AN ALTERNATIVE MATHEMATICAL DESCRIPTION OF A PLAYER IN GAME THEORY

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INTRODUCTION

In standard game theory and mathematical economics, an actor or player is usually represented by a utility function - or, more generally, by a preference preordering. We propose here a new mathematical description of a player which incorporates several features drawn from cognitive psychology. Two key factors are introduced: the environment on which the player acts, and a dynamic (temporal) element. Viability theory is used to add some mathematical flesh to this conceptual skeleton.

We begin by describing a player as a cognitive system, specifying its unknowns and the associated laws of evolution; we then attempt to justify this approach. The discussion concludes with a mathematical description of a player and a presentation of an analog of a standard noncooperative game based on our new approach.

THE COGNITIVE SYSTEM

The unknowns of the cognitive system are described by its state (sensory-motor couple) and a regulatory control (conceptual control). The state of the system (henceforth called the *sensory-motor state*) is described by:

1. The state of the environment on which the player acts.
2. The state of cerebral motor activity of the player, which guides his action on the environment.

The regulatory control of the cognitive system is described by:

3. An endogenous cerebral activity which is not genetically programmed, but acquired by learning and recorded in the memory. The purpose of this activity is to "interpret" (or "illuminate") the sensory perception of the environment, and we shall call it the "*conceptual control*".

We should emphasize that we shall study the *evolution* both of the state of the cognitive system and of its regulatory control. For this purpose, we must identify the laws that govern the evolution of the system. These are as follows:

1. *A recognition mechanism*, with genetically programmed evolution, which matches the conceptual control to be chosen with the sensory perception of the environment and of variations in the environment.

2. *A law for evolution of the environment*: - the velocity of this evolution depends upon the evolutionary history of both the environment and the cerebral motor activity.

3. *A law for evolution of the cerebral motor activity*: - the velocity of this evolution depends upon the evolutionary history of both the sensory perception of the environment and the conceptual control (this law is used as a regulatory mechanism).

4. *A viability condition*, which expresses the fact that at each instant the player *transforms* the environment by acting upon it and consuming scarce resources.

Applying viability theory to this system leads to a nondeterministic feedback map associating a set of conceptual controls (possibly empty) with each sensory-motor couple. This map is determined by the four laws outlined above. Viability theory states that, under certain technical conditions described at the end of this paper, a necessary and sufficient condition for the existence of a solution (state-control) to the cognitive system is that the sets of conceptual controls associated with every sensory-motor couple by the feedback map are nonempty. Furthermore, it states that for every solution of the cognitive system, the conceptual controls depend upon the sensory-motor couples *through* this feedback law. The feedback map can therefore be regarded as the *learning process* of the player.

COMMENTS

There should be no difficulty in accepting the idea of an environment (both external, in terms of air, water, food, etc., and internal, in terms of the body and even the brain) on which the players act, consuming scarce resources and transforming, creating or destroying this environment. (Some four billion years ago the photosynthesis of the first organisms transformed the existing atmosphere of methane and ammonia to the oxygenated one we know today -- this was probably the first example of pollution! -- and since then the ability to transform the environment has been recognized as one of the characteristics of living matter). There should also be no problem in accepting the existence of cerebral activity which operates the internal organs of the body and the muscular activity by which interaction with the environment is possible.

The existence of conceptual controls and their use in a recognition mechanism are more questionable assumptions, which we shall attempt to justify at several levels.

1. The ambiguous concept of perception includes both an "objective" and a "subjective" component. The objective component, which we call sensory perception, is provided by the neuronal circuit activated by the sensory receptors. But everyone knows that there is also a subjective component by which this sensory perception is interpreted: this interpretation may depend on many factors (previous experiences, emotional state, attention level, etc.), i.e., on a state of cerebral activity independent of the sensory inputs. This independent activity represents part of the regulatory control which we called conceptual control.

