$$

and

$$\sum_{j \in L} D_j = Q \,. \tag{28}$$

This says: choose D_j to minimize the square of the differences over all origins between the two ratios (Mayhew, 1980). The use of the 'square' is to eliminate the problems with mixed negative and positive signs. The constraints are on each destination, and they are fixed as appropriate. The total resources, Q, can apply to the whole region, or to a subset L of it. If it is only a subset then the quantity $\sum_i W_i$ should apply over an equivalent subset. By putting

$$\frac{f_{ij}}{\psi_j} = \gamma_{ij} \,, \tag{29}$$

expanding expression (26), and ignoring the constant term $I\alpha^2$, where I is the number of origins, we obtain

$$Z = \tfrac{1}{2} D^{\mathrm{T}} \mathbf{A} D - b^{\mathrm{T}} D \,, \tag{30}$$

where D^{T} is the transpose vector of resources D,

$$D^{\mathrm{T}} = [D_1, \ldots, D_j, \ldots, D_n] \,, \qquad n = J \,, \tag{31}$$

A is a symmetric matrix composed of the following elements

$$\mathbf{A} = \begin{bmatrix} 2\sum_i \gamma_{i1}^2 & 2\sum_i \gamma_{i1}\gamma_{i2} & \cdots & 2\sum_i \gamma_{i1}\gamma_{ij} & \cdots & 2\sum_i \gamma_{i1}\gamma_{in} \\ 2\sum_i \gamma_{i2}\gamma_{i1} & 2\sum_i \gamma_{i2}^2 & & & & 2\sum_i \gamma_{i2}\gamma_{in} \\ \vdots & & \ddots & & & \vdots \\ 2\sum_i \gamma_{ij}\gamma_{i1} & & & 2\sum_i \gamma_{ij}^2 & & 2\sum_i \gamma_{ij}\gamma_{in} \\ \vdots & & & & \ddots & \vdots \\ 2\sum_i \gamma_{in}\gamma_{i1} & & & & & 2\sum_i \gamma_{in}^2 \end{bmatrix} = [a_{ij}] \,, \tag{32}$$

and b^{T} is the transpose of the vector b in which the elements are

$$b = \begin{bmatrix} 2\alpha \sum_i \gamma_{i1} \\ \vdots \\ 2\alpha \sum_i \gamma_{in} \end{bmatrix} = \{b_j\} \,. \tag{33}$$

Similarly expressions (27) and (28) can be written in matrix and vector notation

$$D^{\min} \geqslant D \geqslant D^{\max} \,, \tag{34}$$

$$C^{\mathrm{T}} D = Q \,. \tag{35}$$

where C^{T} is a $1 \times n$ vector transpose with all the elements set equal to 1. Expressions (26), (27), and (28) have now been put into the standard form expected by a general quadratic programming algorithm. The matrix $\mathbf{A}$ is always positive definite or semi-definite, which indicates that global minima are obtainable. In an unconstrained problem the minimum of Z is found when the vector of first derivatives disappears; that is, when

$$g = \nabla(\tfrac{1}{2}D^{\mathrm{T}}\mathbf{A}D - b^{\mathrm{T}}D) = \mathbf{A}D - b = 0 \; . \tag{36}$$

Details of the solution method for this problem with and without constraints are contained in Fletcher (1970, 1971) and briefly in Mayhew (1980).

The equity problem, it should be noted, also has an interesting counterpart. Instead of redistributing the resources between each place of treatment j, the same equitable result may be attained by levying an 'accessibility tax' on each place of residence i to regulate demand. Although such a tax would almost certainly be unpopular, it is of theoretical value since it illustrates the symmetry of the allocation problem. The derivation of the tax and its interpretation are shown in the appendix.

4.3 *Efficiency*

Under the efficiency criterion the objective is to allocate D, so that patient preferences for places of treatment are maximized. These preferences are subject to the same constraints as applied in the equity case; that is, on each place of treatment and on the total resources available, Q. If equation (1) is inserted into equation (7), with summation carried out over i, and if after expansion the constant terms are ignored, then it is found that

$$F = -\sum_j D_j\left[\ln\left(\frac{D_j}{\psi_j}\right) - 1\right] , \tag{37}$$

where 1 in equation (37) replaces the constants without loss of generality. The reformulated problem becomes, therefore,

$$\underset{D_j}{\text{maximize}}\; F \; , \tag{38}$$

subject again to

$$D_j^{\max} \geqslant D_j \geqslant D_j^{\min} \; , \tag{27}$$

and

$$\sum_{j \in L} D_j = Q \; . \tag{28}$$

This is equivalent to finding the saddle point of the Lagrangian function H where

$$H = F + \lambda\left(Q - \sum_j D_j\right) + \sum_j \mu_j [D_j^{\max} - D_j] - \sum_j \eta_j [D_j^{\min} - D_j] \; , \tag{39}$$

and where λ, μ_j, and η_j are the Lagrange multipliers associated with the resources available, Q, and the inequality constraints in expression (27). The solution to this maximization problem is found by solving the $3J+1$ equations

$$\frac{\partial H}{\partial D_j} = 0 \; , \tag{40}$$

$$\frac{\partial H}{\partial \lambda} = 0 \; , \tag{41}$$

$$\frac{\partial H}{\partial \eta_j} = 0 \; , \tag{42}$$

and

$$\frac{\partial H}{\partial \mu_j} = 0 \ , \tag{43}$$

together with the complementarity slackness conditions:

$$\mu_j (D_j^{\max} - D_j) = 0 \ , \qquad \mu_j \geqslant 0 \ , \tag{44}$$

$$\eta_j (D_j^{\min} - D_j) = 0 \ , \qquad \eta_j \geqslant 0 \ , \tag{45}$$

It is easily shown that H is optimal when

$$D_j = \psi_j \exp(\eta_j - \mu_j - \lambda) \ . \tag{46}$$

But, from equation (28),

$$Q = \sum_j D_j = \exp(-\lambda) \sum_j \psi_j \exp(-\mu_j + \eta_j) \ . \tag{47}$$

