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ON THE CONTINUITY OF THE VALUE OF A LINEAR PROGRAM

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ABSTRACT

Results about the continuity of the value of a linear program are reviewed. Particular attention is paid to the interconnection between various sufficient conditions.

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ON THE CONTINUITY OF THE VALUE OF A LINEAR PROGRAM

Roger J.-B. Wets

We are interested in the following function:

$$Q(t) = \inf_{x \in R^n} [cx \mid Ax \geq b \ , \ x \geq 0]$$

where

$$t = (c, A_1, \ldots, A_m, b^T) \quad ,$$

$$= (c_1, \ldots, c_n, a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn}, b_1, \ldots, b_m) \quad .$$

Thus $Q: R^N \rightarrow \overline{R} = [-\infty, +\infty]$ with $N = (n + 1)(m + 1) - 1$. The two following closed-convex-polyhedron-valued multifunctions

$$K(t) = \{x \mid Ax \geq b \ , \ x \geq 0\} \quad ,$$

and

$$D(t) = \{y \mid yA \leq c \ , \ y \geq 0\}$$

play an important role in what follows; they correspond respectively to the set of primal and dual feasible solutions associated with the linear program defining Q. The function Q is finite when both K and D are nonempty, if $K(t) = \emptyset$ but $D(t) \neq \emptyset$ then $Q(t) = \infty$, and if $D(t) = \emptyset$ but $K(t) \neq \emptyset$ then $Q(t) = -\infty$; if both $K(t) = D(t) = \emptyset$ then let us accept the convention that $Q(t) = \infty$. Let

$$T = \{t \in R^N \mid K(t) \neq \emptyset \ , \quad D(t) \neq 0\}$$

denote the set on which Q is finite.

1. PROPOSITION. *The multifunctions* K *and* D *are upper semi-continuous, i.e., if*

$$t = \lim_{\nu\to\infty} t^\nu \quad , \quad x = \lim_{\nu\to\infty} x^\nu \text{ and } x^\nu \in K(t^\nu)$$

then $x \in K(t)$, *and if*

$$t = \lim_{\nu\to\infty} t^\nu \quad , \quad y = \lim_{\nu\to\infty} y^\nu \quad \text{and } y^\nu \in D(t^\nu)$$

then $y \in D(t)$.

PROOF. It clearly suffices to prove the assertion for either K or D. Suppose that for $\nu = 1,\ldots,$ $K(t^\nu) \neq \emptyset$ and $t = \lim_\nu t^\nu$. Then $x^\nu \in K(t^\nu)$ implies that $A^\nu x^\nu \geq b^\nu$ and $x^\nu \geq 0$. Since by hypothesis $(A^\nu - A) \to_{\|\cdot\|} 0$, $(b^\nu - b) \to_{\|\cdot\|} 0$ and $x^\nu - x \to_{\|\cdot\|} 0$, it follows that $Ax \geq b$ and $x \geq 0$, which yields $x \in K(t) \neq \emptyset$. □

Note that the above shows also that T is closed. In general, K and D are not *continuous*, i.e., they are not *lower semicontinuous*, by which one means that if $t = \lim_\nu t^\nu$ and if $x \in K(t)$, there exist $x^\nu \in K(t^\nu)$ such that $x = \lim_\nu x^\nu$, and if $y \in D(t)$, there exist $y^\nu \in D(t^\nu)$ such that $y = \lim_\nu y^\nu$. For example, consider $t^\nu = (c, A^\nu = \nu^{-1}, b^\nu = \nu^{-1})$ with $t = (c, A = 0, b = 0)$. Then $K(t) = R_+$ but for all ν, $K(t^\nu) = [1,\infty($; the point $1/2 \in K(t)$ can not be reached by any sequence $\{x^\nu, \nu = 1,\ldots\}$ with $x^\nu \in [1,\infty($. Later we shall give sufficient conditions for the lower semi-continuity of K and D, that in view of the next theorem also yield sufficient conditions for the continuity of Q.

2. THEOREM. *Suppose the multifunctions* K *and* D *are continuous on* $T \subset \mathcal{T}$. *Then* Q *is continuous on* T.

