LONG-RUN PLANNING FOR CAPITAL AND LABOUR ALLOCATION IN SPACE

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PREFACE

Methodologically, the IIASA research program on Regional Development reflects the general attitude of the majority of regional scientists. Among other things, this means that the models developed deal with discrete sets of regions or locations. For specific planning purposes, this approach is extremely efficient, due to computational advantages. On the other hand, systematic information about regional structures, of the geometric flavor associated with classical location theory, is hard to obtain if one discretizes space from the outset.

To complement this main stream of regional analysis, two scientists currently trying to revive continuous modeling of the space economy, Martin J. Beckamnn and Tõnu Puu, were invited to IIASA in September 1979. They started writing a comprehensive monograph intended to present the state-of-the-art in the field of continuous regional modeling. The completion of such an extensive work was not possible in the brief period of three weeks.

The authors currently continued to work on the project. The present paper by Tõnu Puu is one chapter of the forthcoming monograph, and it was finished during his renewed visit to IIASA in April 1982.

It deals with planning models for the allocation of available labor and capital resources within a continuous two-dimensional space economy. The main results of the paper concern the advantages of specialization and trade, in the absence of even comparative advantages or localized input supplies. So, the usual conditions for trade, as developed in general (spaceless) economic theory, are not needed, as specialization and trade seem to develop from the nature of bounded two-dimensional space itself.

Moreover, the close parallel between the planning and competitive equilibrium solutions is brought out.

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TABLE OF CONTENTS

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Introduction

The model presented here is designed to handle the following planning problem. We deal with a geographical region of given shape and given extension. Considered is a number of different productive activities, represented by linearly homogeneous production functions, allowing smooth substitution among inputs. In order to emphasize the advantages of geographical specialization, even in the absence of localized input supplies, we assume the same production functions to apply at all locations.

There is a local utility function, dependent on the quantities of produced goods available for consumption, and the goal is to maximize the total utility obtained by aggregation with respect to all locations.

The means by which we obtain the maximum are the proper distributions of given aggregates of capital and labour among locations and among productive activities. The third classical input, land, is immobile and hence we only consider the division of land between various activities at each location.

Local consumption may differ from local production for any good and hence we need to specify commodity flows and a production of transportation services. Transportation, of course, also uses up inputs. More specifically, we assume that transportation services, being of a very specific type, are produced by a Leontief technology without substitution and that only capital and labour, but not land are used.

This is fairly realistic if we consider the transportation costs in term of wear of vehicles, fuel and drivers' services. The inputs embodied in the existing network of roads are not taken into explicit consideration, as the planning of a new network has an even more long-run character than the planning of an optimal spatial distribution of production activities.

It should be stressed that housing is included in the productive activities considered. A flow of housing, which might seem to lack sense, simply means that workers live at other locations than those of their occupation. Whether the commodities are physically moved to the consumer or the consumer moves in order to consume housing or public services is of no importance. We can either consider a movement of consumers or a movement of services provided that we account correctly for the costs.

The main outcome of the analysis is a principle of geographical specialization in contrast to the possibility of producing everything locally without any interregional trade. This specialization occurs in the absence of even comparative advantages, as the same productive possibilities are available everywhere.

Noteworthy is that the main conclusions are independent of which utility function we postulate.

The mathematical paradigm is that of a continuous two-dimensionsal space where we consider areal densities of consumption, production and inputs. For land, these areal densities, of course, are fractions that at any location add up to a given constant, at most unity for all space-consuming activities. All these areal densities are assumed to be smooth functions of the space co-ordinates.

In the same way the flows of goods are regarded as continuous flows in the plane. They take paths that minimize transportation costs between any pair of locations. The structure of roads is represented by a location-dependent, but direction-independent, need of capital and labour, and transportation cost is the line integral of the costs for inputs at all the locations traversed by the route. The optimal paths are thus obtained by solutions to Euler equations for well-defined variational problems.

The continuous flow concept also means that knowing the optimal flow directions we can tie the local changes of flow volumes to the local excess supplies.

The Model

Let x_1,x_2 denote the space co-ordinates. We deal with a region A of two-dimensional Euclidean space, bounded by a simple smooth curve ∂A. Unless the contrary is stated, all the variables introduced are functions of x_1,x_2. Surface integrals are taken over all of A and curve integrals along the boundary ∂A, again unless the contrary is stated.

We deal with n different commodities (goods or services, including housing, but not transportation). If the quantities of these commodities available for consumption at a given location x_1,x_2 are $q_1,\ldots,q_n$ then the local utility is $U(q_1,\ldots,q_n, x_1,x_2)$ and the total utility to be maximized is

$$\iint (U(q_1,\ldots,q_n,x_1,x_2)dx_1dx_2 \qquad (1)$$

The explicit inclusion of the space co-ordinates makes it possible to put different weights on consumption in various locations.

For the purpose of exemplification we simply delete these x_1,x_2 and put the utility function into the form $\Sigma\varepsilon_i \ln q_i$. Unless the contrary is stated all summations run over $i=1,\ldots,n$.

Let k_i,l_i,m_i denote the areal densities of capital, labour and land used in the i:th productive process at a given location x_1,x_2. The linearly homogeneous production functions are then

$$f^i(k_i,l_i,m_i) \equiv m_i f^i(k_i/m_i,l_i/m_i,1). \qquad (2)$$

Unless the contrary is stated expressions written for some index i are assumed to hold for all $i=1,\ldots n$. As the space co-ordinates are not explicitly included we assume that the same production possibilities are open everywhere. For exemplification we put $f^i = A_i k_i^{\alpha_i} l_i^{\beta_i} m_i^{\gamma_i}$ where the exponents sum up to unity.

Local excess supplies are

$$f^i(k_i, l_i, m_i) - q_i \tag{3}$$

These excess supplies must enter the commodity flows or, if negative, be withdrawn from them. We denote the commodity flows by Φ_i. These flows are vector fields, i e Φ_i are two-dimensional vectors with the components being functions of the space co-ordinates x_1, x_2. A vector field, of course, has both direction and magnitude. The direction is simply the actual direction of the flow considered and the magnitude is the quantity of commodities shipped in the flow.

Due to one of the basic theorems in vector analysis, Gauss's divergence theorem, the divergence of a vector field represents source density of an incompressible flow such as the transportation of commodities. The source density, of course, is the local excess supply and we may hence write, in view of (3),

$$\text{div } \phi_i = f^i(k_i, l_i, m_i) - q_i \tag{4}$$

As mathematically the divergence of a vector field equals the partial derivative of its first component with respect to the first space coordinate plus the partial derivative of its second component with respect to the second space co-ordinate, (4) are partial differential equations for the magnitudes $|\phi_i|$ of the vectors as soon as the flow directions $\phi_i/|\phi_i|$ and the excess supplies in the right hand sides are known. We will return to the determination of the flow directions.

As stated in the introduction, the transportation of goods uses up capital and labour inputs, say $\kappa_i|\phi_i|$ and $\lambda_i|\phi_i|$ respectively. The κ_i and λ_i are given functions of the space coordinates and reflect the structure of fixed transportation capacity provided by the existing road network. The linear dependence on flow magnitudes means that we presently abstract from congestion. This simplifies analysis a lot. A non-linear dependence on $|\phi_i|$ is not difficult to handle, but the interference of the different flows makes the degree of analytical complication grow disproportionately to the increase in realism.

