RM-74-14

INTERSECTIONS OF CORNER POLYHEDRA

David E. Bell

July 1974

Intersections of Corner Polyhedra*

David E. Bell

Abstract

The aim of this paper is to present some results concerning the intersection of a generalized form of corner polyhedra and its relation to the integer polyhedron for the general linear integer programming problem. In particular a theorem proved earlier concerning such intersections for low dimensional problems will be proved here as a special case of theorems for higher dimensions.

1. Integer Programming Background

For an m x (n + m) integer matrix A and integer m vector $\bar{b}$, let

$$W = \{w \varepsilon R^{n+m} \mid Aw = \bar{b} \quad , w \text{ integral}\} \quad ,$$

then the linear integer programming problem is

$$\begin{array}{l} \text{Minimize } cw \\ \quad w \varepsilon W \\ \quad w \geq 0 \quad . \end{array} \qquad (1.1)$$

Define

$$W(I) = \{w \varepsilon W \mid w_i \geq 0 \quad i \varepsilon I\}$$

where $I \subseteq S = \{1,2,3,\ldots,n + m\}$. Then the convex hull of $W(S)$, denoted $[W(S)]$, is the integer polyhedron for (1·1) which may be written as

$$\begin{array}{l} \text{Minimize } cw \\ \quad w \varepsilon [W(S)] \quad . \end{array} \qquad (1\cdot 2)$$

---

*This is adapted from my dissertation supervised by Professor J.F. Shapiro at the Massachusetts Institute of Technology Operations Research Center.

An algorithm in Bell and Fisher [1] suggested replacing the integer polyhedron in (1.2) by the intersection of polyhedra of the form $[W(I)]$ where $\{w_i,\ i \notin I\}$ forms a linear programming basis for (1.1). A result of that paper was that for the low dimensional cases n = 0,1,2 the two problems were equivalent, that is the integer polyhedron and the intersection of the corner polyhedra are equal.

This paper will present some further results on the connection between the integer polyhedron and intersections of related polyhedra and hopefully will lay a foundation for further work in this area.

2. Generalized Corner Polyhedra

Since the set W lies in an n-dimensional affine space, take a basic set of variables y, where w = (y, x) and y = b - Nx and let

$$X = \{x \mid x \text{ integral},\ b - Nx \text{ integral}\} \quad .$$

The set X completely defines W and the remainder of the paper will be in terms of the x, rather than the w, variables. The sets

$$X(I) \quad , \quad X(w_I \geq 0) \quad , \quad \{x \varepsilon X \mid w_i \geq 0 \quad i \varepsilon I\} \quad ,$$

are equivalent.

Call $[X(I)]$ a <u>generalized corner polyhedron</u> of order $|I|$. Thus the normal corner polyhedra associated with L.P. bases are corner polyhedra of order n. Define a function r(n, m) to be the smallest number such that

$$[X(S)] = \bigcap_{|I|=r(n,m)} [X(I)]$$

for all problems of the form (1.1).

The following properties hold for this function.

Proposition 2.1. The case n = 0, m = 0 is meaningless, otherwise

i) $r(n, m) \leq n + m$

ii) $r(n, m) = n$, $n = 1,2$, $m > 0$

iii) $r(n, m) \geq n + 1$, $n \geq 3$, $m > 0$

iv) $n_1 \geq n_2$, $m_1 \geq m_2 \Longrightarrow r(n_1, m_1) \geq r(n_2, m_2)$.

v) $r(n, 0) = 1$.

Proof. i) is evident since $|S| = n + m$.

ii) follows from the work in [1] but will be proved again later in Corollary 3.6.

iii) Consider the problem

Max $3x + 3y + 4z$

s.t. $45x + 51y + 60z + 8w = 244$

$w,x,y,z \geq 0$ and integral.

Optimal integer solutions are (4,0,0,8), (0,0,3,8), (0,4,0,5) each with value 12. However, the point $(1\frac{1}{2},1\frac{1}{2},1,5)$ with value 13 lies in the intersection of the four corner polyhedra of order 3, with representations

$$\tfrac{1}{2}(4,0,0,8) + \tfrac{1}{2}(-1,3,2,2)$$

$$\tfrac{1}{2}(0,4,0,5) + \tfrac{1}{2}(3,-1,2,5)$$

$$\tfrac{1}{2}(0,0,3,8) + \tfrac{1}{2}(3,3,-1,2)$$

$$\tfrac{1}{4}(4,0,0,8) + \tfrac{1}{4}(0,0,3,8) + \tfrac{1}{4}(0,4,0,5) + \tfrac{1}{4}(2,2,1,-1) .$$

Hence $r(3, 1) \geq 4$. Generalizing the example by substituting $w = w_1 + w_2 +, \ldots, w_{n-2}$ where $(1\frac{1}{2}, 1\frac{1}{2}, 1, 5, 0, 0, \ldots, 0)$ is always in the intersection of all corner polyhedra of order n, shows that $r(n, 1) \geq n + 1$ for all n. Together with part (iv) of this proposition this gives that $r(n, m) \geq n + 1$.

iv) This will require the following lemma.

