NOT FOR QUOTATION
WITHOUT PERMISSION
OF THE AUTHOR

CHAOS IN NONLINEAR DYNAMICS AND THE LOGISTIC SUBSTITUTION MODEL

Bill Keepin

September 1982
WP-82-86

Working Papers are interim reports on work of the International Institute for Applied Systems Analysis and have received only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute or of its National Member Organizations.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

The Tower is as wide and spacious as the sky itself... And within this Tower, spacious and exquisitely ornamented, there are also hundreds of thousands...of towers, each one of which is as exquisitely ornamented as the main Tower itself and as spacious as the sky. And all these towers, beyond calculation in number, stand not at all in one another's way; each preserves its individual existence in perfect harmony with all the rest; there is nothing here that bars one tower being fused with all the others individually and collectively; there is a state of perfect intermingling and yet of perfect orderliness... all is contained in one and each contains all.

D.T. Suzuki (1968)

PREFACE

A rekindled appreciation of an old cliché has touched off a flurry of activity in the field of nonlinear dynamics lately. The truism that nonlinearities often lead to wild and exotic behavior has been known for a long time, but only recently has it been studied carefully, and the discoveries are startling and profound. The simplest equation illuminating these features is the logistic equation (in discrete form), which has a long history of application to growth phenomena in biology and population dynamics. This equation is also the basis for the logistic substitution model developed at IIASA by Cesare Marchetti and Nebojsa Nakicenovic (1979, see also Nakicenovic 1979). This model is a highly effective tool for modeling the dynamics of economic market substitution, and has been extensively applied to primary energy markets (Energy Systems Program Group 1981; Marchetti et al. 1978).

This Working Paper begins with a brief review of the recent developments in nonlinear dynamics, followed by a study of the implications that these phenomena have for the logistic substitution model. The key finding is that only highly unrealistic parameter values can induce chaotic behavior in this model.

CHAOS IN NONLINEAR DYNAMICS AND THE LOGISTIC SUBSTITUTION MODEL

Bill Keepin

INTRODUCTION

Recent developments in the field of nonlinear dynamics have generated quite a stir among scientists in a variety of disciplines. The excitement stems from a new appreciation of an old fact, namely, that nonlinear equations can exhibit wild and exotic behavior. More specifically, the solution trajectories of many nonlinear deterministic systems proceed from regular to chaotic behavior as a system parameter is varied. What's more, this "onset of chaos" evolves at a rate (in parameter space) which is unique for large classes of systems. (Hence the name "universality theory" which is attached to this.)

One particular equation that has been extensively studied recently is the discrete logistic equation, which has a long history of application to various growth phenomena in biology, chemistry, epidemiology, etc. Although the logistic process is inherently deterministic, it is found to behave in an essentially stochastic manner for certain parameter values. This feature is characteristic of a wide variety of discrete and continuous systems. The intent of the present work is to investigate this phenomenon and to determine what implications it might have for the logistic substitution model developed at IIASA by C. Marchetti and N. Nakicenovic (1979). A basic familiarity with this model is assumed. (See Chapter 8 of Energy Systems Group, 1981, or the above reference.)

We begin with a very brief overview of the recent developments in nonlinear dynamics, focusing attention on the properties of the discrete logistic difference equation. Following this we present specific results obtained from implementation of this equation in the logistic substitution model for two competing technologies.

A VERY BRIEF INTRODUCTION TO CHAOTIC PHENOMENA IN NONLINEAR DYNAMICS

The best starting point is a general example. Consider the first order difference equation

$$x_{n+1} = F(x_n) \quad , \tag{1}$$

in which repeated iteration of an initial point x_0 is considered to model the evolution over time of a dynamic process.*) For simplicity we assume that the x_n's are real numbers, and that $F : R \to R$ has a continuous derivative (generalization to R^n is straightforward). A specific sequence of iterates $x_0, x_1, x_2, \ldots$ is called an orbit (or trajectory) of the system.

The first step in the analysis of (1) is to seek special points $\bar{x}$, called fixed points (or equilibrium points), which are time invariant;

$$\bar{x} = F(\bar{x}) \quad . \tag{2}$$

Thus if an evolving orbit ever attains the value $\bar{x}$, it remains there for all future time. An important consideration in this context is the stability of the fixed point with respect to small

*) For this discussion, we do not distinguish between the model and the process.

perturbations. Suppose the n-th term of an orbit is displaced from the fixed point by an amount δ_n,

$$x_n = \bar{x} + \delta_n \quad .$$

In order that the orbit approaches the fixed point, we require $|\delta_{n+1}| < |\delta_n|$. For $|\delta_n|$ sufficiently small, we have (from the Mean Value Theorem),

$$|F(x_n) - F(\bar{x})| \approx |F'(\bar{x})| \cdot |x_n - \bar{x}| \quad .$$

Using (1) and (2) this becomes

$$|\delta_{n+1}| \approx |F'(\bar{x})| \cdot |\delta_n| \quad ,$$

from which we see that

$$\left.\begin{array}{l} \bar{x} \text{ is stable if } |F'(\bar{x})| < 1 \quad , \\ \\ \bar{x} \text{ is unstable if } |F'(\bar{x})| > 1 \quad . \end{array}\right\} \tag{3}$$

As a specific example, consider the logistic difference equation, which has the standard form

$$x_{n+1} = bx_n(1 - x_n) \equiv F_b(x_n) \quad . \tag{4}$$

The set $\{F_b(x)\}_{b\in R}$ is the one parameter family of all parabolas having roots at zero and unity. For this work, we consider only positive values of x_n, which requires that $b > 0$ and $x_n < 1$. To ensure the latter, we restrict $b \leq 4$ (since the maximum of $F_b(x)$ exceeds unity if $b > 4$). Thus two different intervals are involved here. The first is the state space (or phase space) of the system; $0 \leq x_n \leq 1$. The second is the parameter space of the system; $0 < b \leq 4$. A plot of $F_b(x)$ is given in Figure 1,

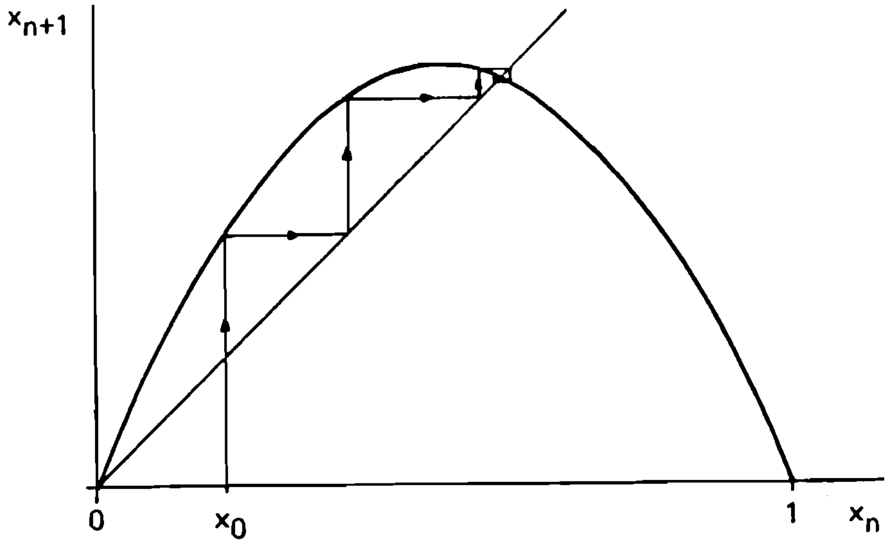


Figure 1.

for b $\approx$ 2.5. Henceforth we drop the subscript b on $F_b(x)$.

Setting $x_{n+1} = x_n$ we find two fixed points at

$$\bar{x} = 0,\ 1 - \frac{1}{b} \quad ,$$

which are indicated in Figure 1 by the intersection of the parabola with the line $x_{n+1} = x_n$. To investigate the stability of the fixed points, we compute

$$F'(0) = b \quad ,$$

$$F'(1-\tfrac{1}{b}) = 2 - b \quad ;$$

and applying the stability criterion (3) we see that zero is stable for $|b| < 1$, and unstable for $|b| > 1$. Similarly, $1 - \frac{1}{b}$ is stable for $1 < b < 3$. These results apply globally in this case. Thus if $|b| < 1$ the orbits (evolving from almost all initial points) eventually approach zero; and we call the point $\bar{x} = 0$ a global attractor. The set of all initial points x_0 that are attracted to $\bar{x}$ form the so-called basin of attraction.

Similarly, if $b \in (1,3)$ the fixed point $\bar{x} = 1 - \frac{1}{b}$ is an attractor (with the basin of attraction $0 < x_0 < 1$). The latter situation is indicated in Figure 1 for a particular orbit. Starting from an initial point x_0, we obtain $x_1 = F(x_0)$. Now, before applying F again, we must transfer x_1 back to the x_n-axis. Horizontal translation to the line $x_{n+1} = x_n$ accomplishes precisely this operation. Thus the orbit evolves as shown in the figure, eventually converging (in an oscillatory manner) to the fixed point.