PROOF. If $(c,A,b) = t \in T \subset \mathcal{T}$, then both K(t) and D(t) are non-empty, and there exist $x \in K(t)$ and $y \in D(t)$ such that

$$yb = Q(t) = cx \quad ,$$

as follows from the duality theorem for linear programs. Since K and D are lower semicontinuous at t, for any sequence

$$\{t^\nu = (c^\nu, A^\nu, b^\nu) \ , \ \nu = 1,\ldots\}$$

in T with $t = \lim_\nu t^\nu$, there exist $\{x^\nu \in K(t^\nu) \ , \ \nu = 1,\ldots\}$ and $\{y^\nu \in D(t^\nu) \ , \ \nu = 1,\ldots\}$ such that $x = \lim_\nu x^\nu$ and $y = \lim_\nu y^\nu$. Moreover, we have that for all ν,

$$y^\nu b^\nu \leq Q(t^\nu) \leq c^\nu x^\nu \quad .$$

From this it follows that

$$Q(t) = cx = \lim_{\nu\to\infty} c^\nu x^\nu \leq \limsup_{\nu\to\infty} Q(t^\nu) \quad ,$$

and

$$Q(t) = yb = \lim_{\nu\to\infty} y^\nu b^\nu \geq \liminf_{\nu\to\infty} Q(t^\nu)$$

which together yield the continuity of Q at t. □

In the remainder of this short note we give some sufficient conditions for the lower semicontinuity of the convex-polyhedron-valued multifunctions K and D.

3. PROPOSITION. *The multifunction* $t \longmapsto D(t)$ *is continuous on* $T \subset \mathcal{T}$ *if and only if the convex-cone-valued multifunction*

$$t \longmapsto \text{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} u \\ \eta \end{pmatrix} \middle| \begin{matrix} u = Ax - Is \\ \eta = cx + \beta \end{matrix}, \; x \geq 0,\; s \geq 0,\; \beta \geq 0 \right\}$$

$$= \{(u,\eta) \in R^{m+1} \,|\, u \leq Ax \;,\; \eta \geq cx \;,\; x \geq 0\}$$

is upper semicontinuous on T.
Similarly t K(t) *is continuous on* T ⊂ T *if and only if the convex-cone-valued multifunction*

$$t \longmapsto \text{pos}\begin{pmatrix} A^T & I & 0 \\ b^T & 0 & -1 \end{pmatrix} = \left\{ \begin{pmatrix} \nu \\ \theta \end{pmatrix} \middle| \begin{matrix} \nu^T = yA + rI \\ \theta = yb - \alpha \end{matrix}, \; y \geq 0 \;,\; r \geq 0 \;,\; \alpha \geq 0 \right\}$$

$$= \{(\nu,\theta) \in R^{n+1} \,|\, \nu^T \geq yA \;,\; \theta \leq yb \;,\; y \geq 0\}$$

is upper continuous on T.

PROOF. For reason of symmetry, it really suffices to prove the assertions involving D. We first note that

$$t \longmapsto C(t) := \text{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$$

is upper semicontinuous if and only if the polar multifunction

$$t \longmapsto \text{pol}\; C(t) = \{(y,\beta) \,|\, yA \leq \beta c \;,\; y \geq 0 \;,\; \beta \geq 0\}$$

is lower semicontinuous [1, Proposition 1]. In turn this multifunction pol C is lower semicontinuous if and only if D is lower semicontinuous as follows from the identity

$$(4) \qquad \text{pol}\; C(t) = \text{cl}\{\lambda(y,1) \,|\, y \in D(t) \;,\; \lambda \in R_+\}$$

where cl denotes closure. The inclusion ⊃ follows directly from the fact that pol C(t) is a closed cone that contains $(D(t) \times \{1\})$. For the converse, let $(y,\beta) \in$ pol C(t). If $\beta > 0$, then $\beta^{-1}y \in D(t)$ and $(y,\beta) = \lambda(\beta^{-1}y,1)$ with $\lambda = \beta$. If $\beta = 0$ then

$$yA \leq 0 \quad \text{and} \quad y \geq 0 \quad .$$

Take any $\bar{y} \in D(t)$; recall that $D(t) \neq \emptyset$ since $t \in \mathcal{T}$. For any $\nu = 1,2,\ldots,$ we have

$$(\bar{y} + \nu y)A \leq c \quad , \quad (\bar{y} + \nu y) \geq 0 \quad ,$$

and thus $(\bar{y} + \nu y) \in D(t)$ for all $\nu = 1,\ldots,$ and hence the sequence of points

$$\{\nu^{-1}(\bar{y} + \nu y, 1) \ , \ \nu = 1,\ldots\}$$

is in the set $\{\lambda(y',1) | y' \in D(t) \ , \ \lambda \in R_+\}$ which implies that $(y,0)$ belongs to its closure. This completes the proof of (4).