Lemma 2.2. If W may be partitioned as $(U \times V, T)$, that is $w^1 = (u^1, v^1, t)$ and $w^2 = (u^2, v^2, t) \in W$ implies (u^1, v^2, t) and $(u^2, v^1, t) \in W$, then

$$[X(u \geq 0, v \geq 0, t_I \geq 0)] = [X(u \geq 0, t_I \geq 0)] \cap [X(v \geq 0, t_I \geq 0)]$$

where t_I is a subset of the t variables.

Proof of the Lemma. The proof will assume that T is empty, but it is easily seen that no generality is lost. Since $X(u \geq 0, v \geq 0)$ is a subset of both $X(u \geq 0)$ and $X(v \geq 0)$ then

$$[X(u \geq 0, v \geq 0)] \subseteq [X(u \geq 0)] \cap [X(v \geq 0)] \quad .$$

Let

$$w^o = (u^o, v^o) \in [X(u \geq 0)] \cap [X(v \geq 0)] \quad .$$

Then

$$(u^o, v^o) = \sum_{i=1}^{k_1} \lambda_i (u^i, \bar{v}^i) \quad \text{with} \quad (u^i, \bar{v}^i) \in X(u \geq 0)$$

and

$$(u^o, v^o) = \sum_{i=1}^{k_2} \mu_i (\bar{u}^i, v^i) \quad \text{with} \quad (\bar{u}^i, v^i) \in X(v \geq 0) \quad ,$$

with

$$\lambda_i, \mu_i \geq 0 \quad \sum_{i=1}^{k_1} \lambda_i = \sum_{i=1}^{k_2} \mu_i = 1 .$$

Now $(u^i, \bar{v}^i)$, $(\bar{u}^j, v^j) \in W$ implies $(u^i, v^j) \in W$ for all i,j since $W = U \times V$. Since

$$w^o = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \lambda_i \mu_i (u^i, v^j)$$

and

$$\lambda_i \mu_i \geq 0 \quad \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \lambda_i \mu_i = 1 \quad \text{and} \quad (u^i, v^j) \geq 0 \quad ,$$

then

$$w^o \in [X(u \geq 0, v \geq 0)]$$

as required.

<u>Proof of (iv)</u>. Take a problem with dimensions n_2, m_2, and add a disjoint set of $m_1 - m_2$ constraints in $n_1 + m_1 - n_2 - m_2$ additional variables that have at least one integer solution. If $m_1 = m_2$ add a zero column for each new variable to the existing constraints. If u is the set of old variables and v the new, then the new W is a cartesian product of the old set W and the solution to the added constraints.

Now

$$[X(S)] = [X(u \geq 0, v \geq 0)]$$

$$= \bigcap_{|I|=r(n_1,m_1)} [X(I)] \qquad \text{by definition of } r(n_1, m_1)$$

$$= \bigcap_{|I|=r(n_1,m_1)} \Big([X(u_I \geq 0)] \cap [X(v_I \geq 0)] \Big) \text{ by the lemma}$$

$$= \bigcap_{|I|=r(n_1,m_1)} [X(u_I \geq 0)] \cap \bigcap_{|I|=r(n_1,m_1)} [X(v_I \geq 0)] \quad . \quad (2.1)$$

If $[X(\bar{S})]$ is the integer polyhedron for the smaller problem then

$$[X(S)] = [X(\bar{S})] \times C \qquad (2.2)$$

for some set C. Comparing (2.1) and (2.2) we have

$$[X(\bar{S})] = \bigcap_{|I|=r(n_1,m_1)} [X(u_I \geq 0)]$$

$$= \bigcap_{|I|=r(n_1,m_1)} [X(I)] \quad .$$

Hence

$$r(n_1, m_1) \geq r(n_2, m_2) \quad .$$

v) The case for m = 0 follows since with no equations restricting the variables all non-negative integer solutions are feasible and

$$[X(w \geq 0)] = \bigcap_{i=1}^{n+m} [X(w_i \geq 0)] \text{ by use of the lemma} \quad ,$$

but this relation is precisely the statement that $r(n, 0) = 1$.

