SUBTRACTION OF CONVEX SETS AND ITS APPLICATION IN ε-SUBDIFFERENTIAL CALCULUS

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ABSTRACT

A new approach to the subtraction of convex sets is presented which inverts in some limited sense the vector addition of sets as defined by Minkovski. It is shown that the notion of subtraction put forward in this paper has a number of useful algebraic properties and can be used to simplify the formulation and proof of some advanced results in convex analysis.

1. Introduction

The basic space that we shall consider is the finite-dimensional euclidean vector space E. We are going to deal mainly with compact convex subsets of E with naturally defined operations of addition and multiplication by real numbers. These sets, however, do not form a linear space due to the absence of an operation analogous to subtraction. From many points of view it would be desirable to introduce such an operation as the inverse of the vector addition of sets, as this is a well-defined operation which is suitable for many applications. This would eventually amount to embedding the space of convex sets into linear space. Some work of this type has already been performed, for instance, in [1]. Here we supply the theoretical embedding procedure with more structure and describe some new applications.

Before we discuss the subtraction of convex sets we should first explain the notation used and define some related concepts.

Minkovski defined the sum of two sets $X + Y$ as follows:

$$X + Y = \{ x + y : x \in X , y \in Y \}$$

We shall denote the product of scalar α and vector x as αx. The product of scalar α and set X is written αX and is interpreted as a collection of products αx where $x \in X$.

The inner product of two vectors x and y is denoted by $x y$ and the euclidean norm of the vector x by $||x|| = \sqrt{x x}$. This notation is naturally extended for sets:

$$||X|| = \sup_{x \in X} ||x||$$

with the triangle inequality

$$||X + Y|| \leq ||X|| + ||Y||$$

obviously holding.

For some set G and vector y, the expression $G y$ denotes the inner product $g y$, where g is any vector from G.

Special notation is used for certain sets :

1. A singleton $\{ 0 \}$ is denoted by 0 .

2. A closed unit ball is denoted by

$$U = \{ x : ||x|| \leq 1 \}$$

3. A ball of radius ρ centered at x' is denoted by

$$U_\rho(x') = \{ x : ||x - x'|| \leq \rho \} = x' + \rho U$$

The Hausdorf distance between sets A and B is defined as

$$d(A,B) = \max \{ d^0(A,B) , d^0(B,A) \}$$

where

$$d^0(A,B) = \max_{a \in A} \min_{b \in B} ||a - b||$$

An alternative definition of $d^0(A,B)$:

$$d^0(A,B) = \inf \{ \varepsilon : A \subset B + \varepsilon U \}$$

makes explicit use of set addition.

The support function of convex set X is denoted by :

$$(X)_p \equiv \sup_{x \in X} p\, x$$

and it is easy to check that $(X)_p$ is a convex positively homogeneous function of p. Support functions also provide an example of positively linear functionals* defined on the space of convex sets X.

Support functions have a number of other useful properties which relate geometrical features of convex sets to analytical properties of $(X)_p$. One useful representation is:

$$||X|| = \sup_{p \in U} (X)_p$$

which suggests another definition of the Hausdorf distance:

$$d^0(A,B) = \sup_{p \in U} (A)_p - \inf_{p \in U} (B)_p$$

for $B \subset A$. This is known as the Hermander equality [2].

Other notation used in the following sections includes int(X) - the interior of set X; co(X) - the convex hull of set X; and cl(X) - the closure of set X.

* A functional $v(X)$ is positively linear if

$$v(X_1 + X_2) = v(X_1) + v(X_2)$$
$$v(\lambda X) = \lambda v(X)$$

for $\lambda \geq 0$.

We also make use of some well-known separation results for convex sets:

Theorem 1 (separation theorem). If X and Y are convex sets and $\mathrm{int}(X) \cap \mathrm{int}(Y)$ is empty, then there exists a vector p such that

$$(X)_p + (Y)_{-p} \leq 0$$

However, a stronger result is more often used in practice.

Theorem 2 (strict separation theorem). If X and Y are convex sets such that $\mathrm{cl}(X) \cap \mathrm{cl}(Y)$ is empty and at least one of these sets is bounded, then there exists a vector p such that

$$(X)_p + (Y)_{-p} < 0$$

This result is often used in the following form: for a closed convex set X to contain 0 it is necessary and sufficient that $(X)_p \geq 0$ for any p.

We shall also consider collections of sets. If we have a collection of sets Ω and a set X we may write

$$X \Subset \Omega$$

to denote that there is a set $X' \in \Omega$ such that $X \subset X'$.

Similarly

$$X \Supset \Omega$$

denotes that there is a set $X' \in \Omega$ such that $X \supset X'$.

For two collections Ω and Ω' we write

$$\Omega \Subset \Omega'$$

iff $X \in \Omega$ implies $X \Subset \Omega'$ and $X' \in \Omega'$ implies $X' \Supset \Omega$. By definition it is assumed that $\Omega = \Omega'$ if simultaneously $\Omega \Subset \Omega'$ and $\Omega' \Subset \Omega$.

To simplify the notation, the family of sets consisting of a single set will be identified with this set itself.

We shall define a collection of sets Ω to be convex if for any C_1, C_2 belonging to Ω there is a $C_3 \in \Omega$ such that

$$C_3 \subset \frac{1}{2}(C_1 + C_2)$$

To justify this definition of convexity notice that it coincides with the usual definition of a convex set when this set is considered as a collection of its elements.

2. Addition of convex sets

Here we briefly review some important properties of the addition of convex sets which will be useful in the discussion of subtraction that follows.

First, it is easy to check that the sum of two convex sets is also a convex set. Also, for nonnegative scalars α and β, we have

$$\alpha X + \beta X = (\alpha + \beta) X$$

for any convex set X. This equality may fail if any of the scalars is negative or if set X is not convex.

Another nice feature of convex sets is that if A, B, and C are bounded convex sets then $A \subset B$ implies $A + C \subset B + C$ and the converse is also true, i.e., $A + C \subset B + C$ implies $A \subset B$.

The simplest way to demonstrate this is using support functions. It is easy to see that

$$(A)_p \leq (B)_p$$

is equivalent to

$$(A+C)_p = (A)_p + (C)_p \leq (B)_p + (C)_p = (B+C)_p$$

3. Subtraction of convex sets

Convex sets have a number of other useful algebraic properties. One of the most important of these for our study is the fact that for closed convex sets it is possible to define an operation which is analogous to subtraction. The introduction of this operation would mean that the space of convex sets could be embedded into linear space.