Now suppose that D is lower semicontinuous at $t \in T \subset \mathcal{T}$. To show that pol C(t) is also lower semicontinuous at t, for any $(y,\beta) \in \text{pol } C(t)$ and $\{t^\nu, \nu = 1,\ldots\}$ any sequence in T we have to exhibit a sequence $\{(y^\nu,\beta^\nu) \in \text{pol } C(t^\nu), \ \nu = 1,\ldots\}$ converging to (y,β). First assume that $\beta > 0$. Then $\beta^{-1}y \in D(t)$ and by lower semicontinuity of D at t there exist $\{\bar{y}^\nu \in D(t^\nu), \ \nu = 1,\ldots\}$ converging to $\beta^{-1}y$. The desired sequence is obtained by setting $y^\nu = \beta\bar{y}^\nu$ and $\beta^\nu = \beta$ for all ν. Next if $\beta = 0$, the previous argument has shown that then there exist $y^k \in D(t)$ such that

$$(y,0) = \lim_{k\to\infty} k^{-1}(y^k,1)$$

Again by lower semicontinuity of D at t, we know that

$$y^k = \lim_{\nu\to\infty} y^{k\nu} \qquad \text{with} \qquad y^{k\nu} \in D(t^\nu) \ , \ \nu = 1,\ldots \quad .$$

The desired sequence is now obtained by a standard diagonalization selection procedure.

If pol C is lower semicontinuous at $t \in T \subset \mathcal{T}$, let $y \in D(t)$ and $\{t^\nu \ , \ \nu = 1,\ldots\}$ be any sequence of points in T. From (4) we know that $(y,1) \in \text{pol } C(t)$ and thus there exist a sequence $\{(y^\nu,\beta^\nu) \in \text{pol } C(t^\nu) \ , \ \nu = 1,\ldots\}$ converging to $(y,1)$. For ν sufficiently large $\beta^\nu > 0$, in which case $((1/\beta^\nu)y^\nu,1) \in \text{pol } C(t^\nu)$, i.e., $(\beta^\nu)^{-1}y^\nu \in D(t^\nu)$ and $y = \lim_{\nu\to\infty} (\beta^\nu)^{-1}y^\nu$. □

5. PROPOSITION. *Suppose* $\mathrm{T} \subset T$ *and for all* $t \in \mathrm{T}$, int $K(t) \neq \emptyset$, *i.e.* $K(t)$ *has nonempty interior, and no row of* (A,b) *is identically* 0. *Then* K *is continuous on* T. *Similarly, if for all* $t \in \mathrm{T}$, int $D(t) \neq \emptyset$ *and no column of* $\begin{pmatrix} A \\ c \end{pmatrix}$ *is identically* 0, *then* D *is continuous on* T.

PROOF. Let $C(t) := \text{pos} \begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$ as in the proof of Proposition 3. If int $D(t) \neq \emptyset$ then as follows from (4), int pol $C(t) \neq \emptyset$. But this in turn implies that $C(t)$ is pointed, i.e., that $C(t) \cap (-C(t)) = \{0\}$. Because suppose otherwise, then there exists $0 \neq v \in C(t)$ such that for all $z \in$ pol $C(t)$

$$vz \leq 0 \quad \text{and} \quad -vz \leq 0 \quad .$$

This means that pol $C(t)$ is contained in the subspace $\{z \mid vz = 0\}$ and int pol $C(t)$ would be empty. The assumptions thus imply that for all t, $C(t)$ is pointed cone and that no column of $\binom{A}{c}$ is identically 0. Corollary 1 of [1] now yields the upper semicontinuity of C on T which in view of Propositions 3 and 1 yields the continuity of D on T.

Naturally the same argument also applies to K. □

Theorem 2 of [1] gives a weaker condition for the upper semicontinuity of the pos map than that used in the proof of Proposition 5. In our context, these conditions can be used to obtain the following stronger version of Proposition 5.