Having described the behavior of the logistic difference equation (4) for $0 < b < 3$, we now ask what happens when $b \geq 3$. For $b = 3$, the fixed point $\bar{x} = 2/3$ is again a (global) attractor, but it is just barely stable, i.e., $F'(\bar{x}) = -1$, and the convergence is very slow. As b increases beyond 3, the fixed point becomes unstable, i.e., $F'(\bar{x}) < -1$, and all orbits tend to a two-point limit cycle (i.e., attractor of period 2). This process is referred to as period doubling or pitchfork bifurcation (Feigenbaum, 1979). To see how it works, we consider the mapping obtained by applying $F(x)$ twice:

$$F^{(2)}(x) = F(F(x)) \quad .$$

The fixed points of this mapping will consist of the fixed points of the original mapping *plus* the points of any period 2 cycle. Now let $\bar{x}$ be the stable fixed point of the original mapping, and let us investigate the stability of $\bar{x}$ for the new mapping $F^{(2)}(x)$. By the chain rule, we readily find

$$F^{(2)\prime}(\bar{x}) = (F'(\bar{x}))^2 \quad .$$

Thus for $b = 3$, since $F'(\bar{x}) = -1$, we have $F^{(2)\prime}(\bar{x}) = 1$, which tells us that at the point $\bar{x}$, the function $F^{(2)}(x)$ is tangent to the line $x_{n+2} = x_n$. As b increases beyond 3, $F^{(2)\prime}(\bar{x})$ increases beyond unity, and this necessarily gives birth to two new fixed points for the function $F^{(2)}(x)$. It is these two points which constitute the stable attractor of period 2 for the original mapping (May, 1976). This process is illustrated in Figure 2. The solid curve is for $b < 3$, and it is clear that $F^{(2)\prime}(\bar{x}) < 1$ in this case. As b increases through 3, the solid curve passes smoothly and continuously to the dashed curve, and by imagining the intermediate stages, it is easy to see that two new fixed points appear just as the slope of $F^{(2)}(\bar{x})$ increases past unity (where $\bar{x}$ becomes unstable). Furthermore, it is clear that the slopes at the new fixed points decrease from unity as b increases beyond 3, hence these points are stable.

It is now not difficult to imagine what happens as b is increased still further. The slopes of the two new fixed points of $F^{(2)}(x)$, which are always equal, continue to decrease until they simultaneously reach the value -1. At this point, the period 2 attractor becomes unstable and bifurcates, giving rise to an initially stable period 4 attractor. We may denote the value of b for which this occurs by b_4. Increasing the value of b still further, we reach a value b_8 at which the period 4 attractor bifurcates to yield a period 8 attractor, and the process continues like this, producing an infinite sequence of period 2^k attractors. The basins of attraction for these attractors are disjoint intervals, often called windows, whose lengths rapidly approach zero as $k \to \infty$. Specifically, if we denote by b_{2^k} the value of b at which the first attractor of period 2^k is born, then the sequence $b_2, b_4, b_8, \ldots$ converges geometrically to a *finite* limit b_∞ (May, 1976; Feigenbaum, 1978). In the case of the logistic equation, this limit has the value $b_\infty = 3.5700$. These ideas are summarized in Figure 3 (adapted from Collet and Eckmann, 1980), where the attractors are displayed as a function of parameter value b. Note the rapid convergence to b_∞.*)

---

*) The map used to generate Figure 3 was not the logistic in standard form, but rather the map $x'_{n+1} = 1 - b'(x'_n)^2$, with $x'_n \in [-1,1]$ and $b' \in (0,2)$. The logistic is easily transformed to this map for $2 < b \leq 4$, via $x'_n = (.25b - .5)\, x_n + 0.5$, where $[-1,1] \to [1-.25b, .25b]$, and the parameter correspondence is $b' = .25b^2 - .5b$. The dynamic structure of the two maps is identical, and the reader need only be aware that the parameter space is represented by [0,2] in the figure, and the state space by [-1,1].

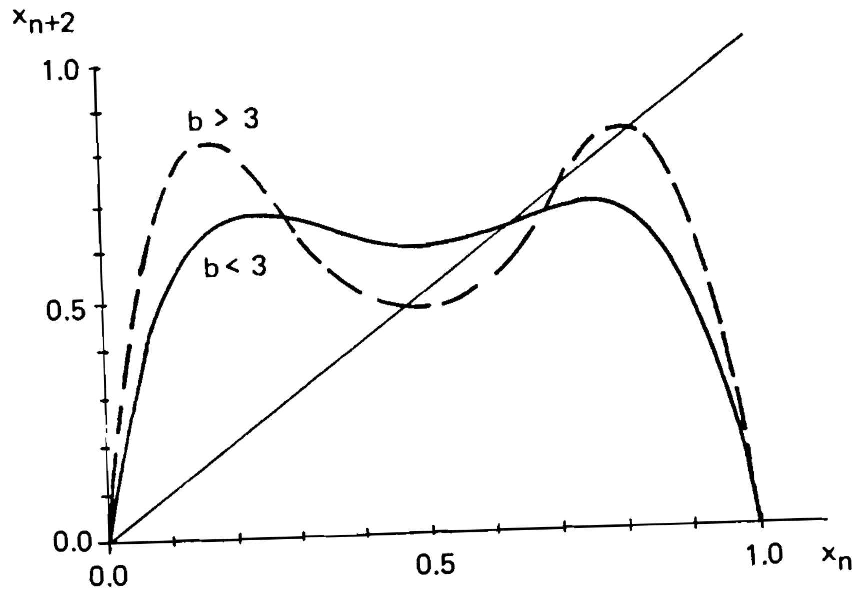
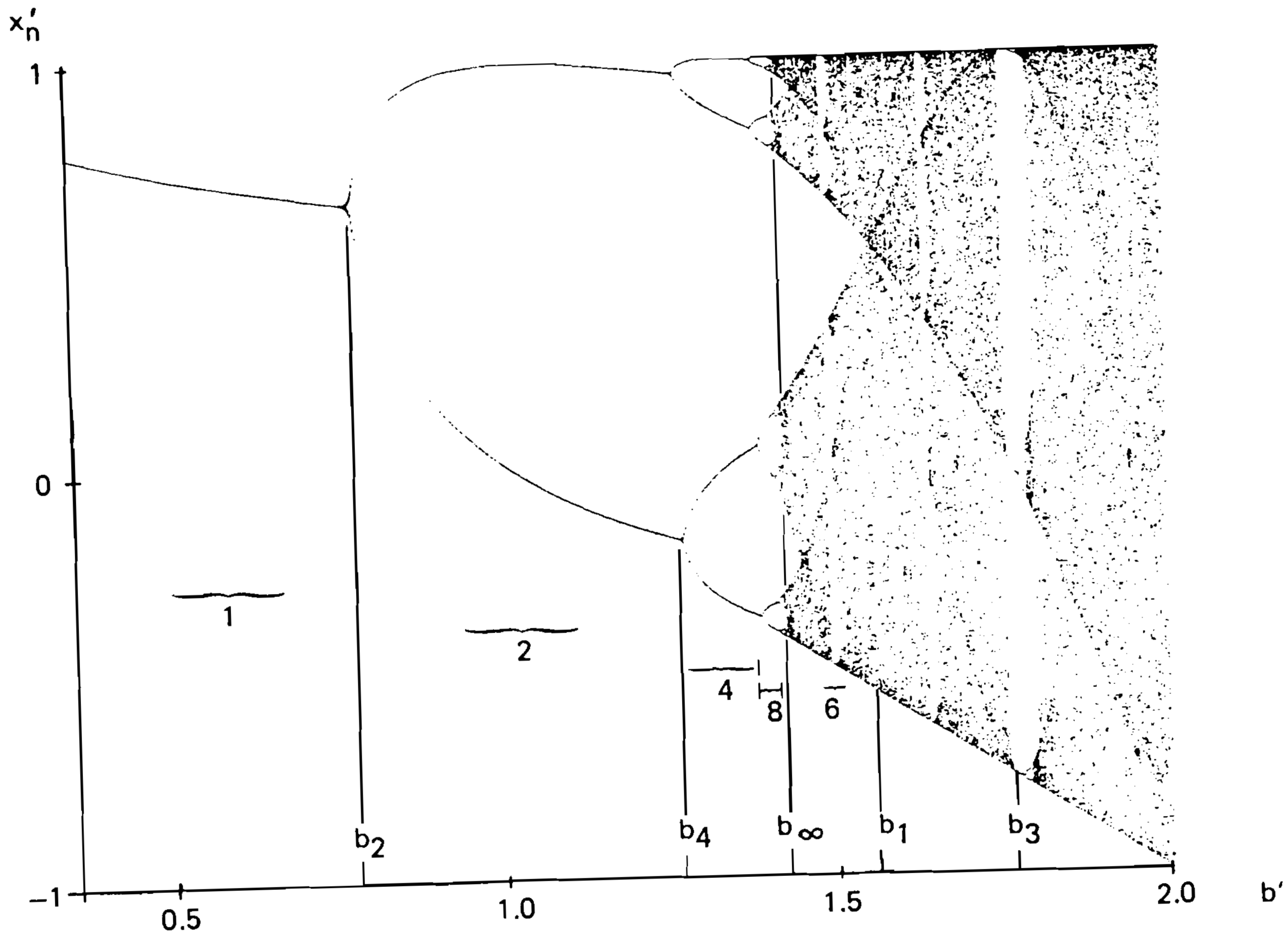


Figure 2.



