MINIMAX THEORY: A SUMMARY

Jean-Pierre Aubin

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

ABSTRACT

We present here classical minimax inequalities as well as more recent ones, as the Ky Fan inequality and its variants, which play an important role not only in game theory, but in convex and non convex analysis.

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TWO PERSON GAMES

Let us introduce the two players, Mike and Nancy, who have to choose "strategies" x and y in "strategy sets" M and N according to certain rules which we are about to describe. The traditional way to ground game theory is to posit that each player classifies pairs of strategies through a real valued function. We can think of such a function as a map that associates to each pair of strategies (x,y) its "cost", measured by a real number. Since the concept of cost involves the notion of money, which is quite difficult to master in economics, we prefer to call it a "loss". So a player uses a *loss function* $f : M \times N \to R$ for defining the *preference preorder* on $M \times N$ as follows:

$$(1) \quad \begin{cases} (x_1, y_1) \text{ is preferred to } (x_2, y_2) \text{ if and only if} \\ f(x_1, y_1) \leq f(x_2, y_2) \quad . \end{cases}$$

Whatever the relevance of this assumption is, we assume from now on that Mike and Nancy select their strategies according to the loss functions $f_M : M \times N \to R$ and $f_N : M \times N \to R$ respectively.

Definition 1

A pair of strategies $(\bar{x},\bar{y})$ is said to be a noncooperative equilibrium of

$$(2) \qquad f_M(\bar{x},\bar{y}) = \min_{x\in M} f_M(x,\bar{y})$$

$$(3) \qquad f_N(\bar{x},\bar{y}) = \min_{y\in N} f_N(\bar{x},y) \quad .$$

▲

So, a noncooperative equilibrium is a situation in which each player optimizes his own criterion, assuming that the choice of his partner is fixed. In other words, this is a situation of *individual stability*.

We consider here the important case of two-person games that satisfy the condition

$$(4) \qquad \forall x \in M,\ y \in N,\ f_M(x,y) + f_N(x,y) = 0 \quad .$$

So, the loss of Nancy is the gain of Mike and vice versa.

For simplicity, we set

$$(5) \qquad f_M(x,y) := f(x,y) \quad , \quad f_N(x,y) := -f(x,y)$$

$$(6) \qquad f^{\#}(x) := \sup_{y\in N} f(x,y), \quad v^{\#} := \inf_{x\in M} \sup_{y\in N} f(x,y)$$

(read f-sharp and v-sharp) and

$$(7) \qquad f^{b}(y) := \inf_{x\in M} f(x,y) \quad ,$$

$$(8) \qquad v^{b} := \sup_{y\in N} \inf_{x\in M} f(x,y)$$

(read f-flat and v-flat).

We assume that the behavior of Nancy consists only in being hurtful to Mike, and that Mike knows it. (Actually, we need only to assume that Mike believes that Nancy is nasty.) So, he assigns to each strategy $x \in M$ the worst loss $f^{\#}(x)$ and he minimizes it; the smallest worst loss is equal to $v^{\#}$. We also assume that Mike behaves in the same way, so that Nancy assigns to each strategy $y \in N$ the worst gain $f^{b}(y)$ and she maximizes it: the largest worst gain is equal to v^{b}. Since $f^{b}(y) \leq f^{\#}(x)$ for all $x \in M$, $y \in M$, we deduce that

$$v^{b} \leq v^{\#} \quad . \tag{9}$$

There are situations where v^{b} is strictly less than $v^{\#}$.

Example

Consider the finite game

$M = \{1,2\}$, $N = \{1,2,3\}$, f is described by the matrix

M \ N	1	2	3
1	-6	2	-3
2	4	-5	-4

Nancy selects columns

Mike selects rows

The entries of this matrix represent the loss of Mike. So, the biggest losses of Mike are 2 and 4 respectively and thus, Mike's conservative strategy is the first row and $v^{\#} = 2$. The least gains for Nancy are respectively -6, -5, -4 and thus, her conservative strategy is the third column and $v^{b} = -4$. Let us try to play that game for ourselves.

First, let Mike implement its conservative strategy (first row). He expects Nancy to choose the second column. But the conservative strategy for Nancy is the third column and she expects Mike to choose the second row. But if Mike is informed of this choice (or guesses it), then we would do better to select his second row (with a loss of -4) instead of the first one. Similarly, if Mike chooses his conservative strategy, then Nancy would do better to play her second row (with a gain of 2) instead of the third.

This "wheels within wheels" situation illustrates the lack of noncooperative equilibrium. The absence of noncooperative equilibria when $v^b < v^\#$ is actually a general fact. The following result shows that its existence requires very stringent conditions.