If there are given aggregate resources of capital and labour, denoted K and L, we arrive at the following constraints:

$$\iint\Sigma(k_i+\kappa_i|\phi_i|)dx_1dx_2 = K \quad (5)$$

$$\iint\Sigma(l_i+\lambda_i|\phi_i|)dx_1dx_2 = L \quad (6)$$

Production uses up k_i units of capital and l_i units of labour and transportation uses $\kappa_i|\phi_i|$ units of capital and $\lambda_i|\phi_i|$ units of labour. Summing over all commodities and integrating over all locations yields the total usage of these inputs.

As mentioned in the introduction we assume that we are completely free to plan to move capital and labour between locations and between activities.

As for land it may only be transferred between activities. So

$$\Sigma m_i = m \quad (7)$$

where m is a positive, at most unitary, location-dependent number. In general it is less than unity, as some space has already been used up in constructing the given fixed transportation capacity or is otherwise not available for further exploitation.

We thus have a well-defined optimization problem, i e to maximize (1) subject to the constraints (4), (5), (6) and (7) by choosing the appropriate scalar fields k_i, l_i, m_i and q_i and the vector fields ϕ_i.

This will be accomplished by a Lagrangean method. We associate Lagrange multipliers: p_i with (4), r with (5), w with (6) and g with (7). At present they are only undetermined multipliers, but the notation indicates that they turn out to be shadow prices for goods, rent of capital, wage rate and land rent respectively. They can also be interpreted as equilibrium prices in a competitive system with individually optimizing agents.

Optimum for Production

We will now derive the optimum conditions, starting with those obtained by maximizing with respect to k_i, l_i and m_i. They obviously are:

$$p_i\ f^i_k(k_i,l_i,m_i) = r \tag{8}$$

$$p_i\ f^i_l(k_i,l_i,m_i) = w \tag{9}$$

and

$$p_i\ f^i_m(k_i,l_i,m_i) = g \tag{10}$$

We recognize them as the common marginal conditions for profit maximizing firms. With production functions, homogeneous of degree one, the marginal productivity functions, f^i_k for capital, f^i_l for labour, and f^i_m for land, become homogeneous of degree zero. So, taking the first two marginal conditions alone we get the system:

$$f^i_k(k_i/m_i,l_i/m_i,1) = r/p_i \tag{11}$$

and

$$f^i_l(k_i/m_i,l_i/m_i,1) = w/p_i \tag{12}$$

This system (11) - (12) certainly is smoothly invertible as the Jacobian in non-zero due to second-order conditions for profit maximization. So, by the inverse function theorem, we get

$$k_i/m_i = F^i_k(r/p_i,w/p_i) \tag{13}$$

and

$$l_i/m_i = F_l^i(r/p_i, w/p_i) \tag{14}$$

As (10) can be put into the form

$$f_m^i(k_i/m_i, l_i/m_i, 1) = g/p_i \tag{15}$$

we obtain, by substituting from (13) and (14)

$$g/p_i = f_m^i(F_k^i(r/p_i, w/p_i), F_l^i(r/p_i, w/p_i), 1) \tag{16}$$

which links product price to the three input prices.

The conclusion is that, if capital rent, wages, and land rent are given, (16) determines the prices of all produced goods at all locations, provided that production is to take place. This is an important conclusion to be used later on.

The equivalence of the optimum conditions presented and the profit-maximizing conditions for an individual firm at a given location are obvious from the following considerations.

Say that a firm has to maximize its profits by choosing an appropriate mix of productive activities. Capital and labour services are freely available at the local prices r and w, whereas the firm owns a fixed amount of land m available for all its activities. For capital and labour the optimum conditions at given product prices p_i are (8)-(9) or equivalently (11)-(12). We can then invert the system to obtain (13)-(14).

Substituting (13)-(14) into the production function and using (2) we get the profits of the firm as

$$\Sigma\{p_i f^i(F_k^i(r/p_i,w/p_i),F_l^i(r/p_i,w/p_i),1) - r\,F_k^i(r/p_i,w/p_i) - w\,F_l^i(r/p_i,w/p_i)\}m_i \tag{17}$$

This is to be maximized subject to the constraint (7) on the total quantity of land available. In view of the fact that both the maximand and the constraint are linear, the solution is to put $m_i = m$ for that i which maximizies

$$p_i f^i - r\,F_k^i - w\,F_l^i \tag{18}$$

and $m_i = 0$ for the other activities. If several activities are to be profitable (18) must be equal for all these. This common value could be called g which is hence the profits imputed to the land-owning firms as land rent. If all activities should take place we get

$$p_i f^i - r\,F_k^i - w\,F_l^i = g \tag{19}$$

for all i.

In view of Euler's theorem for homogeneous functions,

$$f^i = f_k^i\,k_i + f_l^i\,l_i + f_m^i\,m_i \tag{20}$$

and using (7), (11)-(12), and (13)-(14), we see that (19) is exactly the same as (16). This establishes the local equivalence of profit maximization and overall planning.

There, however, is an additional information to be obtained from the conditions (8)-(9), namely, that in view of the fact that (5)-(6) are integral constraints, the associated Lagrange multipliers r and w are to be constant with respect to the space co-ordinates. This means that the efficiency conditions for allocating capital and labour in space require capital rent and wage rate to be constant with respect to location.

This is not true for land rent, g, as it is a Lagrange multiplier for the constraint (7) which is local, i e not in integral form.

The conclusion of all this is that (16) determines all the p_i for which production is to take place, and that the variations of the production opportunity prices in space are determined by the variation of land rent alone, capital rent and wages being spatially invariant due to distributive efficiency requirements.

Optimum for Flows

We next turn to the optimum conditions for the commodity flows, i e, to the maximization of (1) with respect to the ϕ_i, given the constraints (4)-(7). The flows appear in two ways in the constraints, namely by $|\phi_i|$ in (5)-(6), and by div ϕ_i in (4). The Lagrange multipliers associated with these constraints are the p_i and the r and w. The optimum conditions expressed as Euler equations are

$$(r\kappa_i + w\lambda_i)\phi_i/|\phi_i| = \text{grad } p_i \qquad (21)$$

These conditions mean that the flow directions $\phi_i/|\phi_i|$ agree with the directions, grad p_i, of steepest increase of p_i and that along the flow lines the p_i increase at a rate of $(r\kappa_i + w\lambda_i)$. We recall that κ_i and λ_i were the local requirements of capital and labour for transportation of a unit of the i:th commodity. Accordingly $(r\kappa_i + w\lambda_i)$ is the local cost for transportation. As p_i were interpreted as product prices, (21) simply tells that each commodity flow takes the direction of the steepest increase of its price and that in this direction prices increase by transportation cost. This makes good economic sense.

From the previous section we concluded that an efficient distribution of capital and labour on the region requires capital rent and wage rate to be location-independent. In passing, we can note that this can be interpreted in market equilibrium terms by saying that when capital and labour are free to move they seek the place of production where the reward is the highest. In the absence of relocation costs this equalizes factor prices in space.