The phenomenon just discussed, which is referred to as the "onset of chaos", is by no means limited to the logistic equation, but is in fact generic to the process of functional iteration as described by (1). As long as F(x) has a hump whose steepness can be tuned by a parameter b, then there probably exists an infinite sequence of period doubling bifurcations (occurring at $b_2, b_4, b_8, \ldots$), which has a finite point of accumulation b_∞. At each bifurcation, each point of the existing 2^k cycle produces a pair of "twins", as in Figure 2, and the union of all these offspring comprises the 2^{k+1} cycle (Hofstadter, 1981). Furthermore, the sequence $\{b_{2^k}\}$ converges geometrically at a rate that is asymptotically unique for a wide class of functions. Specifically,

$$\lim_{k\to\infty} \frac{b_{2^k} - b_{2^{k-1}}}{b_{2^{k+1}} - b_{2^k}} = \delta = 4.6692016\ldots \quad , \tag{5}$$

where δ is universal for all functions F(x) having a quadratic maximum. This result, discovered by Feigenbaum (1978, 1979), was the spark that prompted much of the recent research in this field. The surprising fact here is that quantitative information can be obtained regarding the behavior of (1) in the absence of specific knowledge of the form of the function F(x). A closely related discovery, also due to Feigenbaum, is that some geometrical feature of the mapping (1) is reproduced at the n-th bifurcation reduced in scale by a factor of approximately α^n, where α is another universal asymptotic constant having the value $\alpha = -2.5029\ldots$ This property of "scale invariance" leads to fascinating hierarchies of self-repeating patterns embedded within one another ad infinitum. Such structure, reminiscent of the Cantor set, will be discussed toward the end of this section. For now, we note that the scale-invariant feature in the logistic process involves the spacing between two newborn twins, which is approximately α times smaller than the spacing between their parent and its twin (Hofstadter, 1981). These sorts of results are sometimes collectively referred to as metric universality theory, much of which has been placed on a firm mathematical foundation by Collet and Eckmann (1980), Guckenheimer (1980), and Lanford (1980). In a mathematical context, the universal constant δ emerges as an eigenvalue of an operator on function space, and α is associated with a nonlinear fixed point problem (Collet and Eckmann, 1980).

Continuing our discussion of the logistic equation, we now ask what happens in the chaotic region, i.e., for $b > b_\infty$? A glance at Figure 3 suggests that things get very complicated; and indeed, a complete answer to this question is not known (Ott, 1981). However, the underlying behavior may be described as follows. For b slightly greater than b_∞, orbits are attracted to "noisy" cycles of period 2^k (with $k \to \infty$ as b decreases to b_∞). This means that the orbit is eventually confined to 2^k disjoint intervals in (0,1) which are visited in a specific sequential order. However, the distribution of visits within any one of these intervals appears to be completely chaotic. As b increases a value $\bar{b}_k$ is reached, at which the 2^k disjoint intervals merge in pairs, causing the noisy 2^k cycle to become an even noisier 2^{k-1} cycle. As b grows further this process of "reverse bifurcation" (Lorenz, 1980) continues in such a way that the sequence $\bar{b}_k$ follows the same scaling relation as in (5). Eventually b reaches the value $\bar{b}_1$ (see Figure 3), beyond which chaotic motion appears over one continuous interval. This interval then widens as b continues to grow, and it becomes the entire phase space, i.e., the unit interval, when b reaches the value 4.*) Indeed, the discrete logistic equation with $b = 4$ is commonly employed as a random number generator.

This is by no means the whole story, however. The underlying structure just described is "interrupted" infinitely often by tiny clusters of infinitely many windows of parameter values in which there are stable periodic orbits. A few of the widest of these clusters appear in the chaotic region of Figure 3 as

*) In Figure 3, this corresponds to $b' = 2$, and the unit interval corresponds to $[-1,1]$. See previous footnote.

thin vertical gaps (or slits). A given such cluster generally begins with a stable period N cycle (called the fundamental), which gives birth, via pitchfork bifurcations, to an infinite sequence of stable cycles (called the harmonics) of period $N2^k$. Each of these cycles occupies a narrow window of parameter values in which it is stable. The windows are adjacent disjoint intervals, and their union forms the cluster. Most clusters are exceedingly narrow; the widest being for $N = 3$ (see Figure 3).

The reader may notice that the structural form of a cluster is identical to that of the original sequence of period doubling bifurcations discussed earlier, and this is indeed the case. The region $1 < b < b_\infty$ is nothing but a very wide cluster whose fundamental cycle has period one ($N = 1$). Thus we see that the parameter space is populated with an infinite number of fundamental periodic orbits; each of which sprouts an infinity of harmonics via period doubling bifurcations. The asymptotic scaling constants δ and α discussed above have different values for different clusters, but they are again "universal", i.e., for all functions $F(x)$ having a quadratic maximum.

We have seen how new stable orbits are born from a fundamental orbit (Figure 2), but it is natural to ask how the fundamental orbit appears in the first place. For example, how did a stable period 3 orbit suddenly emerge from all that chaos, e.g., at the value b_3 in Figure 3? To answer this, we consider the mapping obtained by applying $F(x)$ three times

$$F^{(3)}(x) = F\{F[F(x)]\} \quad .$$

A plot of this function is shown in Figure 4 for $b < b_3$ (solid curve) and $b > b_3$ (dashed curve). As before, we imagine the intermediate stages as the solid curve passes continuously to the dashed curve. At the point $b = b_3$, the first two valleys and the last hump of the function all simultaneously touch (and are tangent to) the fixed line $x_{n+3} = x_n$, giving birth to an unstable period 3 cycle. This phenomenon is referred to as tangent bifurcation. As b increases beyond b_3, the valleys sink, and the hump rises to create six new intersections with

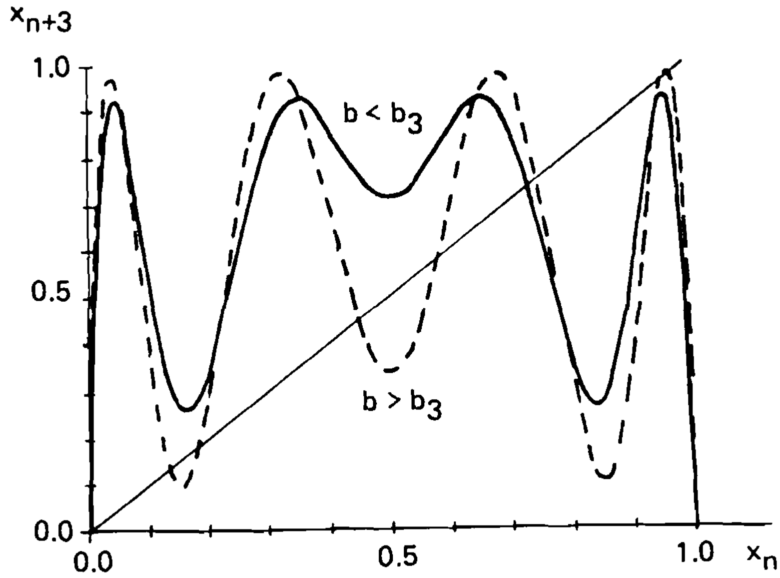


Figure 4.

the line $x_{n+3} = x_n$. The slopes at three of these points are greater than unity, producing an unstable 3 cycle. The slopes at the remaining 3 points decrease from unity, creating a stable 3 cycle. This cycle persists until the slopes reach the value -1, beyond which period doubling bifurcation produces stable cycles of periods 6,12,24,...

