A FORMULA FOR THE LEVEL SETS OF EPI-LIMITS AND SOME APPLICATIONS

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We give a formula for the level sets of the limit function of a sequence of epi-convergent functions. The result is used to characterize the elements of a sequence whose epi-limit is inf-compact. Finally, we examine the implications of these results for the convergence of the infima and the solution (minimizing) sets. We restrict ourselves to the case when the functions are defined on R^n. However, the presentation is such that, either with the Mosco topology for epi-convergence in the reflexive Banach case, or with the De Giorgio topologies in the more general case, the arguments remain similar to those used here. We start with a quick review of epi-convergence which at the same time allow us to introduce some notations.

Suppose $\{S^\nu \subset R^n \ , \ \nu = 1,\dots\}$ is a sequence of sets.

Its *limits inferior* and *superior* are the sets

$$\liminf_{\nu\to\infty} S^\nu = \{x = \lim_{\nu\to\infty} x^\nu \mid x^\nu \in S^\nu \quad \text{for all } \nu = 1,\dots\}$$

and

$$\limsup_{\nu\to\infty} S^\nu = \{x = \lim_{k\to\infty} x^k \mid x^k \in S^{\nu_k},\ k=1,\ldots \text{ for some } \{\nu_k\} \subset N\}.$$

Thus, $\liminf_{\nu\to\infty} S^\nu$ is the set of limit points of all possible sequences $\{x^\nu, \nu=1,\ldots$ with $x^\nu \in S^\nu\}$ and $\limsup_{\nu\to\infty} S^\nu$ is the set of all the cluster points of such sequences. Clearly, we always have that

$$\liminf_{\nu\to\infty} S^\nu \subset \limsup_{\nu\to\infty} S^\nu \quad .$$

The sequence is said to have a *limit*, denoted by $\lim_{\nu\to\infty} S^\nu$, if the inclusion can be replaced by an equality.

Let $\{f^\nu, \nu=1,\ldots\}$ be a sequence of functions defined on R^n and with values in $\bar{R}$, the extended reals. The *epi-limits inferior* and *superior* are the functions $(li_e f^\nu)$ and $(ls_e f^\nu)$ whose epigraphs are respectively the limits superior and inferior of the sequence of sets $\{\text{epi } f^\nu, \nu=1,\ldots\}$ where epi g denotes the *epigraph* of the function g:

$$\text{epi } g = \{(x,\alpha) \mid g(x) \leq \alpha\} \quad .$$

Simply from the definition, and the above inclusion it follows that

$$li_e f^\nu \leq ls_e f^\nu \quad .$$

The sequence $\{f^\nu, \nu=1,\ldots\}$ has an *epi-limit*, denoted by $lm_e f^\nu$, if equality holds, and then

$$lm_e f^\nu = li_e f^\nu = ls_e f^\nu \quad .$$

We then also say that the sequence *epi-converges* to $\mathrm{lm}_e f^\nu$, and we write $f^\nu \to_e (\mathrm{lm}_e f^\nu)$.

Thus a function f is the epi-limit of a sequence $\{f^\nu, \nu=1,\ldots\}$ if

$$\mathrm{ls}_e f^\nu \le f \le \mathrm{li}_e f^\nu .$$

Using the definitions, it is not difficult to see that the second inequality will be satisfied, if for every $x \in R^n$

(i_e) for any subsequence of functions $\{f^{\nu_k}, k=1\ldots\}$ and any sequence $\{x^k, k=1,\ldots\}$ converging to x, we have

$$\liminf_{k\to\infty} f^{\nu_k}(x^k) \ge f(x),$$

and the first inequality, if for every $x \in R^n$

(ii_e) there exists a sequence $\{x^\nu, \nu=1,\ldots\}$ converging to x such that

$$\limsup_{\nu\to\infty} f^\nu(x^\nu) \le f(x).$$

For any decreasing sequence of subsets $\{S^\nu, \nu=1,\ldots\}$ of R^n we have that $\lim_{\nu\to\infty} S^\nu$ exists and is given by the formula

$$\lim_{\nu\to\infty} S^\nu = \bigcap_{\nu=1} \mathrm{cl}\ S^\nu$$

Similarly, if the $\{f^\nu : R^n \to \overline{R}, \nu=1,\ldots\}$ is an increasing sequence of functions, i.e., $f^\nu \le f^{\nu+1}$, then the epi-limit exists and is given by

$$\mathrm{lm}_e f^\nu(x) = \lim_{\nu\to\infty} \mathrm{cl}\ f^\nu(x)$$

where cl g is the *lower semicontinuous closure* of g, or equivalently cl g is the function such that epi cl g = cl epi g.