6. PROPOSITION. *Suppose* $\mathrm{T} \subset T$ *and for all* $t \in \mathrm{T}$

(i_a) *the dimension of* $K(t)$ *is constant on* T,

(i_b) *there exists a neighborhood* V *of* t *such that whenever*

$$K(t) \subset \{x \mid A_i x = b_i \ , \ i \in I\} \cap \{x \mid x_j = 0 \ , \ j \in J\}$$

for index subsets I *and* J *of* $\{i = 1,\ldots,m\}$ *and* $\{j = 1,\ldots,n\}$ *respectively, then for all* $t \in \mathrm{T} \cap V$

$$K(t') \subset \{x \mid A_i' x = b_i' \ , \ i \in I\} \cap \{x \mid x_j = 0 \ , \ j \in J\} \quad .$$

Then K *is continuous on* T.

Similarly if for all $t \in T \subset \mathcal{T}$

(ii_a) *the dimension of* $D(t)$ *is constant on* T

(ii_b) *there exist a neighborhood* W *of* t *such that whenever*

$$D(t) \subset \{y \mid yA^j = c_j \ , \ j \in J\} \cap \{y \mid y_i = 0 \ , \ i \in I\}$$

for J *and* I *index subsets of* $\{1,...,n\}$ *and* $\{1,...,m\}$, *respectively, then for all* $t' \in T \cap W$

$$D(t') \subset \{y \mid y(A')^j = c'_j \ , \ j \in J\} \cap \{y \mid y_i = 0 \ , \ i \in I\} \quad .$$

Then D *is continuous on* T.

PROOF. Again let $C(t) := \text{pos} \begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$. If dim D is constant on T, then the dimension of pol C is also constant on T which in turn implies that the dimension of the lineality space of C is constant on T. This is condition (a) of Theorem 2 of [1]. Condition (b) of this Theorem 2 requires that there exist a neighborhood W of t, such that whenever the linear systems

$$-A^j \leq Ax \ , \ -c_j \geq cx \ , \ x \geq 0$$

for some indices $j \in \{1,...,n\}$, and for fixed $k \in \{1,...,m\}$

$$1 \leq A_k x \ , \ 0 \leq A_i x \quad \text{for} \quad i \neq k \ , \quad 0 \geq cx \ , \ x \geq 0 \quad ,$$

are consistent, then they remain consistent for all $t' \in W \cap T$. From these relations we obtain condition (ii_b) through a straightforward application of Farkas Lemma (Theorem of the Alternatives for Linear Inequalities) using the fact that D is nonempty on $T \subset \mathcal{T}$. The assertions involving K are proved similarly. □

Further sufficient conditions for the lower semicontinuity of D and K are provided by the next result.

7. PROPOSITION. *Suppose that for all* $t \in T \subset \mathcal{T}$,

$$R(t) := \{x \mid Ax \geq 0 \ , \ cx \leq 0 \ , \ x \geq 0\} = \{0\} \quad ,$$

then D *is continuous on* T. *Similarly if for all* $t \in T \subset \mathcal{T}$,

$$S(t) := \{y \mid yA \leq 0 \ , \ yb \geq 0 \ , \ y \geq 0\} = \{0\} \quad ,$$

then K *is continuous on* T.

PROOF. Again for reasons of symmetry it really suffices to prove the first part of the proposition. Again let

$$C(t) := \text{pos} \begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix} = \{(u,\eta) \mid u \geq Ax \ , \ \eta \leq cx \ , \ x \geq 0\} \quad .$$

We show that if $R(t) = \{0\}$ on T, then $C(t)$ is pointed and no column of $\binom{A}{c}$ can be identically 0 on T. Suppose $C(t)$ is not pointed, i.e., there exists $(u,\eta) \neq 0$ such that

$$u \leq Ax^1 \quad , \quad \eta \geq cx^1 \qquad \text{for some } x^1 \geq 0 \quad ,$$

and

$$-u \leq Ax^2 \ , \ -\eta \geq cx^2 \qquad \text{for some } x^2 \geq 0 \quad .$$

This implies that for $(x_1 + x_2) \geq 0$,

$$0 \leq A(x^1 + x^2) \qquad \text{and} \qquad 0 \geq c(x^1 + x^2) \quad .$$

But then $x^1 + x^2 = 0 = x^1 = x^2$ if $t = (c,A,b) \in T$ since $R(t) = \{0\}$. This in turn yields $(u,\eta) = 0$, which contradicts the working assumption that $C(t)$ is not pointed. Also, if some column $\binom{A^i}{c_j}$ is identically 0, then $R(t) \neq \{0\}$ since then any nonnegative multiple of the j-th unit vector u (with $u_l = 0$ if $l \neq j$ and $u_j = 1$) satisfies the inequalities

$$Ax \geq 0 \ , \ cx \leq 0 \ , \ x \geq 0 \quad .$$

This implies the upper semicontinuity of C on T [1, Corollary 1] which gives us the continuity of D via Propositions 3 and 1. □