One final question we may ask is "how many" parameter values give rise to chaotic behavior? In other words, if we take the interval $0 < b \leq 4$, and remove from this all values of b for which there is a stable periodic cycle, what is left over? It is believed that this remaining set, which we denote by C, contains no intervals but has positive Lebesque measure (Collet and Eckmann, 1980). Furthermore, there is a proper subset of C, also believed to have positive Lebesque measure, for which the corresponding trajectories exhibit "sensitive dependence on initial conditions" (Ruelle, 1979). This means that two trajectories starting arbitrarily close together will eventually separate exponentially. (The associated exponent is called the Lyapunov exponent; see Shaw, 1978; Collet and Eckmann, 1980.) Such behavior is characteristic of so-called strange attractors, although the logistic equation does not actually possess such an attractor. However, there are numerous "aperiodic" attractors of varying complexity in between stable periodic orbits and strange attractors. In the case of the logistic equation it is not known whether the aperiodic attractors are truly chaotic, or just periodic attractors having very large periods; but recent research indicates the former (Ott, 1981).

We now conclude our discussion of the discrete logistic equation, but having probed this far into nonlinear chaotic phenomena, it would be a crime not to give the reader a glimpse of more general results and developments along these lines. From the standpoint of classical mechanics, a physical system may be classified as either conservative (e.g., energy is conserved) or dissipative (e.g., friction or viscosity is present). In the latter case, volume elements in phase space shrink to zero as $t \to \infty$, whereas they remain constant for conservative systems. The more dissipative the system, the faster the volume elements shrink, making it generally easier to say something about the long term behavior of the system. For extremely dissipative systems, the attractors are often quite simple, whereas for conservative systems they can be exceedingly complicated. All sorts of possibilities lie in between (Eckmann, 1981).

In three-dimensional phase space, it is natural to assume that attractors for dissipative systems must be surfaces (two-dimensional), curves (one-dimensional), or points (zero-dimensional), since volume elements eventually shrink to zero. However, it often happens that an attractor has non-integer dimension (say, between 2 and 3) so that it occupies no volume but has a very complicated structure. Such attractors are termed strange (or fractal), and they arise in a variety of contexts. The concept of dimension here is the so-called Hausdorf dimension (Ott, 1981), defined for a set S by

$$d(S) = \lim_{\varepsilon\to 0} \frac{\ln N_\varepsilon(S)}{\ln \frac{1}{\varepsilon}} \quad ,$$

where $N_\varepsilon(S)$ is the minimum number of ε-sided cubes needed to cover the set S. This definition yields the usual integer values for "regular" sets and surfaces. For the symmetric Cantor set in the unit interval, the Hausdorf dimension is $(\ln 2) / (\ln 3) \approx .630$. In general, the intersection of a line with a strange attractor yields a Cantor-like set on the line.

The logistic equation does not possess a strange attractor because it is too dissipative. However, Mandelbrot (1980) studied the complex logistic transformation $z_{n+1} = \lambda z_n(1-z_n)$, where z_n and λ are complex, and found that the boundaries of admissible regions in the λ and z planes are fractal sets (having dimension greater than one).

Other phenomena that occur in the presence of strange attractors include wild separatrices. In certain regions of phase space, a separatrix can fold back on itself infinitely often so that two points arbitrarily close together move in opposite directions (which clearly gives rise to sensitive dependence on initial conditions). Or two separatrices may be interlaced in such a way as to intersect in an uncountable number of homoclinic points which are dense in a certain region (giving the feeling of the delicate structure of the double helix in a DNA molecule). These phenomena are often accompanied by the property of scale invariance. Thus a region of very intricate structure will often contain many embedded subregions, each of which upon magnification, has the exact same intricate structure as the original region. This self-generating behavior continues ad infinitum, producing infinite hierarchies of carefully nested patterns, each of which exhibits the essential structure of the whole. Neatly interwoven into these patterns of chaotic behavior are complementary hierarchies of regular regions, in which the trajectories behave in a smooth and continuous way. The strange attractor is the skeletal remains of these structures that emerges as some quantity of the motion is dissipated in time. Thus a strange attractor is rather like a very complicated steady state solution. We remark that the long term behavior of individual orbits is sometimes determined by very unusual criteria, such as, "how irrational" a characteristic parameter is. One example occurs in the physics of intersecting storage rings, where a given trajectory dies out unless its winding number r is "sufficiently irrational," i.e.,

$$|r - n/m| \geq c\ /|m|^{2.5} \text{ for all integers } n, m,$$

where c is a positivie constant (Hellemann, 1980).

One might think that strange or chaotic behavior is pathological, occurring very rarely, but the opposite is actually true. An important mathematical distinction among systems is the notion of separability. Separable systems usually exhibit smooth predictable behavior, whereas non-separable systems generally include chaotic behavior of some kind. It has been shown that the latter are dense in the space of all analytic systems (Hellemann, 1980), so that separable systems are truly the exception rather than the rule. Furthermore, very simple systems can possess highly complicated dynamics. To give an example, one of the most celebrated strange attractors (called the Lorenz Attractor) arises out of a simple first-order linear differential system which is perturbed by a smooth well-behaved nonlinear term:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix},$$

where A is a constant real 3 x 3 matrix.

We briefly remark that for continuous first-order differential systems, at least three dimensions are required in order to observe chaos. Similar systems in one and two dimensions have trajectories which never bend back on themselves in such a way as to produce chaos. However, for discrete systems, only one dimension is required for chaos (as we saw above with the logistic equation). In this case, the associated map is necessarily non-invertible, meaning that it is impossible to proceed backwards in time.

We close this section with a few comments. One point, in regard to metric universality theory, is that the value of the asymptotic constants δ and α are dependent on the presence of a quadratic nonlinearity. Since almost every nonlinear function is locally quadratic (via the Taylor expansion), it is not very surprising that δ and α are "universal." Furthermore, this theory is not completely new, as it is closely related to renormalization group analysis in statistical mechanics (Collet and Eckman, 1980). Another point to bear in mind is that the intricate structure of infinite nested patterns is a mathematical phenomenon, and could never be observed physically. If nothing else, the Heisenberg uncertainty principle is the essential limiting factor. This is not to imply that the results described above do not have physical applications. On the contrary, there are numerous applications, the most dramatic of which is in fluid mechanics. The progression from laminar to turbulent flow has been successfully modeled via period doubling bifurcations leading from stable to chaotic behavior. Indeed, the Feigenbaum constants δ and α have been observed experimentally (Hofstadter, 1981), which greatly corroborates the theory. Other applications occur in various fields such as celestial mechanics, high energy physics, scattering theory, and thermal physics (Eckmann, 1981).

APPLICATION TO THE LOGISTIC SUBSTITUTION MODEL

We now set out to determine what implications, if any, the complex behavior described in the last section has for the logistic substitution model. For this purpose we consider the simplest case, namely, two competing technologies, or market shares. Since the two fractional shares must always sum to unity, there is only one independent variable in the problem. From the logistic in standard form (4), we obtain a more general version via the transformation $f_n \equiv \frac{b}{a} x_n$. This yields the following:

$$f_{n+1} = bf_n - af_n^2 \quad . \tag{6}$$

Now if a given technology ever attains either 100% or 0% of the market it retains that share for all future time. Thus we impose fixed points at 0 and 1, the first of which is present in (6) as it stands. For the fixed point at unity, we set $f_{n+1} = f_n = 1$, which yields $b = a + 1$. Thus (6) becomes

$$f_{n+1} = (a + 1)f_n - af_n^2 \quad ,$$

$$f_{n+1} = f_n + af_n(1 - f_n) \quad . \tag{7}$$

We have $a = b - 1$, so the parameter a in this equation corresponds to one less than the parameter b in (3), and the parameter space is now given by $a \in (-1,3)$. There is now a direct correspondence between this discrete model and the continuous logistic model which has the form

$$f'(t) = \alpha f(t)[1 - f(t)] \quad , \tag{8}$$

where the parameter a in (7) is analogous to the "annual adoption rate" α in (8).

For purposes of modeling economic phenomena such as market substitution, the discrete logistic may be more appropriate than the continuous version. The latter implies continuous adjustment of the market shares, as if data concerning the values of the shares themselves were continuously available. This is contrary to economic reality, where decisions are made periodically (e.g. annually) based on data that are also available only periodically. The discrete model more accurately reflects this situation. Furthermore, the discrete model has a built-in time lag, meaning that some account is taken of prior market conditions. The continuous model, however, ignores past information altogether, which is difficult to justify in view of economic practice.