If $\exp(-\lambda)$ is made the subject of equation (47), then substitution into equation (46) gives

$$D_j = Q \frac{\psi_j \exp(-\mu_j + \eta_j)}{\sum_j \psi_j \exp(-\mu_j + \eta_j)} \ . \tag{48}$$

In the case when there are no bounds on D_j operating [see equation (27)], equation (46) becomes

$$D_j = Q \frac{\psi_j}{\sum_j \psi_j} \ , \qquad Q \geqslant D_j > 0 \ , \tag{49}$$

since

$$\mu_j = \eta_j = 0 \ . \tag{50}$$

Equation (49) is the basic allocation formula that matches the resources in j with patient preferences for treatment in that location. The preference term is ψ, which is the sum of the patient-generating factors discounted by the accessibility costs [equation (22)]. It is a measure of the total demand potential on j after accessibility costs have been paid. Thus, the resources are divided between places of treatment simply by proportioning Q according to the potential on j divided by the sum of all the potentials on all j.

4.4 *Accessibility 1*

The average accessibility cost from i to all j is defined as

$$c_i = \frac{\sum_j T_{ij} c_{ij}}{\sum_j T_{ij}} = \frac{\sum_j (D_j f_{ij} c_{ij} / \psi_j)}{\sum_j (D_j f_{ij} / \psi_j)} \ . \tag{51}$$

Since the criterion requires that c_i be constant, it may be replaced by $\bar{c}$, where $\bar{c}$ is either presumed beforehand or is based on the current average for the system, that is,

$$\bar{c} = \frac{\sum_{i,j} T_{ij} c_{ij}}{\sum_{i,j} T_{ij}} \ . \tag{52}$$

The objective may now be defined. It is

$$\underset{D_j}{\text{minimize}} \ G \ , \tag{53}$$

subject to

$$D_j^{\max} \geqslant D_j \geqslant D_j^{\min} , \tag{27}$$

and

$$\sum_{j \in L} D_j = Q , \tag{28}$$

where

$$G = \sum_i (c_i - \bar{c})^2 . \tag{54}$$

This says: minimize the differences in all i between the average accessibility costs to j and a supplied average, $\bar{c}$, subject to the usual constraints. Equation (54) has an interesting property; it is a homogeneous function of degree 0. Hence, the following property holds

$$G(k\boldsymbol{D}) = \sum_i [c_i(k\boldsymbol{D}) - \bar{c}]^2 = \sum_i [c_i(\boldsymbol{D}) - \bar{c}]^2 = G(\boldsymbol{D}) , \tag{55}$$

where k is a constant ($\neq 0$) and $\boldsymbol{D}$ is a vector with J elements. Equation (55) describes a lined surface in J dimensions with the lines having directional cosines proportional to $\boldsymbol{D}$, where $\boldsymbol{D} = (D_1, ..., D_n)$. Along any line the average cost, and hence G, is unchanged for different values of $\boldsymbol{D}$, which indicates an infinite number of solutions to this problem. However, provided the resource constraint in equation (28) is applied, the problem has a well-defined solution.

4.5 *Accessibility 2*

The variance criterion is constructed in a similar way. The variance in the travel costs from i to all j is defined by

$$v_i = \frac{\sum_j T_{ij}(c_{ij} - \bar{c})^2}{\sum_j T_{ij}} = \frac{\sum_j (D_j f_{ij}(c_{ij} - \bar{c})^2/\psi_j)}{\sum_j (D_j f_{ij}/\psi_j)} . \tag{56}$$

The objective is then written

$$\underset{D_j}{\text{minimize}}\; S , \tag{57}$$

subject to

$$D_j^{\max} \geqslant D_j \geqslant D_j^{\min} , \tag{27}$$

and

$$\sum_{j \in L} D_j = Q , \tag{28}$$

where

$$S = \sum_i v_i . \tag{58}$$

Like the first accessibility criterion, the second is also homogeneous of degree 0, the objective function describing again a lined surface in J dimensions.

4.6 *The two-origin, two-destination problem*

Figure 4 shows sketches of all four criteria in the simplest of possible systems: two origins and two destinations. On the axes in the plane are D_1 and D_2, the two unknowns. On the vertical axis in arbitrary units are the values of the four objective functions. The regional resource constraint is represented by the diagonal AB along which $D_1 + D_2 = Q$. The desired values of D_1 and D_2 are located on AB at the maximum or minimum of the respective functions. When upper and lower bounds on D_j are applied, the plane is divided by vertical and horizontal lines into a feasible and an infeasible region; the optimum value on each criterion is still lying on AB, but inside in the feasible part. Figure 4 also shows the important result that each criterion selects in general a different set of resource allocations from the others, thus drawing attention to their incompatibility. To determine the suitability of these criteria, the results of the application to a planning problem in the United Kingdom are now described.