<u>Proposition 1</u>

The following conditions are equivalent

$$(10) \quad \begin{cases} \text{i)} & (\bar{x},\bar{y}) \text{ is a noncooperative equilibrium} \\ \text{ii)} & \forall (x,y) \in M \times N,\ f(\bar{x},y) \leq f(\bar{x},\bar{y}) \leq f(x,\bar{y}) \\ \text{iii)} & v^\# = v^b = f^b(\bar{x}) = f^\#(\bar{y}) \quad . \end{cases}$$

▲

<u>Definition 2</u>

When $v^b = v^\#$, this common value is called the *value* of the *game* and a noncooperative equilibrium is called a *saddle point*. There are examples where saddle points do exist.

▲

<u>Example</u>

Consider the finite game

$M = \{1,2\}$, $N = \{1,2,3\}$, f is described by the matrix

M \ N	1	2	3
1	-2	-1	-4
2	1	0	-6

We observe that $v = -1$ and that the pair of conservative strategies (1,2) is a noncooperative equilibrium.

THE MINIMAX THEOREMS

We recall that a function f is *inf-compact* if its lower level sets $\{x \mid f(x) \leq \lambda\}$ are relatively compact and that f is lower semicontinuous if its lower level sets are closed.

Theorem 1 (lop-sided minimax). Let M and N be convex subsets of vector spaces, M being supplied with a topology. We assume that

$$(11) \quad \begin{cases} \text{i)} & \forall y \in N,\ x \to f(x,y) \text{ is convex and lower semicontinuous} \\ \text{ii)} & \exists\, y_0 \in N \text{ such that } x \to f(x,y_0) \text{ is inf-compact} \end{cases}$$

and that

$$(12) \qquad \forall x \in M, \quad y \to f(x,y) \text{ is concave} \quad .$$

Then f has a value ($v^b = v^{\#}$) and there exists $\bar{x} \in M$ such that $\sup_{y \in N} f(\bar{x},y) = v^b$. ▲

As a corollary, we obtain the von Neumann minimax Theorem (see von Neumann and Morgenstern (1944)).

Theorem 2 (minimax). Let M and N be convex subsets of vector spaces, supplied with topologies. We assume that

$$(13) \quad \begin{cases} \text{i)} & \forall y \in N,\ x \to f(x,y) \text{ is convex and lower semicontinuous} \\ \text{ii)} & \exists\, y_0 \in N \text{ such that } x \to f(x,y_0) \text{ is inf-compact} \end{cases}$$

and

$$(14) \quad \begin{cases} \text{i)} & \forall x \in M,\ y \to f(x,y) \text{ is concave and upper semicontinuous} \\ \text{ii)} & \exists\, x_0 \in M \text{ such that } y \to f(x,y_0) \text{ is sup-compact.} \end{cases}$$

Then there exists a saddle point $(\bar{x},\bar{y}) \in M \times N$.

▲

MIXED STRATEGIES

We already observed that games with finite strategy sets may not have values. In order to apply the minimax theorem, we can convexify the strategy sets. Namely, let $N := \{1,\dots,n\}$ and $M := \{1,\dots,m\}$ be finite sets. Mike's loss function f is defined by the matrix of losses $f(i,j)$, $i \in N$, $j \in M$. We regard elements of the simplexes

$$(15) \quad S^n := \left\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\right\} , \quad S^m := \left\{y \in \mathbb{R}^m_+ \mid \sum_{j=1}^m y_j = 1\right\} ,$$

as the probabilities on N and M respectively. We extend the function f to the function $\hat{f}$ defined on $S^n \times S^m$ by

$$(16) \quad \hat{f}(x,y) := \sum_{i=1}^n \sum_{j=1}^m f(i,j) x_i y_j \quad .$$

Theorem 2 implies the following corollary.

Corollary 1. Assume that the strategy sets are finite. Then there exist $\bar{x} \in S^n$ and $\bar{y} \in S^m$ such that

$$\forall x \in S^n, \quad y \in S^m, \qquad \hat{f}(x,\bar{y}) \leq \hat{f}(\bar{x},\bar{y}) \leq \hat{f}(x,\bar{y}) \quad .$$

▲

J. von Neumann proposed to interpret elements $x \in S^n$ and $y \in S^m$ as "mixed strategies". In this framework, a player does not play a strategy, but all strategies and chooses only the probabilities. A justification for the use of mixed strategies is the protection a player obtains by disguising his objectives to his partner. By playing all strategies with a given probability, his partner cannot guess the strategy that he will implement.

■

RELAXATION OF THE COMPACTNESS ASSUMPTION

The compactness assumption we made in Theorem 1, and, subsequently, in the lop-sided minimax Theorem 2, happens to be too strong in many problems. We shall relax it when M is a subset of a Banach space.