The next theorem gives a characterization of the level sets of the limit function in terms of the level sets of the functions f^ν. For $\alpha \in R$, the *α-level set* of a function g is the set defined by

$$\mathrm{lev}_\alpha g = \{(x,\alpha) \mid g(x) \leq \alpha\}.$$

In general, if $f = \lim_{\nu\to\infty} f^\nu$, it does not imply that $\mathrm{lev}_\alpha f = \lim_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$. Simply think of the decreasing collection of functions

$$f^\nu(x) = \nu^{-1}x^2 \qquad , \; \nu = 1,\ldots$$

that epi-converge to $f \equiv 0$. The $\mathrm{lev}_0 f^\nu = \{0\}$ for all ν, and thus $\lim_{\nu\to\infty} \mathrm{lev}_0 f^\nu = \{0\}$ but $\mathrm{lev}_0 f = R$. It is even possible for the f^ν to epi-converge to f but for some $\alpha \in R$, $\lim_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$ may not even exist which means that $\liminf_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$ is strictly included in $\limsup_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$. Again take $f^\nu(x) = \nu^{-1}x^2$ for all even ν, and $f^\nu \equiv 0$ for all odd indices ν. Then the f^ν epi-converge to $f \equiv 0$. Clearly

$$\mathrm{lev}_0 f^\nu = \{0\} \qquad \text{if } \nu \text{ is odd}$$

$$= R \qquad \text{if } \nu \text{ is even}$$

and thus $\liminf_{\nu\to\infty} \mathrm{lev}_0 f^\nu = \{0\} \neq R = \limsup_{\nu\to\infty} \mathrm{lev}_0 f^\nu$.

1. THEOREM *Suppose* $\{f^\nu = R^n \to \overline{R},\ \nu = 1,\ldots\}$ *is a sequence of functions. Then for all* $\alpha \in R$,

(2) $$\lim_{\alpha'\downarrow\alpha} \limsup_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu) \subset \mathrm{lev}_\alpha(\mathrm{li}_e f^\nu)$$

and

(3) $$\mathrm{lev}_\alpha(\mathrm{ls}_e f^\nu) \subset \lim_{\alpha'\downarrow\alpha} \liminf_{\nu\to\infty}(\mathrm{lev}_{\alpha'} f^\nu)$$

PROOF. Let $T_{\alpha'} = \limsup_{\nu\to\infty} \mathrm{lev}_{\alpha'} f^\nu$ and $T = \lim_{\alpha' \downarrow \alpha} T_\alpha$. Since the level sets (of any function) are decreasing as $\alpha' \downarrow \alpha$, it follows that the $T_{\alpha'}$ are decreasing as $\alpha' \downarrow \alpha$ and thus

$$T = \lim_{\alpha' \downarrow \alpha} T_{\alpha'} = \bigcap_{\alpha' > \alpha} T_{\alpha'},$$

the sets $T_{\alpha'}$ being closed, as follows directly from the definition of limit superior. It follows that $x \in T$ if and only if $x \in T_{\alpha'}$ for all $\alpha' > \alpha$. The inclusion (2) is trivially satisfied if T is empty. Henceforth, let us assume that T is nonempty. If $x \in T_{\alpha'}$, the definition of limit superior for sequences of sets implies that there necessarily exists a subsequence of functions $\{f^{\nu_k}, k = 1,\ldots\}$ and a sequence $\{x^k, k = 1,\ldots\}$ converging to x such that for all $k = 1,\ldots$

$$x^k \in \mathrm{lev}_{\alpha'} f^{\nu_k}$$

or equivalently such that for all $k = 1,\ldots$

$$(x^k, \alpha') \in \mathrm{epi}\ f^{\nu_k}.$$

Since $\mathrm{epi}\ (\mathrm{li}_e f^\nu) = \limsup_{\nu\to\infty} \mathrm{epi}\ f^{\nu_k}$ it follows that $(x, \alpha') = \lim_{k\to\infty} (x^{\nu_k}, \alpha') \in \mathrm{epi}\ (\mathrm{li}_e f^\nu)$ and thus $x \in \mathrm{lev}_{\alpha'} (\mathrm{li}_e f^\nu)$. Hence if $x \in T_{\alpha'}$ for all $\alpha' > \alpha$ it follows that $x \in \mathrm{lev}_{\alpha'} (\mathrm{li}_e f^\nu)$ for all $\alpha' > \alpha$ which implies that $x \in \mathrm{lev}_\alpha (\mathrm{li}_e f^\nu)$ since for any function g $\mathrm{lev}_\alpha g = \bigcap_{\alpha' > \alpha} \mathrm{lev}_{\alpha'} g$.