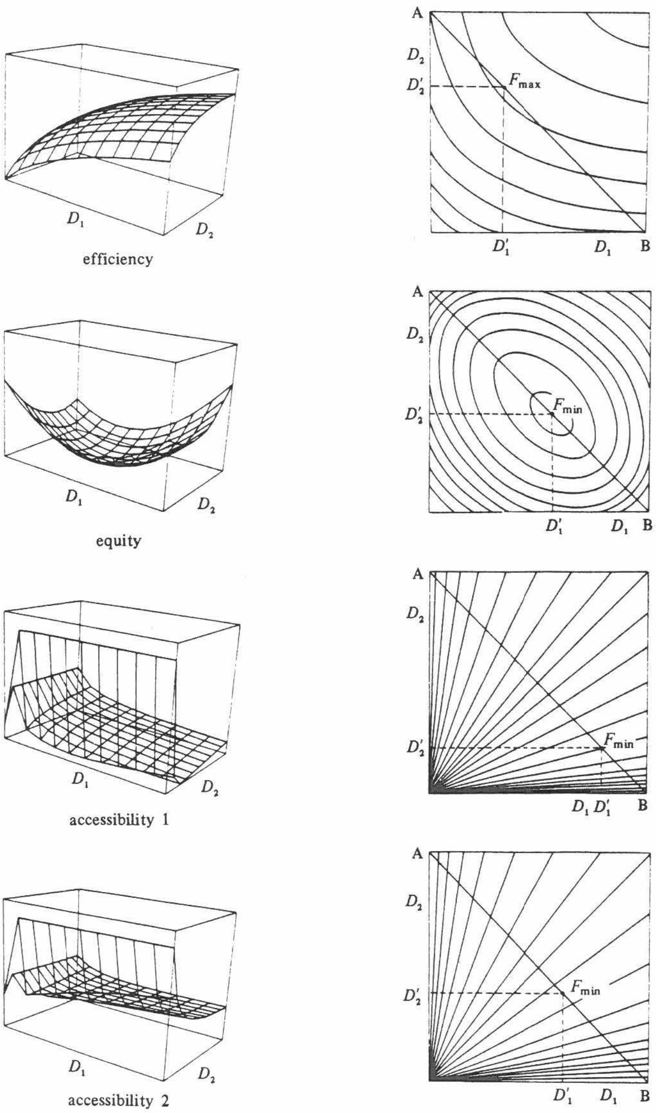


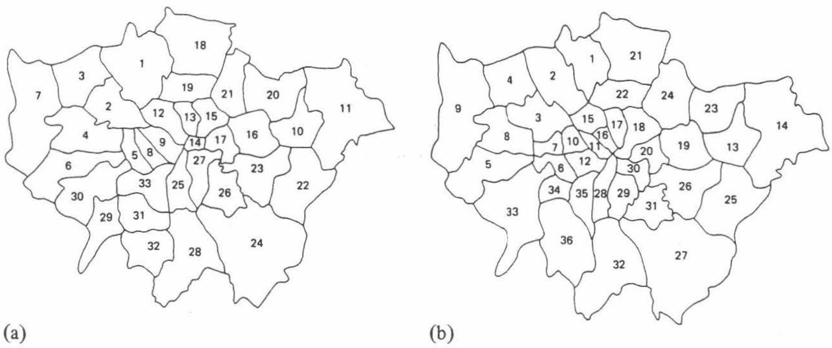
Figure 4. Three-dimensional representations and maps of the objective functions for the two-origin, two-destination case.

5 Application

The methods described above have been applied and tested on 1977 data for the London region in England. London forms a particularly appropriate application since it has especially severe planning problems that have resisted solution by more conventional approaches. Approximately 7 million people live in the area covered, and it is served by about 200 hospitals treating approximately 1 million inpatient cases each year. Because of changes in the size and demographic structure of the population, health authorities are interested to know which facilities to enlarge, reduce in size, or close altogether. The existing pattern of patient flows between areas, however, is complex: this is due to the proximity of facilities (particularly the relative overconcentration in the city center), the ready availability of transport services, and other factors. In addition, there are constraints on change that are imposed by the condition of the existing hospital stock, the availability of land, financing, and other resources. Finally, London is a national and international center for medical education and research whose activities in these fields must be taken into account in the resource-allocation process. To these specific factors must be added the differential trends in treatment that are changing the patient mix and type of care received, with important implications for hospital throughput and hence case-load capacities.

5.1 *Zoning system*

In figure 5 two maps show the thirty-three origin zones (administrative boroughs of the Greater London Council, GLC) and thirty-six destination zones (Health Districts) used in these applications. The names of these zones may be found in table 1 in Mayhew (1980, page 24). In addition to these, there is one external zone to close the system. The model for this region was constructed from an aggregate of twenty-three acute specialties, a list of which is shown in table 1 of Mayhew and Taket (1980, page 16). Details of the calibration procedure are also found in this reference, whereas the results of validation tests to check the predicted capability of the model are given in Mayhew and Taket (1981). Here, all that is essential, in addition to the input data, is a value for the β parameter in equation (1), which was obtained from the above work; this is 0·367.



(a) (b)

Figure 5. The Greater London Council: definition of (a) origin zones and (b) destination zones.

5.2 *Presentation of outputs*

The most convenient ways of illustrating the outputs of these procedures are bar charts, showing the proportionate changes in allocations, and scatter diagrams. Scatter diagrams show the relationships—both before and after the application of the methods—between the numbers of patients generated in i, an origin zone (that is,

$\sum_j T_{ij}$), and the relative needs in i scaled by α—the regional service-demand ratio given in equation (25) (that is, αW_i). A linear equation fitted to this scatter will thus give the extent to which the reallocation process has satisfied the relative needs of the population. In the equity case only, the result should be an equation with a slope coefficient $\hat{b}$ equal to 1 and an intercept term $\hat{a}$ that is not statistically significantly different from 0. When the coefficient of explanation R^2 is also 1, it means the equity criterion has been met exactly. In practice, the value of R^2 is reduced according to the stringency of the constraints applied on the destinations, $D_j^{\min}$ and $D_j^{\max}$. For the other cases, the properties of the resultant scatters are completely different, but as will be seen, they usually provide sufficient information to judge the effectiveness of each criterion. (A straight line in the efficiency case is also obtained when D_j is plotted on $\gamma\psi_j$, where $\gamma = Q/\sum_j \psi_j$. This would be an alternative way of presenting the results.)

5.3 *Tests*

Each criterion has been thoroughly tested by use of the existing and hypothetical data to represent both the current situation and possible development scenarios (changes in supply and demand). Some of these scenarios were deliberately exaggerated to see how the methods performed when they were stretched for particular input sets. Only the results obtained with the current data sets are reported, although all the developmental runs of the methods have been taken into account. For simplicity and brevity, only two tests are shown: one with a lower bound on each destination, and one without. That is,

test 1 $\quad Q \geqslant D_j \geqslant D_j(1-0{\cdot}25)$,

test 2 $\quad Q \geqslant D_j \geqslant 0$.

The upper bound in test 1 has been left open (although Q, of course, is the maximum that can be allocated) to see where the major shortfalls in resources are predicted to occur; the lower bound has been arbitrarily fixed to 75% of the current value. In test 2 the lower bound is simply zero to avoid negative allocations.