The consequence of this, and of the fact that κ_i and λ_i were given functions of the space co-ordinates, is that the increases of prices along the optimal routes are given functions of the space co-ordinates. In fact, we obtain from (21)

$$|\text{grad } p_i| = r\kappa_i + w\lambda_i \qquad (22)$$

These are partial differential equations for the prices p_i with the right hand sides given fucntions of the space co-ordinates.

Specialization

We are now in the position to prove a general specialization theorem. From (16) we see that with r and w given, g and p_i are related by continuous one-to-one mappings as long as the Jacobians of the systems (11)-(12) are non-zero, which we assume according to traditional economic theory. We could write (16) as:

$$p_i = p_i(g) \qquad (23)$$

From these, we obviously get $|\text{grad } p_i| = p_i'(g)|\text{grad } g|$.

In (22), the right hand sides are given functions of the space co-ordinates, say $r\kappa_i + w\lambda_i = \Theta_i h(x_1,x_2)$. The θ_i can be interpreted as characteristic constants for each good. This is so because it is a reasonable simplification to assume that if the shipping of one good costs twice as much as the shipping of another good at one location, the the same relation will hold everywhere in the region.

Hence, equating the two expressions for $|\text{grad } p_i|$, we get:

$$\Theta_i \, h(x_1,x_2) = p_i'(g) \, |\text{grad } g| \tag{24}$$

These conditions can hold for several commodities, say the i:th and the j:th, only if the ratios $p_i'(g)/p_j'(g)$ take the constant value θ_i/θ_j everywhere. But there is no reason whatever why the $p_i'(g)$ functions should be linearly dependent. After all, they were obtained from (16), which depended on the various independent production functions.

So, we conclude that with goods that are transported, only one commodity will be produced in each point of the region. The continuity of the production function and a non-zero Jacobian to system (11)-(12) guarantee that this specialization will not only apply to sets of measure zero like isolated points or curves, but will split the region into a finite collection of subregions of nonzero areas with specialized activity in each. The land rent in each of these regions will be determined by the local revenue shares for these specialized activities.

The reader should note the affinity of our conclusion to v. Thünen's theory, where specialization in concentric rings occurs, despite the fact that there are no localized productivity differences. In general economic theory, trade is supposed to occur only when there are at least comparative localized advantages, due to immobility of inputs. In our model there are no such advantages. Nevertheless, specialization occurs. The reason is that when there are numerous outputs ultimately produced from a few primary inputs, then output prices are tied to the few input prices. In order that production of all the outputs should be equally profitable, their prices must co-vary spatially in a very specific way. On the other hand, the prices of transported goods co-vary in another specific way. The result is a specialization pattern that is inherent in two-dimensional space itself. It is not surprising then that this point is missed in trade theory as general economics lacks the spatial dimension.

Independence of Utility Functions

Before continuing we should just observe the fact that the optimality conditions for production and transportation are independent of the utility function (1). Hence, no matter how we evaluate the availability of the various commodities in different locations, the following conclusions apply: Labour and capital should seek the locations of best reward, which with free mobility equalizes capital rent and wages over space. Production should everywhere be so arranged as if landowning firms tried to maximize their profits, which must equal local land rents. Commodity flows should take the directions in which prices increase most steeply, and the price increases in these directions should equal local transportation costs. The result is such that, if there are commodity flows, then there should be specialization in the production of only one commodity at each location.

These conclusions resulted from the consideration of a planning problem constrained by available resources. But the result could equally well be interpreted in terms of rationally behaving individual workers, capitalists, landowning producers, and transporters in a state of general equilibrium. In particular, the conclusions are independent of which social utility function $U(q_1,\ldots,q_n,x_1,x_2)$ we use.

The only optimality conditions in which this function plays a role are:

$$\partial U/\partial q_i = p_i \tag{25}$$

stating that marginal utility should equal price everywhere. The conditions (25) pose a set of additional constraints on the model relating local commodity prices to local consumption of goods.

A similar result is obtained by considering the behaviour of individual consumers, disposing of their incomes so as to maximize their individual utility functions. The demand functions thus obtained are similar in structure to the inverted system (25), but care should be taken that in the planning case, local budget constraints might not be automatically fulfilled. So if we still wish to admit autonomy of the consumers we might have to consider an interregional income transfer policy as a means of fulfilling the planning purposes. This, however, is the only point where a contradiction between planning and market equilibrium could arise.

Macro Relations

We next establish a number of macro relations of the model. Observe that, due to a general formula in vector analysis,

$$\operatorname{div}(p_i\phi_i) = (\operatorname{grad} p_i)\phi_i + p_i \operatorname{div} \phi_i \tag{26}$$

hold identically for any scalar field p_i and any vector field ϕ_i.

From Gauss's divergence theorem it now follows that

$$\iint \operatorname{div}\,(p_i\phi_i)dx_1\, dx_2 = \int p_i(\phi_i)_n\, ds \tag{27}$$

The left hand double integral is taken on all of the region, whereas the right hand line integral is taken along the boundary of the region. The $(\phi_i)_n$ are the components of the vector fields ϕ_i normal to the boundary. Hence, $p_i(\phi_i)_n$ depending on sign have the simple interpretations of value exports or imports across the boundary. The line integrals take care of all flows across the whole boundary and hence the right hand sides of (27) equal net exports from the region. Let us therefore define:

$$X_i - M_i = \int p_i(\phi_i)_n\, ds \tag{28}$$

Next we should note that, due to (21),

$$(\operatorname{grad} p_i)\phi_i = (r\kappa_i + w\lambda_i)\, |\phi_i| \tag{29}$$

The right hand expression is the product of local transportation costs, as evaluated by the input requirements and the local factor costs, and the quantities of commodities shipped. Taking the double integral of (29) we certainly arrive at the total transportation costs, denoted T_i. Thus

$$\iint (\text{grad } p_i)\phi_i \, dx_1 \, dx_2 = T_i \tag{30}$$

On the other hand (4), along with the well-known fact that with linearly homogeneous production functions all revenues are distributed as factor shares, i e, $p_i f^i = rk_i + wl_i + gm_i$, yields

$$p_i \text{ div } \phi_i = rk_i + wl_i + gm_i - p_i q_i \tag{31}$$

Denoting in aggregate for a given branch, capital incomes by R_i, wages by W_i, the profits of landlords by G_i, and the value of consumption at local prices by C_i, we get:

$$\iint p_i \text{ div } \phi_i \, dx_1 \, dx_2 = R_i + W_i + G_i - C_i \tag{32}$$

Now, integrating both sides of (26) and substituting from (27)-(28), (30) and (32) we finally have:

$$X_i - M_i = T_i + (R_i + W_i + G_i) - C_i \tag{33}$$

i e, net value exports for each branch equals factor incomes plus transportation costs minus consumption.

If we now sum over all the various branches, we can define $X - M = \sum(X_i - M_i)$, $T = \sum T_i$, $G = \sum G_i$ and $C = \sum C_i$. But with capital and labor income we have to remeber that not all of these inputs are accounted for in (33). Due to (5) and (6) some quantities are used in transportation. We have not accounted for the incomes of the transporters yet. Hence, $\Sigma(R_i + W_i) = R + W - T$. The result is then:

$$X - M = R + W + G - C \tag{34}$$

which simply means that in value terms net exports equal factor incomes minus consumption.