3. The Structure of the Set of Integer Solutions

Recall that

$$X = \{x \mid x \text{ integral, } Nx - b \text{ integral}\} .$$

Hence if $x^1,\ldots,x^k \in X$ and $a_1,\ldots,a_k$ are any integers, then $x^o = \sum_{i=1}^{k} a_i x^i$ will be in X if $\sum_{i=1}^{k} a_i = 1$, since x^o is evidently integral and

$$Nx^o - b = \sum_{i=1}^{k} a_i(Nx^i - b) + (\sum_{i=1}^{k} a_i - 1)b$$

is also integral if $(\sum_{i=1}^{k} a_i - 1)b$ is.

A lattice may be defined as a set L in R^n which satisfies

i) $x,y \in L \implies x \pm y \in L$

ii) there exists a basis $x^1,\ldots x^n \in L$ for which

$L = \{x \mid x = a_1x^1 + \cdots + a_nx^n \text{ for integers } a_1,\ldots,a_n\}$

iii) for some number $d > 0$, $x \in L$

$0 \leq ||x - y|| < d \implies y \notin L$.

The set X is not a lattice since (i), (ii) hold only if b is integral (it is however, a quotient lattice).

Proposition 3.1. There exists a linear transformation $T : X \to Y$ for which Y is a lattice. Y may be chosen as the integer lattice.

Proof. The following is essentially the Gram-Schmidt process. Let $x^1,\ldots,x^n \in X$ be any set of independent points, thus defining a unique hyperplane

$$\sum_{i=1}^{n} \alpha_i x_i = \beta .$$

Let

$$\bar{X} = \{x \in X \mid \Sigma\alpha_i x_i < \beta\}$$

and let

$$\bar{\beta} = \sup_{x \in \bar{X}} \Sigma\, \alpha_i x_i \quad . \tag{3.1}$$

If M is the matrix $(x^1,\ldots,x^n)$ then $\alpha = b\underline{1}.M^{-1}$ and $\Sigma\, \alpha_i x_i$ is rational for all $x \in X$ and

$$\beta \geq \bar{\beta} + |\det M|^{-1}$$

so that $\beta > \bar{\beta}$.

Let

$$a^*_{1i} = \frac{\alpha_i}{\beta - \bar{\beta}} \quad , \quad b^*_1 = \frac{\beta}{\beta - \bar{\beta}} \quad ;$$

then all $x \in X$ satisfy

$$\sum_{i=1}^{n} a^*_{1i} x_i \equiv b^*_1 \quad (\text{modulo } 1) \ ,$$

for otherwise it is possible to show the existence of a point $x^o \in X$ such that

$$\bar{\beta} < \Sigma\, \alpha_i x^o_i < \beta \quad \text{which is impossible by (3.1).}$$

Now choose any point x^{n+1} not satisfying

$$\Sigma\, \alpha^*_{1i} x_i = b^*_1$$

and form another set of n independent points with n - 1 points from $\{x^1,\ldots,x^n\}$. The process is repeated so that eventually for some real n x n matrix A^* and vector b^* all elements of X will satisfy

$$A^* x \equiv b^* \quad (\text{mod } 1)$$

and

$$X = \{x \mid A^* x - b^* \text{ is integral}\} .$$

Define

$$Y = \{y \mid y = A^* x - b^* \quad , \quad x \in X\}$$

then Y is the required set and is easily checked to be the lattice of all integer points. ||

An <u>n-cube</u> will be defined as the set of 2^n solutions of the form

$$A^* x = c + \delta$$

where $c \equiv b^*$ (mod 1) is some constant vector and δ is any vector of 0's and 1's.

Define the function $\delta : X \to \{0,1\}^n$ by the relation

$$\delta(x) \equiv A^* x - b^* \quad (\text{mod } 2).$$

<u>Proposition 3.2</u>. For $x^1, x^2 \in X$, $\delta(x^1) = \delta(x^2)$ if and only if

$$\frac{x^1 + x^2}{2} \in X .$$

<u>Proof</u>.

$$A^*\left(\frac{x^1 + x^2}{2}\right) = \tfrac{1}{2}(A^* x^1 + A^* x^2) .$$

Since

$$A^* x^i \equiv b^* + \delta(x^i) \quad (\text{mod } 2) \quad i = 1,2$$

then

$$A^* x^1 + A^* x^2 \equiv 2b^* + \delta(x^1) + \delta(x^2) \quad (\text{mod } 2) . \qquad (3.2)$$