We shall denote the difference between two convex sets A and B by $A \div B$. This difference is defined as a collection of sets constructed in the following way.

Let $\Upsilon_{A,B}$ be the family of all closed convex sets C such that

$$B \subset A + C \tag{1}$$

This family can be partially ordered by inclusion; any partially ordered subset $\bar{\Upsilon}_{A,B}$ of $\Upsilon_{A,B}$ has a lower bound C_*:

$$C_* = \cap C : B \subset A + C \, , \, C \in \bar{\Upsilon}_{A,B}$$

There is therefore at least one minimal element C^* such that there is no $C \subset C^*$ and $C \neq C^*$ for which (1) holds. We denote the collection of minimal elements by $A \div B$.

The properties of $A \div B$ are discussed in some detail in the next section; here we just point out that Figure 1 provides an example of the case when $A \div B$ contains more than one minimal element.

Here set A is taken to be a triangle $MN'N$ (where the vertex M coincides with the origin) and set B its base, the interval MN. The difference $A \div B$ is the collection of line segments connecting the vertex M with any point on the interval $M'N'$, which is parallel to the base and has the same length.

An alternative definition of the difference $A \div B$ can be based on support functions. Consider the difference of the support functions corresponding to two

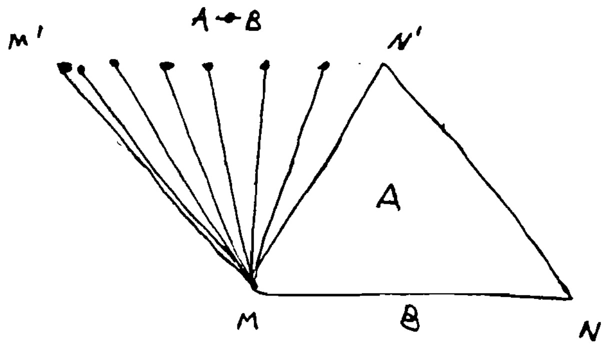

Figure 1
An example of the difference of two sets.

bounded convex sets A and B :

$$\delta_{A,B}(p) = (A)_p - (B)_p$$

The difference $\delta_{A,B}(p)$ is a continuous positively homogeneous function of p, but in the general case it is nonconvex. We shall now define a family $\Phi_{A,B}$ of convex positively homogeneous majorants of $\delta_{A,B}(p)$ with the property $\varphi(p) \in \Phi_{A,B}$ if and only if $\delta_{A,B}(p) \le \varphi(p)$, where $\varphi(p)$ is a convex positively homogeneous function of p. A similar construction was used by Pshenichniy [3] to study optimality conditions in nonconvex nondifferentiable optimization.

In the case under consideration the set of majorants is obviously not empty, as shown by the following example:

$$\overline{\varphi}(p) = (A)_p + (B)_{-p} \ge (A)_p - (B)_p$$

Subsets of $\Phi_{A,B}$ can be ordered by the usual less-or-equal relation, and in any ordered subset $\overline{\Phi}_{A,B}$ there is a least element

$$\overline{\varphi}(p) = \inf_{\varphi \in \overline{\Phi}_{A,B}} \varphi(p)$$

Again, general set theory shows that there exists at least one least element $\varphi^*(p) \in \Phi_{A,B}$ such that there is no $\varphi'(p) \in \Phi_{A,B}$ for which

$$\delta_{A,B}(p) \le \varphi'(p) \le \varphi^*(p)$$

and $\varphi'(p) \neq \varphi^*(p)$ for at least one p .

Denote the family of least elements by $\Phi^*_{A,B}$ and define the difference $A \div B$ as the collection of sets

$$A \div B \equiv \{ \partial\varphi^*(0) \, , \varphi^* \in \Phi^*_{A,B} \}$$

The equivalence of these two definitions of subtraction is obvious when it is recalled that any convex positively homogeneous closed function $f(p)$ is a support function of its subdifferential at zero:

$$f(p) = (\partial f(0))_p$$

For completeness this is demonstrated below.

Consider the conjugate of the convex positively homogeneous closed function $f(x)$:

$$f^*(p) = \sup_x \{ x p - f(x) \} \tag{2}$$

If $p \in \partial f(0)$ then

$$x p - f(x) \le f(0) = 0$$

and hence $f^*(p) = 0$. If $p \notin \partial f(0)$ then according to the strict separation theorem there must be an x^0 such that

$$x^0 p > \partial f(0) x^0 \ge f(x^0) - f(0) = f(x^0)$$

or

$$x^0 p - f(x^0) = \vartheta > 0$$

Then

$$f^*(p) \geq \sup_{\lambda \geq 0} \{ \lambda x^0 p - f(\lambda x^0) \} \geq \vartheta \sup_{\lambda \geq 0} \lambda = \infty$$

Inverting (2) leads to

$$f(x) = \sup_{p} \{ x p - f^*(p) \} = \sup_{p \in \partial f(0)} x p = (\partial f(0))_x$$

4. Some properties of subtraction

It is easy to show that the difference $A \div B$ (a collection of convex sets) is convex in terms of the definition given in Section 1. This collection also has the property described in the following theorem.

Theorem 3. For any p

$$\inf_{C \in A \div B} (C)_p = (A)_p - (B)_p$$

Proof. It is clear that at least

$$\inf_{C \in A \div B} (C)_p \equiv \alpha \geq (A)_p - (B)_p \equiv \beta$$

If this inequality is strict for some p', consider the set C' defined as follows :

$$C' = \{ c : c p' \leq \frac{\alpha + \beta}{2} \}$$

On the one hand

$$A \subset B + C'$$

because

$$(B + C')_p = (B)_p + (C')_p = (B)_p + \infty > (A)_p$$

for $p \neq p'$ and

$$(B + C')_{p'} = (B)_{p'} + (C')_{p'} = (A)_{p'} - \beta + \frac{\alpha + \beta}{2} =$$

$$= (A)_{p'} + \frac{\alpha - \beta}{2} > (A)_{p'}$$

On the other hand, there is no $C^0 \in A \dot{-} B$ and $C^0 \subset C'$. If there were such a set, we would have

$$\frac{\alpha + \beta}{2} = (C')_{p'} \geq (C^0)_{p'} \geq$$

$$\geq \inf_{C \in A \dot{-} B} (C)_{p'} = \alpha > \alpha - \frac{\alpha - \beta}{2} = \frac{\alpha + \beta}{2}$$

and the theorem is proved by contradiction.

