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Working paper

Optimization in age-structured dynamic economic models

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Optimization in age-structured dynamic economic models

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Abstract Age structured optimal control models experience increasing applications in various research fields including e.g., demography, economics, operations research, epidemiology environmental economics. In this paper we present the mathematical theory and potential applications of age-structured optimal control models. We first state the general form of the problem and present the necessary optimality conditions. To illustrate the mathematical theory we introduce a toy model on air pollution, where consumption induces pollution which in turn negatively effects utility, fertility and mortality. We solve the model analytically and present numerical simulations. The potential of age-structure to solve non-standard optimal control models is demonstrated by considering optimal control models with random switches or time-lags and delays.

1 Introduction

In optimal control models (or dynamic optimization models, in general) the dynamics describe the evolution of a system along the direction of an *independent* variable, which typically is either time t, as e.g., in macroeconomic planning problems, or age a, as in microeconomic lifecycle problems. The dynamics are modelled by a system of (first-order) ordinary differential equations (or difference equations). However, in applications that require in-depth models of the dynamics of a population, such as the modelling of social security, labour market and health policies; as well as applications relating to epidemiology, harvesting and the employment of capital vintages, age becomes a crucial variable in addition to and in distinction to time. The key equation that models the dynamics along time and age is a first-order partial differential equation, i.e.,

$$y_t(t,a) + y_a(t,a) = f(\cdot), \qquad y(0,a) = y^0(a), \quad y(t,0) = \varphi(\cdot), \tag{1}$$

known as the McKendrick-von Foerster equation (see e.g., Keyfitz and Keyfitz [45]). Here, $y(t,a)$ denotes the state variable at time t and age a. $y_t(t,a)$ and $y_a(t,a)$ denote the partial derivative of $y(t,a)$ with respect to time and age, respectively. Thus, the left hand side of (1) denotes the directional derivative of $y(t,a)$ in direction $(1,1)$ since time and age evolve at the same pace. Details of the right hand side of the equation, as well as of the initial and boundary conditions will be discussed in the next section. In addition to (1) an age-structured

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optimal control model allows the objective and salvage value functions to be age-structured, as well as *aggregated* state variables $Q(t)$ to be included. These variables aggregate/integrate (system) effects across (all) age groups at any given point in time.

The literature on age-structured optimal control theory evolved as a sequence of papers deriving a Maximum Principle (MP) for a specific problem. The first rigorous proof for a general setup with a nonlinear McKendrick-von Foerster equation was presented by Brokate [12]. Feichtinger et al. [28] generalized the Maximum principle by adding an age-dependent aggregated state variable. Veliov [67] provided a MP for general heterogeneous systems.

Applications of age-structured optimal control theory are broad and originally emerged in population dynamics and population economics, see Arthur and McNicoll [4], Chan and Guo [18, 17], Gurtin and Murphy [36, 37], Medhin [52], Feichtinger et al. [27], Prskawetz and Veliov [56], Feichtinger et al. [25], Prskawetz et al. [55], or Feichtinger and Wrzaczek [29, 30]. In parallel the theory was also applied in the mathematical literature on epidemiology (see Greenhalgh [34], Hethcote [42]) and economics (see Derzko et al. [22], Feichtinger et al. [24], Kuhn et al. [49], Augeraud-Veron et al. [5], Hartl et al. [40]) amongst other fields.

The contribution of this paper to the optimal control literature is twofold. First, we formulate the age-structured MP in a general abstract way and show how it is used in a toy model on air pollution. Within this model we also elaborate how the age-structure enters the necessary conditions (of the MP), how it changes the solution in comparison to a standard (time-dependent) optimal control model, and how it can be understood in an intuitive way. In so doing, we seek to create an understanding of the relevance of the age-time-dynamic in optimal control theory for addressing important policy questions. Second, apart from the importance of age-structure as a dynamic dimension, we demonstrate how the age-structured MP can be used to handle advanced non-standard optimal control models that, otherwise, are difficult to deal with. The transformation uses age-structure as auxiliary dimension, but substantially improves (intuitive) insights and facilitates the solution of model classes that to date are applied to a limited extent only.

The paper is organized as follows. Section 2 presents the age-structured MP, which is applied in Section 3 to a toy model on air pollution. Section 4 discusses how age-structure can be used for non-standard (advanced) optimal control models. Section 5 concludes.

2 The age-structured Maximum Principle

Let us first state the general form of an age-structured optimal control problem. In the following problem (2a) denotes the objective function, (2b) and (2c) the model dynamics, and (2d) and (2e) the initial and boundary conditions:

$$\max_{\substack{u(t,a)\in U\\ v(t)\in V}} \quad \int_0^T \int_0^\omega L\left(y(t,a), Q(t), u(t,a), v(t), t, a\right) \mathrm{d}a\, \mathrm{d}t + \int_0^\omega S\left(y(T,a), T, a\right) \mathrm{d}a \tag{2a}$$

$$\text{s.t.} \quad y_t(t,a) + y_a(t,a) = f(y(t,a), Q(t), u(t,a), v(t), t, a) \tag{2b}$$

$$Q(t) = \int_0^\omega h\left(y(t,a), Q(t), u(t,a), v(t), t, a\right) \mathrm{d}a \tag{2c}$$

$$y(0,a) = y^0(a) \tag{2d}$$

$$y(t,0) = y^b(Q(t), v(t), t). \tag{2e}$$

Here, t and a denote time and age, respectively, with time horizon T and maximal attainable age ω. $y(t,a) \in \mathbb{R}^m$ and $Q(t) \in \mathbb{R}^n$ are distributed and aggregated state variables.[1,2] The corresponding functions f and h depend on time, age, state variables, and the control variables denoted by $u(t,a) \in U \subseteq \mathbb{R}^p$ (distributed) and $v(t) \in V \subseteq \mathbb{R}^q$ (concentrated). While $y^0(a)$ denotes the exogenous[3] initial distribution of $y(t,a)$ across age at time $t = 0$, $y^b(Q(t), v(t), t)$ denotes the boundary condition which can depend on the aggregated state as well as on the concentrated control variables. In contrast to $y(t,a)$, an initial or boundary condition is not required for $Q(t)$, as it is derived from the aggregation of $h(\cdot)$ at every t. The decision maker chooses $u(t,a)$ and $v(t)$ in order to maximize the sum of the aggregated objective $L(\cdot)$ and salvage value function $S(\cdot)$ (see (2a)).

The Lexis diagram shown in Figure 1 illustrates how variables in the general model (2) relate to the time and age dimension in the model. Time and age are plotted on the horizontal and vertical axes, respectively. The characteristic lines (45-degree lines in blue) show that time and age evolve at the same pace, hence, $t - a$ denotes the time at which a specific characteristic line emerges. $y(t,a)$ and $u(t,a)$ are time and age-specific variables. They evolve along characteristic lines and emerge either at the vertical axes according to the initial condition $y^0(a)$ or at the horizontal axes according to the boundary condition $y^b(Q(t), v(t), t)$. $Q(t)$ results from an aggregation across the age domain at t and influences the dynamics (2b)-(2c), the boundary condition (2e), and the objective function (2a). Thus, the Lexis diagram highlights the asynchrony of the variables with respect to the time and age domain, which is the intuitive reason for a separate MP for these problems.