Before continuing further we pause briefly to compare the numerical results obtained from the discrete and continuous logistic models. For this purpose two parameter values are chosen: $\alpha = a = 0.05, 0.30$. The first value is typical for the many experiments carried out by Marchetti and Nakicenovic (1978), and the second value is quite large and rarely observed. The results are presented in Table 1. It is clear from the table that the agreement between the two processes is excellent for $\alpha = 0.05$, and not quite so good for $\alpha = 0.30$. This is explained from the observation that the discrete logistic may be viewed as an approximation to the continuous logistic. Starting with (8), we have

$$f'(t) \approx \frac{f(t + \Delta t) - f(t)}{\Delta t} = \alpha f(t)[1 - f(t)] \quad , \tag{9}$$

which corresponds to

$$f_{n+1} - f_n = \alpha \Delta t \; f_n(1 - f_n) \quad . \tag{10}$$

Comparing this with (7) we see that the parameter a incorporates both the adoption rate α and the mesh size Δt. As $\Delta t \to 0$, $a \to 0$, so the agreement between discrete and continuous models improves for decreasing a. In any case, the agreement is quite good for small a, and we may safely replace the continuous logistic with the discrete version for the great majority of applications of the logistic substitution model.

Now we wish to see what happens as the parameter a increases into the chaotic region. For $a \leq 1$, everything behaves as expected, with the new technology increasing its market share monotonically to unity as time moves forward. However, for $a > 1$, we suddenly encounter a serious problem. Convergence to the fixed point at unity is now oscillatory, meaning that the market share sometimes exceeds one. Formally this means that the model is no longer applicable, but we continue undaunted by this, and "normalize" the equation so as to prevent the market shares from ever exceeding unity. This forces the fixed point (previously at unity) to migrate downward as the parameter a increases beyond one. The model now has the form (for details see the Appendix):

Table 1. Comparison of discrete and continuous logistic processes.

	Typical case		Extreme case	
	$\alpha = 0.05$	$a = 0.05$	$\alpha = 0.30$	$a = 0.30$
	$f(0) = 1/3$	$f_0 = 1/3$	$f(0) = 0.10$	$f_0 = 0.10$
time	continuous	discrete	continuous	discrete
0	.33333	.33333	.10000	.10000
1	.34454	.34444	.13042	.12700
2	.35591	.35573	.16837	.16026
3	.36746	.36719	.21463	.20063
4	.37915	.37881	.26949	.24875
5	.39099	.39058	.33243	.30481
6	.40296	.40248	.40198	.36838
7	.41505	.41450	.47571	.43818
8	.42723	.42664	.55052	.51204
9	.43951	.43887	.62311	.58699
10	.45186	.45118	.69057	.65972
20	.57612	.57552	.97818	.98313
30	.69144	.69163	.99889	.99951
40	.78699	.78816	.99994	.99999
50	.85898	.86083	1.00000	1.00000
100	.98670	.98762	1.00000	1.00000

$$f_{n+1} = \begin{cases} (a + 1)f_n - af_n^2 & -1 < a \leq 1 \\ (a + 1)f_n - \left(\frac{a + 1}{2}\right)^2 f_n^2 & 1 < a \leq 3 \end{cases} \quad . \tag{11}$$

For $|a| < 1$, this model is identical to the original model (7); and for $a > 1$ it has the same dynamic structure as the original model while ensuring that the market shares always remain within the unit interval.

The discrete logistic process in the form (11) has been implemented within the market penetration software developed by Nakicenovic (1979). Presented below are several plots for two hypothetical competing technologies, which were generated for selected values of the annual adoption rate parameter a. In all cases, the new technology begins with a 1% share of the market, and 200 iterations are performed (as indicated by the time span of 200 years, 1900 - 2100). To anyone who is familiar with market penetration curves, these plots will appear completely ridiculous, but they do demonstrate what the time-honored logistic process will do when sufficiently provoked by large parameter values.

Figures 5 through 7 present hypothetical market shares for $a = 1.9$, 2.0, and 2.1, respectively. In all cases, the declining technology simply mirrors the dynamics of the growing technology (since they add up to one); thus we focus attention on the latter only. For values of $a < 2$, convergence is oscillatory to a single fixed point, as shown in Figure 5. At $a = 2$, we again have convergence to a single fixed point (Figure 6), but the convergence is extremely slow since this is the threshold of bifurcation (recall that a corresponds to $b - 1$). For $a > 2$, the orbit is attracted to a period 2 cycle, as shown in Figure 7. Increasing the value of parameter a further, this period 2 cycle widens, eventually bifurcating into a period 4 cycle, as seen in Figure 8 (for $a = 2.5$). For clarity, only the curve for the new technology is shown here. At the threshold of chaos ($a_\infty = 2.57$), Figure 9 displays an interesting pattern. At first glance it looks like a period 4 cycle. Closer inspection reveals that it is more nearly a period 32 cycle, but this too is not correct if one looks very carefully. Since the value of a_∞ is not precisely 2.57, this curve could be either an aperiodic orbit or a periodic cycle with a very long period; the distinction cannot be made from the figure.

Moving into the chaotic region, for a = 2.603, we have what appears to be a noisy period 18 cycle, as shown in Figure 10. Increasing the value of a to 2.8284, we encounter the stable period 3 cycle shown in Figure 11. The graph looks like a simple two point oscillation, but close inspection reveals it to be a period 3 cycle, as emphasized by the three representative dots in the figure. Finally, in Figure 12, we see fully chaotic orbits spread over the entire phase space (a = 3.0).

The basic question which we finally need to address is: what does all this mean for the logistic substitution model? Earlier we glossed over an important fact: the discrete logistic substitution model breaks down for $a > 1$ because it permits market shares to exceed unity. As the parameter a increases, we may take this to mean either the adoption rate is growing, or the time intervals are growing. Choosing the latter interpretation, it may be expected that any discrete model for the prediction of time series data will eventually exceed its limits of applicability as the time mesh becomes increasingly coarse. In the present case, as Δt grows, there comes a point when the model predicts a market share exceeding unity at time $t + \Delta t$, based on information available at time t. Since this is unacceptable, we conclude that the model is applicable only if the time mesh is suitably restricted, which in this case means $|a| \leq 1$. As it happens, this is not a severe limitation since actual values for the annual adoption rate are typically in the range from 0.01 to 0.30.

A natural question to ask is why the discrete logistic exhibits chaos whereas the continuous logistic does not. To answer this, we must consider the mathematical relationship between discrete and continuous dynamical systems, which may be illustrated as follows. Consider a continuous curve in three-dimensional space defining the solution trajectory of a

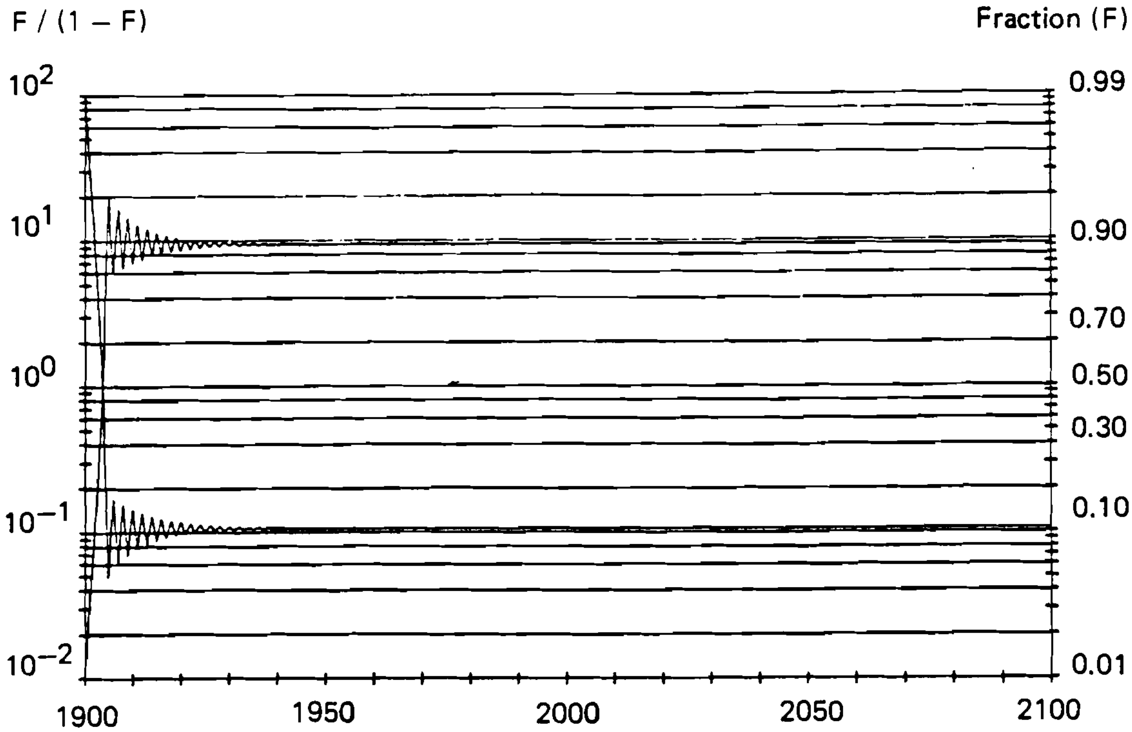


Figure 5. Hypothetical market shares using discrete logistic; a = 1.90.