Theorem 3 demonstrates that the infimum of the support functions of sets which belong to $A \dot{-} B$ can be written as the difference of two support functions. The interesting question is whether *any* positively homogeneous continuous function can be expressed as the difference of convex positively homogeneous continuous functions. The answer is generally no but a partial result going some way toward such a decomposition is given by Theorem 4.

Theorem 4. Let $\psi(p)$ be a positively homogeneous continuous function with Lipschitz constant Λ and let Φ be a set of convex positively homogeneous functions $\varphi^+(p), \varphi^-(p)$ such that $\varphi^+(p) - \varphi^-(p) \geq \psi(p)$. Then

$$\inf_{\varphi^+, \varphi^- \in \Phi} \sup_{p \in U} \{ \varphi^+(p) - \varphi^-(p) - \psi(p) \} = 0$$

Proof. Assume, contrary to the assertion of the theorem, that

$$\inf_{\varphi^+, \varphi^- \in \Phi} \sup_{p \in U} \{ \varphi^+(p) - \varphi^-(p) - \psi(p) \} = \varepsilon > 0$$

Then for any $\delta > 0$ there are two functions $\varphi_\delta^+(p)$, $\varphi_\delta^-(p) \in \Phi$ such that

$$\sup_{p \in U} \{ \varphi_\delta^+(p) - \varphi_\delta^-(p) - \psi(p) \} \geq \varepsilon - \delta$$

The set $P_{\varepsilon,\delta}$ of vectors $p \in U$ such that

$$\psi'(p) = \varphi_\delta^+(p) - \varphi_\delta^-(p) - \psi(p) \geq \frac{\varepsilon - \delta}{2}$$

is not empty and compact. Because $\psi'(p)$ is positively homogeneous it is sufficient to consider only vectors p for which $||p\,||= 1$.

For any $p^0 \in P_{\varepsilon,\delta}$ consider a function

$$\psi(p,p^0) = \max\{\, 4\Lambda||p - p^0(p\,p^0)\,||, \psi(p^0)p\,p^0\,\} - 4\Lambda||p - p^0(p\,p^0)\,||=$$

$$= \psi^+(p,p^0) - \psi^-(p,p^0)$$

where $\psi^+(p,p^0)$, $\psi^-(p,p^0)$ are convex positively homogeneous functions by construction. It is evident that $\psi(p,p^0)$ is also a continuous positively homogeneous function with the Lipschitz property and that

$$\psi(p^0,p^0) = \psi(p^0) > 0$$

Less obvious but equally true is the fact that

$$\psi'(p) \geq \psi(p,p^0) \tag{3}$$

for any p. The proof of (3) is based on the equality

$$||p - p^0\,||\;|p\,(p + p^0)| = ||p - p\,(p\,p^0)\,||\;||p + p^0\,|| \tag{4}$$

which can easily be checked directly. The nontrivial part of (3) concerns p such that

$$0 \leq 4\Lambda||p - p^0(p\,p^0)\,|| \leq \psi(p^0)(\,p\,p^0) \leq \psi(p^0)$$

when

$$\psi(p,p^0) = \psi(p^0)(\,p\,p^0) - 4\Lambda||p - p^0(p\,p^0)\,||$$

If it is assumed that there is a p such that

$$\psi(p) < \psi(p,p^0) = \psi(p^0)(\,p\,p^0) - 4\Lambda||p - p^0(p\,p^0)\,|| \leq$$

$$\leq \psi(p^0) - 4\Lambda||p - p^0(p\,p^0)\,||$$

then

$$\psi(p^0) - \psi(p) \geq 4\Lambda||p - p^0(p\,p^0)\,||$$

Using (4) it is possible to obtain the following chain of inequalities:

$$\psi(p^0) - \psi(p) \geq 4\Lambda ||p - p^0|| \frac{|p(p + p^0)|}{||p + p^0||} >$$

$$> 2\Lambda ||p - p^0|| \, |1 + p p^0| > 2\Lambda ||p - p^0||$$

which contradicts the Lipschitz property of $\psi(p)$ with constant Λ.

Further, an open set $\Delta(p^0)$ can be associated with every $\psi(p, p^0)$ such that

$$\psi(p, p^0) > \frac{\varepsilon - \delta}{4}$$

for $p \in \Delta(p^0)$.

These sets cover the set $P_{\varepsilon,\delta}$ and due to the compactness of $P_{\varepsilon,\delta}$ it is possible to single out a finite collection of sets $\Delta(p_i^0)$, $i = 1,2,\ldots,N$ that still covers $P_{\varepsilon,\delta}$. These sets are naturally associated with functions $\psi(p, p_i^0)$, $i = 1,2,\ldots,N$.

Consider now

$$\psi''(p) = \psi'(p) - \sum_{i=1}^{N} \psi(p, p_i^0) \leq \psi'(p)$$

$$\sup_{p \in U} \psi''(p) = \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi''(p), \sup_{p \notin P_{\varepsilon,\delta}} \psi''(p) \} \leq$$

$$\leq \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi''(p), \sup_{p \notin P_{\varepsilon,\delta}} \psi'(p) \} \leq$$

$$\leq \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi''(p), \frac{\varepsilon - \delta}{2} \} =$$

$$= \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi'(p) - \sum_{i=1}^{N} \psi(p, p_i^0), \frac{\varepsilon - \delta}{2} \} \leq$$

$$\leq \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi'(p) - \inf_{p \in P_{\varepsilon,\delta}} \sum_{i=1}^{N} \psi(p, p_i^0), \frac{\varepsilon - \delta}{2} \} \leq$$

$$\leq \max \{ \sup_{p \in P_{\varepsilon,\delta}} \psi'(p) - \frac{\varepsilon - \delta}{4}, \frac{\varepsilon - \delta}{2} \} =$$

$$= \max \{ \varepsilon - \frac{\varepsilon - \delta}{4}, \frac{\varepsilon - \delta}{2} \} < \varepsilon$$

for δ sufficiently small. This contradiction proves the theorem.

This theorem means that it is possible to approximate any Lipschitz continuous positively homogeneous function by the difference of two convex positively homogeneous continuous functions with arbitrary degree of accuracy. An example in which it is not possible to obtain an exact equality can be constructed as follows. Let $\{ \varphi_k(p) \}$ be a sequence of convex positively homogeneous functions like that used to prove Theorem 4:

$$\varphi_k(p) = \Lambda \| p - p^k (p p^k) \|$$

where $\{ p^k \}$ is a sequence of distinct unit vectors converging to some vector $\bar{p}$. Then the Lipschitz continuous positively homogeneous function

$$\psi(p) = \min_{k=1,2,\ldots} \{ \varphi_k(p) \}$$

cannot be expressed as a difference of only two convex functions. As a consequence, it cannot be expressed as a finite linear combination of convex continuous positively homogeneous functions.