5.4 *Allocative behavior*

Figures 6 and 7 show the predicted percentage change in allocations for each test. In test 1, the influence of the 75% lower bound shows up strongly in the negative part of the charts, whereas in test 2 it is seen that the allocations can give extreme solutions with emphasis on allocations to only one or two locations. In the experiments carried out, the equity criterion is always the least susceptible to this behavior, whereas efficiency and accessibility are the most susceptible. In the efficiency case, for example, the results are especially sensitive to the measurement of the local accessibility costs; the reasons for the very unusual large allocations in test 2 to zones 14 and 23 by accessibility 1 are unclear, however. It was generally found that the spatial patterns of reallocations are more intuitive in the cases of equity and efficiency than for accessibility tests 1 and 2, and this empirical feature makes them more practical as allocative criteria. For example, the charts both in test 1 and in test 2 show that the equity and efficiency criteria tend to decentralize the available resources to zones lying closer to the perimeter of the urban region. This is consistent with other findings (for example, LHPC, 1979a) which show that the central area is relatively overprovided with resources.

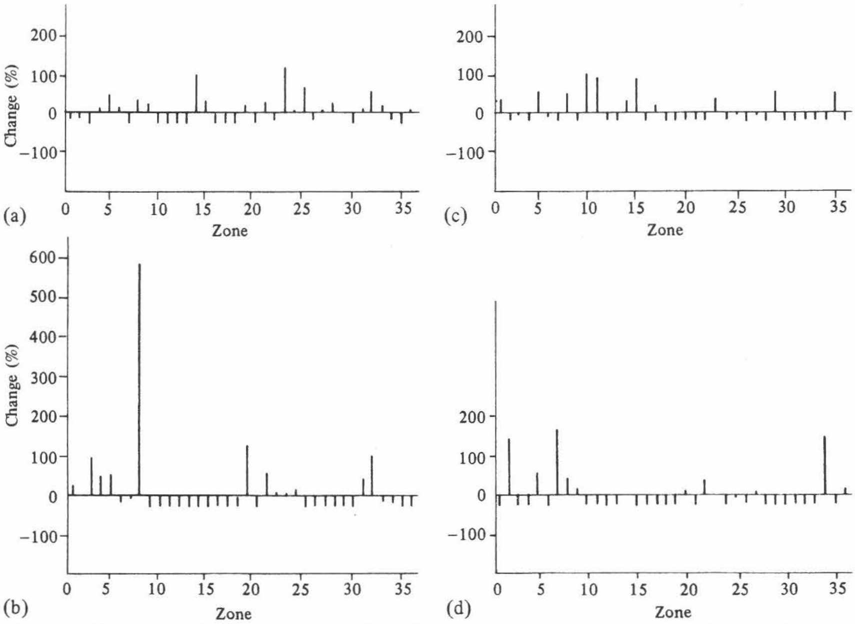


Figure 6. Percentage changes in resource allocations to destination zones under test 1 for (a) equity, (b) efficiency, (c) accessibility 1, and (d) accessibility 2.

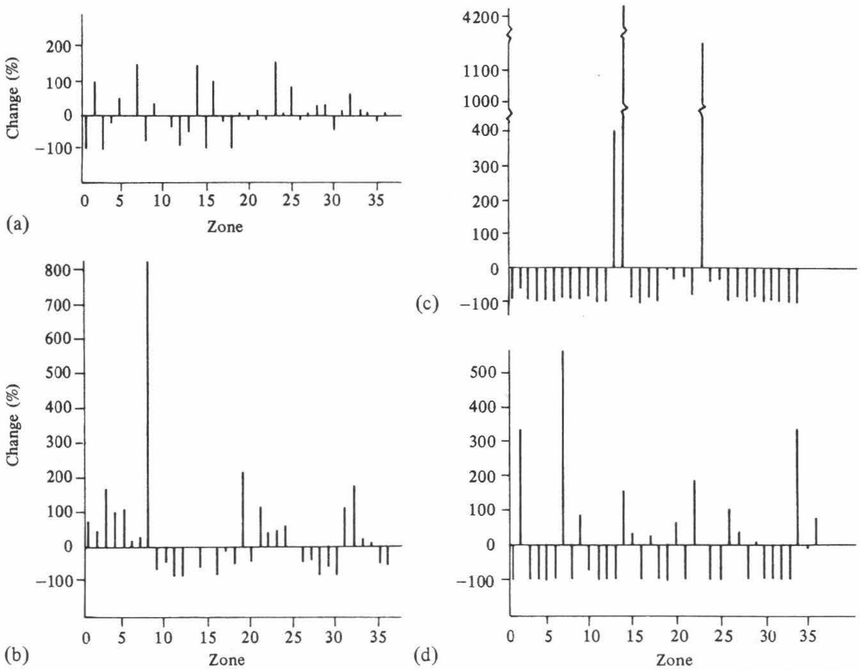


Figure 7. Percentage changes in resource allocations to destination zones under test 2 for (a) equity, (b) efficiency, (c) accessibility 1, and (d) accessibility 2.

5.5 *Patient behavior*

The effects of these reallocations on the service levels (numbers treated) of the population in each place of residence i are shown in figures 8 and 9 for tests 1 and 2, respectively. (Figure 10 shows the existing service levels plotted on relative needs.) As is seen in figures 8 and 9 the equity criterion reproduces the straight line as desired in both tests. An encouraging feature in all the experiments is the stability of the slope and intercept terms (which is necessary under the equity definition) even during some very severe tests of the method. Furthermore, it was found that large gains in equity were attainable even when the constraints on change were very tight (say D_j ±5%). In test 2, the unconstrained case (figure 9), an outlier among the data points is observed for the equity case: fortunately this behavior never arises in more realistic applications that use constraints.

The other criteria do not have the above slope property, and the values of R^2 they give are, as is seen in figures 8 and 9 always less than in the equity case for the same sets of constraints. This underlines the fact that equity, efficiency, and accessibility 1 and 2 are incompatible goals in that it is impossible with these data and this model to achieve all four simultaneously.