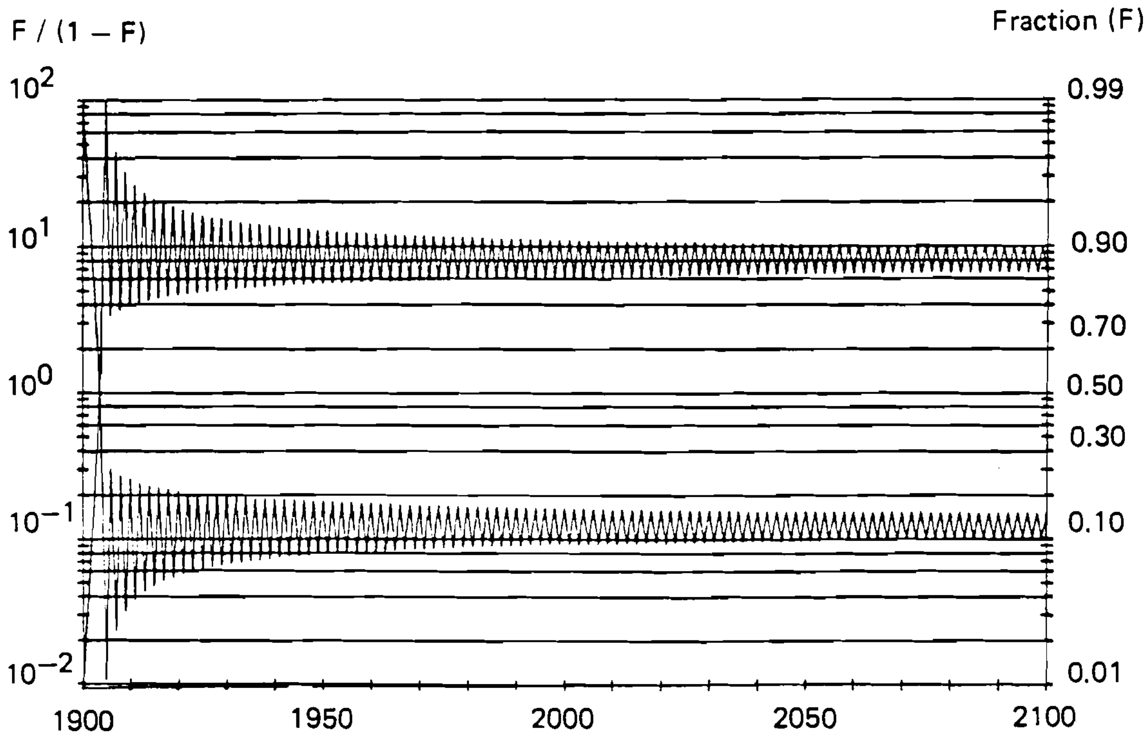


Figure 6. Hypothetical market shares using discrete logistic; a = 2.00.

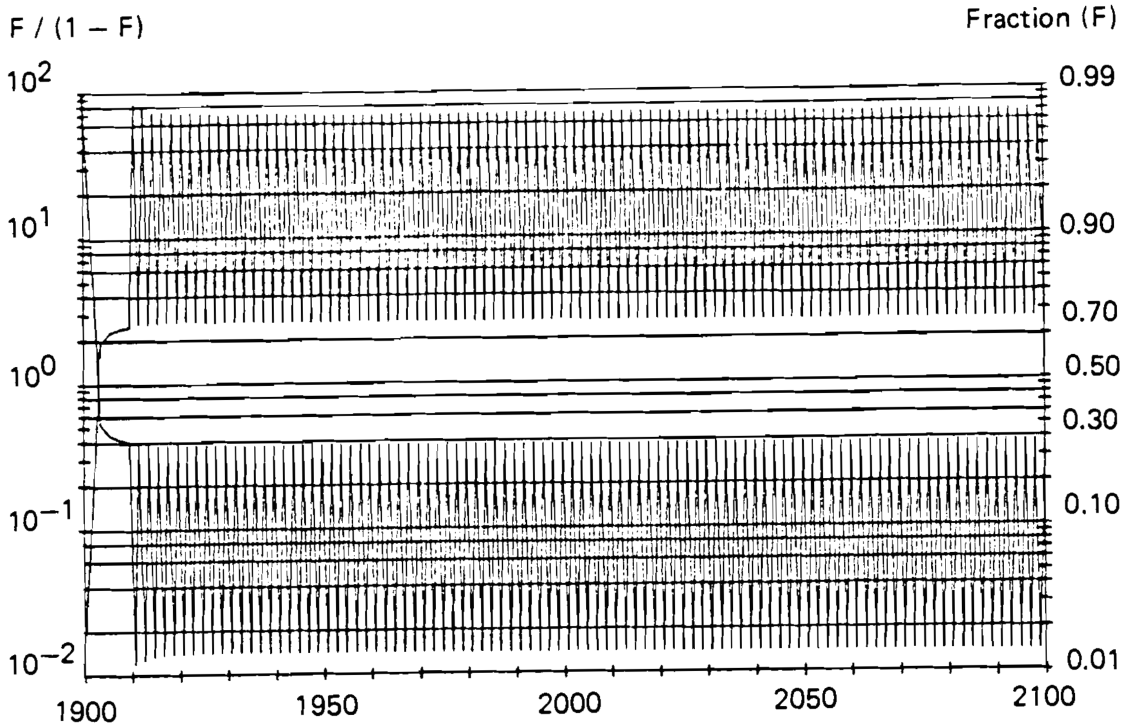


Figure 7. Hypothetical market shares using discrete logistic; a = 2.10.

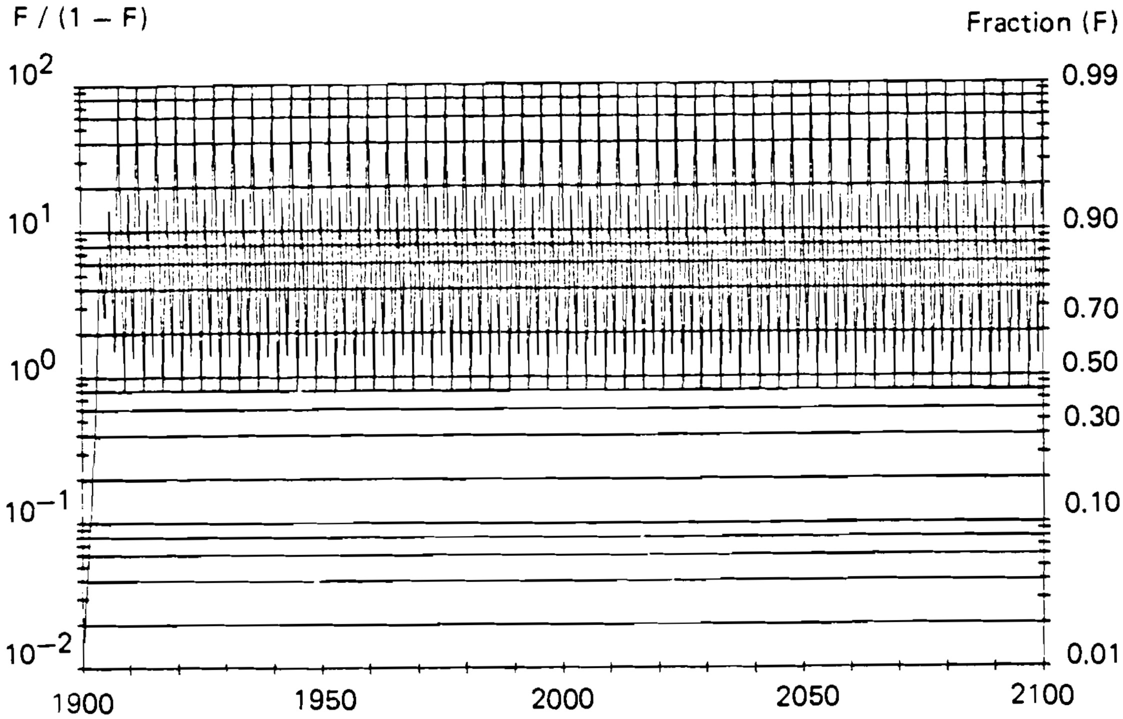


Figure 8. Hypothetical market shares using discrete logistic; a = 2.50.