In the two-dimensional case the graph of this function along the circumference of the unit circle is as shown in Figure 2.

5. Algebra of subtraction

This section describes some algebraic properties of subtraction and the relation of this operation to the Hausdorf distance between sets. It is shown that subtraction of convex sets has quite standard algebraic properties, although some of these (for instance monotonicity) are weaker than the corresponding properties for real numbers.

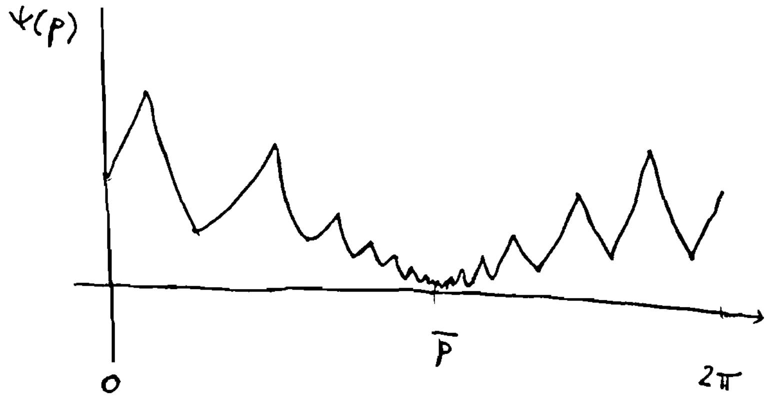

Figure 2
An example of a Lipschitz continuous positively homogeneous function which cannot be expressed as a difference of two convex functions.

Lemma 1. For convex sets A, B, C

$$A \dot{-} B = (A + C) \dot{-} (B + C)$$

Proof. This follows from the fact that for convex sets $A + C \subset B + C$ implies $A \subset B$ and vice versa.

Lemma 2 (distributive law).

$$A \dot{-} \gamma A = (1 - \gamma)A \ , 0 \leq \gamma \leq 1$$

Proof. This follows immediately from

$$(A)_p - (\gamma A)_p = (A)_p - \gamma (A)_p = (1 - \gamma)(A)_p$$

where the last term is a convex positively homogeneous function.

Lemma 3 (invertability).

$$0 = A \dot{-} B$$

if and only if $B = A$.

Proof. The equality $A \dot{-} A = 0$ follows from Lemma 2 with $\gamma = 1$. That this condition is sufficient can be proved in the following way:

$$0 = A \dot{-} B$$

immediately implies that

$$A \subset B$$

Furthermore, if there is a vector b such that $b \in B$ but $b \notin A$, then there exists a vector p such that

$$(A)_p < p\, b$$

and

$$0 > (A)_p - p\, b > (A)_p - (B)_p = \inf_{C \in A \dot{-} B} (C)_p = (0)_p = 0$$

which proves the lemma by contradiction.

Lemma 4 (monotonicity). If $B \subset A$ then $0 \in A \dot{-} B$.

Proof. Under the given conditions, for any $C \in A \dot{-} B$

$$(C)_p \geq \inf_{C \in A \dot{-} B} (C)_p = (A)_p - (B)_p \geq 0$$

for any p, and this proves the lemma.

Notice that the lemma in fact states that $0 \in C$ for any $C \in A \dot{-} B$. However, the counter-example given in Figure 3 demonstrates that the generalization of this lemma for $A \subset A'$ and differences $A \dot{-} B$ and $A' \dot{-} B$ is not correct.

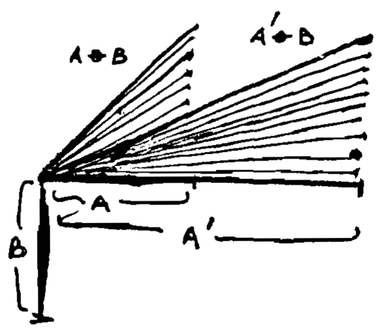

Figure 3
Example showing that Lemma 4 cannot be generalized.

It is necessary to add a few more definitions relating to subtraction. We assume that, by definition,

$$(A \dotminus B) + (C \dotminus D) = (A + C) \dotminus (B + D)$$

We define the multiplication of a difference of sets by a real number in the following manner:

$$\gamma(A \dotminus B) = \begin{cases} \gamma A \dotminus \gamma B & , 0 \le \gamma \\ |\gamma| A \dotminus |\gamma| B & , 0 > \gamma \end{cases}$$

The norm of the difference we define as

$$||A \dotminus B|| = \inf_{C \in A \dotminus B} ||C||$$

Notice that for a convex set A there is a difference between

$$||A|| = ||A \dotminus 0||$$

and the same set considered as a collection of its elements. It is easy to see that this collection can be represented as $0 \dotminus A$ and so

$$||0 \dotminus A|| = \inf_{a \in A} ||a||$$

To prove the triangle inequality for the above definitions we need one auxiliary result which we will use again further on.

Theorem 5. If $A \supset B$ then

$$\inf_{C \in A \dotminus B} \sup_{p \in U} (C)_p = \sup_{p \in U} \inf_{C \in A \dotminus B} (C)_p$$

Proof. Evidently

$$\inf_{C \in A \dotminus B} \sup_{p \in U} (C)_p = \sup_{p \in U} (C^*)_p = \vartheta^* \ge$$

$$\ge \vartheta_* = \sup_{p \in U} \inf_{C \in A \dotminus B} (C)_p = \sup_{p \in U} \{ (A)_p - (B)_p \}$$

Assume, however, that $\vartheta^* > \vartheta_*$, and consider the function

$$\varphi_\varepsilon(p) = \vartheta^*(1-\varepsilon)\,||p\,||$$

with $0 < \varepsilon < 1$. It is easy to show that in the nontrivial case $\vartheta^* > 0$ and therefore the function $\varphi_\varepsilon(p)$ is a convex positively homogeneous function of p.