2. If we accept the existence of an endogenous cerebral activity which "interprets" the sensory perception of the environment, we must postulate the existence of a recognition mechanism which tells us whether a conceptual control and the sensory perception of the environment and its variations are consistent. It seems that brains have evolved systems which transform information on bodily needs and environmental events into cerebral activity producing either pleasure (comfort) or pain (discomfort). These systems are known by psychologists as *motivational systems*, and are naturally more sophisticated than strictly pleasure-seeking or pain-avoiding systems. They include the emotional system and the homeostatic drive systems, which basically keep the organism functioning (for example, the hunger drive). These systems reveal the relation between the perception of the environment and the conceptual controls: if these are not consistent the situation can be remedied by:

(a) acting on the environment (for example, by looking for and consuming food in the case of hunger);
(b) changing the conceptual control when action on the environment cannot induce a perception of the environment consistent with the existing conceptual control.

The latter strategy (change of conceptual controls) appears to be less frequent than the first and, for many subsystems (such as the homeostatic systems), is quite impossible.

3. The idea of a recognition mechanism based on conceptual controls is consistent with the concept of epigenesis. The recognition mechanism outlined above is basically a *selection* mechanism with a definite Darwinian flavor, choosing conceptual controls as a function of the environment and changes in the environment. By representing the cerebral activity as the flux of neurotransmitters in individual synapses (see below), one could suppose that the synapses used most frequently would be stabilized, while those used less frequently would deteriorate. But the mere description of the synapses which are stabilized after a period of activity is capable of explaining epigenesis only to the extent that a road

network can determine the routes taken by cars - in this case existing travel patterns require the maintenance of commonly used routes while the others can be neglected.

4. We also postulated that the evolution of the recognition mechanism is programmed genetically. This recognition mechanism is probably rather simple: it may just open or close (activate or deactivate) a number of neuronal circuits during one or several specified periods of time, allowing both the neurotransmitters released by the perception of the environment and the conceptual controls to pass through.

It seems likely that some components of this mechanism (which should obey the laws of biochemistry) are periodic. These components are the many biological clocks involved in maintaining the homeostatic equilibrium of the organism. [It may be postulated that the recognition of the periodicity of the sun and the moon by periodic components of the recognition mechanism in combination with suitable conceptual controls leads to the concept of time.] These periodic components of the recognition mechanism probably lie at the heart of the ability to recognize regularities and extrapolate them, as well as the desire to look for causal relations.

Other components of this mechanism are not periodic, but are active only during a certain period. This may be illustrated by the phenomenon of "imprinting" in ethology: in animal species where the young are able to walk almost immediately after birth, the new-born animals follow the first moving object that they perceive, whatever this may be. (In practice, it is usually a parent.) However, this susceptibility does not last indefinitely. For example, ducklings can be imprinted only during the first twenty-four hours of their life, with sensitivity at a maximum between the 14th and 17th hours. The crucial factor in imprinting is the *mobility* of the object to be imprinted, and this reveals the importance of the perception of variations in the environment.

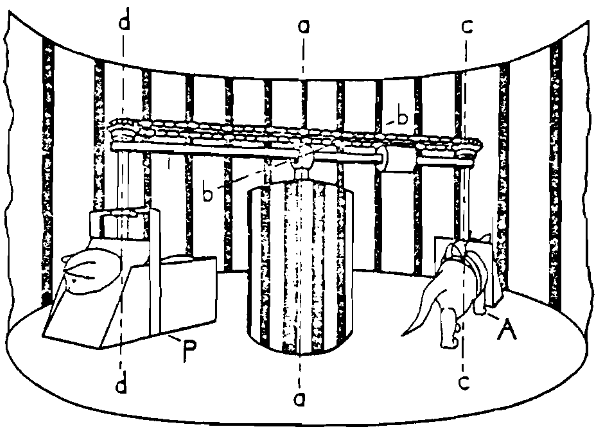
5. The assumption of a recognition mechanism using conceptual controls allows us to explain the adaptability and redundancy of cerebral activities. A player can recognize the same sensory perception using different conceptual controls at different times -- this is redundancy. Then, thanks to the periodic nature of many components of cerebral activities, this sensory perception can be "interpreted" in several ways, provoking different actions (since we have assumed that the action taken depends upon the conceptual controls) - and this is adaptability.