Thus if

$$\delta(x^1) = \delta(x^2) ,$$

then

$$A^*x^1 + A^*x^2 \equiv 2b^* \pmod 2 \tag{3.3}$$

and

$$\tfrac{1}{2}(A^*x^1 + A^*x^2) \equiv b^* \pmod 1$$

so that

$$\frac{x^1 + x^2}{2} \in X .$$

Conversely if $\frac{x^1 + x^2}{2} \in X$, then (3.3) is true so that in (3.2)

$$\delta(x^1) + \delta(x^2) \equiv 0 \pmod 2$$

hence

$$\delta(x^1) = \delta(x^2) . \qquad ||$$

<u>Theorem 3.3</u>. Let $C \subseteq R^n$ be a convex set containing an $(n-1)$-cube of X in a hyperplane $tx = t_o$. If k^* is the smallest positive integer such that

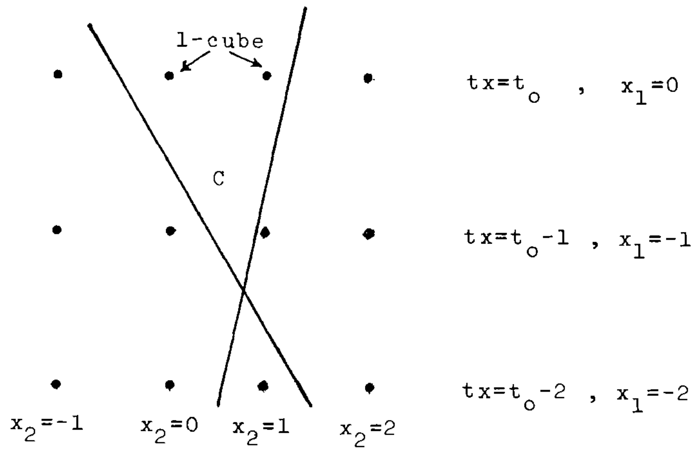
$$X_{k^*} = \{x \in X \mid tx = t_o - k^*\}$$

is non empty and $C \cap X_{k^*} = \emptyset$, then

$$tx \geq t_o \quad \text{for all } x \in C \cap X .$$

<u>Proof</u>. Without loss of generality it may be assumed that X is the integer lattice (Proposition 3.1), that the hyperplane is $x_1 = 0$, and that the $n-1$ cube is that consisting of points

$x^1, \ldots, x^{2^{n-1}}$ with 0 and 1 entries.



Suppose there is a point $x^o \in C \cap X$ satisfying $tx^o = t_o - k$ $(x_1 = -k)$ for some $k \geq 2$. Consider the convex hull of the points $\{x^o, x^1, \ldots, x^{2^{n-1}}\}$ and its intersection with the plane $x_1 = -1$. This consists of all points x satisfying

$$\frac{x_i^o}{k} \leq x_i \leq \frac{k-1}{k} + \frac{x_i^o}{k} \quad , \quad i=2,\ldots,n.$$

Since x^o is integral, each of these intervals must contain an integer and hence the intersection must contain an integral point. This is a contradiction to the assumption that $C \cap X_{k^*}$ was empty and so the point x^o cannot exist. ||

If the convex set of the theorem is a corner polyhedron this means that if $tx = t_o - 1$ contains no elements of $X(I)$ then $tx = t_o$ is a face of the polyhedron. If a face of the integer polyhedron is to be a face of the intersection polyhedron then it must be the face of some corner polyhedron. Hence if it contains an $(n-1)$-cube, then if a subset $I \subseteq S$ can be found such that

$$\{x \in X \mid tx = t_o - 1 \quad , \quad x_I \geq 0\}$$

is empty, then the corner polyhedron $[X(I)]$ will contain that face.

Theorem 3.4. If F is a feasible region defined by $n+m$ linear constraints and $F \cap X = \emptyset$, and if $n+m > 2^n$ then there exists a subset I of these constraints having a feasible region $\bar{F}$ such that

i) $F \subseteq \bar{F}$

ii) $|I| \leq 2^n$

iii) $\bar{F} \cap X = \emptyset$.

Proof. It can be seen readily for the case $n = 1$. In general, label each element of X with the indices of the constraints which it violates. Let I be the set of all constraints for which there is some point which violates only that one constraint. Then all points in X which violate no constraint in I must have at least two labels. Thus any one constraint not in I may be removed entirely without any point in X becoming feasible. Now there may be some new points with only one label, I is updated and the process is repeated until all the constraints are either in I or have been removed.

The claim is that $|I| \leq 2^n$. Suppose the contrary is true. The proof will show a contradiction by constructing an infinite sequence of distinct elements of X lying in a bounded region, which is clearly impossible.

Note that for each constraint remaining (that is, in I) there exists at least one point of X which violates only that constraint. Choose $|I|$ such points, one for each constraint forming a set P. Since $|I|$ is finite, the set $[P]$ is bounded and $[P] \cap X$ is finite.