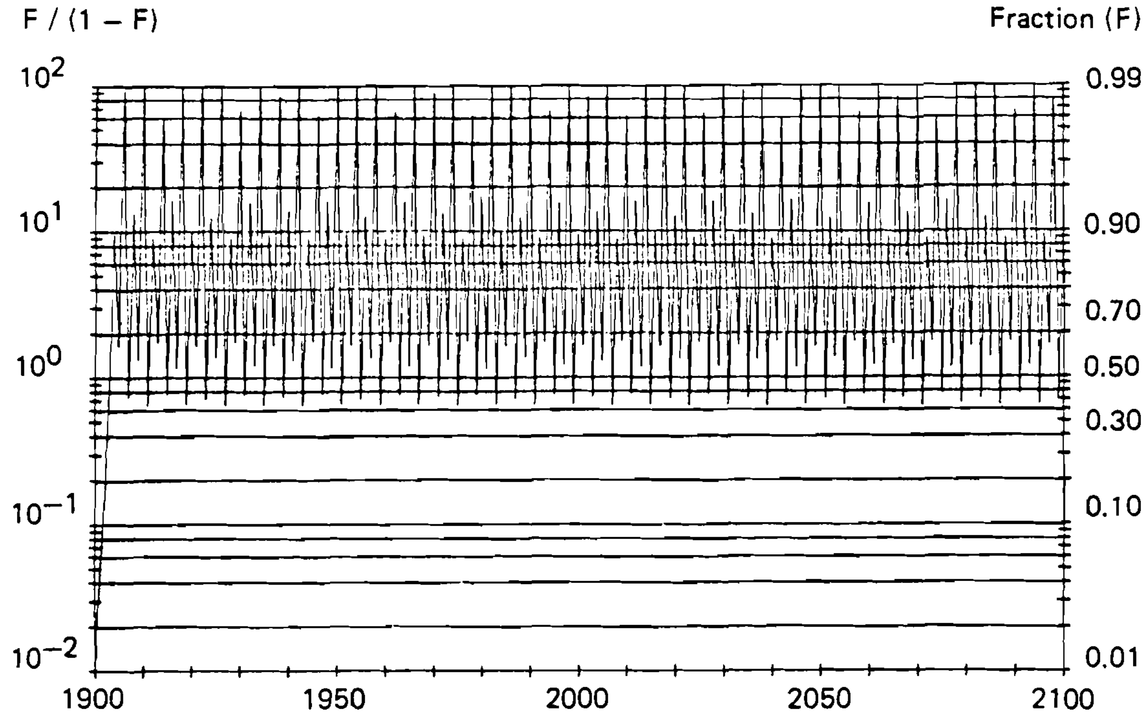


Figure 9. Hypothetical market shares using discrete logistic; $a = 2.57$.

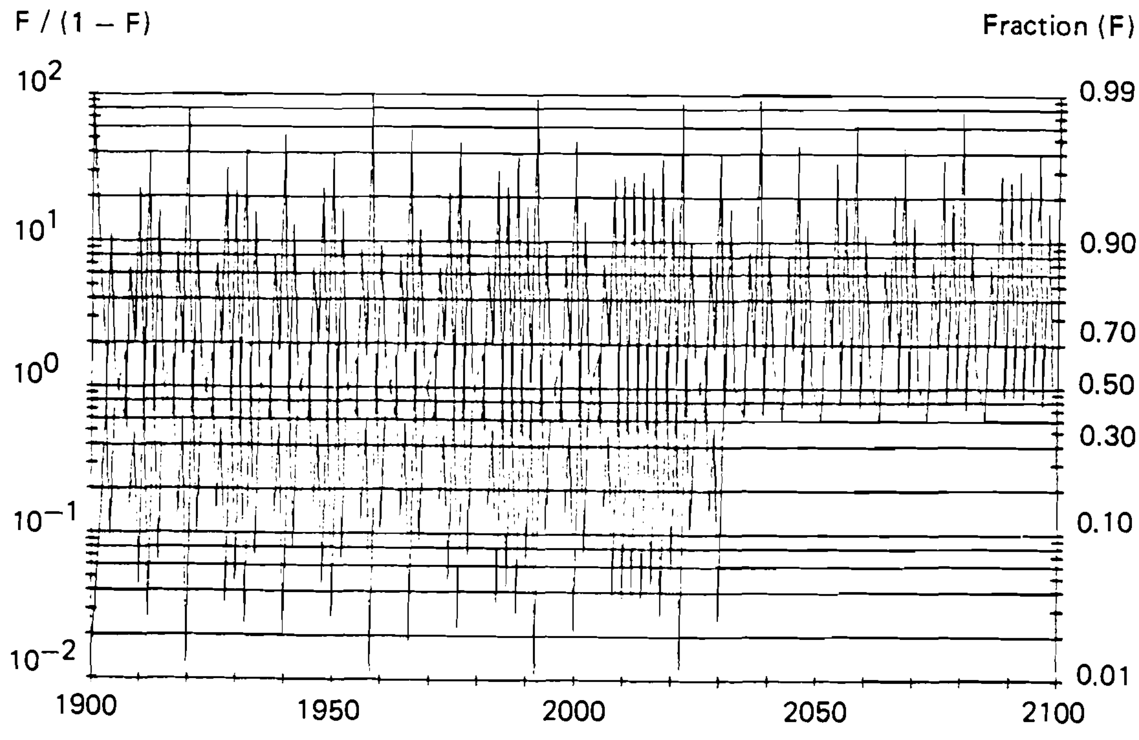


Figure 10. Hypothetical market shares using discrete logistic; $a = 2.603$.

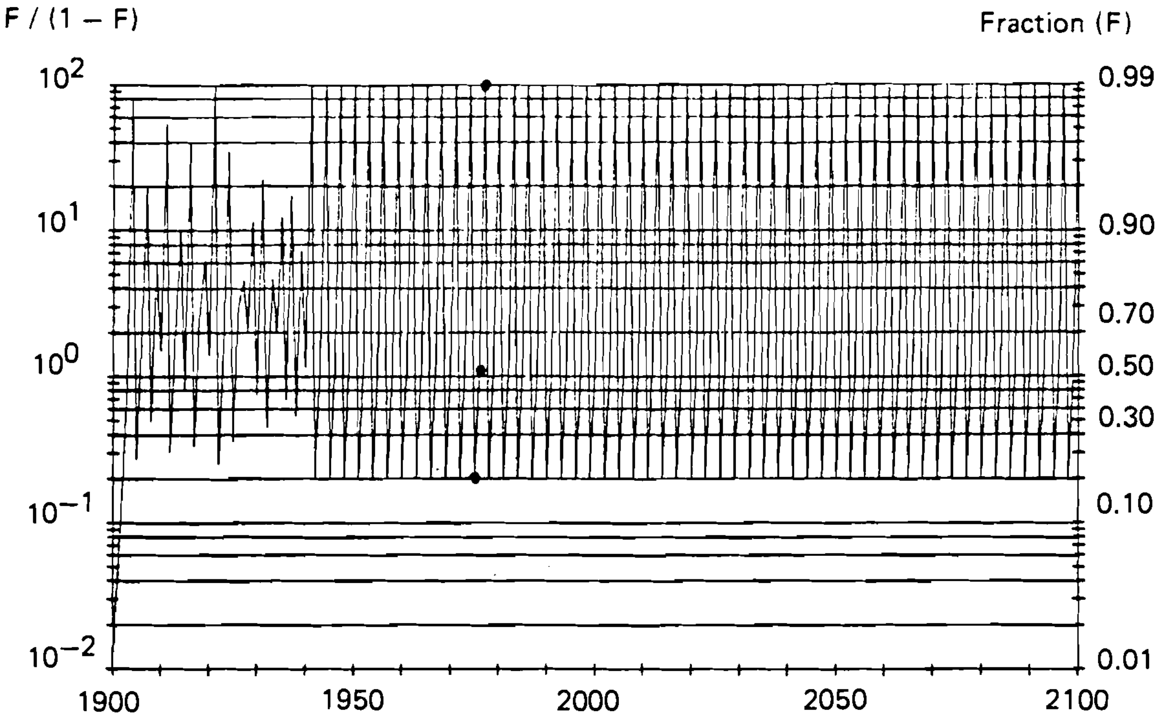


Figure 11. Hypothetical market shares using discrete logistic; a = 2.8284.