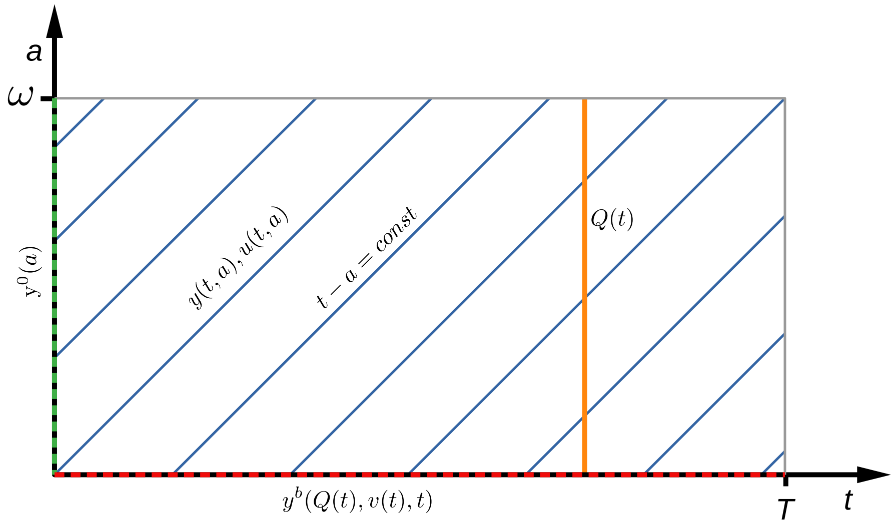

Figure 1: Lexis diagram: showing variables and conditions along time and age dimension.

[1]Note that the MP presented in Feichtinger et al. [28] also allows for distributed aggregate state variables. For an application see e.g., Almeder et al. [2].

[2]Note that the above problem (2) can easily be extended to include a concentrated state variable $x(t)$, whose dynamic is describe by and ordinary differential equation (ODE) . This is important in a number of applications such as the employment of age-structure in multi-stage optimal control models with stochastic switches and optimal control models with time-lag as discussed in Section 4. For the extended necessary conditions and a sketch of a proof we refer to Feichtinger and Wrzaczek [29].

[3]Note that the MP of Feichtinger et al. [28] also allows the control of the initial condition $y^0(a)$ by a purely age-dependent control variable. This is similar to a control of the initial condition in a standard optimal control model. Due to its infrequent usage, a control variable of this type is not presented here.

The length of the time horizon T and the maximal attainable age ω are both finite and define the intervals $D^T := [0,T]$, $D^A := [0,\omega]$, as well as the domain $D := D^T \times D^A$ within which the distributed state and control variables are defined. Note that this is in line with most theoretical and applied works based on age-structured optimal control models. In contrast to time-dependent optimal control models, the formulation of general limiting transversality conditions for the adjoint variables is difficult for age-structured optimal control models and, therefore, implies the absence of a general MP for the above problem (2) with infinite time horizon.[4]

Table 1 summarizes the variables and functions that define (2). We assume the admissible control sets to be compact and convex, and the involved functions to be twice continuously differentiable.[5]

Independent variables	time	$t \in D^T$
	age	$a \in D^A$
Control variables	distributed	$u(t,a) : D \mapsto U$
	concentrated	$v(t) : D^T \mapsto V$
State variables	distributed	$y(t,a) : D \mapsto \mathbb{R}^m$
	aggregated	$Q(t) : D^T \mapsto \mathbb{R}^n$
Functions	objective functional	$L : \mathbb{R}^m \times \mathbb{R}^n \times U \times V \times D \mapsto \mathbb{R}$
	salvage value	$S : \mathbb{R}^m \times D \mapsto \mathbb{R}$
	distributed system dynamic	$f : \mathbb{R}^m \times \mathbb{R}^n \times U \times V \times D \mapsto \mathbb{R}^m$
	aggregation	$h : \mathbb{R}^m \times \mathbb{R}^n \times U \times V \times D \mapsto \mathbb{R}^n$
Initial and boundary conditions	Initial distribution of $y(0,a)$	$y^0 : D^A \mapsto \mathbb{R}^m$
	Boundary condition of $y(t,0)$	$y^b : \mathbb{R}^n \times V \times D^T \mapsto \mathbb{R}^m$

Table 1: Variables, functions and conditions of an age-structured optimal control model (2).

The age-structured MP formulates necessary optimality conditions for problem (2).

Theorem 1 *Let $(y^*(t), Q^*(t), u^*(t,a), v^*(t))$ be an optimal solution of (2). Then there exist a unique solution $\xi(t,a)$ and $\eta(t)$ of the adjoint system*

$$\xi_t(t,a) + \xi_a(t,a) = -\mathcal{H}_y(\cdot), \qquad \xi(t,\omega) = 0, \quad \xi(T,a) = \frac{\partial S(\cdot)}{\partial y}, \quad a \in D^A, t \in D^T \tag{3a}$$

$$\eta(t) = \xi(t,0)\frac{\partial y^b(\cdot)}{\partial Q} + \int_0^\omega \frac{\partial \mathcal{H}(\cdot)}{\partial Q}\,\mathrm{d}a \tag{3b}$$

and the control variables satisfy

$$\mathcal{H}(y^*, Q^*, u^*, v^*, \xi, \eta, t, a) \geq \mathcal{H}(y^*, Q^*, u, v^*, \xi, \eta, t, a), \quad \forall u \in U \tag{4a}$$

$$\left(\xi(t,0)\frac{\partial y^b}{\partial v}(Q^*, v^*, t) + \int_0^\omega \frac{\partial \mathcal{H}}{\partial v}(y^*, Q^*, u^*, v^*, \xi, \eta, t, a)\,\mathrm{d}a\right)(v^* - v) \geq 0, \quad \forall v \in V \tag{4b}$$

[4] For a discussion on a MP for infinite time horizon for the specific case of the PDE being linear in $y(t,a)$ see e.g. Skritek and Veliov [63] or Buratto, Cesaretto and Grosset [13].

[5] Note that these assumptions are stronger than necessary. We refer to Brokate [12] or Feichtinger et al. [28] for weaker assumptions that are sufficient to prove the MP.

for a.e. $t \in D^T$ *and* $(t,a) \in D$*, where the Hamiltonian is defined by*

$$\mathcal{H}\left(y^*, Q^*, u^*, v^*, \xi, \eta, t, a\right) := L\left(y(t,a), Q(t), u(t,a), v(t), t, a\right) + \xi(t,a) f(y(t,a), Q(t), u(t,a), v(t), t, a) + \\ + \eta(t) h\left(y(t,a), Q(t), u(t,a), v(t)\right). \quad (5)$$

Here $\xi(t,a)$ and $\eta(t)$ denote the adjoint variables corresponding to the state variables $y(t,a)$ and $Q(t)$, respectively. They share the dimension and the same dependencies with their corresponding state variables.[6] A strict proof of Theorem 1 can be found in Brokate [12], Feichtinger et al. [28] (including a distributed aggregated state variable), or Veliov [67] (for the case of more general heterogeneous systems). Feichtinger and Wrzaczek [29] explicitly formulate the optimality conditions with an additional concentrated state variable (which is a special case of the previous papers) and provide a sketch of the proof. Wang [69] and Brogan [11] derive the MP by using a dynamic programming approach, which allows interpreting the adjoint variables as a shadow price.

The MP in Theorem 1 is formulated in a general way. For an interior solution of the control variables Equation (4) reduces to

$$\frac{\partial \mathcal{H}(\cdot)}{\partial u} = 0 \quad (6a)$$

$$\xi(t,0)\frac{\partial y^b(\cdot)}{\partial v} + \int_0^\omega \frac{\partial \mathcal{H}(\cdot)}{\partial v}\,\mathrm{d}a = 0. \quad (6b)$$

To obtain the maximum of the Hamiltonian (where (2) is formulated as a maximization problem) the (static) second order condition, not explicitly formulated here, has to be fulfilled as well. Note that (6a) has to hold for every (t,a), whereas (6b) corresponds to t alone, implying that the derivative of the Hamiltonian with respect to $v(t)$ is aggregated across the age domain.

Analogous to the MP for time-dependent optimal control models the age-structured MP provides a set of conditions that are necessary for optimality. General sufficiency conditions akin to the Arrow, the Mangasarian or the Leitmann-Stalford conditions (cf. Seierstad and Sydsaeter [59] and Leitmann and Stalford [50]) are not available so far and have to be derived in relation to the specific problem.