The effects of the unusual allocations on service levels by accessibility 1 found in test 2 (see figure 7) are shown in figure 9. The result is clearly extreme in that, as is shown, no attempt is made to reconcile the resources allocated with the relative needs of the population ($R^2 = 0{\cdot}002$). On this basis and on the basis of other experiments, it thus seems unreasonable to proceed with this criterion. The case for rejecting accessibility 2, however, is much less clear-cut. The main problems with it seem to be, first, its somewhat unpredictable behavior in sensitivity tests carried out

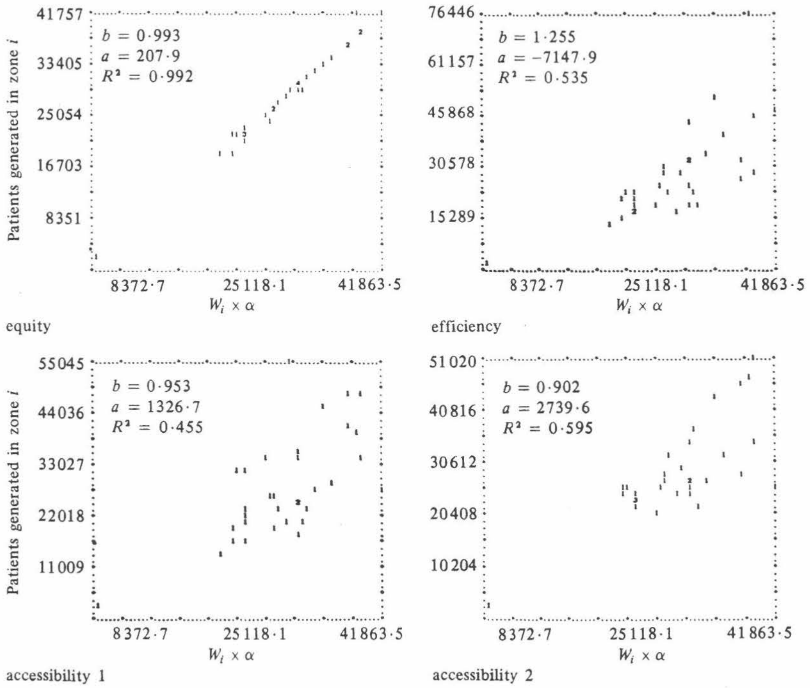


Figure 8. Plot of number of patients generated in zone i on the relative needs of zone i (test 1).

on the constant $\bar{c}$ in equation (56), and, second, the often counterintuitive results obtained. These make it difficult to understand the precise mechanisms of this method. Nevertheless, further applications are needed to settle these points.

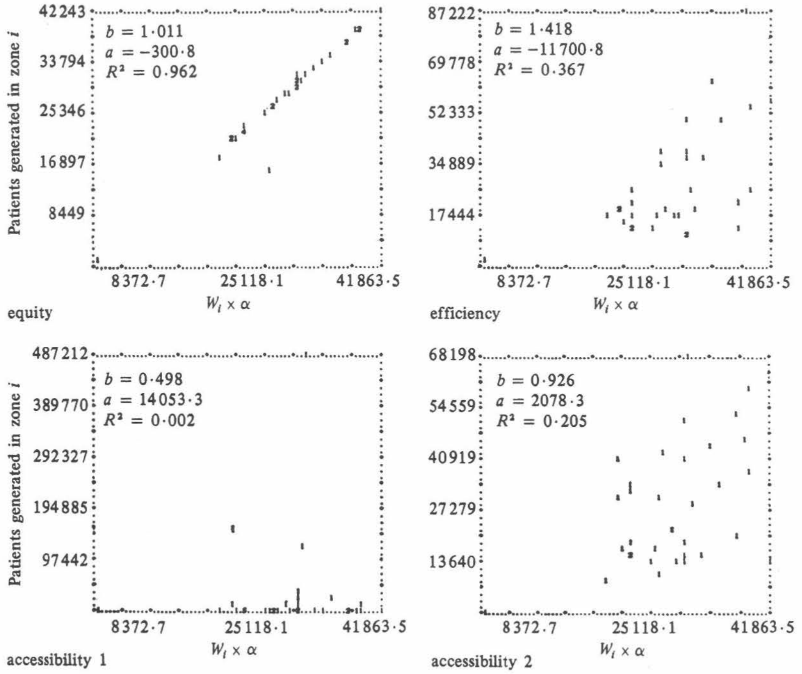


Figure 9. Plot of number of patients generated in zone i on the relative needs of zone i (test 2).

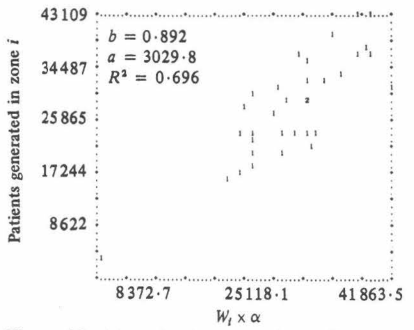


Figure 10. Plot of existing numbers of patients generated in zone i on the relative needs of zone i.

5.6 *Sensitivity analysis*

The equity and efficiency cases were thus selected for further sensitivity analysis. This involves an unconstrained model of the type used in test 2, but in which the β parameter is allowed to vary over a wide range. Although in practice this parameter is expected to change very little, the experiment is necessary to test the logic of the allocations when the criteria are exposed to extremes of behavior. For instance, a value of β equal to 0 implies that there are no accessibility costs to pay, whereas a large value implies very large costs and therefore a high space discount premium. Tables 1 and 2 indicate facility behavior in each treatment district for different β values. A dot indicates that all the facilities in a district have been closed. Tables 3 and 4 show the regression coefficients and values for R^2.