In a regional economy with zero balance of payments where it holds that $X = M$, so that net imports of some goods are bought by net exports of other ones, we conclude that aggregate factor incomes sum up to the value of aggregate consumption. This is not a trivial conclusion because both incomes and consumption are evaluated at local prices.

The result establishes an aggregate budget constraint for the economy and hence the model is consistent with consumer autonomy and locally fulfilled budget constraints. Consistency, however, does not guarantee local fulfillment of budget constraints for any state we wish to consider. But it establishes that, if a socially desirable spatial organization of the region does not lead to local fulfillment of budget constraints, we can always design an appropriate completely internal income transfer policy that makes budget constraints hold locally and admits free choice for the consumers.

Examples

We now supply two examples of spatial organization patterns possible with the model outlined.

Assume first that the fixed transportation capacity is equally distributed in space so that all the κ_i and λ_i are constants. Due to the constancy of r and w, we conclude that the local transportation costs $\Theta_i = (r\kappa_i + w\lambda_i)$ as well are invariants in space.

Put all $\phi_i/|\phi_i| = \pm \text{grad } \rho = \pm(x_1/\rho, x_2/\rho)$, where $\rho = \sqrt{(x_1^2 + x_2^2)}$. If $p_i = \bar{p}_i + \Theta_i|\rho - \rho_i|$, we see that equation (21) is fulfilled. The flows all become radial and the constant price contours become concentric circles. This suggests a production specialization structure in concentric rings, as in the familiar v. Thünen case. The difference is that there is not a single CBD in the centre to which all commodities flow. Rather the whole region is supplied by commodities produced in each ring.

The case is illustrated in Figure 1 where we, for illustrative purposes, show a four-commodity model with acitivities called: public services (S), industry (I), housing (H), and agriculture (A).

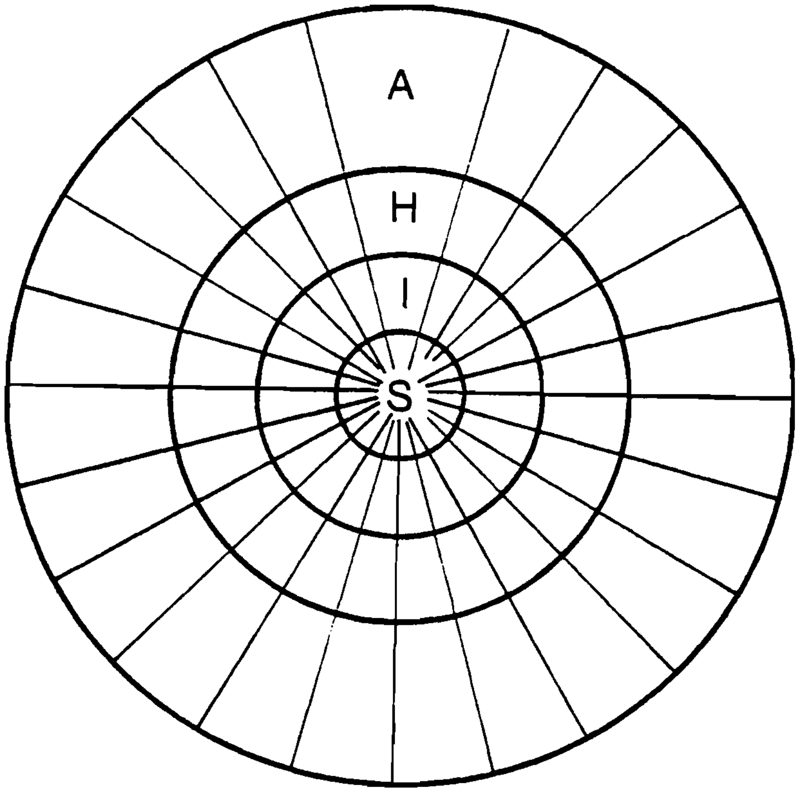

Figure 1. Ring-shaped spatial organization.

For the second example, we suppose that fixed transportation capacity is not equally distributed in space but rather concentrated to the central parts of the region. Suppose that all the κ_i and λ_i are proportionate to ρ, where again $\rho = \sqrt{(x_1^2 + x_2^2)}$. Thus we can write local transportation cost as $(r\kappa_i + w\lambda_i) = \Theta_i\rho$ where again the Θ_i are constants.

We can now put all $\phi_i/|\phi_i| = \pm \text{grad } \frac{1}{2}(x_1^2 - x_2^2) = \pm (x_1,-x_2)$. If we let $p_i = \bar{p}_i + \Theta_i|\frac{1}{2}(x_1^2 - x_2^2) - \bar{\sigma}_i|$ we see that (21) is again fulfilled. The flow lines integrate to hyperbolas, xy = constant, and the constant price lines are hyperbolas, $(x^2 - y^2)$ = constant, rotated by 45° in comparison to the paths.

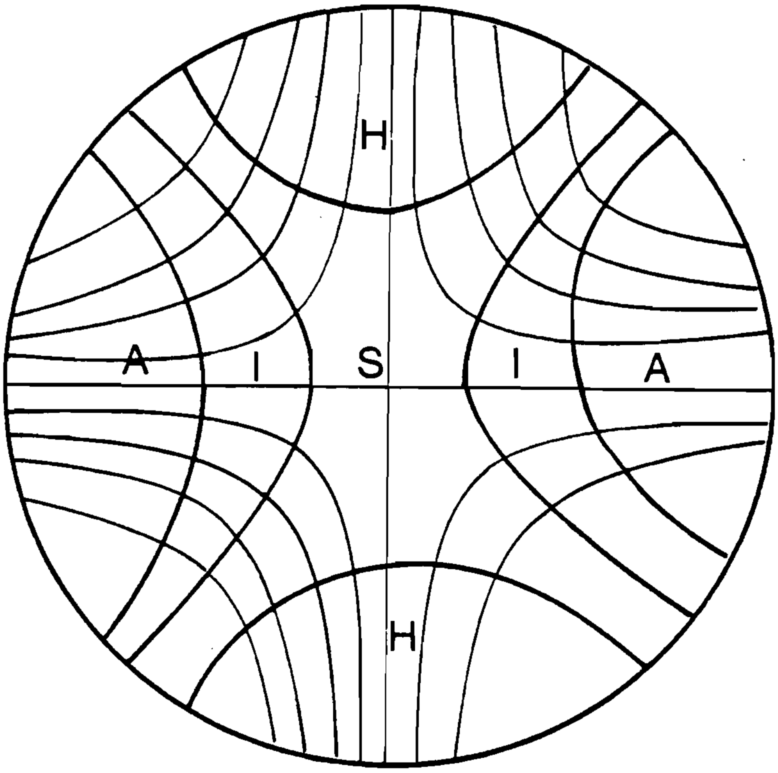

Figure 2. Sectoral spatial organization.

These illustrations are in perfect agreement with the optimality conditions stated. They are not chosen at random, but represent spatial organization around singularities of the only types admitted under the assumption of structural stability. We are not going to repeat the discussion about this from the preceding chapter. The considerations of structural stability apply to the planning case, as do the considerations in this chapter on specialization to the equilibrium case.