Let $S_{\alpha'} = \liminf_{\nu\to\infty} \mathrm{lev}_{\alpha'} f^\nu$ and $S = \lim_{\alpha' \downarrow \alpha} S_{\alpha'} = \bigcap_{\alpha' > \alpha} S_{\alpha'}$. Again the inclusion (3) is trivial if $\mathrm{lev}_\alpha (\mathrm{ls}_e f^\nu) = \emptyset$, there only remains to consider the case when $\mathrm{lev}_\alpha (\mathrm{ls}_e f^\nu)$ is nonempty. If $x \in \mathrm{lev}_\alpha (\mathrm{ls}_e f^\nu)$ it implies that there exist (x^ν, α^ν) converging to (x, α) such that

$$(x^\nu, \alpha^\nu) \in \mathrm{epi}\ f^\nu$$

since by definition $\mathrm{epi}(\mathrm{ls}_e f^\nu) = \liminf_{\nu\to\infty} \mathrm{epi}\, f^\nu$. Since $\alpha = \lim_{\nu\to\infty}\alpha^\nu$, to any $\alpha' > \alpha$ there corresponds ν' such that $\alpha^\nu \leq \alpha'$ for all $\nu \geq \nu'$. This implies that $x^\nu \in \mathrm{lev}_{\alpha'} f^\nu$ for all $\nu \geq \nu'$ and consequently $x \in S_{\alpha'}$. The above holds for every $\alpha' > \alpha$ from which it follows that $x \in S$. This yields the inclusion (3). □

4. COROLLARY. *Suppose* $\{f; f^\nu, \nu = 1,\ldots\}$ *is a collection of functions defined on* R^n, *with values in the extended reals* $\overline{R}$, *and such that* $f = \mathrm{lm}_e f^\nu$. *Then for all* $\alpha \in R$

$$\text{(5)}\qquad \mathrm{lev}_\alpha f = \lim_{\alpha'\downarrow\alpha} \limsup_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu)$$

$$= \lim_{\alpha'\downarrow\alpha} \liminf_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu) \quad .$$

PROOF. Since $f = \mathrm{lm}_e f^\nu = \mathrm{li}_e f^\nu = \mathrm{ls}_e f^\nu$, it follows from the Theorem that

$$\lim_{\alpha'\downarrow\alpha} \limsup_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu) \subset \mathrm{lev}_\alpha f \subset \lim_{\alpha'\downarrow\alpha} \liminf_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu)$$

The relations (5) now simply follow from the fact that for any α', $\liminf_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu) \subset \limsup_{\nu\to\infty} (\mathrm{lev}_{\alpha'} f^\nu)$. □

Equipped with his formulas, we now turn to the characterization of the elements of a sequence of functions $\{f^\nu, \nu = 1,\ldots\}$ whose epi-limit (exists and) is inf-compact. The first couple of propositions are proved in [1].

6. PROPOSITION. *Suppose* $\{S^\nu, \nu = 1\ldots\}$ *is a consequence of subsets of* R^n. *Then* $\limsup_{\nu\to\infty} S^\nu = \emptyset$, *or equivalently* $\lim_{\nu\to\infty} S^\nu = \emptyset$, *if and only if to any bounded set* D *there corresponds an index* ν_D *such that*

$$S^\nu \cap D = \emptyset \text{ for all } \nu \geq \nu_D \quad .$$

7. PROPOSITION. *Suppose* S *and* $\{S^\nu, \nu = 1,\ldots\}$ are *subsets of* R^n *with* S *closed. Then*

$$S \subset \liminf_{\nu\to\infty} S^\nu \text{ if and only if for all } \varepsilon > 0,\ \lim_{\nu\to\infty} S \setminus \varepsilon^\circ S^\nu = \emptyset,$$

and

$$S \supset \limsup_{\nu\to\infty} S^\nu \text{ if and only if for all } \varepsilon > 0,\ \lim_{\nu\to\infty} S^\nu \setminus \varepsilon^\circ S = \emptyset.$$

where

$\varepsilon^\circ D$ *denotes the (open)* ε-enlargement *of the set* D, *i.e.*

$$\varepsilon^\circ D = \{x \in R^n \mid \text{dist}(x,D) < \varepsilon\} \quad .$$

The next proposition improves somewhat a result of [2] concerning the convergence of connected sets.