5.6.1 *Equity* For $\beta = 0{\cdot}005$ the only facilities open are at the city center itself (zone 18). This seems most logical as this zone is a focus for the whole region. The first facilities in outer zones appear when $\beta = 0{\cdot}1$. When $\beta = 0{\cdot}2$, the facilities in the center close because as costs get higher, needs are better served locally rather than centrally. As β increases further, more suburban facilities open until a maximum of thirty-two out of thirty-six zones have resources allocated to them. The special case when $\beta = 0$ should also be noted (that is, no accessibility costs at all). From equations (22) and (26), we see that the coefficients γ_{ij} become constant and that the objective function reduces to

$$Z = \sum_i \left(\frac{\sum_j D_j}{\sum_j W_i} - \alpha \right)^2 . \tag{59}$$

Table 1. Sensitivity of facility behavior with respect to β: the equity case (zones where all facilities have been closed are indicated by black dots).

Zone	β															
	0·005	0·01	0·05	0·1	0·15	0·2	0·25	0·3	0·35	0·4	0·45	0·5	1·0	3·0	5·0	8·0
1	●	●	●	●	●	●	●		●							●
2	●	●	●	●	●										●	
3	●	●	●	●	●	●			●	●	●	●				
4	●	●	●													
5	●	●	●	●												
6	●	●	●													
7	●	●	●	●	●	●	●	●					●	●	●	
8	●	●	●	●	●	●	●	●								
9	●	●														
10	●	●	●	●												
11	●	●														●
12	●									●	●	●				
13	●	●	●	●	●											
14	●	●														
15	●	●	●				●	●	●	●		●	●	●	●	●
16	●	●	●													
17	●	●	●													
18						●	●	●	●	●	●	●	●	●	●	
19	●	●	●	●	●	●										
20	●	●														
21	●	●	●	●												
22	●	●	●		●											
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30	●	●								●	●	●	●	●	●	●
31	●	●														
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33	●	●	●													
34	●	●	●	●	●									●	●	
35	●	●	●	●	●	●										
36	●	●														

Since $\sum_j D_j = Q$ and since $\alpha = Q/\sum_i W_i$, Z will be a minimum no matter how the resources are allocated. Thus there are an infinite number of equitable solutions to this case.

5.6.2 *Efficiency* Facility behavior under the efficiency criterion is the opposite of equity. When β is zero, equation (49) reduces to

$$D_j = \frac{Q}{J}, \tag{60}$$

where J equals the number of treatment zones. Thus each district receives an identical one-Jth share of the available resources Q. As β increases, the more accessible locations to demand (that is, those with high potentials ψ_j) begin to dominate the solution, so that gradually the zones with less potential become ignored and the facilities in them are closed. Another major difference with the equity solution is that the central facilities (zone 18) always remain open, whereas in the equity case they are closed ($0 \cdot 2 \leqslant \beta \leqslant 8 \cdot 0$).

Table 2. Sensitivity of facility behavior with respect to β: the efficiency case (zones where all facilities have been closed are indicated by black dots).

Zone	β															
	<1·0	1·5	1·75	2·0	2·25	2·5	2·75	3·0	3·5	4·0	4·5	5·0	5·5	6·0	7·0	8·0
1												•	•	•	•	•
2									•	•	•	•	•	•	•	•
3																
4																
5																
6														•	•	•
7					•	•	•	•	•	•	•	•	•	•	•	•
8																
9							•	•	•	•	•	•	•	•	•	•
10								•	•	•	•	•	•	•	•	•
11		•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
12			•	•	•	•	•	•	•	•	•	•	•	•	•	•
13											•	•	•	•	•	•
14				•	•	•	•	•	•	•	•	•	•	•	•	•
15									•	•	•	•	•	•	•	•
16		•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
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27										•	•	•	•	•	•	•
28		•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
29						•	•	•	•	•	•	•	•	•	•	•
30			•	•	•	•	•	•	•	•	•	•	•	•	•	•
31																•
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33																•
34						•	•	•	•	•	•	•	•	•	•	•
35								•	•	•	•	•	•	•	•	•
36									•	•	•	•	•	•	•	•

Table 3. Sensitivity analysis of β: regression results for the equity case.

β	Regression results [a] R^2	$\hat{b}$	$\hat{a}$	Total number of zones with open facilities
0·005	1·00	0·64	−18·2	1
0·01	0·99	0·64	−3·15	2
0·05	0·96	1·06	−1734	11
0·1	0·95	1·07	−2018	20
0·15	0·96	1·06	−1552	25
0·2	0·98	1·03	−884	28
0·25	0·98	1·02	−424	30
0·3	0·98	1·00	−122	31
0·35	0·9	1·01	−256	32
0·4	0·95	1·02	−427	31
0·45	0·93	1·03	−698	32
0·5	0·91	1·04	−1022	31
1·0	0·76	1·16	−4367	32
3·0	0·67	1·28	−7895	31
5·0	0·67	1·29	−8158	30
8·0	0·67	1·30	−8385	32

[a] R^2—coefficient of explanation; $\hat{b}$—slope; $\hat{a}$—intercept.

Table 4. Sensitivity analysis of β: regression results for the efficiency case.

β	Regression results [a] R^2	$\hat{b}$	$\hat{a}$	Total number of zones with open facilities [b]
1·0	0·17	2·40	−39000	36
1·5	0·13	2·87	−52000	33
1·75	0·12	3·00	−56000	31
2·0	0·11	3·10	−58000	30
2·25	0·10	3·17	−60000	29
2·5	0·09	3·21	−61000	27
2·75	0·09	3·23	−62000	26
3·0	0·08	3·25	−63000	24
3·5	0·07	3·25	−62000	19
4·0	0·07	3·24	−62000	16
4·5	0·06	3·22	−62000	15
5·0	0·06	3·20	−61000	13
5·5	0·06	3·18	−61000	12
6·0	0·05	3·17	−61000	11
7·0	0·05	3·16	−60000	11
8·0	0·05	3·15	−60000	9

[a] R^2—coefficient of explanation; $\hat{b}$—slope; $\hat{a}$—intercept.