We consider two Banach spaces X and Y and a function $\tilde{f}$ from $X \times Y$ to $\overline{\mathbb{R}} := [-\infty,+\infty]$. We set

(17) $$M := \{x \in X \mid \forall y \in Y ,\ \tilde{f}(x,y) < +\infty\}$$

and

(18) $$N := \{y \in Y \mid \forall x \in X ,\ \tilde{f}(x,y) > -\infty\} .$$

We shall say that $M \times N$ is the *domain of* $\tilde{f}$.

We assume that M and N are *nonempty*; we set:

(19) f is the restriction of $\tilde{f}$ to $M \times N$.

Thus, f maps $M \times N$ to $\mathbb{R}$.

We begin by stating a corollary to Theorem 1 that uses the conjugate functions $\tilde{f}_y^*$ from X^* to $]-\infty,+\infty]$ defined by

(20) $$\tilde{f}_y^*(p) := \sup_{x \in X} [\langle p,x\rangle - \tilde{f}(x,y)] .$$

We set

(21) $$\text{Dom } \tilde{f}_y^* := \{p \in X^* \mid \tilde{f}_y^*(p) < +\infty\} .$$

Corollary 2. We assume that X is a reflexive Banach space supplied with its weak topology. Theorem 1 remains true when we replace assumption (11)ii) by assumption

(22) $$\begin{cases} \exists y_0 \in N \text{ such that } 0 \in \text{Int}(\text{Dom } \tilde{f}_{y_0}^*) \text{ (for the} \\ \text{strong topology of the dual } X^*). \end{cases}$$

▲

Assumption (22) can be considerably relaxed, as the following theorem shows.

Theorem 3 (relaxed lop-sided minimax Theorem). Let X and Y be reflexive Banach spaces and $\tilde{f}$ be a function from $X \times Y$ to $\overline{\mathbb{R}}$ whose domain $M \times N$ is nonempty. We assume that

(23) $\forall y \in N,\quad x \to \tilde{f}(x,y)$ is convex lower semicontinuous,

that

(24) $\forall x \in M,\quad y \to \tilde{f}(x,y)$ is concave upper semicontinuous,

and that

(25) $0 \in \text{Int}\left(\bigcup_{y \in N} \text{Dom}\ \tilde{f}^*_y\right)$ (for the strong topology) .

Then f has a value $v := v^b = v^\#$ and there exists $\bar{x} \in M$ such that $\sup_{y \in N} f(\bar{x},y) = v$. (See Aubin (1979), chapter 13.)

▲

<u>PLAYING DECISION RULES</u>

Let M and N be the strategy sets of Mike and Nancy and f be Nancy's gain function. She can use it to assign to each decision rule $C_N : M \to N$ a gain defined by

(26) $$f^b(C_N) := \inf_{x \in M} f(x, C_N(x)) \quad .$$

This represents the *worst* gain she can expect using the decision rule C_N, assuming that Mike's behavior is noncooperative.

Note that this definition is consistent with the definition of the worst gain yielded by a strategy $\bar{y}$, regarded as a constant decision rule $x \to \bar{y}$:

$$f^b(\bar{y}) := \inf_{x \in M} f(x,\bar{y}) = \inf_{x \in M} f(x,\bar{y}(x)) \quad .$$

Consequently, if $\mathcal{C}_N$ is a set of continuous decision rules containing the set N of constant decision rules,

(27) $$\begin{cases} v^b := \sup_{y \in N} \inf_{x \in M} f(x,y) \le \sup_{C_N \in \mathcal{C}_N} f^b(C_N) \\ \quad \le \inf_{x \in M} \sup_{y \in N} f(x,y) := v^\# \end{cases} \quad .$$

Symmetrically, Mike assigns to each decision rule $C_M : N \to M$ the worst loss

(28) $$f^{\#}(C_M) := \sup_{y \in N} f(C_M(y),y) \quad .$$

If $\mathcal{C}_M$ is a set of continuous decision rules containing the set M of constant decision rules, we have

(29) $$v^b \leq \inf_{C_M \in \mathcal{C}_M} f^{\#}(C_M) \leq v^{\#} \quad .$$

We shall present a set of assumptions (weaker than the assumptions of Theorem 2) under which

$$\inf_{C_M \in \mathcal{C}_M} f^{\#}(C_M) = \sup_{C_N \in \mathcal{C}_N} f^b(C_N) \quad .$$

Theorem 4. Let M be a topological space, N be a convex subset of a topological vector space and f be a function from $M \times N$ to $\mathbb{R}$. Let us suppose that