Compared to time-dependent optimal control models the numerical treatment of age-structured optimal control problems is substantially more involved. As presented above, the necessary optimality conditions consist of a set of partial differential equations (PDEs) combined with boundary conditions for state and co-state variables as well as algebraic equations for all (t,a). In general, the solution process for a set of PDEs is highly complex already. However, $t-a$ =const for the PDEs in an age-structured optimal control model allows the use of the methods of characteristics (see Zachmanoglou and Thoe [73]). This solution technique reduces each PDE to a set of ordinary differential equations (ODEs). Each ODE represents a cohort and it can be solved numerically with a wide range of established solution techniques, significantly reducing the degree of difficulty/complexity of the numerical problem.

Nevertheless, the difficulties resulting from the mixed boundary conditions (initial conditions for the state variables/end conditions for the co-state-variables) and the algebraic optimality conditions remain. We will now briefly discuss two potential iterative approaches to solve these issues: (i) shooting algorithm and (ii) gradient-based algorithms.

[6] Note that the multidimensional adjoint variables should be read as row vectors while the state and control variables are column vectors by definition.

Shooting algorithms[7] start with a guess for the initial values of the co-state variables. The state and co-state variables are then solved forward in time with the algebraic equations being solved (analytically or numerically) at each point in time to obtain values for the control variables. According to the discrepancy between the end values of the co-state variables and the target values according to the boundary conditions, the initial values for the co-state variables are adjusted iteratively until the end-conditions for the co-state variables are fulfilled (within a given margin of error).

Conversely the gradient-based algorithm[8] starts with an initial guess for the control variables over the whole domain. Using this guess the state variables are calculated iterating forward in time. Given the state profiles the end-constraints for the state variables can be evaluated and co-state dynamics are solved backwards in time (starting at $t = T$ and ending at $t = 0$). Given this solution for state, co-state and controls, the gradient of the Hamiltonian is evaluated and the controls are adjusted in the direction of the gradient to find an improvement in the objective function. These steps are iterated until no further improvement in the state-variables is found.

Each approach has its own advantages and disadvantages with respect to computation times or stability and range of convergence, but both can provide a solution of the full system.

3 Toy model on air pollution

We take inspiration from recent work on the pathways and welfare impacts of consumption-based air pollution (e.g., Zhao et al. [74], Almetwally et al. [3], Rao et al. [57], Peszko et al. [54]) for the purpose of illustrating the advantages of considering an age-structured population within an optimal control model. Specifically, we employ the model to derive the welfare maximizing allocation of consumption across a population and over time when taking into account negative impacts of consumption-driven pollution on health and productivity.

The economy consists of an age-structured population $N(t, a)$ the dynamics of which are driven by an age-specific mortality rate $\mu(\cdot)$ and fertility rate $\nu(\cdot)$ both of which depend on pollution $P(t)$. In line with the above cited literature, pollution is assumed to increase mortality for all age-groups, fertility is negatively affected by pollution (e.g., Conforti et al [20] and Jurewicz et al. [44] on bio-medical channels (fecundity) and Gao et al. [33] on socio-economic channels). The initial population structure at time $t = 0$ is exogenously given by $N_0(a)$, the number of births $B(t)$ defines the population of age $a = 0$ at every t and results from the total fertility of the population. These population dynamics are summarized in equations (7b), (7c), and (7f).

The cohort of age a at time t holds a total value of $A(t, a)$ in assets. These assets generate interest at the rate r and are adjusted at every point in time t by the difference between age-specific (per capita) earnings $w(a, P(t))$, also assumed to depend negatively on pollution (e.g., Aguilar-Gomez et al. [1], Neidell et al. [53]), and consumption $c(t, a)$. Individuals start their lives with zero assets ($A(t, 0) = 0$) and have to possess zero assets at their maximum age of survival ω.[9] For the dynamic and boundary equations for cohort assets see equations (7d) and (7e).

[7] See Bonnans [7] for an overview of shooting algorithms for optimal control problem.

[8] See Veliov [66] for the theoretical proof of convergence of the Newton's method for age-structured optimal control problems.

[9] Note that defining $A(t, a)$ as cohort assets (rather than per-capita assets) allows us to easily incorporate the redistribution of assets that are held by deceased individuals. In our toy-model the assets automatically get redistributed between the surviving individuals of the same cohort. This fact becomes obvious when examining the differential equation for per-capita assets $S(t, a) :=$

Air pollution is assumed to be a flow variable in our toy model and results from the total consumption across all cohorts (see (7g)).

The planner's objective is to maximize social welfare, which is defined by the total utility aggregated across time and cohorts. The per-capita period utility function $u(c(t,a),P(t))$ increases with per capita consumption $c(t,a)$ and decreases with the total pollution in the economy, the latter reflecting direct negative effects on physical or mental health (e.g., Almetwally et al. [3], Shi and Yu [62]). The objective function in (7a) is of the Benthamite type and counts the utility of every individual at t.[10] The model can be summarized as follows.

$$\max_{c(t,a)} \quad \int_0^T \int_0^\omega e^{-\rho t} N(t,a) u(c(t,a),P(t))\,\mathrm{d}a\,\mathrm{d}t \tag{7a}$$

$$\text{s.t.} \quad N_a(t,a) + N_t(t,a) = -\mu(a,P(t))N(t,a), \tag{7b}$$

$$N(0,a) = N_0(a), \quad N(t,0) = B(t) \tag{7c}$$

$$A_a(t,a) + A_t(t,a) = rA(t,a) + (w(a,P(t)) - c(t,a))N(t,a), \tag{7d}$$

$$A(0,a) = A_0(a), A(t,0) = 0, A(T,a) = A(t,\omega) = 0 \tag{7e}$$

$$B(t) = \int_0^\omega \nu(a,P(t))N(t,a)\,\mathrm{d}a \tag{7f}$$

$$P(t) = \int_0^\omega c(t,a)N(t,a)\,\mathrm{d}a \tag{7g}$$

3.1 Analysis and economic insight

We now demonstrate the age-structured MP by following Theorem 1. The current value Hamiltonian (Equation (5)) reads (ignoring t and a for simplicity)

$$\mathcal{H} \quad = \quad Nu(c,P) + \xi^N(-\mu(P)N) + \xi^A(rA + (w(P) - c)N) + \eta^B \nu(P)N + \eta^P cN, \tag{8}$$

where $\xi^N(t,a)$ and $\xi^A(t,a)$ denote the adjoint variables of the (distributed) population and asset states, and where $\eta^B(t)$ and $\eta^P(t)$ denote the adjoint variables of the (aggregated) births and pollution states, respectively. As an implication the necessary first order conditions for age-structured consumption follow from equations (4a) and (6a), i.e.,

$$\begin{aligned} \mathcal{H}_c \quad &= \quad Nu_c - \xi^A N + \eta^P N = 0 \\ \Longrightarrow \quad & u_c = \xi^A - \eta^P, \quad (t,a) \in D. \end{aligned} \tag{9}$$

Equations (4b) and (6b) are not used, since the toy model does not include a concentrated control variable. Equation (9) reflects the standard *marginal utility* = *marginal costs*-criterion in economics. The left hand side (lhs) equals the marginal utility of an individual (aged a at t). The right hand side (rhs) comprises the value of

$A(t,a)/N(t,a)$.

$$S_t(t,a) + S_a(t,a) = (r + \mu(a,P(t)) \cdot S(t,a) + w(a,P(t)) - c(t,a).$$

This equation shows that assets get redistributed equivalently to an annuity market that covers the mortality risk.

[10] While in the Benthamite setting, per-capita utility is scaled with the cohort size $N(t,a)$, the Millian social welfare function is based on the average utility across the whole population (see e.g., Kuhn et al. [48]) and can straightforwardly be obtained by dividing utility by the total population size $\bar{N}$ at time t.

assets, reflecting alternative future consumption possibilities, and the shadow cost of air pollution, embracing an immediate effect on instantaneous utility and an intertemporal effect on mortality and fertility. Note here that typically ξ^A is positive and η^P is negative. Here, the rhs also illustrates the interaction of the independent dimensions *age* and *time*. Whereas ξ^A (depending on age and time) depicts the intertemporal effect along the life-course of a cohort, η^P evaluates the cost of pollution across all cohorts at t and intertemporally. The social optimum, thus, includes cross-cohort pollution damages to the optimization nexus of the age-structured consumption decision. This feature cannot be obtain in a standard (i.e., concentrated parameter) optimal control model that is simplified by neglecting the age dimension.