The components of the recognition mechanism based on one or a small number of conceptual controls operate the automatic biological systems (the automatic nervous system, etc.), since in this case the subsystem inherits the genetic program of the component of the recognition mechanism.

6. The concept of a recognition mechanism reflects the dichotomy between "conceptually-driven processes" and "data-driven processes" introduced by specialists in cognitive psychology and pattern recognition. In this case the data-driven process is the cerebral activity provoked by the sensory perception of the environment while the conceptually-driven process takes the form of conceptual controls (this is the origin of our terminology). The idea of a recognition mechanism is also consistent with the concept of metaphor, regarded as a combination of a sensory perception of the environment and a conceptual control recognized by the recognition mechanism. A feeling of understanding, which amounts to a feeling of pleasure, occurs when a metaphor is recognized by the recognition process. Perhaps thought processes also fit into this representation, since they involve setting up conceptual controls in the form of assumptions and then comparing them with the perception of the environment. This dynamical process of making and matching seems to be quite universal.

7. The proposed mathematical metaphor suggests the existence of a learning process described by a feedback relation which associates a set of conceptual controls with each sensory-motor couple. The larger the set of conceptual controls, the less deterministic the learning process. This is consistent with

several observed facts. For instance, studies of the imprinting phenomenon have shown that the greater the effort made by the young animal to follow the moving object, the stronger is the imprint. When one of the components of the sensory-motor couple is suppressed, the learning mechanism does not work normally. For instance, if kittens are raised in a visual environment composed of black and white vertical lines, they are unable to "see" horizontal stripes later in life. In another experiment, two kittens from the same litter spend several hours a day in a contraption which allows one kitten fairly complete freedom to explore and perceive its environment while the other is suspended passively in a "gondola" whose motion is controlled by the first kitten. Both animals receive the same visual stimulation, but the active kitten learns to interpret these signals to give it an accurate picture of its environment while the passive kitten learns nothing and is, in practical terms, "blind" to the real world.



Apparatus for equating motion and consequent visual feedback for an actively moving kitten *(A)* and a passively moved one *(P)* (Held and Hein, 1963).

DESCRIPTION OF THE MATHEMATICAL METAPHOR

We represent the environment by a finite-dimensional vector space X. We emphasize the phenomenon of chemical communication in the description of cerebral activity, regarding hormones and neurotransmitters as chemical messengers released by endocrine glands or the axons of neurons and received by receptors in various organs or the dendrites of other neurons.

At this level, the endocrine system and the nervous system may be distinguished by the mode of transport of the chemical messenger. This transport is slow and non-specific in the case of the endocrine system: hormones are carried by the blood; it is fast and specific in the case of the nervous system: neurotransmitters have to cross a synapse (the place where the axon of a "presynaptic" neuron meets the dendrite of a "postsynaptic" one), which is only 0.02 μ wide. When the pulse-coded information sent by the postsynaptic neuron reaches a certain threshold value, it releases neurotransmitters in the synapse, inducing an electrical response on the postsynaptic membrane after about 10^{-3} seconds. For simplicity, we shall neglect both the endocrine system and the so-called electrical synapses. We then assume that *the state of cerebral activity is described by the evolution of neurotransmitters in each synapse.* We denote by S the set of synapses (about one hundred thousand billion of them) and by $Y := \mathbb{R}^S$ the finite-dimensional vector space of cerebral activity. The component y_s of an element $y := (y_s)$ of this space denotes the number of neurotransmitter molecules passing through synapse s.

We shall describe the temporal cerebral activity by several functions of time into the space $\mathbb{R}^S$. The complex mechanism describing the processing of the presynaptic signals by each neuron will not be taken *explicitly* into account. The usual description of the brain as a network is given in terms of the "trace" of the functions involved; a given synapse is "weighted" by the total number of neurotransmitters crossing it during each period.