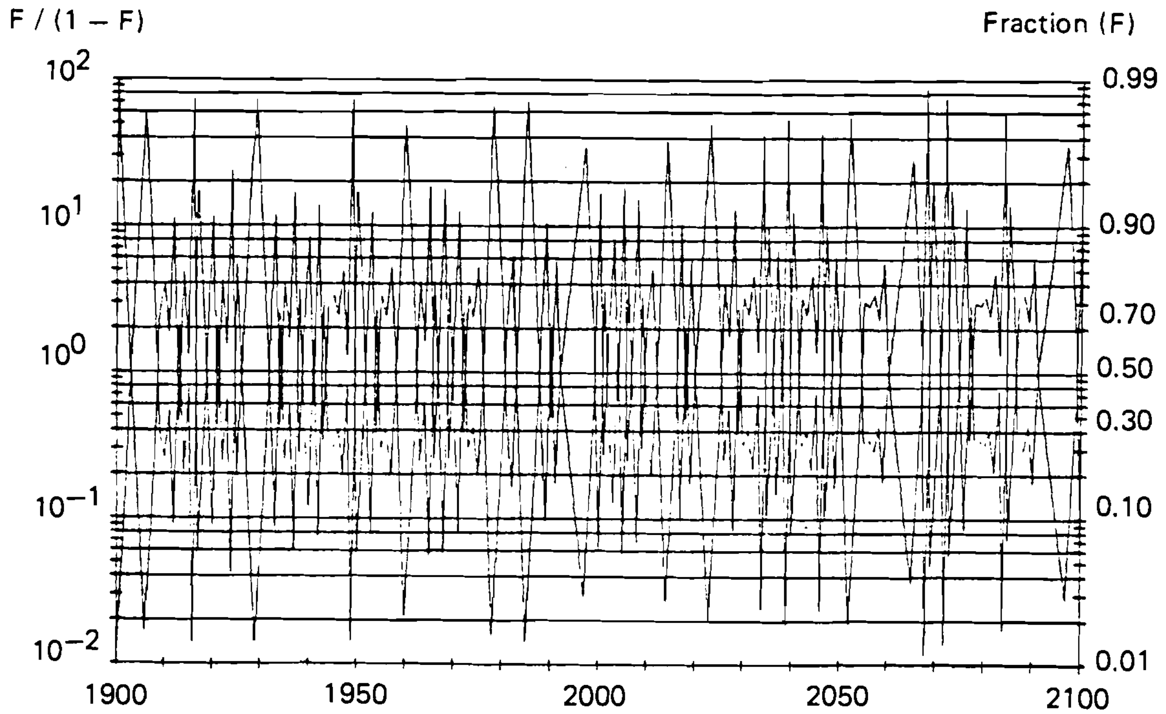


Figure 12. Hypothetical market shares using discrete logistic; a = 3.00.

differential system. Now take a surface in this space, and consider the intersections of the continuous trajectory with this surface (called the "surface of section"). These intersections are points which define the orbit of a discrete dynamical system known as a Poincaré map (Hirsch and Smale, 1974). Clearly, there are many possible Poincaré maps depending on the choice of the surface of section. In any case, such a map is a discrete representation of the original continuous system, and has similar dynamic structure. In the case at hand, the discrete logistic process is a Poincaré map not for the continuous logistic equation, but for the Volterra differential system (for two competing species) with a time delay added (Shibata and Nobuhiko, 1980). It is the time delay that gives rise to bifurcation and chaos, and this complicated behavior is preserved in the Poincaré map. Thus, although the discrete logistic equation can be easily obtained from its continuous counterpart (see Eqs. (8) through (10) above), the two equations represent very different dynamic processes, because the former incorporates a time lag whereas the latter does not (see May, 1980). This is a good illustration of the fact that discretization of continuous systems often leads to models having drastically different dynamic structures.

In conclusion, the chaotic behavior of the logistic process occurs for parameter values that are completely unrealistic for market substitution processes. In addition, the discrete logistic equation ceases to be an appropriate model for market penetration phenomena long before any unusual behavior, such as bifurcation, is encountered. Although one can certainly imagine the possibility of some type of chaos in real market systems, it is unreasonable to expect that a highly aggregated and simplified model such as the logistic equation could be directly applied in such a case. However, one should be cautious here, because the structure of logistic chaos is common to many physical systems, e.g., turbulent flow, and could find new application in a variety of fields, including economics.

REFERENCES

Collet, P., and J.P. Eckmann. 1980. Iterated Maps on the Interval as Dynamical Systems. Basel: Birkhäuser.

Eckmann, J.P. 1981. Roads to turbulence in dissipative dynamical systems. Review of Modern Physics 53(4) Part 1, October, 1981.

Feigenbaum, M.J. 1978. Quantitative universality for a class of nonlinear transformations. Journal of Statistical Physics 19(1).

Feigenbaum, M.J. 1979. The universal metric properties of nonlinear transformations. Journal of Statistical Physics 21(6).

Guckenheimer, J. 1980. One Dimensional Dynamics. Annals of the New York Academy of Sciences 357:343-347.

Energy Systems Program Group. 1981. Energy in a Finite World: A Global Systems Analysis. Report by the Energy Systems Program of the International Institute for Applied Systems Analysis, Wolf Häfele, Program Leader. Cambridge, Mass.: Ballinger Publishing Co.

Helleman, R.H.G. 1980. Self-generated chaotic behavior in nonlinear mechanics. Fundamental Problems in Statistical Mechanics 5:165-233, edited by E.G.D. Cohen. Amsterdam: North Holland.

Hirsch, M.W., and S. Smale. 1974. Differential Equations, Dynamical Systems, and Linear Algebra. New York: Academic Press.

Hofstadter, D.R. 1981. Metamagical themas. Scientific American 245(5).

Lanford, O.E. 1980. Remarks on the accumulation of period doubling bifurcations. Lecture Notes in Physics, 116. Berlin: Springer Verlag.

Lorenz, E.N. 1980. Noisy periodicity and reverse bifurcation. Annals of the New York Academy of Sciences 357:282-291.

Mandelbrot, B.B. 1980. Fractal aspects of the iteration of $z \to \lambda z(1 - z)$ for complex λ and z. Annals of the New York Academy of Sciences 357:249-259.

Marchetti, C., and N. Nakicenovic. 1978. The Dynamics of Energy Systems and the Logistic Substitution Model, Volume 1: Phenomenological Part. Report prepared for the Stiftung Volkswagenwerk. AR-78-18. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Marchetti, C., and N. Nakicenovic. 1979. The Dynamics of the Logistic Substitution Model. RR-79-13. Laxenburg, Austria: International Institute for Applied Systems Analysis.

May, R.M. 1976. Simple mathematical models with very complicated dynamics. Nature 261, June 10, 1976.

May, R.M. 1980. Nonlinear Phenomena in Ecology and Epidemiology. Annals of the New York Academy of Sciences 267:267-281.

Nakicenovic, N. 1979. Software Package for the Logistic Substitution Model. RR-79-12. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Ott, E. 1981. Strange attractors and chaotic motions of dynamical systems. Review of Modern Physics 53(4) Part 1, October 1981.

Ruelle, D. 1979. Sensitive Dependence on Initial Conditions and Turbulent Behavior of Dynamical Systems. Annals of the New York Academy of Sciences 316, 1979.

Shaw, R. 1978. Strange atractors, chaotic behavior, and information flow. Santa Cruz, Ca: University of California. Preprint.

Shibata, A., and S. Nobuhiko. 1980. Time delays and chaos in two competing species. Mathematical Biosciences 51:199-211.

Suzuki, D.T. 1968. On Indian Mahayana Buddhism. Edited by E. Conze. New York: Harper and Row.

APPENDIX

We describe here the normalization procedure followed to obtain the modified logistic model (11).

Since we are forced to sacrifice the fixed point at unity, we return to the logistic equation in the general form (6);

$$f_{n+1} = F(f_n) = b\,f_n - a\,f_n^{\,2} \quad . \tag{6}$$

A simple calculation shows that

$$\max_{0 \le f \le 1} F(f) = \frac{b^2}{4a} \quad .$$

Dividing by this quantity, a new function F(f) is obtained which never exceeds unity:

$$\begin{aligned} F(f) &= \frac{4a}{b^2}\,(bf - af^2) \quad , \\ &= 4\lambda f(1 - \lambda f) \quad , \end{aligned} \tag{12}$$

where $\lambda \equiv a/b$. A simple stability analysis shows that there is a fixed point at $(4\lambda - 1)/4\lambda^2$, which is stable for $1/4 < \lambda < 3/4$.

We define a simple linear transformation of parameter λ so that the parameter space corresponds with that used in the text. The transformation is $a \equiv 4\lambda - 1$, and the stability interval of the fixed point becomes $0 < a < 2$ as desired. In terms of parameter a, Eq. (12) becomes

$$F(f) = (a + 1)f - \left(\frac{a + 1}{2}\right)^2 f^2 \quad ,$$

which is (11) in the text (for $a > 1$). We remark that as a function of a, the composite function (11) is continuously differentiable at $a = 1$.

ACKNOWLEDGMENTS

I would like to thank Professor Wolf Häfele for his suggestion and support of this investigation. Thanks are also due to Nebojsa Nakicenovic for several interesting discussions as well as his help with the computer work. In addition, Professors Heiner Müller-Krumbhaar and Gert Eilenberger of the Kernforschungsanlage Jülich offered helpful ideas and suggestions, many of which are incorporated herein. Special thanks are due to Helene Pankl for cheerfully and carefully typing the manuscript, to Maria Bacher-Helm for help in editing, and to Helmut Frey for preparing the figures.

Finally, thanks are extended to the Birkhäuser publishers for permission to reproduce the figure appearing here as Figure 3.