The adjoint equations and transversality conditions are derived straightforwardly (equations (3)). We obtain

$$\xi^N_t + \xi^N_a = (\rho + \mu)\,\xi^N - u - \xi^A\,(w - c) - \eta^B\nu - \eta^P c \tag{10a}$$

$$\xi^A_t + \xi^A_a = (\rho - r)\,\xi^A \tag{10b}$$

$$\eta^B = \xi^N(t, 0) \tag{10c}$$

$$\eta^P = \int_0^\omega \left(N u_P - \xi^N \mu_P N + \xi^A w_P N + \eta^B \nu_P N\right) \mathrm{d}a, \tag{10d}$$

with

$$\xi^N(T, a) = 0, \qquad a \in D^A \tag{11a}$$

$$\xi^N(t, \omega) = 0, \qquad t \in D^T. \tag{11b}$$

Regarding the adjoint variables for the aggregated state variables (10c)-(10d) we would like to emphasize the structural difference between η^B on the one hand and η^P on the other hand, although all of these are derived from the general expression (3b). Births $B(t)$ do not enter the objective function and the system dynamics, but only the boundary condition of the population. This means that $B(t)$ is affecting $N(t, a)$ only once (i.e., at $a = 0$) and is covered by the first term of (3b). Pollution $P(t)$, on the other hand, enters the objective function and/or the system dynamics but not the boundary condition of the population. Therefore (10d) draws on the second term of (3b) covering the effect across all cohorts at t and intertemporally.

To explore the dynamics of an optimal allocation, the derivative of the control variable (starting from the first order condition), in economics referred to as *Euler equation*, can be used. In the case of an age-structured control variable, the derivative has to be taken along time and age. Using (9) and (10) we obtain the following general expression:

$$\frac{c_t + c_a}{c} = -\frac{u_c}{u_{cc} \cdot c}\left((r - \rho) + \frac{(r - \rho)\,\eta^P + \eta^P_t}{u_c} + P_t \frac{u_{cP}}{u_c}\right). \tag{12}$$

The equation determines whether it is better to postpone or advance consumption. The first term on the rhs outside the parenthesis shows the social planner's absolute risk aversion or, equivalently, the inverse of the elasticity of intertemporal substitution. A more risk-averse social planner is less responsive to changes in the economy and less willing to shift consumption over time. The first term inside the parenthesis shows the difference between the current valuation of savings by the market (r) and the social planner (ρ). If the market values savings more (or less) than the social planner, i.e., $r > \rho$ (or $r < \rho$), the social planner has an incentive to defer (or advance) consumption, implying an increase (decline) in consumption over time. The second term inside the parenthesis depicts how the social planner values the evolution of pollution. Noting that pollution typically carries a negative value, i.e., that $\eta_P < 0$, a further decrease (increase) towards a more (less) negative value, i.e., $(r-\rho)\eta^P + \eta^P_t < (>)0$, implies that the social planner chooses to advance (postpone) consumption and reduce (or increase) savings. This is to reduce future (present) pollution damage. The third term inside the parenthesis accounts for the negative impact on the utility from consumption of increasing pollution. Assuming a negative impact $u_{cP} < 0$, an increase (decrease) in the pollution flow over time implies an advancement (deferral) of consumption.

As the shadow price of pollution η_P itself is partially determined by the shadow price of the population, it is helpful to consider the analytic expression of $\xi^N(t,a)$ obtained by backward integration of (10a). Using (11b) we obtain the following expression for an individual that dies before T (i.e., $t-a \leq T-\omega$)[11]

$$\xi^N(t,a) = \int_a^\omega e^{-\int_a^s(\rho+\mu)\,\mathrm{d}s'} \Bigg(\underbrace{u}_{(i)} + \underbrace{\left(u_c + \eta^P\right)(w-c)}_{(ii)} + \underbrace{\xi^N(t,0)\nu}_{(iii)} + \underbrace{\eta^P c}_{(iv)} \Bigg) \mathrm{d}s. \tag{13}$$

The integral (13) aggregates the marginal effects on social welfare over the remaining life-span of an individual aged a at time t discounted by ρ and the conditional survival probability.[12] Therefore the closed-form (13) can also be referred to as the *expected* present value of a consumer aged a at time t akin to similar expressions derived in Kuhn et al. [48, 49].

The present value of an additional consumer aged a can be decomposed into four substantive parts (each discounted and weighed by the respective survival function). (i) denotes the increase in utility associated with this consumer. (ii) depicts the marginal effect (positive for utility, negative for pollution) of redistributing income over the consumer's remaining life-time. This 'cohort-redistribution' effect obviously cannot be obtained in a dynamic model without age-structure and includes the consumption path (along the life-cycle), as well as age-structured mortality and fertility. Term (iii) is a population dynamic effect that again can only be derived for an age-structured population. It captures the value of the expected progeny (as consumers) born to an individual born at $t-a$ over its own remaining life-cycle (including the effect of a newborn cohort on their offspring etc). This is a generalization of the demographic reproductive value and can be proven to appear in all age-structured optimal control models that model population via the McKendrick-von Foerster equation with endogenous births (see Kuhn et al. [48], Wrzaczek et al. [72] and Feichtinger et al. [26]). (iv) assigns to the individual the (negative) value of the pollution it causes over its remaining life-course. In this way the decision maker is able to internalize the cross-cohort pollution externality within and across cohorts (and over time).

3.2 Numerical solution

For the numerical solution of the toy model we use specific functional forms for the utility, the mortality rate, and the wage rate. The fertility rate is assumed not to depend on pollution and to be equal to a standard baseline rate. In the following we briefly discuss the functional choices. The specific parameters can be found in Table 2.

[11] For an individual that is alive at T (i.e., $t-a > T-\omega$) the expression is analogous, only with the upper bound of the interval now defined by the individual age at T instead of ω and the employment of (11a) instead of (11b).

[12] Note that $e^{-\int_a^s(\rho+\mu)\,\mathrm{d}s'}$ can also be written in terms of state variables $\frac{N(t-a+s,s)}{N(t,a)}$.

For the utility we use a standard constant relative risk aversion (CRRA) function, which is multiplicatively reduced according to $e^{-\kappa_1(a)P^\gamma}$. The non-negativity of the exponential function guarantees a non-negative utility. The mortality rate is the baseline mortality rate, which follows a Gompertz law with a modal age at death of 80 years and a senescence rate of 0.10 (see Horiuchi et al. [43]), augmented by the effect of pollution. For the illustration purpose of the toy model a linear form is sufficient. The wage rate is structured analogously, i.e., a baseline rate is reduced by a linear effect of pollution.

The initial population distribution is determined by stable population theory (see Coale [19]), using the population growth rate as the ratio of the logarithm of the net reproduction rate to the average childbearing age. The wage rate is modeled by a standard Mincerian equation.

Although an age-pattern of the pollution effects would be realistic and one reason for age-dependence of the consumption profile, we assume age-independence, i.e., $\frac{\partial \kappa_i(a)}{\partial a} = 0$, $i = 1, 2, 3$. Already in this simplified setup a non-trivial dynamic consumption path is optimal due to the interaction of the time and age domains within the problem. This observation would potentially be overlayed by an age-dependence.

The model is calculated for 250 years with a maximal lifetime of 100 years. The remaining model parameters and the complete set of specific functional forms are listed in Table 2.