We distinguish two classes of temporal functions:

1. The functions $t \to a(t) \in Y$ which describe the evolution of *cerebral motor activity*.
2. The function $t \to c(t) \in Y$ which describes the endogenous evolution of the *conceptual controls*.

The cerebral activity induced by sensory perception of the environment (both external and internal) is not included explicitly.

The states of the systems are therefore sensory-motor couples (x,a) ranging over the vector space $X \times Y$; the controls of the system are elements c of Y.

We represent the history of a trajectory $t \to z(t)$ (a continuous function from $[0,\infty[$ to a finite-dimensional vector space Z) by the function $T(t)\tau$ from $]-\infty,0]$ to Z, where

$$T(t)z(\tau) = z(t+\tau) \quad \text{for all } \tau \leq 0 \quad .$$

The derivative of a function $z(\cdot):t \to z(t)$ is denoted by $z'(\cdot)$.

The recognition mechanism

The recognition mechanism compares the perception of the history of the environment, the perception of its variation, and the state of the conceptual control at each instant. We shall describe it by a family of subsets $\mathcal{R}(t)$ of the space $Y \times C(-\infty,0;Y) \times X$. There is recognition if, for all instants t, we have

$$(c(t),T(t)x,x'(t)) \in \mathcal{R}(t) \quad . \tag{1}$$

The evolution of the environment

We assume that the evolution of the environment is governed by a differential inclusion with memory of the form

$$x'(t) \in F(t,T(t)x,T(t)a) \quad , \tag{2}$$

where F is a set-valued map from $\mathbb{R}_+ \times C(-\infty,0;X) \times C(-\infty,0;Y)$ to X.

The evolution of the cerebral motor activity

The evolution of the cerebral motor activity is governed by a differential inclusion with memory, and is regulated by the conceptual controls:

$$a'(t) \in G(t,T(t)x,T(t)a,c(t)) \quad , \tag{3}$$

where G is a set-valued map from $\mathbb{R}_+ \times C(-\infty,0;X) \times C(-\infty,0;Y) \times Y$ to Y.

The viability condition

The viability condition states that the player consumes scarce resources in order to be able to act on the environment. We translate this into mathematics by requiring that

$$\forall t \geq 0 \quad , \qquad a(t) \in K(t,x(t)) \quad , \tag{4}$$

where K is a set-valued map from $\mathbb{R}_+ \times X$ to Y.

Note that the recognition mechanism is actually a differential inclusion with memory, since the family of subsets $\mathcal{R}(t)$ can be regarded as the graph of a set-valued map R

$$\text{from } \mathbb{R}_+ \times \mathcal{C}(-\infty,0;X) \times Y \text{ to } X \quad .$$

(We say that z belongs to $R(t,\varphi,c)$ if and only if (c,φ,z) belongs to $\mathcal{R}(t)$.) Therefore, the evolutionary laws of the cognitive system of the player can be written as follows:

$$\begin{aligned} &\text{(i)} \quad x'(t) \in F(t,T(t)x,T(t)a) \cap R(t,T(t)x,c(t)) \\ &\text{(ii)} \quad a'(t) \in G(t,T(t)x,T(t)a,c(t)) \quad . \end{aligned} \tag{5}$$

We shall use the viability theorem for differential inclusions with memory (see Haddad 1981a). We shall assume that the set-valued maps F, R and G are bounded upper-semicontinuous maps with compact convex images and that the graph of the set-valued map K is closed. We adopt the concept of the *contingent derivative* of a set-valued map introduced in Aubin (1981), and consider the map L

$$\text{from } \mathbb{R}_+ \times \mathcal{C}(-\infty,0;X) \times \mathcal{C}(-\infty,0;Y) \text{ to } Y$$

defined by

$$\forall (\xi,\alpha) \in \mathcal{C}(-\infty,0;X) \times \mathcal{C}(-\infty,0;Y) \quad ,$$

$$L(t,\xi,\alpha) := \{c \in Y \mid G(t,\xi,\alpha,c) \cap DK(t,\xi(0),\alpha(0))(1,F(t,\xi,\alpha) \cap R(t,\xi,c)) \neq \emptyset\}. \tag{6}$$