8. PROPOSITION. *Suppose* $\{S^\nu, \nu = 1,\ldots\}$ *is a sequence of connected subsets of* R^n *such that* $\limsup_{\nu\to\infty} S^\nu$ *is bounded. Then there exists* ν' *such that for* $\nu \geq \nu'$, *the sets* S^ν *are uniformly bounded.*

PROOF. Let $S = \limsup_{\nu\to\infty} S^\nu$. For all $\varepsilon > 0$, we have that

$$S^\nu = (S^\nu \setminus \varepsilon^\circ S) \cup (S^\nu \cap \varepsilon^\circ S) \quad .$$

From Proposition 7, it follows $\lim_{\nu\to\infty}(S^\nu \setminus \varepsilon^\circ S) = \emptyset$. In view of Proposition 6, this implies that for any $\beta > \varepsilon$,

$$(S^\nu \setminus \varepsilon^\circ S) \cap \beta^\circ S = \emptyset$$

for all ν sufficiently; recall that S is bounded by assumption and thus so is $\beta^\circ S$. Hence for ν sufficiently large $S^\nu \subset \varepsilon^\circ S$ since otherwise the sets S^ν would have to be disconnected since we could write $S^\nu = (S^\nu \cap \varepsilon^\circ S) \cup (S^\nu \setminus \beta^\circ S)$ with $\beta > \varepsilon$. □

9. THEOREM. *Suppose* $\{f^\nu : R^n \to \bar{R}, \nu = 1,\ldots\}$ *is a sequence of lower semicontinuous functions with connected level sets and such that the epi-limit inferior* $\text{li}_e f^\nu$ *is inf-compact. Then the functions* f_ν *are uniformly inf-compact, in the sense that for all* α *there exists* ν_α *such that for all* $\nu \geq \nu_\alpha$, *the level sets* $\text{lev}_\alpha f^\nu$ *are uniformly compact.*

PROOF. We first note that for all $\alpha \in R$, we have

$$\limsup_{\nu\to\infty} \mathrm{lev}_\alpha \subset \lim_{\alpha'\downarrow\alpha} \limsup_{\nu\to\infty} \mathrm{lev}_{\alpha'} f^\nu \quad .$$

The inclusion is certainly true if $\limsup_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$ is empty. Otherwise $x \in \limsup_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu$ implies that there exists a subsequence $\{\nu_k, k=1,\ldots\}$ and $\{x^k, k=1,\ldots\}$ a sequence converging to x such that $x^k \in \mathrm{lev}_\alpha f^{\nu_k}$ for all $\alpha' > \alpha$. Hence $x \in \lim_{\alpha'\downarrow\alpha} \limsup_{\nu\to\infty} \mathrm{lev}_{\alpha'} f^\nu$ which completes the proof of the inclusion.

We now combine the above with (2) to obtain

$$\limsup_{\nu\to\infty} \mathrm{lev}_\alpha f^\nu \subset \mathrm{lev}_\alpha(\mathrm{li}_e f^\nu) \quad .$$

By assumption for all α, $\mathrm{lev}_\alpha(\mathrm{li}_e f^\nu)$ is compact. A straightforward application of Proposition 8 completes the proof, recalling that for all ν the $\mathrm{lev}_\alpha f^\nu$ are closed since the functions f^ν are lower semicontinuous. □

10. COROLLARY. *Suppose* $\{f^\nu : R^n \to \overline{R},\ \nu = 1,\ldots\}$ *is a sequence of lower semicontinuous functions with connected level sets, that epi-converges to* f. *Then* f *is inf-compact if and only if the* f^ν *are uniformly inf-compact.*

PROOF. If the f^ν epi-converge to f, then $\mathrm{li}_e f^\nu = f$ and thus the only if part follows from the Theorem. The if part follows from (5). The uniform inf-compactness of the f^ν implies that the $\{S_{\alpha'} = \liminf_{\nu\to\infty} \mathrm{lev}_{\alpha'} f^\nu, \alpha' > \alpha\}$ form a decreasing sequence of compact sets as $\alpha'\downarrow\alpha$ and thus $\mathrm{lev}_\alpha f = \lim_{\alpha'\downarrow\alpha} S_{\alpha'}$ is compact. □