Function	Form	Parameters	Value
Base Parameters	Discount rate	ρ	0.02
	Market interest rate	r	0.02
	Time horizon	T	250
	Maximal age	ω	100
Utility	$u(c,P) = \left(b + \frac{c^{1-\sigma}}{1-\sigma}\right) e^{-\kappa_1(a)P^\gamma}$	b	3
		σ	1.0
		γ	1.1
		$\kappa_1(a)$	$1.5 \cdot 10^{-5}$
Mortality rate	$\mu(a,P) = \tilde{\mu}(a)(1 + \kappa_2(a)P)$	$\kappa_2(a)$	$1.5 \cdot 10^{-5}$
		$\tilde{\mu}(a)$	Calibrated
Fertility rate	$\nu(a) = \tilde{\nu}(a)$	$\tilde{\nu}(a)$	Calibrated
Wage rate	$w(t,a) = \tilde{w}(t,a) \cdot (1 - \kappa_3(a)P)$	$\kappa_3(a)$	$2.0 \cdot 10^{-4}$
		g	$1.5 \cdot 10^{-3}$
		β_0	$-6.66 \cdot 10^{-1}$
	$\tilde{w}(t,a) = e^{\beta_0 + gt + \beta_1 a + \beta_2 a^2}$	β_1	$6.02 \cdot 10^{-2}$
		β_2	$-9.0 \cdot 10^{-4}$

Table 2: Summary of functions and parameters for numerical solution.

Figure 2 shows the age-distribution of the population $N(t,a)$. The red line refers to the density of the cohort born at $t = 50$. The shape naturally follows the survival profile of an individual, which in our model corresponds to the base mortality rate augmented by negative pollution effects. The blue dashed line shows the population density across ages at the time of birth of the cohort shown with the red line, i.e., $t = 50$. As we assume a stable increasing population, the dashed blue line lies below the red one. The figure illustrates the possibility of modeling state (and control) variables in the two independent directions time and age, along which they develop differently. Whereas the red line corresponds to the 45-degree line in the Lexis diagram (Figure 1), the blue dashed line resembles a vertical line in the Lexis diagram. Both of them start at the same time and age specific value $(t, a) = (50, 0)$.

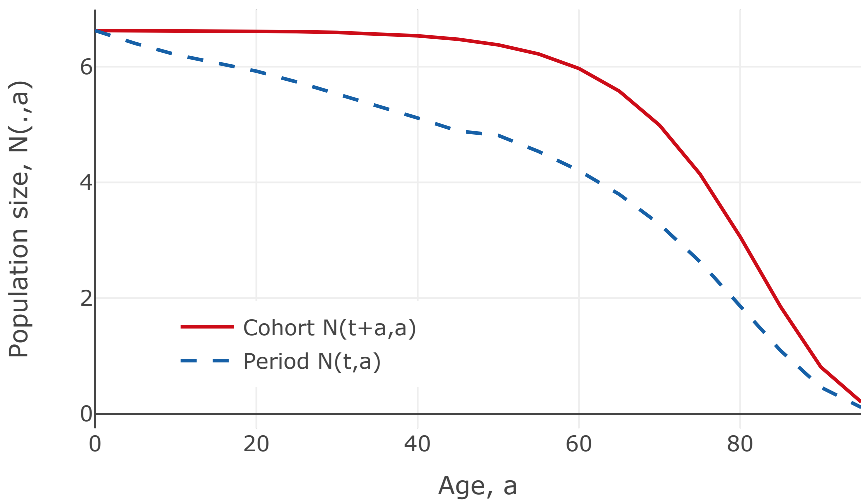

Figure 2: Population along a cohort (red line) and across cohorts for a fixed t (blue dashed line).

Figure 3 plots the optimal consumption profile along the same dimensions as in Figure 2. I.e., along the lifetime of one cohort (red line) and across cohorts for fixed t (blue dashed line).

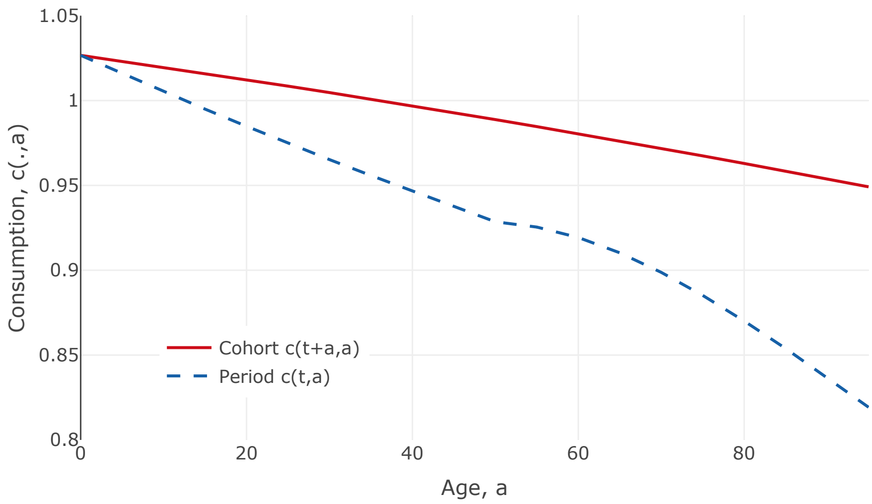

Figure 3: Optimal consumption along a cohort (red line) and across cohorts for a fixed t (blue dashed line).

To discuss the shape of the red line we refer back to the Euler equation (12). As the market interest rate and the social planer time discount rate coincide ($r = \rho$), Equation (12) simplifies to

$$\frac{c_t + c_a}{c} = \frac{1}{\sigma} \cdot \left(\frac{\eta_t^P}{u_c} - \kappa_1(a) \cdot \gamma \cdot P^{\gamma-1} \cdot P_t \right). \tag{14}$$

Equation (14) shows, that the consumption profile would be flat, if pollution had none of the three externality effects, i.e., if it would not impact on utility, wage and vitality rates of individuals.[13] This allows us to directly identify the impact of pollution on the timing of consumption. As pollution increases over time (see Figure 5), the negative effects through all channels become stronger, which is reflected by a decrease in the corresponding negative shadow price η^P. Moreover, the marginal utility w.r.t. consumption decreases in pollution, as implied by the specific functional form shown in the second term in (14). Both effects give an incentive to shift consumption to younger ages and to decrease it continuously over the life-cycle. For the consumption profile across cohorts the explanation is similar, but the profile here combines the consumption values of different cohorts. Earlier cohorts face a lower productivity (lower wage profile) and a higher shadow price $\xi^N(t,a)$, which enters in the definition of η^P and is based on a longer remaining time horizon. Therefore, the consumption profile along the time dimension decreases stronger as compared to the consumption profile along the life-cycle.

As the shadow price of the population plays a decisive role for the development of η^P and consequently the optimal consumption allocation, Figure 4 provides more insights into the development of $\xi^N(t,a)$ across time and age. The shadow price is illustrated in two different ways. The left panel shows several cross sectional age schedules (i.e., vertical lines in the Lexis diagram, Figure 1) of the shadow price for different time periods t. These schedules correspond to cross sectional slices in the figure of the three dimensional shadow price plotted in the right panel. In the following we offer a more detailed intuition about the hump shaped pattern of the cross sectional age distribution and the decrease in the level of the shadow price.

First, to understand the shape we use the explicit solution (13), which is an aggregation of immediate and indirect effects. Since the remaining lifetime of a cohort becomes shorter as time evolves the overall shape is decreasing until it reaches zero at the maximal age ω. The initial increase until age ≈ 25 is due to the inclusion of the effect of expected progenities of the cohort as already discussed above. Second, the left panel shows a noticeable decrease over time, which is again a result of the moving time. Shadow prices always include the aggregation of future effects of a state variable. Therefore, a shorter remaining time horizon implies less effects and an overall decrease in the toy model. Finally, note that the value at the end of the time horizon (i.e., at the maximal age ω and at the end of the planning period T) equals zero, which is due to the absence of a salvage value function. Important to notice in this respect is an anticipative behavior of the control variables implied by the overall decrease of the shadow prices. For a fairly long time horizon that means that the system stabilizes after some transitional initial period before the nearing end of the time horizon leads to a deviation which, in some cases, can be quite counter intuitive. To avoid those effects and caveats, solutions are often plotted on a truncated time horizon.

