ON THE PRODUCTIVITY CRITERIA OF LEONTIEF MATRICES AND THE CONCEPTUAL VALIDITY OF LABOR VALUES

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ABSTRACT

This paper generalizes some well-known productivity (nonnegative L-invertibility) crtieria defined for nonnegative quadratic input-output coefficient matrices. The new economic criteria cover both the reducible and irreducible cases, treated seaprately until now, and are based on the absence of self-serving production and/or complete automation, which can be viewed as dual concepts. Detailed investigation of these concepts also reveals that their presence is incompatible with the idea of pure market commodity production. In particular, it is shown that the fundamental assumptions of the pure market economy and the indispensibility of labor are sufficient to rigorously prove the existence, uniqueness, and strict positivity of labor values.

CONTENTS

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1. INTRODUCTION

The Leontief-inverse of an input-output coefficient matrix, i.e., $(I - A)^{-1}$, occupies a central position in most input-output analyses. The various possible interpretations of the Leontief-inverse (hereafter abbreviated as the L-inverse) usually require that its elements be nonnegative. Much effort has therefore been devoted to finding sufficient (or necessary and sufficient) conditions for the existence and nonnegativity of the L-inverse. These conditions, generally referred to as the Hawkins-Simon conditions [4], have been discussed fairly comprehensively by Nikaido [7].

From the mathemtical point of view, the nonnegative L-invertibility of a nonnegative input-output coefficient matrix, A, is equivalent to saying that the dominant eigenvalue of A is less than unity [see the Perron-Frobenius theorems on eigenvalues of nonnegative quadratic matrices, for example in Nikaido [7]). Economists, however, are generally more interested in finding economically interpretable and meaningful criteria that guarantee the existence and nonnegativity of the L-inverse. The *productivity* criteria fall into this category.

The term productivity in relation to an input-output coefficient matrix A seems to have been used first by Gale [3] with the following meaning: A is productive if there is some (nonnegative) production vector such that the inequality $x > Ax$ holds.[1] It has been shown that A is productive if and only if A has a nonnegative Leontief-inverse. Thus, from the mathematical point of view, the productivity of A and its nonnegative L-invertibility are again equivalent statements. Indeed, they are used interchangeably in recent works. The productivity criteria can be formulated both in terms of production (primal fashion) and in terms of prices (dual fashion); following Robinson [8], we can refer to these as *technological* (primal) and *economic* (dual) productivity conditions.

Our special interest in the productivity criteria is that they provide a set of conditions whose fulfillment seems, on economic and empirical grounds, to be simpler to check than the ones of more mathematical character. Thus, for example, the technological productivity condition says that if an observed production system is such that it gives rise to a final product in each sector, then its input-output coefficient matrix is necessarily productive. This condition is usually satisfied in

[1] We adopt here the convention of using $>$, $\geq$, and $\geqq$ to denote, respectively, strict, semistrict, and weak inequality relations between vectors or matrices.

the case of aggregated input-output tables. If, however, we think of rather detailed tables, this condition may not hold. Due to foreign trade or the presence of pure intermediaries the total output of some commodities may not exceed their total use in production. Therefore, it is not a general enough condition, if applied only to observed production patterns.

Observe that the technological productivity condition implicitly asserts that there exists a production vector (observed or imaginary) that gives rise to a final product $(x - Ax)$ in *each* sector of the production system. In practice, however, due to foreign trade (imports) and the presence of purely intermediate commodities, this condition is satisfied only in the case of highly aggregated input-output tables. Therefore, it is not a *general* enough condition. On the other hand, if we think of an *imaginary* production bundle (assuming constant input-output coefficients), then the criterion becomes purely *tautological*, in the sense that to check whether such a production bundle can exist poses the same degree of difficulty as to check whether A is nonnegatively L-invertible. It is also worth pointing out that the production vector in question (x) must in fact be *strictly positive* $(x > 0)$, although the condition seemingly assumes only its nonnegativity.

Probably these weaknesses of the productivity conditions and the Perron-Frobenius theorems lead to the formulation of alternative conditions based on the assumption of the *irreducibility* of the input-output matrix (or the production system). This latter concept is quite well known (it is sometimes referred to as indecomposability) and implies an essential and complete interconnectedness of the production system: there is no subsystem capable of operating without the rest of the system. A more formal definition is as follows: $A = (a_{ij})$ is irreducible if and only if, for any pair of indices i,j, there exists a chain of indices $i = k_0, k_1, k_2, \ldots, k_s = j$ such that $a_{k_t, k_{t+1}} > 0$, $\forall$ $t = 0,1,\ldots,s-1$. That is, in economic terms, each sector relies directly ($s = 1$) or indirectly ($s > 1$) on the production of each of the other sectors. Again, in very detailed (commodity-by-commodity) models, such complete interrelatedness of various production activities is not generally a justifiable assumption. Nevertheless, if A is irreducible one can relax the productivity criterion described above somewhat. Namely, A is productive if and only if there is a nonnegative (observed or imaginary) production vector x such that $x \geq Ax$. In addition, the L-inverse is strictly positive. It turns out that in this case, too, x must in fact be strictly positive (see Gale [3]).