[b] Allocations for which $D_j \leqslant 0$ are impossible with the efficiency criterion [equation (49)]. Thus a 'closed' facility is said to occur when $D_j < 10$.

6 The equity–efficiency trade-off model

In view of the different resource configurations produced by the equity and efficiency criteria, it seems reasonable for certain types of HCSs to design a model that permits the user to trade off one goal against the other. To analyze these trade-offs the following mathematical programming problem is constructed

$$\underset{D}{\text{maximize}}\ F(D) = \theta V_1(D) + (1-\theta)V_2(D)\ , \tag{61}$$

subject to

$$\sum_{j \in L} D_j = Q \, , \tag{28}$$

$$D_j^{\max} \geqslant D_j \geqslant D_j^{\min} \, , \tag{27}$$

where

$$D = \{D_j\} \, , \qquad j = 1, \ldots, J \, ,$$

$$V_1(D) = -\sum_j D_j \left(\ln \frac{D_j}{\psi_j} - 1 \right) \, , \tag{37}$$

$$V_2(D) = -\sum_i \left(\sum_j \frac{D_j f_{ij}}{\psi_j} - \alpha \right)^2 \, , \tag{26}$$

and θ is a trade-off parameter. Equation (61) is a mixture of the equity and efficiency objective functions. It is to be maximized subject to the usual constraints (27) and (28). This is a concave programming problem with simple linear constraints. A well-known method to solve it is the Frank–Wolfe method (Frank and Wolfe, 1956), which in this case takes a simple form. The iterations of the method are based on the use of linear approximation to equation (61) to find best directions of increase. The linear subproblem for equation (61) and constraints (27) and (28) is written

$$\underset{D}{\text{maximize}} \sum_j D_j F'(D^0) \, , \tag{62}$$

where D^0 is the best guess solution so far and $F'(D^0)$ are the derivatives evaluated at the point D^0.

This is derived by expanding $F(D)$ in a Taylor expansion around D^0, truncated to the first-order terms. These terms describe the tangent plane to equation (61), and if the constant terms are ignored the result simplifies to expression (62). Subproblem (62) is now a simple continuous knapsack problem, which is easily solved for this special case (for example, see McMillan, 1975).

The solution to subproblem (62), and constraints (27) and (28) are now used to determine the best direction for an improvement in equation (61). That is,

$$d = D^* - D^0 \, , \tag{63}$$

where D^* is the solution just obtained. The best guess solution to problem (61) and constraints (27) and (28) is now found by solving the univariate maximization problem

$$\underset{0 \leqslant \lambda \leqslant 1}{\text{maximize}} \; F(D^0 + \lambda d) \, . \tag{64}$$

Once λ is obtained, D^1, the improved guess to the solution, is given by

$$D^1 = D^0 + \lambda d \, . \tag{65}$$

Problem (64) can be solved, for instance, by the Newton–Raphson method. These steps, subproblem (62), constraints (21) and (28), and problem (64), may then be repeated until convergence. The method is usually fast in the first few iterations, although it is difficult to reach a much higher level of precision in further steps. However, it is well suited to the type of sensitivity analysis required in the trade-off model whose application is described in the next section.

6.1 *Trade-off results*

Figure 11 shows the results for the service levels in the origin zones based on different values of the trade-off parameter (P^{to}), which range from pure efficiency ($P^{\text{to}} = 1 \cdot 0$) to pure equity ($P^{\text{to}} = 0 \cdot 0$). No constraints, only $D_j \geqslant 0$, have been applied in this example, although the algorithm developed has the capability of incorporating constraints. As is seen, by reducing the effect of the efficiency component, the scatter of points gradually assumes the characteristic straight-line form with a slope b becoming closer to 1·0. Notice that the trade-off parameter must first be very small ($<0{\cdot}5 \times 10^{-5}$) before the equity criterion takes effect. This is simply a reflection of the different ways the individual functions are constructed and a reflection of their component values. The general form of the trade-off curve is shown in figure 12. Since each part of the function is measured in different units and since each has a range of values dependent on the input variables, it was found useful to standardize the axes in this figure in the range 0–100.

The result is the smooth curve in figure 11, points of which indicate the indexed values (0–100) of the component functions. We have not yet examined how to infer from a given set of allocations the percentage efficiency or equity that would be