Note that the right hand side of Figure 4 also demonstrates that all control, state and corresponding adjoint variables are derived in the full time-age-spectrum, as a PDE. Paths along a cohort or cross cohort (as in Figures 2 and 3 or in the left panel of Figure 4) are only slices of the variable in the full time-age-domain, but often more suitable to highlight specific effects.

In the discussion of the consumption profile we already mentioned that pollution increases over time. Figures 5 and 6 presents a sensitivity analysis of the pollution effect and compares a high, low and zero pollution

[13] No impact of pollution would imply $\kappa(a) = 0$, eliminating the second term in (14) as well as $\mu_P = w_P = \nu_p = 0$. Following Equation (10d) this directly leads to $\eta^P = 0$ and consequently $\eta^P_t = 0$.

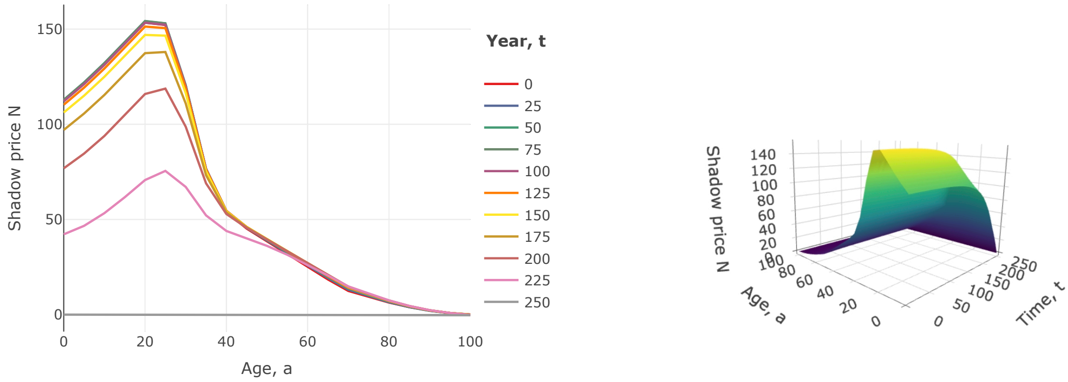

Figure 4: Adjoint variable of population $\xi^N(t, a)$. The left panel shows a cross cohort spectrum (i.e., $\xi^N(t, \cdot)$) for specific t. The right panel shows the full trajectory in the full time-age-domain D.

impact. The left panel of Figure 5 shows the pollution over time for the three cases. The gradual increase of pollution is due to the increasing population (i.e., more people with the same wage profile consume more products) and productivity (i.e., higher income). The decrease at the end of the time horizon is due to the end condition for assets $A(T, a) = 0$. This implies that young individuals earning a small income cannot smooth the consumption (see discussion of Figure 3) since their most productive age is only reached beyond the end of the planning period. Thus, these cohorts can only consume less, which also means fewer air pollution. A higher pollution effect clearly implies a higher total pollution by effecting the mortality and fertility rates, as well as the wage. Comparing the lines shows that shifting the consumption optimally according to the damages from air pollution decreases the total pollution especially shortly before T. The corresponding income effect is shown in the right panel, again plotted over the life-cycle of a specific cohort and accross ages. The blue lines (including cohorts born earlier) are below the red ones because of a lower life-time income due productivity growth over time.

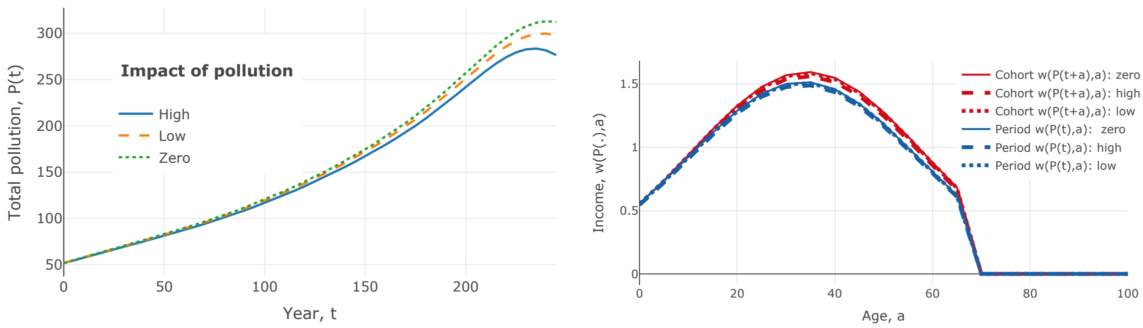

Figure 5: Impact of pollution. The left panel plots pollution over time with (red dashed) and without (blue) an effect on the model. The right panel plots the income along a cohort (red lines) and cross cohort for fixed t (blue lines).

The comparison of the optimal consumption profiles of the scenarios where pollution has an effect with the one where pollution does not affect the model as illustrated in Figure 6 is probably the most interesting and discussed for the optimal consumption along a cohort (red lines). If pollution has no effects (solid lines) a complete consumption smoothing along the life-cycle is optimal (red solid line). The effect of pollution implies a shift to younger ages, i.e., a decreasing consumption path as discussed before. At very young ages (between 0 and ≈ 20) the consumption path for a high pollution impact (red dotted line) even lies above that without a pollution and below afterwards. The loss in the lifetime income (see right panel of Figure 5) is not strong enough to shift the entire consumption path below the one without pollution effects as for the case with a low pollution impact (red dashed line). The arguments carry over to the cross cohort consumption paths (blue lines) where the consumption at young ages is also increased if pollution has an effect. The decrease across the age-groups for fixed t again goes along with the productivity growth over time.

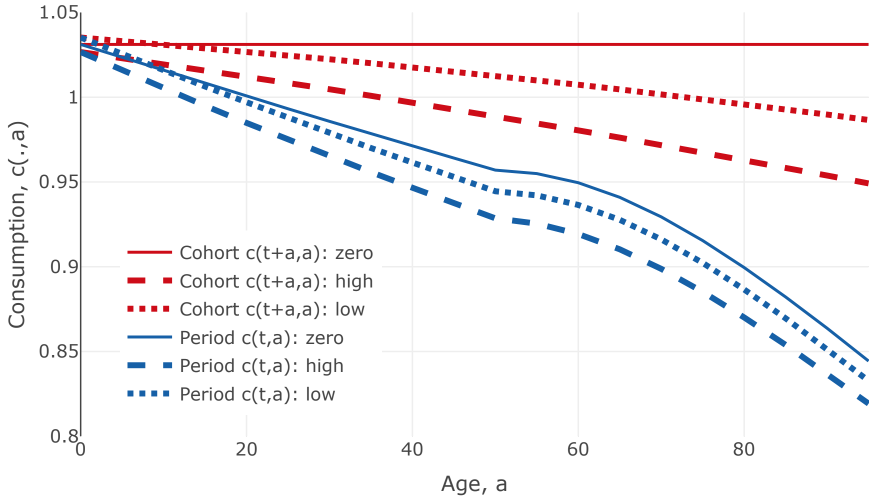

Figure 6: Impact of pollution on optimal consumption along a cohort (red lines) and cross cohort for a fixed t (blue lines).

4 Age-structure as a toolkit for non-standard optimal control models

Apart from modeling age-structure in the classical sense as in the previous section, age-structured optimal control models can also be used to handle advanced types of optimal control problems. In the following we consider two of them: optimal control models with random switches (subsection 4.1) and optimal control models including a delay (subsection 4.2). For both types respective MPs are available, but these are notoriously involved and difficult to use. The transformation to an age-structured optimal control model presents a promising option.

4.1 Optimal control models with random switches

Optimal control models with multiple stages (referred to as multi-stage optimal control models) are well developed if the stage switches are endogenously determined by the decision maker. At the switching point so-called *switching conditions* (see Tomiyama [64], Tomiyama and Rossana [65] or Makris [51]) have to hold in addition to the standard MP, and a solution can be found by a standard numerical approach.

For stochastic switches the conditions are different and the analysis is more involved. There is no unique second stage, but infinitely many, starting at all possible instants of time. Consequently, the decision maker has to consider all possible switching times (with corresponding stages) and anticipate them in the optimization nexus. She thereby has to consider the effects of her decisions on (i) the level of preparedness for the impacts of a switch and (ii) the likelihood of the switch occurring (as is described by the hazard rate of the switch).