Thus, none of the above criteria is general enough; therefore, it is interesting to see whether it is possible to provide, from the economic point of view, less restrictive criteria to guarantee the nonnegative invertibility of the L-matrices. In addition to its general economic-theoretical interest, however, the question is closely related to recent reformulations of various Marxian concepts and propositions concerning labor values, production prices, the balanced production processes, and so on,[2] making use of an input-output framework. In these analyses the productivity, and often also the irreducibility of the input-output matrices involved are postulated without any attempt to

[2] A long list of authors has contributed to this process of "modernizing" Marx, starting with Dmitriev [2] and culminating in the late sixties and early seventies with the outstanding work of Brody [1] and Morishima [5].

justify them in terms of more general and valid assumptions. This is an obvious weakness of these otherwise very sophisticated and elegant analyses, since it means that their generality can be questioned at the outset.

In this paper we fill the perceived gaps by providing a complete justification for assuming the productivity[3] of input-output coefficient matrices (defined by the common, i.e., nonlabor, commodities) in a Marxian economic analysis. It will be shown that, starting from two fundamental assumptions (axioms) of Marx's economics, one can rigorously prove the existence, uniqueness and positivity (i.e., the conceptual validity) of labor values in an input-output type of framework. These axioms are rather simple and straightforward: we analyze a human production system, and in addition a perfect (competitive) market commodity production system. As by-products, we derive new economic criteria that guarantee the productivity of an input-output matrix, and it will be shown that these sufficient conditions can be viewed as generalizations of those provided by Gale and discussed above.

2. PRODUCTIVITY AND "SELF-SERVING PRODUCTION"

First we will examine the economic criteria for the productivity of an input-output coefficient matrix, A. Recall that Gale's criteria postulated a production system that produces at

[3]In a related paper [9] I have discussed the irreducibility assumption in detail. I have shown that the common input-output coefficient matrix cannot be assumed irreducible without substantial loss of generality. The irreducibility of the *complete* input-output coefficient matrix (the one that includes labor power as a special commodity) is, however, an assumption justifiable on economic grounds.

least as much of each commodity as is used in the production system itself: $x \geqq Ax$. In addition, in both criteria it was assumed that $x > 0$. Hereafter this positivity will be explicitly assumed, not only because it was implicit in the previous criteria, but also for the simple reason that we want to provide criteria based on *observed* (and not hypothetical) production systems.

DEFINITION 1: A production system characterized by x and A will be called a weakly, semistrongly, or strongly *self-sufficient system*, depending on the relation of x to Ax $(\geqq, \geq, >)$.

The assumption of a self-sufficient production system is rather strong in view of foreign trade possibilites and international specialization. However, in less open economies and in particular at high levels of aggregation, the observed input-output tables show just such self-sufficient production systems, so that in practice this assumption is not so binding as might be thought on theoretical grounds. Also, it is clear that weak self-sufficiency is not a sufficient assumption to ensure the productivity of the input-output coefficient matrix (if $x = Ax$, one of the eigenvalues of A is 1, so A cannot be productive).

Therefore, it is no surprise that one has to postulate semi-strong or strong self-sufficiency in the technological criteria of productivity. The input-output coefficient matrix of such a production system is at least '*quasi-productive*' [3]; in other words, their dominant eigenvalue is not larger than 1. In order to guarantee their *strong productivity* Gale assumed either strong self-sufficiency or semistrong self-sufficiency together with irreducibility.

In what follows, we will introduce the notion of *self-serving production* and show that by means of this new concept one can generalize Gale's criteria.

DEFINITION 2: In a given production system, characterized by $x > 0$ and $A \geqq O$, self-serving production is present if there is a group of sectors (activities) whose output does not exceed the total input of the same commodities into the group. Let I_2 denote the set of indices of those sectors belonging to such a group, and assume they are placed after the other sectors in the complete list of sectors. The existence of self-serving production means that the proper partitioning of x and A according to $N - I_2$ and I_2 (where N is the full set of sector indices) results in the following inequality

(1) $$x_2 \leqq A_{22}x_2 \qquad (x_2 > 0)$$

A few remarks may be useful to illustrate and explain this new concept. It should be noted, for example, that in a closed economy (with no end use), self-serving production is a natural state of equilibrium. Thus, the concept does not necessarily have any prejorative connotation. If, however, final output is considered as the real purpose of production, then self-serving production is clearly a waste of resources (nonproductivity).

Next, observe that inequality (1) implies (see the Perron-Frobenius theorems) that the dominant eigenvalue of A_{22} is larger than or equal to 1. From this it follows that the same holds for matrix A too; therefore *productivity* and *self-serving production* are mutually *exclusive*.

