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RESEARCH MEMORANDUM

A NOTE ON THE GENERAL INVERSE PROBLEM OF OPTIMAL CONTROL THEORY

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1. Introduction

A problem commonly faced by the designer of a control system is how to choose a criterion function which accurately reflects the functional objectives of the system and which yields a "simple" optimal control law. Since two different control laws may require radically different physical structures to implement, the designer obviously desires to use that law which imposes the minimal weight, cost, and complexity requirements upon his system. Another way of looking at this question is to ask: "given a particular feedback control law, what is the family of criterion functions for which this law is optimal?" This is the inverse problem of optimal control theory.

Compared with the direct problem, the inverse problem has not been actively studied, although in the past ten years there has been a growing body of literature on the important special case of the problem when the systems dynamics and the control law are linear, the so-called "optimal regulator problem" [1-6] and a few results have been obtained for certain classes of nonlinear problems [7-8].

In this note, we present equations describing equivalent

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criteria for a very general class of control processes. Specifically, we consider the problem of minimizing

$$J = \int_0^T g(u,v)\, dt \quad ,$$

where u and v are related by the differential equation

$$\frac{du}{dt} = h(u,v) \quad , \qquad u(0) = c \quad .$$

The problem we study is the determination of all g which minimize J when the functions h and v are given. Throughout this paper, we shall assume that v is given in "feedback" form, i.e. $v = v(u,t)$. As will be indicated below, our results constitute the natural generalization of those in [1-6] to the nonlinear case and include the previous results as special cases.

2. The Basic Equations

The precise statement of the problem we consider is: let u and v be n, m-dimensional vector functions of t, respectively and let $h(u,v)$ be an n-dimensional vector-valued function of u and v which is continuously differentiable in each component of v. Assume that $g(u,v)$ is an unknown scalar function of u and v which is continuously differentiable in each argument and that v is a given feedback control law. To avoid degenerate situations, we further assume that g belongs to a class of functions for which the variational problem has a nontrivial solution, e.g. g strictly convex in v.

We desire to classify all functions g which minimize the functional

$$J = \int_0^T g(u,v)\, dt \quad , \qquad 0 \leq T \leq \infty \quad .$$

The first task is to obtain an equation satisfied by all "optimizing" functions g, then we will impose additional structure on the allowable g in order to utilize the theoretical solution.

We begin by observing that under the foregoing assumptions on h and g, any optimal g must be related to the system dynamics h by the Hamilton-Jacobi-Bellman equation

$$\frac{\partial f}{\partial T} = \min_{v} \left[g(c,v) + (\mathrm{grad}_c\ f,\ h(c,v)) \right] \quad ,$$

where $\mathrm{grad}_c\ f(c,T)$ denotes the gradient of f relative to the vector c, (,) denotes the usual vector inner product, and f(c,T) is the optimal value function, i.e. f = min J. Since, a fortiori, v is the optimal control for the unknown function g, the Hamilton-Jacobi-Bellman equation is in reality equivalent to two equations

$$\frac{\partial f}{\partial f} = g(c,v) + (\mathrm{grad}_c\ f,\ h(c,v)) \tag{1}$$

for f, and

$$\mathrm{grad}_v\ g = -\left(\frac{dh}{dv}\right)'\ \mathrm{grad}_c\ f \quad , \tag{2}$$

expressing the fact that v is the minimizing control. In Eq. (2), $\frac{dh}{dv}$ is the Jacobian matrix of h relative to v and ' denotes the matrix transpose.

Equations (1) and (2) may be utilized to give the first basic result:

Theorem 1. All functions g which are optimal relative to a given h and v satisfy the differential equation

$$0 = \frac{d}{dT} \{p(c,v)\} + \mathrm{grad}_c \left[g(c,v) - (p(c,v), h(c,v))\right] \quad ,$$

where $p(c,v) = A^{\#} \,\mathrm{grad}_v\, g + (I - A^{\#}A)\, y$, $A = \left[\left(\frac{dh}{dv}\right)'\right]$, y is an arbitrary vector, and $^{\#}$ denotes the pseudo-inverse operation.