Define

$$\overline{\varphi}_\varepsilon(p) = \mathrm{conv}\,\{\, \varphi_\varepsilon(p)\,,\,(C^*)_p\,\} \le (C^*)_p$$

Clearly $\overline{\varphi}_\varepsilon(p)$ is also a convex positively homogeneous continuous function. It is possible to choose $\varepsilon > 0$ small enough to guarantee that

$$(A)_p - (B)_p \le \overline{\varphi}_\varepsilon(p) = (\partial\overline{\varphi}_\varepsilon(0))_p$$

This means that there is $C^{**} \in A \div B$ such that

$$C^{**} \subset (\partial\overline{\varphi}_\varepsilon(0))_p$$

It is clear, on the other hand, that

$$C^* \supset \partial\overline{\varphi}_\varepsilon(0)$$

and so

$$\sup_{p\in U}(C^{**})_p = ||C^{**}\,|| \le (1-\varepsilon)\,||C^*|| <$$

$$< ||C^*\,|| = \inf_{C\in A\div B}\ \sup_{p\in U}(C)_p$$

and the theorem is proved by contradiction.

Another form of this theorem is the relation

$$d(A,B) = ||B \div A\,||$$

for $A \supset B$, which links the Hausdorf distance and the norm of the difference of two sets as defined in this paper.

In the general case, for two arbitrary convex sets A and B

$$d(A,B) = \max \{ \|co(A,B) \div A \|, \|co(A,B) \div B \| \}$$

Theorem 5 also provides a means of proving the triangle inequality. If $A \supset B$ and $C \supset D$, then

$$\|A \div B + C \div D \| = \|(A + C) \div (B + D) \| =$$

$$= \sup_{p \in U} \{ (A + C)_p - (B + D)_p \} = \sup_{p \in U} \{ (A)_p - (B)_p + (C)_p - (D)_p \} \leq$$

$$\leq \sup_{p \in U} \{ (A)_p - (B)_p \} + \sup_{p \in U} \{ (C)_p - (D)_p \} = \|A \div B \| + \|C \div D \|$$

6. Applications

A useful application of this notion of the difference of convex sets lies in the study of the analytical properties of ε-subdifferential mappings.

The concept of ε-subdifferential mapping proposed by Rockafellar [4] has proven to be very useful in convex nondifferentiable optimization. This mapping is defined for any convex function $f(x)$ and its value for a fixed point x is a convex set $\partial_\varepsilon f(x)$ of vectors g such that

$$\partial_\varepsilon f(x) \equiv \{ g : f(y) - f(x) \geq g(y - x) - \varepsilon \}$$

for any y where ε is a nonnegative constant.

There are a number of practical advantages in using ε-subgradients in computational methods, but the most interesting and promising feature of ε-subdifferential mapping lies in its richer analytical properties compared to the subdifferential.

One of the earlier observations [5] was that this mapping has stronger continuity properties than subdifferential mapping. Later results demonstrated that ε-subdifferential mapping also has the stronger property of Lipschitz continuity [6,7] and it was finally proved that it is even in some sense differentiable

[8], leading to the hope that second derivatives will eventually be described in a satisfactory way. Here we show again, using our definition of the subtraction of convex sets, that $\partial_\varepsilon f(x)$ is Lipschitz continuous for positive ε. The result itself is known but the proof is elegant and remarkably similar to that demonstrating the Lipschitz property of convex single-valued functions.

Theorem 6. For $\varepsilon > 0$, $\partial_\varepsilon f(x)$ is locally Lipschitz continuous in Hausdorf metric with respect to ε.

Proof. For a fixed x we can write

$$D(\varepsilon) \equiv \partial_\varepsilon f(x)\ ,\ D(0) \equiv \partial f(x)$$

Let $0 < \varepsilon < \varepsilon'$. Then

$$D(0) \subset D(\varepsilon) \subset D(\varepsilon')$$

Using convexity arguments

$$D(\varepsilon) \supset \left(1 - \frac{\varepsilon' - \varepsilon}{\varepsilon'}\right) D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(0)$$

Adding $\frac{\varepsilon' - \varepsilon}{\varepsilon'} D(\varepsilon')$ to both sides yields

$$D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(\varepsilon') \supset D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(0)$$

which can be rewritten as follows:

$$D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(0) \subseteq D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(\varepsilon') \subset$$

$$\subset D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'} D(0) + \frac{\varepsilon' - \varepsilon}{\varepsilon'} \left(D(\varepsilon') \dot{-} D(0) \right)$$

Now we can drop $\frac{\varepsilon' - \varepsilon}{\varepsilon'} D(0)$ from both sides and obtain

$$D(\varepsilon') \subseteq D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'} \left(D(\varepsilon') \dot{-} D(0) \right)$$

By definition there is a set $C \in D(\varepsilon') \dot{-} D(0)$ such that

$$C \subset (\, ||D(\varepsilon') \dot{-} D(0)\,|| + \delta\,)\, U$$

for any $\delta > 0$ and hence

$$D(\varepsilon') \subset D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'}(\, ||D(\varepsilon') \dot{-} D(0)\,|| + \delta\,)\, U$$

The norm of the difference $D(\varepsilon') \dot{-} D(0)$ is bounded from above for bounded ε' :

$$||D(\varepsilon') \dot{-} D(0)\,|| \leq L < \infty$$

As $D(\varepsilon) \subset D(\varepsilon')$ we have

$$||D(\varepsilon') \dot{-} D(\varepsilon)\,|| \leq L\, \frac{\varepsilon' - \varepsilon}{\varepsilon} + \delta$$

for arbitrary $\delta > 0$, which implies that

$$||D(\varepsilon') \dot{-} D(\varepsilon)\,|| \leq L\, \frac{\varepsilon' - \varepsilon}{\varepsilon}$$

i.e., $\partial_{\varepsilon} f(x)$ displays the Lipschitz property as a multivalued function of ε. Furthermore, for $\varepsilon > 0$ it is easy to obtain inclusions of the kind

$$\partial_{\varepsilon - C||y||} f(x) \subset \partial_{\varepsilon} f(x + y) \subset \partial_{\varepsilon + C||y||} f(x)$$

for a sufficiently large constant C and hence to transform these results into a statement of continuity with respect to space variable x.

There is another interesting property of $\partial_{\varepsilon} f$ which can be proved in a similar way.