11. COROLLARY. *Suppose* $\{f^\nu : R^n \to \overline{R}, \nu = 1,\ldots\}$ *is a sequence of lower semicontinuous convex functions that epi-converges to the (necessarily lower semicontinuous and convex) function* f. *Then* f *is inf-compact if and only if the* f^ν *are uniformly inf-compact.*

PROOF. The level sets of convex functions are convex and thus connected. □

Inf-compactness is usually used to prove the existence of a minimum. It is well-known that a number of weaker conditions can actually be used to arrive at existence. An easy generalization is *quasi-inf-compactness*. A function f is *quasi-inf-compact* if there exists $\alpha \in R$ such that $\mathrm{lev}_\alpha f$ is nonempty and for all $\beta \leq \alpha$, $\mathrm{lev}_\beta f$ is compact. The argument that shows that inf-compact functions have a minimum can also be used in the context of quasi-inf-compact functions. It is not difficult to see how Theorem 9 can be generalized to the case when $\mathrm{li}_e f^\nu$ is quasi-inf-compact. All of this, just to point out that the subsequent results about convergence of infima are not necessarily the sharpest one could possibly obtain by relying on the preceding arguments and results. Thus the next propositions are meant to be illustrative (rather than exhaustive).

12. PROPOSITION. *Suppose* $\{f^\nu : R^n \to \overline{R},\ \nu = 1,\dots\}$ *is a sequence of functions uniformly inf-compact that epi-converges to* f. *Then*

(13) $$\lim_{\nu\to\infty}(\inf f^\nu) = \inf f.$$

PROOF. The inequality

$$\limsup_{\nu\to\infty}\ (\inf f^\nu) \leq \inf f$$

is well-known as it follows directly from epi-convergence in particular condition (ii_e). To see this let us assume (without loss of generality) that $\inf f < \infty$ and that $\{x^k, k = 1,\dots\}$ is a sequence in R^n such that $\lim_{k\to\infty} f(x^k) = \inf f$. From ($\mathrm{ii}_e$) it follows that to every x^k there corresponds a sequence $\{x^{k\nu}, \nu = 1,\dots\}$ converging to x^k such that for all k

$$\limsup_{\nu\to\infty}\ f^\nu(x^{k\nu}) \leq f(x^k)$$

Since $\inf f^\nu \leq f^\nu\ (x^{k\nu})$, for all k it follows that

$$\limsup_{\nu\to\infty}\ (\inf f^\nu) \leq f(x^k)$$

Taking limits on both sides, with respect to k yields the desired relation.

There remains to show that

$$\liminf_{\nu\to\infty} (\inf f^\nu) \geq \inf f$$

There is nothing to prove if $\inf f = -\infty$, so we shall only deal with the case when $\inf f > -\infty$. We restrict our attention to the subsequence of indices for which the $\inf f^\nu$ converge to $\liminf_{\nu\to\infty} (\inf f^\nu)$, say

$$\lim_{k\to\infty} (\inf f^{\nu_k}) = \liminf_{\nu\to\infty} (\inf f^\nu) \quad .$$

Now, the f^{ν_k} are inf-compact and thus their infima are attained. Let $\{y^k, k=1,\ldots\}$ be a sequence of points such that for all k, $f^{\nu_k}(y^k) = \inf f^{\nu_k}$. The sequence $\{y^k, k=1,\ldots\}$ is bounded. To see this first observe that $\limsup_{\nu\to\infty} (\inf f^\nu) \leq \inf f$ implies that for any $\delta > 0$

$$f^{\nu_k}(y^k) = \inf f^{\nu_k} \leq \inf f + \delta$$

for k sufficiently large. Thus for those k, $y^k \in \operatorname{lev}_{\delta+\inf f} f^{\nu_k}$. The uniform inf-compactness of the f^ν implies that the compact sets $\operatorname{lev}_{\delta+\inf f} f^\nu$ are uniformly bounded. Hence the $\{y^k, k=1,\ldots\}$ admit a cluster point, say y. It now follows from epi-convergence, in particular condition (ii_e), and the above that

$$\lim_{\nu\to\infty} (\inf f^{\nu_k}) = \lim_{k\to\infty} f^{\nu_k}(y^k) \geq f(y) \geq \inf f \quad ,$$

which completes the proof. □

As corollary to this proposition, we obtain a companion to Theorem 7 of [3] and Theorem 1.7 of [4] which were derived via completely different means.