The literature of optimal control models with stochastic switches emerged in the 1970's in environmental economics (see Cropper [21], Reed [58], etc.) and spilled over to economics (Guo et al. [35]), epidemiology (Brock and Xepapadeas [10]) and other fields. Most of these models use the so-called *backward approach*, i.e., the deterministic reformulation of the stochastic optimal control model (Boukas et al. [9]), where the value function for the second stage after the switch is derived either analytically or numerically. In the following we introduce the general formulation of an optimal control model with random switching time and present the transformation to an age-structured optimal control model (see Wrzaczek et al. [71]).

Let $x(t)$ and $u(t) \in U$ denote the state and control vectors at t in a standard optimal control model where $F(x(t), u(t), t)$ and $f(x(t), u(t), t)$ are the objective functional and system dynamics. The time horizon is separated into two stages by the switching time τ (random variable out of the sample space $\Omega = [0, \infty)$, with probability space $(\Omega, \Sigma, \mathbb{P})$), which is stochastic according to the hazard rate η that depends on the state and control vectors at t, i.e.,

$$\eta(x(t), u(t), t) \quad = \quad \frac{\mathcal{F}'(t)}{1 - \mathcal{F}(t)}, \qquad \mathcal{F}(t) = \mathbb{P}(\tau \leq t). \tag{15}$$

At the switch, the model changes disruptively according to three possibilities: (i) change of the objective functional, (ii) change of the system dynamics (including the addition of further or removal of existing state variables), and/or (iii) jump in a state variable. Combining these three possibilities makes it possible to model a broad variety of different effects associated with disruptive regime-shifts.

We denote (i) and (ii) by adding subscripts to the corresponding functions, (iii) is modelled via a function $\varphi(x(t), u(t), t)$ that embraces a possible jump. In case of a switch at τ, the limit of φ from the left defines the initial state value of the second stage at τ (see Equation (17b)).

We adopt the standard notation and introduce the value function of the second stage problem as the salvage value of the first stage and arrive at the following general model[14]

$$\max_{u(t) \in U} \qquad \mathbb{E}_{\tau \in [0, \infty)} \left[\int_0^\tau e^{-\rho t} F_1(x(t), u(t), t) \, \mathrm{d}t + e^{-\rho \tau} S(x(\tau), \tau) \right] \tag{16a}$$

$$\text{s.t.} \quad \dot{x}(t) = f_1(x(t), u(t), t), \quad x(0) = x_0 \tag{16b}$$

$$\eta(t) = \eta(x(t), u(t), t), \tag{16c}$$

[14] Note the difference in notation in comparison to multi-stage optimal control models (with endogenous switch): In these models the objective function of both stages can be written together as only one switch occurs. In the case of a random switch the control of the second stage will differ for every single switch and therefore depend on the realization of τ. The control variable $u(t)$ cannot therefore be put within one maximization operator.

where

$$S(x(\tau),\tau) \quad := \quad \max_{u(t)\in U} \int_{\tau}^{\infty} e^{-\rho t} F_2(x(t),u(t),t)\,\mathrm{d}t \tag{17a}$$

$$\text{s.t.} \quad \dot{x}(t) = f_2(x(t),u(t),t), \quad x(\tau) = \lim_{t'\nearrow\tau} \varphi(x(t'),u(t'),t'). \tag{17b}$$

In general, (16) is a stochastic optimal control problem w.r.t. the time horizon. However, by introducing an auxiliary state variable $z_1(t)$ (see Boukas et al. [9]) that evolves according to

$$\dot{z}_1(t) = -\eta(x(t),u(t),t)z_1(t), \qquad z_1(0) = 1, \tag{18}$$

it is possible to formulate (16) as a deterministic optimal control model. $z_1(t)$ can be interpreted as the probability that the switch has not set in during the interval $[0,t)$ (analogously to a survival probability). Exploiting this deterministic formulation of (16) and the auxiliary state variable Wrzaczek et al. [71] propose the transformation into an age-structured optimal control problem by considering every possible switching time to generate a new 'cohort' and by denoting the corresponding state and control variables (for a second stage initiated by a switch at $t-a$) by $y(t,a)$ and $v(t,a)$.

The full model in age-structured form reads

$$\max_{u(t)\in U, v(t,a)\in V} \quad \int_0^{\infty} e^{-\rho t}\left[z_1(t)F_1(x(t),u(t),t) + Q(t)\right]\mathrm{d}t \tag{19a}$$

$$\text{s.t.} \quad \dot{x}(t) = f_1(x(t),u(t),t), \quad x(0) = x_0 \tag{19b}$$

$$\dot{z}_1(t) = -\eta(x(t),u(t),t)z_1(t), \quad z_1(0) = 1 \tag{19c}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t,a) = f_2(y(t,a),v(t,a),t), \quad y(t,0) = \varphi(x(t),u(t),t) \tag{19d}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) z_2(t,a) = 0, \quad z_2(t,0) = z_1(t)\eta(x(t),u(t),t) \tag{19e}$$

$$y(0,a) = z_2(0,a) = 0, a \geq 0 \tag{19f}$$

$$Q(t) = \int_0^{\infty} z_2(t,a)F_2(y(t,a),v(t,a),t)\,\mathrm{d}a. \tag{19g}$$

Here, (19a) denotes the deterministic formulation of the objective function, (19b) and (19c) the system dynamics (state variable and probability that the switch has not set in at t) during stage 1 (before the switch), and (19c) and (19d) the system dynamics[15] (state variable and auxiliary state variable $z_2(t,a)$ which is the probability density that the switch happened at $t-a$[16]). The aggregated state $Q(t)$ covers the objective functionals of all possible switches that may have happened before t weighted by the corresponding density.[17]

The advantages of our age-structured formulation in comparison to the backward approach are twofold. Firstly, problem (19) can be treated with established numerical solution methods (as briefly discussed in section 2) offering a structured (while not always easy) way to solve concrete models. Freiberger [31] provides a specialised toolbox for the numerical solution of this type of problem implemented in the Julia programming

[15] Note that the limit can be neglected in (19d) as the notation of the state variable changes from $x(\tau)$ to $y(\tau,0)$ at the switch at τ.

[16] As argued in Wrzaczek et al. [71], the auxiliary state $z_2(t,a)$ avoids having to deal with a time lag in (19g).

[17] Note that the time-horizon here is infinite although we introduced the MP only for age-structured optimal control problems in finite time in Theorem 1. However, the present model (19) is a special case as the different "cohorts" do not interact with each other and their aggregation only enters the objective function. It can be shown for this case that the MP also holds for problems with infinite time horizon.

language. In contrast to the age-structured transformation approach the backward approach is often plagued by the curse of dimensionality. As the number of state variables increases, the dimension of the slice manifold along which the value function of stage 2 has to be evaluated increases, too. Consequently, the number of grid points (for which the value function has to be calculated) increases exponentially with the number of state variables.

Secondly, analytical and structural insights are limited in case of the backward approach, because (and in analogy to a dynamic programming approach) the second stage has to be solved first in order to subsequently obtain a solution for stage 1. In contrast, the age-structured formulation allows to solve stage 1 and stage 2 simultaneously within a single set of optimality conditions and by way of a single (numerical) optimization routine. The solution then reveals the links between the two stages explicitly and allows to characterize the mechanisms of the model in a natural and intuitive way.

The idea of considering optimal control models with a random switch as age-structured optimal control models is rather new still, but has already been used in a number of applications. Kuhn and Wrzaczek [47] consider a model of rational experimentation with an addictive good where the switch to addiction is modelled as a stochastic shock (that embraces a Skiba point in the second stage). Wrzaczek [70] includes the risk of catastrophic climate change in an OLG model on pollution control. Buratto et al. [15] consider the development of a vaccine protecting against COVID-19 as a positive stochastic shock and analyze anticipative behaviour in the stage without vaccination and the optimal adaptation of pandemic countermeasures shortly after the start of the vaccination roll out. Freiberger et al. [32] analyzes the optimal patterns of consumption and health care utilization over the individual life-cycle in view of large shocks to health. This paper exploits the advantages of the age-structured approach and carefully disentangles different channels of the optimal allocation of preventive, acute and chronic care. Buratto et al. [14] consider an advertising model with an abrupt change of the production costs, where the random switch depends positively on demand.