There are a number of equivalent ways to express the conditions of Proposition 7. For example: $R(t) = \{0\}$ if and only if

(8) if $0 \neq \hat{x} \in \{x \geq 0 | Ax \geq 0\}$ then $c\hat{x} > 0$,

or still

(8') $c \in \text{int pos } (A^T, I)$

where int denotes interior.
Similarly $S(t) = \{0\}$ if and only if

(9) if $0 \neq \hat{y} \in \{y \geq 0 | yA \leq 0\}$ then $\hat{y}b < 0$,

or still

(9') $b \in \text{int pos } (A, -I)$

10. COROLLARY. *Suppose that for all* $t \in T \subset \mathcal{T}$, $K(t)$ *is bounded, then* D *is continuous on* T. *Similarly if all* $t \in T \subset \mathcal{T}$, $K(t)$ *is bounded then* D *is continuous on* T.

PROOF. The convex polyhedron $K(t)$ is bounded if and only if $\{x | Ax \geq b\ ,\ x \geq 0\} = \{0\}$. This implies that $R(t) = \{0\}$ with $R(t)$ as defined in Proposition 7. The lower semicontinuity of R now follows from Proposition 7. One argues similarly for K using this time the boundedness of K to conclude that $S(t) = \{0\}$. □

11. COROLLARY. *Suppose that for all* $t \in T \subset \mathcal{T}$, *either all columns* A^j *of* A *are nonpositive and* $A^j \neq 0$ *or* $c < 0$. *Then* D *is continuous on* T. *Similarly if for all* $t \in T$, *either all rows* A_i *of* A *are nonnegative and* $A_i \neq 0$ *or* $b < 0$, *then* K *is continuous on* T. *Hence, if for all* $t \in T$, $A < 0$ *and* $b < 0$ *or* $A > 0$ *and* $b > 0$, *then* Q *is continuous on* T.

PROOF. If $A^j \leq 0$ and $A^j \neq 0$ then $\{x \geq 0 | Ax \geq 0\} = \{0\}$ and thus $K(t)$ is bounded for all $t \in T$. The lower semicontinuity of D then follows from Corollary 10. If $c > 0$ then for every $0 \neq x$, $cx > 0$ and from (8) it follows that $R(t) = \{0\}$ and in turn the lower semicontinuity of D follows from Proposition 7. Again, the lower semicontinuity of K is obtained by arguing similarly using $A_i \geq 0$ and $b < 0$. The assertions about Q now follow from the above using naturally Theorem 2. □

There is another way to prove Corollary 11, which also shows how to generalize it. The proof of Proposition 5 shows that many of the sufficient conditions for the lower semicontinuity of D boil down to checking if

$$C(t) := \mathrm{pos} \begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$$

is pointed. The last m + 1 columns $\begin{pmatrix} -I & 0 \\ 0 & 1 \end{pmatrix}$ of the matrix that generate C(t) determine an orthant and this cone will certainly be pointed if the remaining columns $\{\begin{pmatrix} A^j \\ c_j \end{pmatrix}, j = 1,\ldots,n\}$ belong to this orthant or are such that when added to $\begin{pmatrix} -I & 0 \\ 0 & 1 \end{pmatrix}$ they keep the cone pointed. Sufficient conditions of this type are provided by Corollary 11, but they clearly do not exhaust the realm of possibilities. For example, if there exist a vector $\pi \in R^m$ with $\pi_i > 0$ for all $i = 1,\ldots,m$ such that $\pi A < c$ then C(t) is pointed since then all the columns of $\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$ have strictly positive inner product with the vector $(-\pi, 1) \in R^{m+1}$. Here we are naturally very close to the conditions of Proposition 5 and 8.