Let

$$0 < \varepsilon < \varepsilon' < \varepsilon''$$

Again by convexity

$$D(\varepsilon') \supset (\, 1 - \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon}\,)\, D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} D(\varepsilon'')$$

By adding to both parts

$$\frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} D(\varepsilon)$$

it is possible to obtain the following inclusion:

$$D(\varepsilon) + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} D(\varepsilon'') \subset D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} D(\varepsilon) \subseteq$$

$$\subseteq D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} D(\varepsilon'') + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} \left(D(\varepsilon) \overset{*}{-} D(\varepsilon'') \right)$$

or

$$D(\varepsilon) \subseteq D(\varepsilon') + \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} \left(D(\varepsilon) \overset{*}{-} D(\varepsilon'') \right)$$

This means that

$$D(\varepsilon) \overset{*}{-} D(\varepsilon') \subseteq \frac{\varepsilon' - \varepsilon}{\varepsilon'' - \varepsilon} \left(D(\varepsilon) \overset{*}{-} D(\varepsilon'') \right)$$

or

$$\frac{D(\varepsilon) \overset{*}{-} D(\varepsilon')}{\varepsilon' - \varepsilon} \subseteq \frac{D(\varepsilon) \overset{*}{-} D(\varepsilon'')}{\varepsilon'' - \varepsilon}$$

The monotone dependence of the quotient

$$\frac{D(\varepsilon) \overset{*}{-} D(\varepsilon + \vartheta)}{\vartheta}$$

upon the $\vartheta > 0$, demonstrated above, and its boundedness, which follows from the Lipschitz continuity of $\partial_\varepsilon f(x)$ also demonstrated earlier, suggest the existence of a limit:

$$D^{\delta}(\varepsilon) \equiv \lim_{\vartheta \to +0} \frac{D(\varepsilon + \vartheta) \overset{*}{-} D(\varepsilon)}{\vartheta}$$

understood as a collection of limits of sets C_n where

$$C_n \in \frac{D(\varepsilon + \vartheta_n) \overset{*}{-} D(\varepsilon)}{\vartheta_n}$$

It would be of great interest to characterize this limit more precisely. By Theorem 3, any collection of the kind $A \dot{-} B$ is fully characterized by

$$\inf_{C \in A \dot{-} B} (C)_p = (A)_p - (B)_p$$

This leads us to consider

$$\lim_{\vartheta \to +0} \inf_{C \in \frac{D(\varepsilon+\vartheta) \dot{-} D(\varepsilon)}{\vartheta}} (C)_p = \lim_{\vartheta \to +0} \frac{(\partial_{\varepsilon+\vartheta} f(x))_p - (\partial_\varepsilon f(x))_p}{\vartheta}$$

To prove the theses put forward below, we need an alternative definition of $\partial_\varepsilon f(x)$. This can be provided by the following expression for the support function of $\partial_\varepsilon f(x)$:

$$(\partial_\varepsilon f(x))_p = f'_\varepsilon(x,p) =$$

$$= \sup_{g \in \partial_\varepsilon f(x)} g\,p = \inf_{\tau > 0} \frac{f(x + \tau p) - f(x) + \varepsilon}{\tau} = \tag{5}$$

$$= \sup_{f(x) + f^{*}(g) - x\,g \le \varepsilon} g\,p$$

where $f^{*}(g)$ denotes a conjugate of $f(x)$. Following Hiriart-Urruty [9], we denote the set of solutions of (5) as $\partial_\varepsilon f(x)_p$.

We consider here first the differentiability of a support function of an ε-subdifferential mapping with respect to ε and show that the formula for the directional derivative of a support function of an ε-subdifferential mapping is the result of what can be called a chain rule.

We shall therefore consider

$$\lim_{\vartheta \to +0} \frac{f'_{\varepsilon+\vartheta}(x,p) - f'_\varepsilon(x,p)}{\vartheta} = \psi_\varepsilon^+(x,p) \tag{6}$$

on the assumption that this limit exists. Further on we show that this is indeed the case and obtain an expression for $\psi_\varepsilon^+(x,p)$.

For symmetry we also consider

$$\lim_{\vartheta \to +0} \frac{f'_{\varepsilon}(x,p) - f'_{\varepsilon-\vartheta}(x,p)}{\vartheta} = \psi_{\varepsilon}^{-}(x,p) \tag{7}$$

and obtain an expression for $\psi_{\varepsilon}^{-}(x,p)$ as well.

To derive $\psi_{\varepsilon}^{+}(x,p)$ and $\psi_{\varepsilon}^{-}(x,p)$ from other characteristics of a given convex function $f(x)$ we need some additional results. First define a set $T_{\varepsilon}(x,p)$ such that

$$f'_{\varepsilon}(x,p) = \sup_{g \in \partial_{\varepsilon} f(x)} g\,p =$$

$$= \inf_{\tau > 0} \frac{f(x+\tau p) - f(x) + \varepsilon}{\tau} = \frac{f(x+\tau_{\varepsilon} p) - f(x) + \varepsilon}{\tau_{\varepsilon}}$$

for $\tau_{\varepsilon} \in T_{\varepsilon}(x,p)$. Notice that $T_{\varepsilon}(x,p)$ is upper semicontinuous with respect to ε and x. If the above infimum is not attainable we assume $T_{\varepsilon}(x,p) = \{ +\infty \}$ and $1/\infty = 0$. We consider the general case in which there exists at least one $\tau_{\varepsilon} < \infty$ with the other case regarded as a trivial exception.

It can easily be demonstrated that

$$\partial_{\varepsilon} f(x)_p \subset \partial f(x + \tau_{\varepsilon} p)$$

for $\tau \in T_{\varepsilon}(x,p)$. Now we are ready to have a closer look at (6) and (7).

Theorem 7. For any p

$$\psi_{\varepsilon}^{+}(x,p) = \inf_{\tau \in T_{\varepsilon}(x,p)} \frac{1}{\tau} \tag{8}$$

and

$$\psi_{\varepsilon}^{-}(x,p) = \sup_{\tau \in T_{\varepsilon}(x,p)} \frac{1}{\tau} \tag{9}$$

Proof. We prove (8) first. On the one hand

$$\frac{f(x+\tau_\varepsilon p)-f(x)+\varepsilon+\vartheta}{\tau_\varepsilon}=f'_\varepsilon(x,p)+\frac{\vartheta}{\tau_\varepsilon}\geq f'_{\varepsilon+\vartheta}(x,p)$$

Taking the upper limit as $\vartheta \to +0$ yields

$$\overline{\lim_{\vartheta\to+0}}\ \frac{f'_{\varepsilon+\vartheta}(x,p)-f'_\varepsilon(x,p)}{\vartheta}\leq\frac{1}{\tau}$$

for any $\tau \in T_\varepsilon(x,p)$. Taking the infimum of the right-hand side now yields

$$\overline{\lim_{\vartheta\to+0}}\ \frac{f'_{\varepsilon+\vartheta}(x,p)-f'_\varepsilon(x,p)}{\vartheta}\leq\inf_{\tau\in T_\varepsilon(x,p)}\frac{1}{\tau} \tag{10}$$