Consider the following construction. Since $|P| > 2^n$ there must exist two points $x^1, x^2 \in P$ for which $\delta(x^1) = \delta(x^2)$. By Proposition 3.2 $x^o = \frac{x^1 + x^2}{2} \in X$. Suppose that x^1 violates constraint i, x^2 violates constraint j (remember $j \neq i$) then x^o must violate constraint i or j or both. It cannot violate a third constraint k since then x^1 or x^2 must also.

Now perform the following updating process.

i) If x^o violates only i, let $P^1 = P - \{x^1\} + \{x^o\}$.

ii) If x^o violates only j, let $P^1 = P - \{x^2\} + \{x^o\}$.

iii) If x^o violates i and j replace the ith constraint in the problem by

$$w_i \geq x_i^o$$

(or by $w_i \geq b_i - N_i x^o$ in some cases).

Every point which satisfies $w_i \geq 0$ satisfies $w_i \geq x_i^o$ because $x_i^o < 0$ but x^1 still violates it because $x_i^2 \geq 0$ and

$x_i^1 = 2x_i^o - x_i^2 < x_i^o$. In this case let $P^1 = P - \{x^2\} + \{x^o\}$. The changed constraint is only a temporary construction for the proof.

Now $|P^1| = |P| > 2^n$ and each element of P^1 violates exactly one constraint of I and each constraint of I is violated by exactly one point in P^1. Hence a set P^2 may be constructed in the same manner and, since there is no stopping condition the sequence $\{P^k\}$ will be infinite with $P^{k+1} \subset [P^k]$ which is impossible by the finiteness of $[P] \cap X$. This contradicts the basic assumption that $|I| > 2^n$. The feasible region defined by the original constraints of I is $\bar{F}$. ||

The following result now combines the last two theorems to give conditions under which a face of the integer polyhedron will be a face of a lower order corner polyhedron.

Theorem 3.5. If a face of the integer polyhedron contains an $(n-1)$-cube, then it is also a face of some corner polyhedron of order $|I|$, satisfying

$$|I| \leq \min (n+m, 2^{n-1}) \quad .$$

Proof. Since the integer polyhedron has order $n+m$, $|I| \leq n+m$ is clear. If $n+m > 2^{n-1}$ and if the face is the hyperplane $tx = t_o$ then if F is the L.P. feasible region for the problem,

$$F \cap X \cap \{x \mid tx = t_o - 1\} = \emptyset \quad ,$$

and by Theorem 3.4 there is a subset of 2^{n-1} of the constraints which also gives a feasible region $\bar{F}$ satisfying

$$\bar{F} \cap X \cap \{x \mid tx = t_o - 1\} = \emptyset \quad .$$

Consider the corner polyhedron which enforces these 2^{n-1} constraints. All the points in the $(n-1)$-cube are feasible in it but no elements of X satisfying $tx = t_o - 1$ are. By Theorem 3.3 this implies that $tx \geq t_o$ for all points of X in the corner polyhedron and thus this face of the integer polyhedron is a face of the corner polyhedron. ||

Corollary 3.6. $r(1, m) = 1$, $r(2, m) = 2$.

Proof. For any problem satisfying $n \geq 2^{n-1}$ all the faces of a non-degenerate integer polyhedron contain $(n-1)$-cubes. Hence Theorem 3.5 applies and in these cases $r(n, m) \leq 2^{n-1}$. ||

4. Conclusions

The rank function $r(n, m)$, suggested here, is likely to be a gross overestimate of the order of corner polyhedra required to give an intersection which is the integer polyhedron. However, precise numerics was not the aim of this paper. What has been demonstrated are some of the possibilities that can result from making use of the lattice structure of the set of integer solutions. The analysis here could lead to the construction of a class of "difficult" integer programs, that is, problems with a high intersection order.

What has not been studied here is the further connection between the lattice X and the linear programming constraints $\{Nx \leq b, x \geq 0\}$. What information about the constraints can be deduced from a knowledge of X?

From an algorithmic point of view it is difficult to judge how useful an approximation an intersection polyhedron is likely to be without some computational experience but from a structural point of view it is hoped that the approach of this paper will lead to further insights into the nature of integer programs.

References

[1] Bell, D.E. and Fisher, M.L. "Improved Bounds for Integer Programs Using Intersections of Corner Polyhedra," University of Chicago, Graduate School of Business Working Paper, October 1973.

[2] Gomory, R.E. "Some Polyhedra Related to Combinatorial Problems," Linear Algebra, 2 (1969), 451-558.