This set-valued map represents the learning process we mentioned earlier. We now specify the initial conditions: these are the sensory-motor couples $(\xi,\alpha) \in C(-\infty,0;X) \times C(-\infty,0;Y)$ such that

$$\alpha(0) \in K(0,\xi(0)) \quad . \tag{7}$$

We require that the sensory-motor couple satisfies the initial condition

$$T(0)x = \xi \quad \text{and} \quad T(0)a = \alpha \quad . \tag{8}$$

The viability theorem states that the necessary and sufficient condition for the existence of a solution $(x(\cdot),a(\cdot),c(\cdot))$ *of the cognitive system* (1),(2),(3),(4) *for every initial state* (ξ,α) *satisfying* (7) *is that*

$$\forall t,\xi \ , \quad \forall \alpha \in K(t,\xi(0)) \quad , \quad L(t,\xi,\alpha) \neq \emptyset \quad . \tag{9}$$

In this case, the evolution of the conceptual control is related to the evolution of the sensory-motor couple by the feedback law

$$\forall t \geq 0 \quad , \quad c(t) \in L(t,T(t)x,T(t)a) \quad . \tag{10}$$

GAME THEORY AND VIABILITY THEORY

This section presents a simple analog of classical noncooperative game theory in which players are represented by utility functions. We consider n players, each player i being described by a cognitive system defined by set-valued maps (F_i,G_i,R_i,K_i).

Assuming that the overall action of the n players on the environment is the sum of the actions of each player, we find that the evolution of the environment and of the cerebral motor activity of the n players is governed by a differential inclusion with memory of the following type:

$$\begin{aligned} &\text{(i)} \quad x'(t) \in \sum_{i=1}^{n} F_i(t,T(t)x,T(t)a_i) \cap \bigcap_{i=1}^{n} R_i(t,T(t)x,c_i(t)) \\ &\text{(ii)} \quad a_i'(t) \in G_i(t,T(t)x,T(t)a_i,c_i(t)) \qquad (i=1,\ldots,n) \end{aligned} \tag{11}$$

satisfying the viability conditions

$$\forall t \geq 0 \quad , \quad a_i(t) \in K_i(t,T(t)x) \quad , \quad i=1,\ldots,n \quad . \tag{12}$$

Here again, most of the mechanisms for perception and communication among the players are implicitly described in each cognitive system. For example, communication among players takes place via the state of the environment. If the structure of the environment is very detailed it can reveal the existence of an explicit communication system among players.

We can apply the viability theorem for differential inclusions with memory to this system: the result is a learning process L which associates the conceptual controls of each player at each time t with the evolutionary histories of the environment and the cerebral motor activities of the players.

For this purpose we set

$$\left.\begin{aligned} &\text{(i)} \quad \vec{G}(t,\xi,\vec{\alpha},\vec{c}) = \prod_{i=1}^{n} G_i(t,\xi,\alpha_i,c_i) \\ &\text{(ii)} \quad \vec{K}(t,\xi,\vec{\alpha}) = \prod_{i=1}^{n} K_i(t,\xi,\alpha_i) \quad . \end{aligned}\right\} \tag{13}$$

The learning process L is described as follows:

$$(c_1,\ldots,c_n) \text{ belongs to } L(t,\xi,\alpha_1,\ldots,\alpha_n)$$

if and only if the intersection

$$\vec{G}(t,\xi,\vec{\alpha},\vec{c}) \cap D\vec{K}(t,\xi(0),\vec{\alpha}(0))\left(1, \sum_{i=1}^{n} F_i(t,\xi,\alpha_i) \cap \bigcap_{i=1}^{n} R_i(t,\xi,c_i)\right)$$

is nonempty.

An open problem is the *decentralization* of this mechanism: under what conditions affecting the cognitive mechanisms and the structure of the environment can the map L be written

$$L(t,\xi,\vec{\alpha}) = \prod_{i=1}^{n} L_i(t,\xi,\alpha_i) \quad .$$

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