On the other hand

$$f'_{\varepsilon+\vartheta}(x,p)=\frac{f(x+\tau_{\varepsilon+\vartheta}p)-f(x)+\varepsilon+\vartheta}{\tau_{\varepsilon+\vartheta}}\geq$$

$$\geq f'_\varepsilon(x,p)+\frac{\vartheta}{\tau_{\varepsilon+\vartheta}}\geq f'_\varepsilon(x,p)+\vartheta\inf_{\tau\in T_{\varepsilon+\vartheta}(x,p)}\frac{1}{\tau}$$

and taking the lower limit as $\vartheta \to +0$ yields

$$\underline{\lim_{\vartheta\to+0}}\ \frac{f'_{\varepsilon+\vartheta}(x,p)-f'_\varepsilon(x,p)}{\vartheta}\geq\underline{\lim_{\vartheta\to+0}}\ \inf_{\tau\in T_{\varepsilon+\vartheta}(x,p)}\frac{1}{\tau}\geq\inf_{\tau\in T_\varepsilon(x,p)}\frac{1}{\tau} \tag{11}$$

due to the upper semicontinuity of $T_\varepsilon(x,p)$ as a multifunction of ε. Comparison of (10) and (11) proves (8).

The expression for $\psi_\varepsilon^-(x,p)$ can be obtained in a similar way. On the one hand

$$f'_\varepsilon(x,p)=\frac{f(x+\tau_\varepsilon p)-f(x)+\varepsilon}{\tau_\varepsilon}\geq f'_{\varepsilon-\vartheta}(x,p)+\frac{\vartheta}{\tau_\varepsilon}$$

and taking the supremum of the right-hand side for $\tau \in T_\varepsilon(x,p)$ together with the lower limit of the left-hand side yields

$$\underline{\lim_{\vartheta\to+0}}\ \frac{f'_\varepsilon(x,p)-f'_{\varepsilon-\vartheta}(x,p)}{\vartheta}\geq\underline{\lim_{\vartheta\to+0}}\ \sup_{\tau\in T_\varepsilon(x,p)}\frac{1}{\tau} \tag{12}$$

On the other hand

$$f'_\varepsilon(x,p) \le \frac{f(x+\tau_{\varepsilon-\vartheta}p) - f(x) + \varepsilon}{\tau_{\varepsilon-\vartheta}} =$$

$$= f'_{\varepsilon-\vartheta}(x,p) + \frac{\vartheta}{\tau_{\varepsilon-\vartheta}} \le f'_{\varepsilon-\vartheta}(x,p) + \vartheta \sup_{\tau\in T_{\varepsilon-\vartheta}(x,p)} \frac{1}{\tau}$$

Taking the upper limit as $\vartheta \to +0$ yields

$$\overline{\lim_{\vartheta\to+0}} \frac{f'_\varepsilon(x,p) - f'_{\varepsilon-\vartheta}(x,p)}{\vartheta} \le \overline{\lim_{\vartheta\to+0}} \sup_{\tau\in T_{\varepsilon-\vartheta}(x,p)} \frac{1}{\tau} \le \sup_{\tau\in T_\varepsilon(x,p)} \frac{1}{\tau} \quad (13)$$

again due to the upper semicontinuity of $T_\varepsilon(x,p)$ as a multifunction of ε. As before, comparison of (12) and (13) proves (9).

Theorem 7 provides an easier way of deriving the formula for the directional derivative of $f'_\varepsilon(x,p)$ obtained earlier by Lemarechal and Nurminski [8], and also yields an interesting interpretation of it.

Theorem 8.

$$\lim_{\tau\to+0} \frac{1}{\tau}(f'_\varepsilon(x+\tau d,p) - f'_\varepsilon(x,p)) = \quad (14)$$

$$= \inf_{\tau\in T_\varepsilon(x,p)} \frac{1}{\tau}(\sup_{g\in\partial_\varepsilon f(x)_p} g\,d - f'(x,d))$$

Proof. Let d be some direction and consider $\bar{g} \in \partial_\varepsilon f(x)_p$ such that

$$f(x+\tau_\varepsilon p) = f(x) + \tau_\varepsilon \bar{g} p - \varepsilon$$

where $\tau_\varepsilon \in T_\varepsilon(x,p)$. As $\bar{g} \in \partial f(x+\tau_\varepsilon p)$, the linear form

$$l(y) = f(x+\tau_\varepsilon p) + \bar{g}(y - x - \tau_\varepsilon p)$$

represents a supporting hyperplane for the epigraph of the function $f(x)$ at the point $x + \tau_\varepsilon p$, and also $\bar{g} \in \partial_{\varepsilon-\Delta\varepsilon} f(x+\tau d)$ where

$$\Delta\varepsilon = \tau(\bar{g}\,d - f'(x,d)) + o(\tau)$$

and the remainder $o(\tau)$ goes to zero faster than τ: $o(\tau)/\tau \to 0$ when $\tau \to +0$.

Then

$$f'_{\varepsilon-\Delta\varepsilon}(x + \tau d, p) \geq \bar{g}\, p = f'_{\varepsilon}(x, p)$$

and

$$\frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon}(x, p)\,) \geq \frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon-\Delta\varepsilon}(x + \tau d, p)\,)$$

Because this inequality holds for all $\bar{g} \in \partial_{\varepsilon} f(x)_p$ it implies that

$$\frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon}(x, p)\,) \geq \frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon-\overline{\Delta\varepsilon}}(x + \tau d, p)\,)$$

where

$$\overline{\Delta\varepsilon} = \tau(\sup_{g \in \partial_{\varepsilon} f(x)_p} g\, d - f'(x, d)\,) + o(\tau)$$

We assume that

$$\sup_{g \in \partial_{\varepsilon} f(x)_p} g\, d - f'(x, d) > 0$$

so $\overline{\Delta\varepsilon} > 0$ for small $\tau > 0$. The reverse case can be considered using (9) instead of (8).