Intermediate Goods

The preceding analysis does not consider intermediate products. In particular, it would be interesting to know whether the specialization theorem would still hold even if it implies that an output could be shipped to another place to produce something that is re-imported rather than produced at the place itself.

In fact, it still holds, as will be shown now for the case of a Cobb-Douglas technology. Put:

$$f^i = A_i\, k_i^{\alpha_i}\, l_i^{\beta_i}\, m_i^{\gamma_i} \prod_j (f_i^j)^{\varepsilon_{ij}} \qquad (35)$$

where f_i^j denotes the quantity of the output j used as input in the production of output i. The product in (35) is taken over all indices j from 1 to n. Linear homogeneity now means that:

$$\alpha_i + \beta_i + \gamma_i + \sum_j \varepsilon_{ij} = 1 \qquad (36)$$

The optimum conditions corresponding to (8)-(10) obviously are

$$\alpha_i\, p_i\, f^i/k_i = r \qquad (37)$$

$$\beta_i\, p_i\, f^i/l_i = w \qquad (38)$$

$$\gamma_i\, p_i\, f^i/m_i = g \qquad (39)$$

and

$$\varepsilon_{ij}\, p_i\, f^i/f_i^j = p_j \qquad (40)$$

We can substitute back from (37)-(40) into (35) and obtain, in view of (36),

$$A_i\, \alpha_i^{\alpha_i}\, \beta_i^{\beta_i}\, \gamma_i^{\gamma_i} \prod_j (\varepsilon_{ij})^{\varepsilon_{ij}}\, p_i = r^{\alpha_i}\, w^{\beta_i}\, g^{\gamma_i} \prod_j (p_j)^{\varepsilon_{ij}} \qquad (41)$$

Taking logarithms we get a set of n linear equations in the logarithms of the (n + 3) prices. Regarding r, w, and g as given, we can solve for the logarithms of all the p_i, as the matrix of the system, $[\varepsilon_{ij} - \delta_{ij}]$, where δ_{ij} is the Kronecker delta, is non-singular. Accordingly, the ln p_i are obtained as explicit linear expressions of ln r, ln w, and ln g. After taking exponentials and substituting, we transform (41) into the explicit form:

$$p_i = B_i\, r^{\hat{\alpha}_i}\, w^{\hat{\beta}_i}\, g^{\hat{\gamma}_i} \tag{42}$$

where the B_i, as well as the exponents, are constants that can be calculated from the original constants in (35).

Consider now a proportional change of r, w, g, and all the p_i. Then obviously the solution to (37)-(40), whatever it is, is unchanged and so (35) is still fulfilled. This demonstrates that (42) must hold for proportionate changes in all prices, i e, that:

$$\hat{\alpha}_i + \hat{\beta}_i + \hat{\gamma}_i = 1 \tag{43}$$

We can hence substitute back from (37)-(39), disregarding (40) altogether, and obtain:

$$f^i = \hat{A}_i\, k_i^{\hat{\alpha}_i}\, l_i^{\hat{\beta}_i}\, m_i^{\hat{\gamma}_i} \tag{44}$$

due to (43). Now these are Cobb-Douglas production functions in the primary inputs only, and they are linearly homogenous in them.

Accordingly, as (44) fulfills the condition (2), the whole reasoning about specialization still holds. This, of course, does not preclude that, if there is a certain hierarchy, so that goods produced at a certain stage are never used in the production of any of their inputs, then the flows should simply take a one-way route to higher levels. With a more complicated interdependence, however, it is possible that goods are re-imported at a later stage of refinement.

Local and Global Optima

We should note that the optimality conditions stated so far are local in character. The determination of the global optimum is a matter whose outcome is likely to change with the boundary conditions.

Our specialization theorem states that in every location of the region there is complete specialization in the production of traded goods. But, if the utility function does not include the space coordinates as explicit arguments, i e, if a certain consumption is equally valued at all locations, then local production and no trade is a solution that fulfills all local optimality conditions. And, since the goods are not traded, the specialization theorem does not exclude this possibility. For certain cases the solution is probably a global optimum as the given input quantities are most efficiently used when no part of them is "wasted" in moving commodities around.

To illustrate, we could as well simplify the model. As trade rather than specialization is at issue we can discuss a one-commodity economy. We can also put the production function for this commodity in a Cobb-Douglas form, and assume the utility function to be logarithmic, and without explicit dependence on the space coordinates. Finally, we do not specify any production technology, but assume in the traditional v Thünen way that the product may be used up in transportation. We normalize the unit of distance so that the cost of moving one unit of goods one distance unit uses up exactly one unit of them.

So, we have the following problem: Maximize

$$\iint \ln q \, dx_1 dx_2 \tag{45}$$

subject to

$$K = \iint k \, dx_1 dx_2 \tag{46}$$

$$L = \iint l \, dx_1 dx_2 \tag{47}$$

and

$$k^{\alpha} l^{\beta} - q - |\phi| = \text{div}\ \phi \tag{48}$$

The optimality conditions are then:

$$1/q = p \tag{49}$$

$$\alpha\ k^{\alpha} l^{\beta}/k = r/p \tag{50}$$

$$\beta\ k^{\alpha} l^{\beta}/l = w/p \tag{51}$$

and

$$\phi/|\phi| = \text{grad}\ \ln p \tag{52}$$

In these conditions r and w are independent of the location coordinates, whereas p is not. The conditions state that:

(i) Local marginal utility is everywhere equal to opportunity cost for goods in the flow.

(ii) Production is everywhere so arranged that marginal value products of the inputs equal their local opportunity costs. With constant r and w, these opportunity costs are equal in space and there is no incentive to relocate inputs.

(iii) The flow of traded goods is in the direction of steepest price increase and the rate of increase in this direction is exponential as moving one unit of goods uses up one unit of its own value.

We see that (49)-(51) determine inputs, k and l, output, $k^{\alpha} l^{\beta}$, and consumption, q, once r, w, and p are known. As r and w take constant values, determined by the constraints (46)-(47), we see that p completely determines the spatial densities.

So, let us pick any function $p(x_1,x_2)$ such that $|\text{grad } \ln p| = 1$. Then (52) is fulfilled and $\phi/|\phi| = (\cos\theta, \sin\theta)$ is a known unit vector field. As $\text{div } \phi = \text{grad } |\phi| \cdot (\cos\theta, \sin\theta) + |\phi| \text{ div}(\cos\theta, \sin\theta)$, (48) becomes a partial differential equation in the flow intensity $|\phi|$. The solution of this differential equation solves the whole problem. Hence we have seen that any price structure such that:

$$|\text{grad } \ln p| = 1 \tag{53}$$

holds can represent a sensible local optimum. We will now illustrate the matter by two different solutions.