(30) $$\begin{cases} \text{i)} & \exists y_0 \in N \text{ such that } x \to f(x,y_0) \text{ is inf-compact} \\ \text{ii)} & \forall y \in N,\ x \to f(x,y) \text{ is lower semicontinuous} \end{cases}$$

and that

(31) $$\forall x \in M, \quad y \to f(x,y) \text{ is concave.}$$

Let $\mathcal{C}_M := \mathcal{C}(N,M)$ and $\mathcal{C}_N := \mathcal{C}(M,N)$ denote the set of continuous decision rules of Mike and Nancy. Then there exists $\bar{x} \in N$ such that

(32) $$\begin{cases} \sup_{y} f(\bar{x},y) = v^{\#} \\ = \inf_{C_M \in \mathcal{C}_M} \sup_{y \in M} f(C_M(y),y) \\ = \sup_{C_N \in \mathcal{C}_N} \inf_{x \in N} f(x,C_N(x)) \end{cases}$$

(See Aubin (1979), chapter 7.)

▲

This is a very powerful theorem, equivalent to the Brouwer fixed point Theorem and all the many equivalent results of nonlinear analysis. A consequence of it is the very important Ky Fan's inequality (Ky Fan (1972)). It happens that it is a more versatile tool of nonlinear analysis than the Brouwer or Schauder fixed point theorems or the Kakutani fixed point theorem. (See Aubin (1979), chapters 8, 9 and 15.)

Theorem 5 (Ky Fan). Let K be a compact convex of a topological vector space and $\varphi: K \times K \to \mathbb{R}$ be a function satisfying

$$(33) \quad \begin{cases} \text{i)} & \forall y \in K, \; x \to \varphi(x,y) \text{ is lower semicontinuous} \\ \text{ii)} & \forall x \in K, \; y \to \varphi(x,y) \text{ is concave.} \end{cases}$$

Then there exists $\bar{x} \in K$ satisfying

$$(34) \quad \sup_{y \in K} \varphi(\bar{x},y) \leq \sup_{y \in K} \varphi(y,y) \quad .$$

▲

THE FINITE TOPOLOGY

Still, despite their apparent generality, assumption (33)i) of Ky Fan's Theorem is not satisfied in several instances. We shall replace it by another set of assumptions involving the *finite topology*, which is not generally a vector space topology, but which is stronger than any vector space topology. Let N be a convex subset of a vector space.

We associate with any finite set $K := \{y_1,\ldots,y_n\}$ of n elements y_i of N the map β_K from S^n to N defined by

$$(35) \quad \forall \lambda \in S^n \quad , \quad \beta_K(\lambda) := \sum_{i=1}^{n} \lambda_i y_i \quad .$$

The *finite topology* on a convex subset N is the strongest topology for which the maps β_K are continuous when K ranges over the family $\mathcal{S}$ of finite subsets of N.

▲

So, a map C from N, supplied with the finite topology, to a topological space M is continuous, if and only if

(36) $\forall K \in S$, the maps $C\beta_K$ from S^n to M are continuous.

Also, any map C from a topological space M to N of the form

(37) $$C(x) := \beta_K \vec{p}(x) = \sum_{i=1}^{n} p_i(x) y_i$$

where $\vec{p}$ is a continuous map from M to S^n, is continuous from M to N supplied with the finite topology. Any affine map from a convex set M to a convex set N is continuous when they are both supplied with the finite topology.

We begin by generalizing Theorem 4.

Theorem 4 bis. Theorem 4 holds true when N is supplied with the finite topology.

▲

We now present an inequality due to Brézis-Nirenberg-Stampacchia (1973) which is very useful in the theory of monotone operators.

Theorem 6 (Ky Fan's inequality for monotone functions). Let $K \subset X$ be a convex subset of a topological vector space and $\varphi : K \times K \to \mathbb{R}$ be a function satisfying

(38) $$\begin{cases} \text{i)} & \forall y \in K,\ x \to \varphi(x,y) \text{ is lower semicontinuous for the finite topology} \\ \text{ii)} & \forall x \in K,\ y \to \varphi(x,y) \text{ is concave and upper semicontinuous} \end{cases}$$

We also assume that

(39) $\exists\ y_0 \in K$ such that $x \to \varphi(x,y_0)$ is inf-compact

and that φ is "monotone" in the sense that

(40) $$\begin{cases} \text{i)} & \forall y \in K,\quad \varphi(y,y) \leq 0 \\ \text{ii)} & \forall x,y \in K,\quad \varphi(x,y) + \varphi(y,x) \geq 0 \end{cases}$$

Then there exists $\bar{x} \in K$ such that

$$\sup_{y \in K} \varphi(\bar{x}, y) \leq 0 \quad . \tag{41}$$

▲

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