This short note was essentially an attempt at organizing the available results about the continuity of Q; we conclude by giving the pertinent references. Theorem 2 and Proposition 3 come from [2, Theorem 2]*. The continuity of Q with the special conditions given by Proposition 7, more exactly with relations (8) and (9), is proved by Bereanu [3, Theorem 2.2]. He also exhibits the sufficient conditions of Corollary 11. Conditions (8') and (9') are those of Robinson [4] when applied to linear programs in the form considered here. He also shows that these conditions are equivalent to having the set of optimal solutions of the primal and the dual bounded. Propositions 5 and 6 can be traced back to [1] and to Dantzig, Folkman and Shapiro [5] and have been used by Salinetti [6] in the study of the distribution of the optimal

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value of random linear programs. The general continuity results for the optimal value function (of an optimization problem depending on parameters) provide us with the following results [7]:

12. THEOREM. *Let* T ⊂ $\mathcal{T}$. *Suppose that* K *is lower semicontinuous on* T; *then* Q *is upper semicontinuous. If* K *is uniformly compact on* T, *then* Q *is lower semicontinuous on* T.

The first hypothesis is one of the two used to prove Theorem 2. The uniform compactness is stronger than needed since simply K bounded on T yields the lower semicontinuity of D, cf. Corollary 10, and that is what we used to prove the lower semicontinuity of Q in Theorem 2.

If the coefficient of A are not variable, then Q is always continuous. In particular we get

13. THEOREM. *Suppose that for all* $t \in T \subset \mathcal{T}$ *the matrix* A *is constant. Then* Q *is continuous on* T.

PROOF. In this case, the multifunctions K and D are not only continuous on T but in fact Lipschitz continuous on T as follows from [8, Theorem 1]. The continuity of Q resulting again from Theorem 2. □

In fact in this case Q is actually Lipschitz continuous. This can be demonstrated using the Lipschitz continuity of K and D, or as is more usual by using the fact that on T (for fixed A), the value of a linear program is a piecewise linear function of (c,b), convex in b and concave in c [9, Basis Decomposition Theorem]. If only b varies, then clearly Q is continuous but in this case there also exist a continuous function $t \longmapsto x^*(t): \mathcal{T} \to R^n$ such that for all t, $x^*(t) \in K(t)$ and $cx^*(t) = Q(t)$ [10, Theorem], [11]. If only c varies a similar statement can be made, viz., there exists $y^*(\ ): \mathcal{T} \to R^m$, continuous such that $y^*(t) \in D(t)$ and $y^*(t)b = Q(t)$.

Note: Robinson [4] formulates his pair of dual linear programs to take into account problems involving both equalities and constraints. For such cases there are also appropriate versions of Theorem 2 and Propositions 1 and 3. For example, if

$$Q(t) = \inf_{x\in R^n} [cx \mid Ax = b,\ x \geq 0]$$

then we should study the continuity of the maps

$$t \longmapsto \text{pos}\begin{pmatrix} A & 0 \\ c & 1 \end{pmatrix} \qquad \text{and} \qquad t \longmapsto \text{pos}\begin{pmatrix} A^T & -A^T & I & 0 \\ b^T & -b^T & 0 & -1 \end{pmatrix} \quad .$$

Continuity results of a similar nature are then readily available.

SUMMARY

1. $K = \{x \geq 0 \mid Ax \geq b\}$ continuous
   $D = \{y \geq 0 \mid yA \leq c\}$ continuous
   $\Rightarrow$ Q continuous.

2. D cont. $\Leftrightarrow$ $C = \text{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$ cont. (=u.s.c.)

$\Uparrow$

$\left\langle \text{dim.D = constant + active constraints cond. (Prop.6)} \right\rangle$ $\Rightarrow$ $\left\langle \text{dim } C \cap (-C) = \text{constant cond. on columns } \begin{pmatrix} A \\ c \end{pmatrix} \right\rangle$

$\Uparrow$

int $D \neq \emptyset$ $\begin{pmatrix} A \\ c \end{pmatrix}^j \neq 0$ $\Rightarrow$ C pointed $\begin{pmatrix} A \\ c \end{pmatrix}^j \neq \emptyset$

$\Uparrow$

$c > 0$ $\Rightarrow$ $\{x \geq 0 \mid Ax \geq 0 \ , \ cx \leq 0\} = \{0\}$
equiv. $c \in \text{int pos } (A^T, I)$

$\Uparrow$

$A \leq 0$ $A^j \neq 0$ $\Rightarrow$ K bounded

3. K cont. $\Leftrightarrow$ $\text{pos}\begin{pmatrix} A^T & I & 0 \\ b^T & 0 & -1 \end{pmatrix}$ cont. (=u.s.c.)

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