First, put $\rho = \sqrt{(x_1^2 + x_2^2)}$ and $\phi/|\phi| = \text{grad } \rho$. This flow obviously satisfies (52) for $p = e^{\rho}$. Assuming now that $\alpha = \beta = r = w = 1/4$ we get from (49)-(51): $k^{\alpha}l^{\beta} = p$, and $q = 1/p$. Thus: $k^{\alpha}l^{\beta} - q = e^{\rho} - e^{-\rho} = 2 \sinh \rho$. This result can be substituted into (48). But $\text{div } \phi = \text{grad } |\phi| \cdot \text{grad } \rho + |\phi| \text{ div grad } \rho$. Using polar coordinates, $x_1 = \rho \cos\omega$ and $x_2 = \rho \sin\omega$, we easily get $\text{grad } |\phi| \cdot \text{grad } \rho = d|\phi|/d\rho$. Moreover, $\text{div grad } \rho = 1/\rho$. Thus (48) becomes an ordinary linear differential equation:

$$\frac{d|\phi|}{d\rho} + \left(1 + \frac{1}{\rho}\right)|\phi| = 2 \sinh \rho \tag{54}$$

Given a simple boundary constraint the equation is readily solved. The spatial organization associated with this solution is one where goods flow radially outward and price increases at an exponential rate in that direction, whereas consumption is decreasing outwards and production is increasing outwards.

As in the example chosen, excess supply is zero at the origin and decreases outward, the case is imcompatible with an outward flow unless there is a singularity with net outflow at the origin.

Second, we easily see that, putting k, l, $k^{\alpha}l^{\beta} = q = 1/p$ constant, and $|\phi|$ equal to zero identically, we get all the equations fulfilled and so this case of no trade and local production is another local optimum. It is hard to tell which of the two cases is global optimum.

The reader might ask whether there are always just two local optimum: one with trade and one without. In fact, it is easy to find cases with more than two local optima. Let us change the model (45)-(48) by assuming that the cost of movement is not the same everywhere in the region, but that it increases in proportion to the distance from the origin, so that the communications are best in the center and become worse at the periphery. Thus, we suppose that $\rho|\phi|$ units of the goods are used up in moving one unit of goods one distance unit. Then (48) is changed to:

$$k^{\alpha} l^{\beta} - q - \rho|\phi| = \text{div } \phi \qquad (55)$$

Only (52) in the optimality conditions is changed by this and takes the form:

$$\rho \ \phi/|\phi| = \text{grad } \ln p \qquad (56)$$

In accordance with this, (53) is changed to:

$$|\text{grad } \ln p| = \rho = \sqrt{(x_1^2 + x_2^2)} \qquad (57)$$

We can now easily find at least three different solutions to (55), namely p = constant and $p = \exp((\pm x_1^2 \pm x_2^2)/2)$. The latter are actually four cases, but discarding only reversals of flow directions we are left with two qualitatively different flows, one radial and one saddle flow. It is easy to see that the nontrade, the radial, and the saddle flows are all local optima.

Again it is hard to tell which one is a global optimum without considering the boundary conditions.

This multiplicity of local optima did not occur in our equilibrium model in the preceding chapter, as a price-flow distribution on the boundary was taken as given from world market conditions. To the extent one finds it reasonable to use an analogous boundary condition in the planning problem the arbitraryness will be removed. This might be reasonable, as an acceptance of the trade conditions, determined by trade outside the region studied, might lead to a maximum benefit from trade with the exterior.

Boundary Constraints

Let us consider this from a more formal point of view. From (55) we see that

$$pk^{\alpha}l^{\beta} - pq = p \text{ div } \phi + p\rho|\phi| \tag{58}$$

But, from (56), $p\rho|\phi|$ = grad $p \cdot \phi$. Substituting this, and using the identity div$(p\phi)$ = grad $p \cdot \phi + p$ div ϕ, we get

$$\iint(pk^{\alpha}l^{\beta} - pq)dx_1dx_2 = \iint \text{div}(p\phi)dx_1dx_2 \tag{59}$$

The right hand side, due to Gauss's theorem, equals the curve integral $\int p(\phi)_n$. This, however, is zero in two cases: When $(\phi)_n$ vanishes identically on the boundary, and when it does not vanish, but trade with the exterior balances. Obviously, we only need to be concerned with the two cases of either insulation or balancing interregional trade.

Putting the right hand side of (59) equal to zero yields

$$\iint pk^{\alpha}l^{\beta}\ dx_1dx_2 = \iint pq\ dx_1dx_2 \tag{60}$$

Accordingly the aggregate value of output equals the aggregate value of consumption. Now, the optimality condition (49) states that marginal utility equals product price. With our logarithmic utility function we have $pq = 1$ on the whole region. So, the integrand in the right hand side of (60) being unitary, we conclude that the integral equals the area of the region. Denoting this (the total quantity of land) by M, we get

$$\iint pk^{\alpha}l^{\beta}\ dx_1dx_2 = M \tag{61}$$

Let us next substitute from (50)-(51) into the production function, and solve for

$$k^{\alpha}l^{\beta} = \left(\frac{\alpha}{r}\right)^{\alpha/\gamma} \left(\frac{\beta}{w}\right)^{\beta/\gamma} p^{(\alpha+\beta)/\gamma} \tag{62}$$

where $\gamma = 1-\alpha-\beta$. We see that, r and w being spatial constants, local output is proportionate to a power function of the price p. We can also solve for k and l from (50)-(51) and integrate to obtain:

$$\frac{\alpha}{r}\iint pk^{\alpha}l^{\beta}\ dx_1dx_2 = K \tag{63}$$

and

$$\frac{\beta}{w}\iint pk^{\alpha}l^{\beta}\ dx_1dx_2 = L \tag{64}$$

Substituting from (61) into (63)-(64) we get $\alpha/r = K/M$ and $\beta/w = L/M$, which can be substituted into (62). The result is

$$k^{\alpha}l^{\beta} = \left(\frac{K}{M}\right)^{\alpha/\gamma} \left(\frac{L}{M}\right)^{\beta/\gamma} p^{(\alpha+\beta)/\gamma} \tag{65}$$

Local output thus is a Cobb-Douglas function of the average areal densities of capital and labour, multiplied by the aforementioned power function of local price.

The relations derived in this section must hold in any case where either there is no trade with the exterior or the exterior trade balances.

We note that output $k^{\alpha} l^{\beta}$ is an increasing function of price. This function is given and identical in all cases that may be considered as candidates for a global optimum. From (49) we, on the other hand, know that consumption q is a decreasing function of price. So, excess supply

$$z = k^{\alpha} l^{\beta} - q \tag{66}$$

certainly is an increasing function of p. Thus, considering two different cases, distinguished by subscripts, we conclude that

$$(p_i - p_j)(z_i - z_j) > 0 \tag{67}$$

must hold at all locations.

Let us now consider two alternative price-flow patterns fulfilling the optimality conditions. Consider the value flows

$$p_i k_j^{\alpha} l_j^{\beta} - p_i q_j = p_i \mathrm{div}\ \phi^j + p_i \rho |\phi^j| \tag{68}$$

It is true that

$$|\phi^j| > (\phi^i / |\phi^i|) \cdot \phi^j \tag{69}$$

as by projecting the vector ϕ^j on the direction $\phi^i / |\phi^i|$ at most results in the norm $|\phi^j|$. So, using the optimality condition (56) for the flow ϕ^i we get from (69)

$$p_i \rho |\phi^j| > \mathrm{grad}\ p_i \cdot \phi^j \tag{70}$$

If we substitute from (70) into (68) we see that the right hand side must at least equal $\mathrm{div}(p_i\phi^j)$. Using the notation z_j for excess supply from (66) we thus get

$$p_i z_j \geq \mathrm{div}(p_i\phi^j) \tag{71}$$

and by integration and use of Gauss's theorem

$$\iint p_i z_j \, dx_1 dx_2 \geq \int p_i(\phi^j)_n \tag{72}$$

with equality when i = j as seen from (66).