Proof. From Eq. (2), $\mathrm{grad}_c\, f = -p(c,v)$. Substituting this result into Eq. (1), we have

$$\frac{\partial f}{\partial T} = g(c,v) - (p(c,v),h(c,v)) \quad . \tag{3}$$

Differentiating $\mathrm{grad}_c\, f$ with respect to T, taking the gradient of $\frac{\partial f}{\partial T}$ relative to c, and equating the two expressions so obtained, we arrive at the equation of the Theorem.

Remarks. (i) Although of theoretical interest, the equation of Theorem 1 is of limited value in the absence of further structure on g. However, as will be shown by examples later, when we do parametrize the class of allowable g, then the equation may be effectively utilized.

(ii) The equation for g is a linear second-order partial differential equation in n+1 variables. Consequently, its general solution poses serious computational (and analytic) difficulties. Utilizing the fact that knowledge of v determines f, and conversely, we now show how to derive an alternative first-order partial differential equation whose solution can be approached through the theory of characteristics.

The result we wish to establish is

Theorem 2. The optimal value function $f(c,T)$ satisfies the first-order partial differential equation

$$\frac{d}{dT}\left[\frac{\partial f}{\partial T} - (\mathrm{grad}_c\ f,h)\right] = 0 \quad . \tag{4}$$

Proof. From Eq. (1), we have

$$\frac{d}{dT}\left[\frac{\partial f}{\partial T} - (\mathrm{grad}_c\ f,h)\right] = \frac{\partial g}{\partial T} + (\mathrm{grad}_v\ g,\frac{\partial v}{\partial T})$$
$$= (\mathrm{grad}_v\ g,\frac{\partial v}{\partial T}) \quad ,$$

since g does not explicitly depend on T. Using the expression in Equation (2) for $\mathrm{grad}_v\ g$, we have

$$\frac{d}{dT}\left[\frac{\partial f}{\partial T} - (\mathrm{grad}_c\ f,h)\right] = - \left((\frac{dh}{dv})'\ \mathrm{grad}_c\ f,\frac{\partial v}{\partial T}\right)$$

or, upon carrying out the differentiation with respect to T on the left hand side of the equation,

$$\frac{\partial^2 f}{\partial T^2} - (\mathrm{grad}_c\ \frac{\partial f}{\partial T},h) = 0 \quad ,$$

i.e.

$$\frac{d}{dT}\left[\frac{\partial f}{\partial T} - (\mathrm{grad}_c\ f,h)\right] = 0 \quad ,$$

which was to be established.

Equation (4) may be utilized to obtain f, making use of the obvious initial condition

$$f(c,0) = 0 \quad .$$

Once the function f is known, we obtain g by Eq. (1) as

$$g(c,v) = \frac{\partial f}{\partial T}(c,T) - (\mathrm{grad}_c\ f,h) \quad .$$

3. Examples

(a) We illustrate Theorem 1 and 2 by applying them to the standard linear regulator problem. In this case we have $T = \infty$, $h(c,v) = Ac + Bv$, $v(c,T) = -Kc$, where for the problem to be well-posed we demand that (A,B) be controllable and K be such that $A - BK$ is a stability matrix. To parametrize g, let us assume that g has the quadratic structure

$$g(c,v) = (c,Qc) + (v,v) \quad ,$$

where Q is an unknown positive semi-definite matrix. In view of the assumed linear structure of v, we may write

$$g(c,v) = [c,(Q + K'K)c] \quad .$$

The objective is to characterize all Q satisfying Theorem 1.