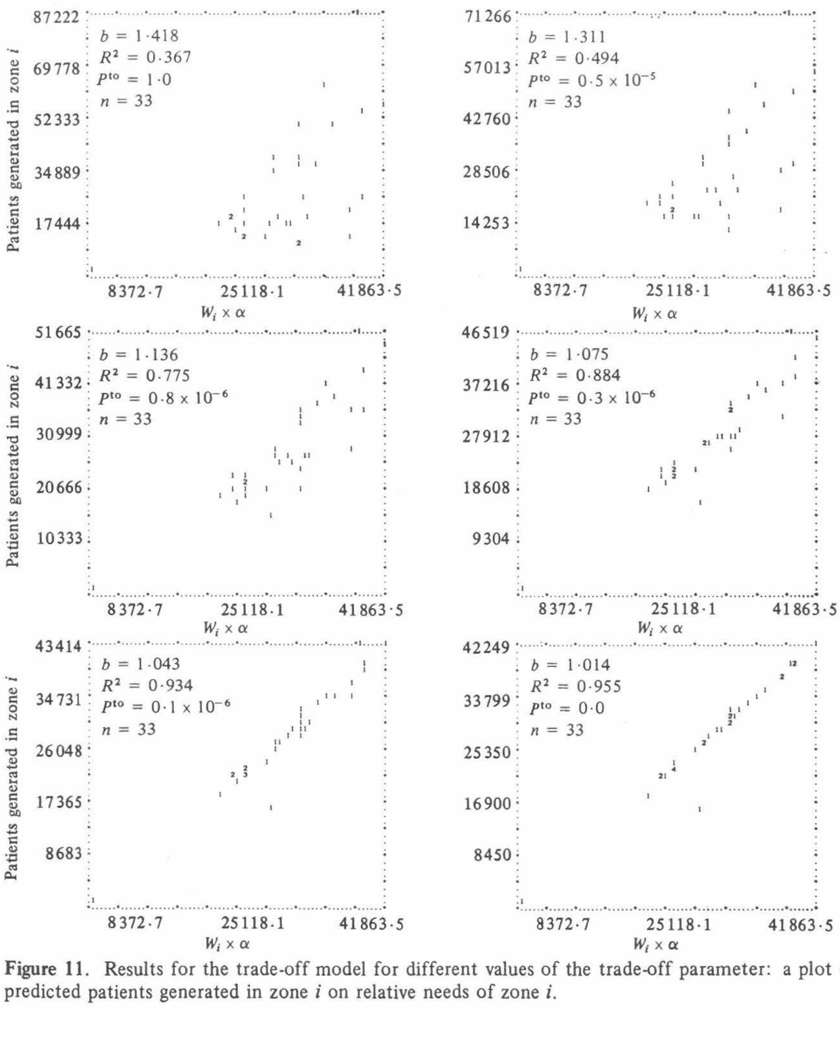

Figure 11. Results for the trade-off model for different values of the trade-off parameter: a plot of predicted patients generated in zone i on relative needs of zone i.

implied by the data. The main advantage of this approach is to allow a user to test a wider range of planning options that are not based purely on notions of efficiency or equity (as they have been defined here) and to see how the predicted resource configuration changes with the size of the trade-off parameter.

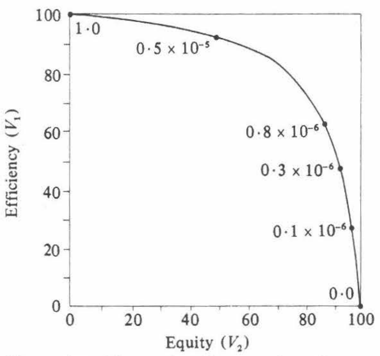

Figure 12. The trade-off curve for efficiency versus equity for different values of the trade-off parameter.

7 Conclusions

This paper has considered four criteria of resource allocation in a health-care system, where size and structure of the population and the availability of resources can change over time and space. These criteria are based on simple notions of the choice behavior of patients that can be described by a simple attraction-constrained gravity model. This model assumes that there are insufficient resources in the health-care system to supply all needs, and that service levels in areas of residence would be strongly influenced by the local availability of resources. The methods are designed with the strategic planning of health-care services in mind, in which planners are interested mainly in the broad distributional effects of different spatial resource configurations and not in the detailed pattern of service provision. The criteria considered are based on measures of equity, efficiency, and two types of accessibility, with bounds on the sizes of the facilities allocated in each place of treatment. They have been thoroughly tested on data from the London area in England, which is known to have a very complex distributional problem. As a result of these considerations, accessibility as an operational allocative criterion has been rejected in favor of the equity and efficiency measures. But because it was shown that a regional health-care system cannot attain an equitable and efficient allocation of resources simultaneously, it was suggested that the criteria could be merged into a biobjective trade-off function that allowed the user to test resource configurations by trading off one criterion against the other by means of a trade-off parameter. This was successfully tested on the same data by use of a purpose-designed algorithm based on a modified Frank–Wolfe method. A problem that was not examined with this approach, however, was how to interpret from a given set of data at what point on the trade-off curve (figure 12) the data lie. This aspect needs further work for the multiobjective allocative approach to be completely successful. Currently, the equity and efficiency methods are ready to be used by themselves, wherever health authorities have a reasonably clear idea of their main goals. The equity case, for instance, is related to the 'RAWP' formula (RAWP, 1976) for sharing resources in England and Wales, but it takes automatically into consideration problems of cross-boundary flows and interactions between supply and demand. For more detailed planning purposes, the methods are being further developed so that they can apply to multilevel systems, structured in an hierarchical way, that explore equity and efficiency problems when there are multiple services and a range of facility sizes.

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APPENDIX: Accessibility tax

The basic model is

$$T_{ij} = B_j D_j W_i \exp(-\beta c_{ij}) \ . \tag{A1}$$

The service-need ratio is given by

$$\alpha_i = \sum_j \frac{T_{ij}}{W_i} = \sum_j B_j D_j \exp(-\beta c_{ij}) \ , \tag{A2}$$

where

$$B_j = \left[\sum_i W_i \exp(-\beta c_{ij}) \right]^{-1} \ . \tag{A3}$$

The equity criterion requires that α_i = constant, $\forall i$ (that is, $\alpha_i = \alpha$). Define an accessibility tax ρ_i, then

$$\alpha = \sum_j B_j D_j \exp(-\beta c_{ij}) \phi_i \ , \tag{A4}$$

where

$$\phi_i = \exp(-\rho_i) \ , \tag{A5}$$

and

$$B_j = \left[\sum_i W_i \exp(-\beta c_{ij}) \phi_i \right]^{-1} \ . \tag{A6}$$

From equation (A4),

$$\phi_i = \alpha \left[\sum_j B_j D_j \exp(-\beta c_{ij}) \right]^{-1} \ . \tag{A7}$$

In effect, equation (A7) means that zones with a higher accessibility to services will be charged more 'tax' than those with lower accessibilities. As ϕ_i occurs on both sides of equation (A7), it must be found by the iterative sequence

$$\phi_i^{(n+1)} = \alpha \left[\sum_j B_j D_j \exp(-\beta c_{ij}) \phi_i^{(n)} \right]^{-1} \ , \tag{A8}$$

where n is the iteration number. The tax is expressed in the same units as c_{ij}. A problem, however, is to give it an operational meaning. In fact, on closer examination, the tax need not be a monetary tax in the traditional sense at all. Nonmonetary costs, for example, are incurred by people who are forced to 'queue' for treatment on waiting lists. Thus ϕ_i may be used to determine annual patient quotas from different origin zones with the usual provisions of giving emergency cases priority. Such a scheme, it may be argued, would distribute the burden of waiting time more fairly among the population as a whole. However, although the idea of a tax is of theoretical interest, there might be political and administrative difficulties associated with its implementation.