For the right hand side of (72) we conclude that it is zero if i = j, as already seen. This results from the trade balance condition. We also conclude that it is zero if both cases considered are cases with trade across the boundary, as then $p_i = p_j$ are determined by the "world market" on the boundary, and the trade balance condition requires the integrals $\int p_i(\phi^i)_n$ and $\int p_j(\phi^j)_n$ to be zero. The same is true when both cases represent insulation, as then $(\phi^i)_n$ and $(\phi^j)_n$ are identically zero.

The only situation where the right hand side of (72) can be nonzero is when the case i represents insulation and the case j represents balancing trade. Then we evaluate the non-zero flow across the boundary in the case of trade at the prices in the case of insulation. We have no reason to expect that an integral like this should be zero.

But let us postpone the discussion of this case for a moment and consider the situation where both cases represent insulation or balancing trade. Then all the right hand sides of (72) are zero however we permute i and j. Recalling that (72) hold as equalitites when i = j we get

$$\iint (p_i - p_j)(z_i - z_j) dx_1 dx_2 \leq 0 \tag{73}$$

The only way a non-positive integral can be obtained from a non-negative integrand according to (67), is by having an integrand that is identically zero, i e,

$$(p_i-p_j)(z_i-z_j) \equiv 0 \qquad (74)$$

As our excess supply function is strictly increasing we conclude that $p_i = p_j$ and $z_i = z_j$ must hold identically. The conclusion hence is that any two solutions, where the optimum conditions are fulfilled, along with the boundary condition, stating either that there is no trade with the exterior or that trade balances, are identical. So, the solution is unique. More specifically, there is a unique solution with trade and a unique solution with insulation.

Let us return to the case where one case, say i, represents insulation and the other, j, represents trade. Then one of the right hand side integrals of (72) need not be zero and accordingly the zero in (68) is replaced by the expression

$$- \int p_i (\phi^j)_n \qquad (75)$$

Should this curve integral be strictly negative, then we are in trouble, as (73) does not hold, and the discussion leading to uniqueness would no longer be valid.

How likely is it that the curve integral in (75) is negative? Negativity obviously means that insulation prices p_i are lower than world market prices p_j where the flow ϕ^j leaves the region, and higher where it enters. The world market prices, on the other hand, are low where the flow enters and high where it leaves the region. This is so as the flow of trade adjusts to the direction of increasing prices. We conclude that spatial price differences in the case of insulation must be smaller than the differences in world market prices.

But, the price differences in the case of insulation are obtained as accumulated transportation costs. As thus world market price differences, between various points on the boundary, are greater than the costs of transportation between them, there seems to be a profit to be obtained from arbitrage across the region. This profit can be converted into an increased consumption in the region.

So, it seems that the planning authorities should open up trade with the exterior when boundary price differences exceed transportation costs. As this case was the only one leading to trouble with the uniqueness proof, we conclude that it holds when the planning authority takes due consideration of trade opportunities with the exterior to the benefit of interior consumption.

Relocation Costs for Capital and Labour

Let us now return to the problem of planning the use of capital and labour in a region, but relax the assumption that relocations of capital and labour are costless. We still have initially given quantities of capital and labour. Now there are not only aggregates, but spatial distributions of these aggregates given. From these initial distributions the future distribution can differ in two ways. First, capital wears out and if it is not completely replaced by new equipment the stock of capital will change, whereas labour stock normally changes with the net reproduction rate. Second, labour and capital can actually be transferred in space by the application of transportation services.

We have to make the assumptions more precise. Suppose we consider only one commodity produced, and that this commodity can be used as consumers' goods or equally well be invested as capital stock.

Capital stock wears out exponentially at a given depreciation rate. Accordingly, local production, minus local consumption, minus local capital depreciation, minus local net capital accumulation is the quantity entered into the flow of capital goods, or, if negative, withdrawn from it. As we focus the interest on capital flows and accumulation, we disregard flows of consumers' goods. If we wished to include them there would be no difficulty in doing so as the model does not distinguish between consumers' goods and capital goods.

Labour stock accumulates with a given net reproduction rate, and the quantity entered into the flow of labour, or, if negative, withdrawn from it is local labour reproduction, minus local accumulation of labour. Again, we disregard short-run phenomena like commuting, and focus interest on migration and labour accumulation.

Production is thus determined by the local labour and capital stocks, per unit land area, or rather, what remains of them after the fixed coefficient transportation technology has withdrawn what is needed for the transportation of capital goods and migrants.

The goal function is now a utility index dependent on local consumption, aggregated on both space and time.

Accordingly, we maximize

$$\iiint U(q)\, dx_1 dx_2\, dt \tag{76}$$

By introducing the space and time coordinates as arguments in the uitility function we can account for temporal and spatial discounting. Of course, q denotes consumption. The rate of consumption as well as the utility index is a continuous and differentiable function of the space and time coordinates.

The production technology is again represented by a neo-classical production function:

$$f(k,l) \tag{77}$$

where k is capital stock used in production of goods and l is labour stock used for the same purpose.

Transportation services are again produced by a Leontief technology of fixed coefficients. We suppose that each unit of flow uses up κ units of capital and λ units of labour. To simplify notation we normalize the units of measurement of capital and labour so that transportation costs for one unit are the same for both flows.

Denoting the flow of capital by ϕ and the flow of labour by ψ, we know that $\kappa(|\phi|+|\psi|)$ units of capital and $\lambda(|\phi|+|\psi|)$ units of labour are withdrawn from the local stocks for production of transportation services. What remains is used in production of goods (for consumption and investment). Thus, denoting the local stocks of capital and labour by K and L respectively, we have:

$$k = K - \kappa(|\phi|+|\phi|) \tag{78}$$

and

$$l = L - \lambda(|\phi|+|\psi|) \tag{79}$$

Suppose capital wears out in proportion to the accumulated stock at the rate α. We then deal with a need of replacement that is αK. Denoting net capital accumulation by $\dot{K}$, where the dot represents a derivative with respect to time, we see that the quantity $\alpha K + \dot{K}$ is withdrawn for investments. As the quantity q is withdrawn for consumption the difference $f(k,l) - q - \alpha K - \dot{K}$ is added to the flow at each location, or, if negative, withdrawn from it. Accordingly,

$$\text{div}\ \phi = f(k,l) - q - \alpha K - \dot{K} \tag{80}$$

For labour the stock increases with the net reproduction rate, denoted β. Thus, the local increase of labour due to reproduction is βL, whereas the local accumulation of labour is denoted $\dot{L}$. The difference enters the migration flow or, if negative, is withdrawn from it. Formally:

$$\text{div}\ \psi = \beta L - \dot{L} \tag{81}$$

The optimization is now a well defined problem. We seek the maximum of (76) subject to the constraints (78)-(81). As a preliminary step we substitute for k and l from (78)-(79) into (80). In this way we dispose of two constraints and of the two substituted variables. Only the constraints (80)-(81) remain (with the substitutions being made). We have to choose consumption, q, capital stock, K, labour stock, L, and the flows of capital and migrants, ϕ and ψ respectively. What we seek are optimal function forms defined on space and time. So, we deal with a variational problem whose solution is obtained in terms of Euler equations. We associate Lagrangean multipliers, p and w with (80) and (81) respectively. It is to be noted that the Lagrangean multipliers are not constants, but change over space and time, due to the fact that the constraints are in local, not aggregate, form.