Then

$$\frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon}(x, p)\,) \geq \frac{f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon-\overline{\Delta\varepsilon}}(x + \tau d, p)}{\overline{\Delta\varepsilon}} \frac{\overline{\Delta\varepsilon}}{\tau}$$

and

$$\underline{\lim}_{\tau \to +0} \frac{1}{\tau}(\, f'_{\varepsilon}(x + \tau d, p) - f'_{\varepsilon}(x, p)\,) \geq$$

$$\geq \overline{\lim}_{\tau \to +0} \sup_{\tau \in T_{\varepsilon}(x + \tau d, p)} \frac{1}{\tau}(\sup_{g \in \partial_{\varepsilon} f(x)_p} g\, d - f'(x, d)\,) \geq$$

$$\geq \underline{\lim}_{\tau \to +0} \inf_{\tau \in T_{\varepsilon}(x + \tau d, p)} \frac{1}{\tau}(\sup_{g \in \partial_{\varepsilon} f(x)_p} g\, d - f'(x, d)\,) \geq$$

$$\geq \inf_{\tau \in T_\varepsilon(x,p)} \frac{1}{\tau}\Big(\sup_{g \in \partial_\varepsilon f(x)_p} g\, d - f'(x,d) \Big) \tag{15}$$

due to the upper semicontinuity of $T_\varepsilon(x,p)$ as a multifunction of x.

The reverse inequality can be obtained in a similar way by considering $x + \tau d$ instead of x. Consider $\bar{g} \in \partial_\varepsilon f(x + \tau d)_p$ such that

$$f(x + \tau d + \tau_\varepsilon p) - f(x + \tau d) - \bar{g}(\tau d + \tau_\varepsilon p) + \varepsilon = 0$$

for $\tau_\varepsilon \in T_\varepsilon(x + \tau d, p)$. Then

$$\bar{g} \in \partial f(x + \tau_\varepsilon p + \tau d)$$

and $\bar{g} \in \partial_{\varepsilon + \Delta\varepsilon} f(x)$ where

$$\Delta\varepsilon = \tau\Big(\sup_{g \in \partial_\varepsilon f(x + \tau d)_p} g\, d - f'(x,d) \Big) + o(\tau) \leq$$

$$\leq \Big(\sup_{g \in \partial_\varepsilon f(x)_p} g\, d - f'(x,d) \Big) + o(\tau) = \overline{\Delta\varepsilon}$$

due to the upper semicontinuity of $\partial_\varepsilon f(x + \tau d)_p$ as a multifunction of τ. Then $f'_\varepsilon(x + \tau d, p) = \bar{g}\, p \leq f'_{\varepsilon + \Delta\varepsilon}(x,p)$ and

$$\frac{1}{\tau}\big(f'_\varepsilon(x + \tau d, p) - f'_\varepsilon(x,p) \big) \leq \frac{1}{\tau}\big(f'_{\varepsilon + \Delta\varepsilon}(x,p) - f'_\varepsilon(x,p) \big) \leq$$

$$\leq \sup_{g \in \partial_\varepsilon f(x + \tau d)} \frac{\big(f'_{\varepsilon + \Delta\varepsilon}(x,p) - f'_\varepsilon(x,p) \big)}{\tau}$$

Taking the upper limit as $\tau \to +0$ and $\Delta\varepsilon \to 0$ yields

$$\overline{\lim_{\tau \to +0}}\, \frac{1}{\tau}\big(f'_\varepsilon(x + \tau d, p) - f'_\varepsilon(x,p) \big) \leq$$

$$\leq \overline{\lim_{\tau \to +0}} \sup_{g \in \partial_\varepsilon f(x + \tau d)_p} \frac{\big(f'_{\varepsilon + \Delta\varepsilon}(x,p) - f'_\varepsilon(x,p) \big)}{\tau} \leq$$

$$\leq \overline{\lim_{\tau \to +0}} \sup_{g \in \partial_\varepsilon f(x)_p} \frac{\big(f'_{\varepsilon + \Delta\varepsilon}(x,p) - f'_\varepsilon(x,p) \big)}{\tau} \leq$$

$$\leq \overline{\lim_{\tau \to +0}} \frac{(f'_{\varepsilon+\overline{\Delta\varepsilon}}(x,p) - f'_{\varepsilon}(x,p))}{\tau} \leq$$

$$\leq \overline{\lim_{\tau \to +0}} \frac{(f'_{\varepsilon+\overline{\Delta\varepsilon}}(x,p) - f'_{\varepsilon}(x,p))}{\overline{\Delta\varepsilon}} \frac{\overline{\Delta\varepsilon}}{\tau} =$$

$$= \inf_{\tau \in T_\varepsilon(x,p)} \frac{1}{\tau} (\sup_{g \in \partial_\varepsilon f(x)_p} g\, d - f'(x,d))$$

which, combined with (15), proves the theorem.

The proof demonstrates that (14) is in fact an analogue of the chain rule of classical differential calculus. Hiriart-Urruty [9] gave another interpretation of Theorem 8 in terms of the sensitivity of problem (5) to perturbations in x.

7. Related work

The notion of subtraction has been studied by several authors in connection with differential calculus for multivalued mappings. Hukuhara [10] defined the difference of two sets in a way which corresponds to the case in which $A \dot{-} B$ is a family consisting of a single set, i.e., the least element in the proposed definition is unique. This leads to a differential calculus very similar to the usual one for single-valued mappings but the conditions are so restrictive that the concept is not widely applicable.

Early work by Radstrom [1] established an embedding relation between the space of convex sets and normed linear space. This allows the use of results related to normed linear spaces in the study of convex sets and Banks and Jacobs [11] have developed a corresponding differential calculus.

Work by Tjurin [12] should be mentioned here - he also made use of the embedding result and studied the differentiability of set-valued maps given by systems of inequalities. Further advances in this field were made by Bradley and Datko [13] who studied the differentiability properties of set-valued measures.

Demyanov and Rubinov [14] have also done work closely related to the subject of this paper. They examined the properties of a class of quasidifferentiable functions defined as follows:

A function $f(x)$ is called quasidifferentiable at point x if it is directionally differentiable and if there are two convex sets $\partial^* f(x)$ and $\partial_* f(x)$ such that

$$f'(x,d) = \max_{g \in \partial^* f(x)} g\, d - \max_{g \in \partial_* f(x)} g\, d$$

This pair ($\partial^* f(x)$, $\partial_* f(x)$) of sets is called the quasidifferential of f at x. It is easy to see that the above definition is equivalent to

$$f'(x,d) = \inf_{C \in \partial^* f(x) \overset{*}{-} \partial_* f(x)} (C)_d$$

and thus the quasidifferential of $f(x)$ can be associated with the difference $\partial^* f(x) \overset{*}{-} \partial_* f(x)$. This demonstrates another application of our definition of subtraction.

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