13. COROLLARY. *Suppose* $\{f^\nu : R^n \to \overline{R}, \nu = 1,\ldots\}$ *is a sequence of lower semicontinuous convex functions that epi-converge to the (necessarily lower semicontinuous and convex) function* f. *Moreover suppose that either the* $\{f^\nu, \nu = 1,\ldots\}$ *are uniformly inf-compact or* f *is inf-compact. Then*

$$\lim_{\nu\to\infty}(\inf f^\nu) = \inf f \quad .$$

PROOF. When the f^ν are convex, the inf-compactness of f yields the uniform inf-compactness of the f^ν as follows from Corollary 11. We are thus in the setting which allows us to apply the Proposition. □

The assumptions of Proposition 12 are not strong enough to allow us to conclude that the solution sets argmin f^ν converge to argmin f. Indeed consider the situation when the f^ν are defined as follows:

$$f^\nu(x) = \begin{cases} \nu^{-1}[|x| - 1] & \text{if } x \in [-1,1] \\ +\infty & \text{otherwise.} \end{cases} ,$$

The f^ν epi-converge to the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,1] \\ +\infty & \text{otherwise,} \end{cases}$$

and satisfy all the hypotheses of Proposition 12, even those of Corollary 13, and indeed the infima converge. But the solution sets, argmin $f^\nu = \{0\}$ for all ν do not converge to argmin $f = [-1,1]$. The same situation prevails even if the inf f^ν converge to inf f from above. For example, let

$$f(x) = \max\ [0\ ,\ |x| - 1]$$

and for all $x \in R$,

$$f_\nu(x) = \begin{cases} f(x) & \text{if } \nu \text{ is odd} \\ \max\ [\nu^{-1}x^2, f(x)] & \text{if } \nu \text{ is even.} \end{cases}$$

Then the f^ν epi-converge to f, the infima converge but

$$\liminf_{\nu\to\infty} \text{argmin } f^\nu = \{0\}$$

$$\limsup_{\nu\to\infty} \text{argmin } f^\nu = [-1,1] = \text{argmin } f$$

and thus the limit does not exist.

There does not appear to exist easily verifiable conditions that will guarantee the convergence of the argmin sets. We always have the following, cf. [4] for example.

14. PROPOSITION. *Suppose* $\{f^\nu : R^n \to \overline{R}, \nu = 1,\ldots\}$ *is a sequence of functions that epi-converges to* f. *Then*

(15) $$\limsup_{\nu\to\infty} \text{argmin } f^\nu \subset \text{argmin } f.$$

The preceding example has shown that in general, even in very "regular" situations, one cannot expect the inclusion

$$\text{argmin } f \subset \liminf_{\nu\to\infty} \text{argmin } f^\nu$$

to hold. The simple example that follows has all of the following properties: the functions f^ν are convex, uniformly inf-compact, inf f^ν converges to inf f from above and for all $\alpha \in R$

$$\liminf_{\nu\to\infty} \text{lev}_\alpha f^\nu = \limsup_{\nu\to\infty} \text{lev}_\alpha f^\nu \quad .$$

And nonetheless we still do not have that argmin f is the limit of the argmin f^ν. Again let $f(x) = \max\,[0,|x|-1]$ and for all ν

$$f_\nu(x) = \max\,[\nu^{-1}x^2, f(x)] \quad .$$

It thus appears that the search for characterizations of the points that minimize f, should be mostly in terms of formula (15). In particular one should seek conditions which guarantee that

$\limsup_{\nu\to\infty}$ argmin f^ν is nonempty. Sufficient conditions are provided by the assumptions of Proposition 12 (or Corollary 13) as can be gathered from its proof. Formulas (5) however suggest another direction, namely to replace argmin f^ν by ε-argmin $f^\nu =$ $\{x \in R^n | f^\nu(x) \leq \inf f^\nu + \varepsilon\}$. Indeed this allows us to obtain argmin f as an inferior limit of the ε-argmin f^ν. The proposition below is essentially proven in [5].

16. PROPOSITION. *Suppose* $\{f^\nu : R^n \to \overline{R}, \nu = 1, \ldots\}$ *is a sequence of functions that epi-converge to* f, *and* inf f *is finite. Then*

$$\lim_{\nu\to\infty}(\inf f_\nu) = \inf f$$

if and only if

$$\text{argmin } f = \lim_{\varepsilon\downarrow 0} \liminf_{\nu\to\infty} \varepsilon\text{-argmin } f_\nu ,$$

$$= \lim_{\varepsilon\downarrow 0} \limsup_{\nu\to\infty} \varepsilon\text{-argmin } f_\nu .$$

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