4.2 Optimal control models with time-lags or delays

Optimal control models with time-lags or delays are other advanced extensions of standard optimal control theory. These models can be divided into two classes: (i) Models with continuous time-lags correspond to a class of models where the system dynamics and/or the objective function at t depend on the previous path – or a part of it – of the control or state variable. (ii) Models with delay include the dependence of the system dynamics and/or the objective function on the state and control variables at one specific past point in time $t - \tau$, where $\tau > 0$ denotes the delay.

Formal proofs for (i) can be found in Bate [6] or Vinokurov [68], sufficiency conditions have been shown by Sethi [60]. For a text book representation we refer to Feichtinger and Hartl [23] and applications can be found e.g., in Sethi and McGuire [61], Arthur and McNicoll [4], Hartl and Sethi [41], Caulkins et al. [16], or Boucekkine et al. [8]. The first proof for (ii) goes back to Kharatishvili [46], which has been extended by Halany [39] to the case of multiple delays (equal for state and control). Göllmann et al. [38] add mixed control-state constraints to the problem.

The literature on applications of optimal control models with time-lag is still relatively scarce in spite of the many new developments, extensions and applications of optimal control theory over the past decades. The reason appears to be twofold: First, although the theoretical contributions provide necessary optimality conditions, the theory is advanced, and it is more difficult to obtain analytical as well as numerical results. Second, a time-lag in state and control variables is often modeled as an aggregated state variable that approximates a certain effect at t. Given the complex nature of optimal control problems in general, the second argument then usually implies the use of an approximation to guarantee tractability of the model. This is striking, given the importance of applications in which the time-lag is crucial, e.g., the construction of a dam, which requires several years of planning and construction, followed by multiple years of rising water levels until the benefits of the investment can be realised (hydro-power, flood control, ...).

Although models with a continuous time-lag and models with delay are treated separately in the literature we will work with a continuous time-lag and argue why it can also be used for a delay, at least as an arbitrarily close approximation. Let us thus consider the following optimal control model with continuous delay.

$$\max_{u(t)\in U} \quad \int_0^T e^{-\rho t} F(x(t), u(t), \phi(t), t)\,\mathrm{d}t + e^{-\rho T} S(x(T), \phi(T), T) \tag{20a}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), \phi(t), t), \quad x(0) = x_0 \tag{20b}$$

$$\phi(t) = \int_{-\infty}^{t} g(x(s), u(s), s, t)\,\mathrm{d}s \tag{20c}$$

$$x(t) = \tilde{x}(t), \quad u(t) = \tilde{u}(t), \quad \text{for } t < 0, \tag{20d}$$

where control and state variables and functions are denoted in the same way as above. The function $\phi(t)$ is now governing the continuous time-lag by aggregating the density of the effects of all past (i.e., in general for $s \in (-\infty, t)$) control and state variables. Thus, $\phi(t)$ enters the objective functional, the salvage value function and the system dynamics. To capture past effects reaching back before the start of the planning period (i.e., $t < 0$) condition (20d) adds the part of the trajectory before the planning horizon.

In case the time-lag is a fixed delay, $\phi(t)$ does not depend on the entire history of the control and state variables, but on a specific time alone. This dependence is described by the function $\tilde{g}$:

$$\phi(t) = \tilde{g}(x(t-\tau), u(t-\tau), t-\tau, t), \qquad x(t) = \tilde{x}(t), u(t) = \tilde{u}(t), \quad \text{for } t \in [-\tau, 0). \tag{21}$$

This definition can be extended straightforwardly to cover the case of multiple fixed delays. Note that (21) can be transformed to fit into framework (20) by employing the Dirac delta function $\delta(t)$,

$$\begin{aligned} \phi(t) &= \tilde{g}(x(t-\tau), u(t-\tau), t-\tau, t) \\ &= \int_0^{\infty} \delta(\tau - s')\tilde{g}(x(t-s'), u(t-s'), t-s', t)\,\mathrm{d}s' \\ &= \int_{-\infty}^{t} \underbrace{\delta(s-(t-\tau))\tilde{g}(x(s), u(s), s, t)}_{=:g(x(s),u(s),s,t)}\,\mathrm{d}s \end{aligned}$$

such that state and control enter $\phi(t)$ only by the delay.[18]

[18] Note that this argument is not entirely correct in mathematical terms as the delta distribution is not integrable in a Riemann- and Lebesgue-sense. However, the delta function can be defined as the limit of a series of probability density functions $(\delta_k)_{k\in\mathbb{N}}$ (imagine a series of normal distributions with mean 0 and a variance converging to 0). Each function in the series is integrable and can be used to approximate the Dirac function to an arbitrary degree of freedom, which translates to an arbitrarily close approximation of the case of fixed delay, based on a probability density function.

In order to formulate (20) as an age-structured optimal control problem, we consider two new auxiliary (age-structured) state variables emerging at every t with zero dynamics at the corresponding boundary condition. One of the 'cohorts' accounts for the control and the other for the state variable at t. The initial distributions for these auxiliary states cover the state and control trajectories before the planning horizon (denoted by $\tilde{x}(t)$ and $\tilde{u}(t)$ in (20)). The function $\phi(t)$ in turn is represented by an aggregate state variable, as standard in the general model (2).

The full model in age-structured form then reads

$$\max_{u(t)\in U} \quad \int_0^T e^{-\rho t} F(x(t), u(t), \phi(t), t)\, \mathrm{d}t + e^{-\rho T} S(x(T), \phi(T), T) \tag{22a}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), \phi(t), t), \quad x(0) = x_0 \tag{22b}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y^1(t,a) = 0, \qquad y^1(t,0) = x(t) \tag{22c}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y^2(t,a) = 0, \qquad y^2(t,0) = u(t) \tag{22d}$$

$$\phi(t) = \int_0^\infty g(y^1(t,a), y^2(t,a), t-a, t)\, \mathrm{d}a \tag{22e}$$

$$y^1(0,a) = \tilde{x}(a), \quad \text{for } a \geq 0 \tag{22f}$$

$$y^2(0,a) = \tilde{u}(a), \quad \text{for } a \geq 0. \tag{22g}$$

To the best of our knowledge this transformation has not been investigated (at least) explicitly in the literature. However, as discussed above, the analytical treatment and the numerical methods are developed for such a model. The advantages of these approaches are similar to the ones of the age-structured formulation of optimal control models with random switches.

5 Discussion and conclusions

Age-structured optimal control models are important frameworks to take into account the interplay between cohort and period effects in many applications in economics, environmental science, epidemiology, and many more disciplines. Setting up a generic age structured optimal control model we have shown how such models allow for aggregated and distributed state variables as well as concentrated and age and time dependent control variables. We introduced the analytical results of the age structured Maximum Principle and sketched the numerical solutions of these models.

Based on a toy model of a social planer aiming to reduce aggregate pollution generated by consumption when maximizing the discounted stream of future utility for a society, we demonstrate the application of the age-structured optimal control model. Within our toy model we can show how inter and intra cohort pollution effects interact and how the social planner can internalize these effects. Considering the present value of an additional consumer, i.e., the shadow value of an individual, we show how cohort redistribution effects and population dynamic effects are intertwined with optimal redistributions across the life-cycle.

We end our review by showing two examples where age-structured optimal control models can be applied to solve non-standard optimal control models. First, we introduce control models with random switches. By defining every possible switching time to generate a new cohort the toolkit of age-structured optimal control models allows to apply established numerical solution methods. Second, we introduce a rather novel transformation of an optimal control model with time lags or delays into an age-structured optimal control model.

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