We can now state the Euler equations for optimality. For consumption we obtain:

$$U'(q) = p \tag{82}$$

For production we obtain the two conditions:

$$p\, f_k = \alpha p - \dot{p} \tag{83}$$

and

$$p\, f_l = -\beta w - \dot{w} \tag{84}$$

For transportation we obtain:

$$p(\kappa f_k + \lambda f_l) \frac{\phi}{|\phi|} = \text{grad } p \tag{85}$$

and

$$p(\kappa f_k + \lambda f_l) \frac{\psi}{|\psi|} = \text{grad } w \tag{86}$$

These conditions are obtained as solutions to a planning problem. But, again, it is not too hard to find interpretations of the conditions in terms of market equilibrium. We note that the left hand sides of (83)-(84) are marginal value productivities of capital and labour. Hence, we expect the right hand sides to be input costs. Due to the fact that the goods produced are also capital goods, p is related to the price of capital stock as well. Compared to the static optimum conditions, we might be surprised to find, not the input prices, but their time derivatives (and the reversed signs).

However, if we assume that the firms are not maximizing their momentary profits, but their accumulated profits on a time interval, then it is obvious that a decrease of input prices in the future should be an incentive to postpone accumulation of capital stock, and so the negative of the time derivative of price is a reasonable measure of temporary input cost. Likewise, the depreciation and the consequent need of replacement of capital is an obvious cost item.

As for labour the net reproduction plays the same role as depreciation of capital, but the sign is reversed. This may seem a bit odd at first. But, a natural increase of the local stock of labour makes it possible to avoid wage raises in the future to attract more immigrants.

To some extent the firm is protected by transportation costs from the surrounding competitors. A local surplus of labour may be assumed to emigrate only if the wage difference is greater than the transportation cost. Likewise, in order to attract immigrants the local wage rate ought to be higher than in the surroundings, the difference again being greater than transportation costs.

So, in terms of dynamic optimum, the conditions (83)-(84) are not without sense in a market economy setting. These conditions come pretty close to those found in the recent theory of "micro-economic foundations of macro-economics", where the firms are supposed to plan their stocks of inputs by designing an appropriate dynamic price policy.

The conditions (85)-(86) are even more easy to interpret in market equilibrium terms. Each unit of flow uses up κ units of capital and λ units of labour. By the marginal productivities we know the sacrifice in terms of goods not produced due to this withdrawal of inputs. The opportunity cost in terms of commodities is $(\kappa f_k+\lambda f_l)$. If we multiply by commodity price p we obtain the monetary opportunity cost $p(\kappa f_k+\lambda f_l)$. This naturally is the local cost of transportation, and so it is natural to find it in the left hand sides of (85)-(86). Accordingly these conditions again tell the familiar story that flows take the direction of steepest price increase, and that the price increases in these directions equal transportation costs. Thus (85)-(86) are conditions of efficient trade and spatial equilibrium.

We notice that we can take squares of both sides of the vector equations (85)-(86) and equate. Then the unit flow fields multiply up to unit scalars and we obtain the equations:

$$p^2(\kappa f_k + \lambda f_l)^2 = (\text{grad } p)^2 = (\text{grad } w)^2 \qquad (87)$$

Next, we see that we can substitute for the marginal productivities from (83)-(84), so that (87) becomes:

$$(\alpha\kappa p - \beta\lambda w - \kappa\dot{p} - \lambda\dot{w})^2 = (\text{grad } p)^2 = (\text{grad } w)^2 \qquad (88)$$

This is a pair of differential equations in the price of commodities and the wage rate. Solving it we know the development of price and wage in space and over time.

By substituting the solutions for price and wage into the right hand sides of (83)-(84) we can then solve for capital, k, and labour, l, used in production. Next, (77) gives the resulting output.

But, knowing capital and labour stocks used in production. we see that total capital and labour stocks, K and L, only depend on flow volumes. Accordingly, the right hand sides of (80)-(81) only depend on flow volumes.

On the other hand, the directions of the flows are gradient to price and wage, as we learn from (85)-(86). Knowing p and w we also know the unit flow fields $\phi/|\phi|$ and $\psi/|\psi|$. But, we know that div ϕ = grad $|\phi| \cdot \phi/|\phi| + |\phi|$ div $(\phi/|\phi|)$ and likewise for ψ. Thus, the left hand sides of (80)-(81) too depend on flow volumes and their gradients only. This means that (80)-(81) supply us with another pair of partial differential equations. Solving them for flow volumes we know all the variables of the model.

Thus, the differential equations (88) contain the initial information from which we can calculate everything else. As soon as we know price and wage rates we know the essential facts of structure in terms of the flow lines and the corresponding potentials, including their development over time. This means that the solution to (88) is extremely interesting in the context of the present model.

In fact, these equations are easy to discuss if we introduce an artifice to separate spatial and temporal aspects of price-wage changes. Let us define a new scale for time and space by putting $t = \delta\tau$, $x_1 = \varepsilon\xi_1$ and $x_2 = \varepsilon\xi_2$. We note that this coordinate change does not distort space, it only introduces a linear change of scale. By letting ε approach zero we magnify the scale so that we are in the limit dealing with a point only. Likewise, by letting δ approach zero we magnify the time scale so that in the limit we are only dealing with conditions at a certain moment of time.

Let us now change the system (88) so that we let the time derivatives be taken in the τ coordinate, whereas the gradients are taken in the ξ_1, ξ_2 coordinates. As a result of this:

$$(\alpha\kappa p - \beta\lambda w - \delta\kappa\dot{p} - \delta\lambda\dot{w})^2 = (\varepsilon \text{grad } p)^2 = (\varepsilon \text{grad } w)^2 \qquad (89)$$

If we now let $\varepsilon \to 0$, $\delta = 1$, then

$$\kappa\dot{p} + \kappa\dot{w} = \alpha\kappa p - \beta\lambda w \qquad (90)$$

whereas if we let $\delta \to 0$, $\varepsilon = 1$, we get:

$$(\text{grad } p)^2 = (\text{grad } w)^2 = (\alpha\kappa p - \beta\lambda w)^2 \tag{91}$$

Equation (90) is a pair of dependent linear differential equations in p and w. It is very easy to solve as we deal with ordinary linear differential equations with constant coefficients. The only fact we need notice is that we can choose one of the functions arbitrarily as far as (90) is concerned.

Likewise, equations (91) are easy to deal with in terms of the qualitative features of two-dimensional flows. This equation separates the spatial aspect so that we can study flow patterns as we do in the stationary cases. Likewise, the former equations separate the temporal aspects, so that we can study the price-wage dynamics in a point economy without spatial extension as in traditional economic theory.