In the notation of Theorem 1, we have

$$A = B' \quad , \qquad \mathrm{grad}_v\ g = -2Kc \quad ,$$

$$p(c,v) = -2B^{\#\prime}Kc + (I - B^{\#\prime}B')y$$

$$\frac{d}{dT}\ p(c,v) = 0 \quad ,$$

$$\mathrm{grad}_c\ g(c,v) = 2(Q + K'K)c$$

$$[p(c,v),\ h(c,v)] = [\ -2B^{\#\prime}Kc + (I - B^{\#\prime}B')(A - BK)c]$$

$$= -2(c, K'B^{\#}(A - BK)c)$$
$$+ (c, (A - BK)'(I - B^{\#\prime}B')y)$$

$$\text{grad}_c\ [p(c,v), h(c,v)] = -2K'B^{\#}(A - BK)c$$
$$+ (A - BK)'(I - B^{\#\prime}B')y \quad .$$

Thus, we see that Q must satisfy the equation

$$2(Q + K'K)c = -2K'B(A - BK)c$$
$$+ (A - BK)'(I - B^{\#\prime}B')y \quad ,$$

or

$$2(Q + K'K + K'B^{\#}(A - BK))c = (A - BK)'(I - B^{\#\prime}B')y \quad ,$$

for arbitrary y and c.

Different choices of y yield different equivalent Q's. For example, y = 0 gives

$$Q + K'K + K'B^{\#}(A - BK) = 0 \quad ,$$

while if y = c, we have

$$Q + K'B^{\#}(A - BK) - (A - BK)'(I - B^{\#\prime}B') = 0 \quad .$$

The only situation in which there exists a unique Q corresponding to K is when $B^{\#} = B^{-1}$, i.e. when m = n and B is non-singular.

(b) Theorem 2 may be illustrated on a trivial modification of the foregoing problem. Let g(c,v) have the parametrized form

$$g(c,v) = \frac{1}{2}\,[(c,Qc) + (v,v)] \quad ,$$

with, as before, $v = -Kc$. Since, in this case, $f(c)$ has the form

$$f(c) = \frac{1}{2}(c,Lc) \quad , \qquad L > 0 \quad ,$$

we have

$$\text{grad}_c\ f = Lc \quad ,$$

$$(\text{grad}_c\ f,h) = (c,L(A - BK)c) \quad .$$

Thus, applying Theorem 2 and keeping in mind that $T = \infty$ implies $\frac{\partial f}{\partial T} \equiv 0$, we see that

$$\frac{-d}{dT}(\text{grad}_c\ f,h) = -(c,L(A - BK)c) = 0$$

for all c. Hence,

$$L(A - BK) = \frac{1}{2}\ (Q + K'K) \quad .$$

Upon symmetrizing the quantity 2LA and using the fact that any optimizing control law K must have the structure $K = B'L$, the above equation reduces to the well known expression for Q,

$$Q - K'K + LA + A'L = 0 \quad .$$

Of course, we want to eliminate L from the foregoing equation and deal directly with the given data K. This is accomplished by solving the equation $K = B'L$ for L, yielding the solution

$$L = B^{\#'}K + (I - B^{\#'}B')C \quad ,$$

where C is an arbitrary symmetric nxn matrix. Substituting this result into the above equation gives

$$Q = K'K - [B^{\#'}K - (I - B^{\#'}B')C]A - A'[B^{\#'}K - (I - B^{\#'}B')C]$$

for determining equivalent Q.

4. Discussion

We have presented some new equations relating a specified feedback control law to all integral criteria for which it is the optimal law under fixed dynamics. The examples presented for the linear regulator case may be extended to non-quadratic criteria by a suitable parametrization of the cost function g, e.g. expansion of g into a finite power series, a Fourier expansion, and so forth. The coefficients of the expansion are then determined by the equations of Theorems 1 or 2.

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Errata Sheet

Please note the following changes:

Page 4: Theorem 1 is footnoted to read: *To avoid degeneracies in the problem statement, we assume throughout that $\text{grad}_v\, g \in \text{Range}\, (\frac{dh}{dv})'$.

Page 7: 1. The equation on line 6 should read: $2(Q + K'K)c = -2K'B^{\#}(A - BK)c + \ldots\ldots$

2. The equation on line 8 should read: $2(Q + K'K + K'B^{\#}(A - BK))c = (A - BK)'(I - B^{\#}B')y$

3. Below the last equation of the page, the sentence "Of course, the only choices of y that are interesting are those which lead to a positive semi-definite Q." should be added.

Page 8: The last sentence should read: "where C is any nxn matrix such that $L > 0$.* The footnote reads: *Necessary and sufficient conditions in terms of K and B are given in [3].