But it also follows that there does not exist any p_2 that fulfills the following inequality:

(2) $$p_2 > p_2A_{22} \qquad (p_2 \geqq 0)$$

Let us assume that, in the economy concerned, there is some positively priced primary resource (or factor of production) that is directly or indirectly required for the production of every commodity. Then, the lack of a vector, p_2 fulfilling inequality (2), implies that, under any nonnegative price system, at least one of the sectors in the self-serving producing group will operate at a loss. Thus, *self-serving production* and *pure market* (competitive) commodity production are, in general, once again mutually *exclusive.*[4]

From these observations alone it should be clear that Gale's criteria exclude the possibility of self-serving production, and this can easily be formally demonstrated. Strong self-sufficiency *ab ovo* excludes the possibility of self-serving production. Semistrong self-sufficiency together with irreducibility also lead to the same situation. Why this is so is explained by the following theorem.

THEOREM 1: Given a (semi)strongly self-sufficient production system characterized by $x > 0$ and $A \geqq 0$, self-serving production can be present if and only if there are sectors whose production is not required either directly or indirectly to produce the given final output, $y = x - Ax$.

PROOF: Clearly, it suffices to show that a self-serving producing group is composed of those and only those sectors

[4]By pure market commodity production we mean here, above all, that there is no government intervention compensating for possible losses incurred by individual producers. This is a much weaker concept than the usual profit-maximization principle associated with competitive production.

whose production is not needed either directly or indirectly to produce the given final output. Let I_2 denote the set of indices of these sectors. We will define them indirectly, beginning with a definition of the complementary set I_1: $I_1 = N - I_2$. Clearly, $j \in I_1$ only if $y_i > 0$ or if there exists a chain of indices, $j = j_0, j_1, \ldots, j_k = k$ such that $y_k > 0$ and $a_{j_t, j_{t+1}} > 0$, $\forall\ t = 0,1,\ldots,k-1$. Obviously, if $j \in I_1$, then sector j cannot be a member of a self-serving producing group, because of the assumed self-sufficiency ($x \geqq Ax$) and the fact that it supplies (directly or indirectly) at least one final-output producing sector. Also, $I_1 \neq \phi$, because of the assumed (semi)strong self-sufficiency.

Next we show that the group of sectors defined by $I_2 = N = I_1$ is a self-serving producing group. First, observe that, by definition, $a_{ij} = 0$ for all $i \in I_2$ and $j \in I_1$. Let us now partition x and A according to I_1 and I_2:

$$A_{22} = \{a_{ij} : i,j \in I_2\}$$

$$x_2 = \{x_j : j \in I_2\}$$

Because $A_{21} = O$ and $x \geq Ax$, $x_2 \geqq A_{22}x_2$. Since, however, the sectors belonging to I_2 do not produce final output (by definition), in fact we obtain:

$$x_2 = A_{22}x_2 \qquad\qquad x_2 > 0$$

which means self-serving production.

(q.e.d.)

Thus, if $I_2 \neq \phi$, then the self-sufficient production system is only quasi-productive. If A is irreducible, then, of course, each sector contributes directly or indirectly to the production of every other sector, as is well known. This is why in Gale's second criterion semistrong self-sufficiency is enough to exclude the possibility of self-serving production. However, irreducibility and the absence of self-serving production are quite different things. One can, for example, imagine an economy decomposable into completely independent groups of sectors, each of them irreducible in itself and producing final output. Such an economy is clearly reducible as a whole, but self-serving production is absent. Thus, the latter notion is a more general concept than the irreducibility of the whole production system. In the next theorem we show that it is also a more general sufficiency criterion for the productivity of an input-output coefficient matrix.

THEOREM 2: In a (semi)strongly self-sufficient production system ($x > 0$, $A \geqq 0$, $x \geq Ax$), in which self-serving production is absent, the input-output coefficient matrix is productive, i.e., nonnegatively L-invertible. On the other hand, if the input-output coefficient matrix of a production system is productive, then self-serving production is impossible.

PROOF: Since $x > 0$, for any $k > 1$ we obtain:

$$kx > Ax$$

Thus (see the Perron-Frobenius theorems) the dominant eigenvalue of A is less than or equal to 1. Next, we show that it cannot be equal to 1. This can be proved indirectly. Suppose the dominant eigenvalue of A is 1; therefore there is a semipositive x^o such that

$$x^o = Ax^o \tag{3}$$

x^o cannot be proportional to x (because of the assumed (semi) strong self-sufficiency); therefore, we can choose an $\alpha > 0$ such that

$$x_\alpha = x - \alpha x^o \geq 0$$

but at least one component of x_α is equal to 0. For such x_α we have

$$x_\alpha - Ax_\alpha = x - Ax = y$$

This means that an amount αx^o of the total production was not needed to achieve the given final output. This already implies self-serving production, but we will now prove this more formally. By definition, x_α has zero components. This means that the production of these sectors is not needed at all to achieve the given final output. Thus (see Theorem 1), these sectors form a self-serving producing group. But this contradicts our initial assumption; therefore, the dominant eigenvalue of A must be less than 1, that is, A is productive.

To prove the second part of our theorem, we have to show that, if A is productive, there is no index set, I_2, such that for the matrix

$$A_{22} = \{a_{ij} : i,j \in I_2\}$$

there exists an $x_2 > 0$ for which

$$x_2 \leqq A_{22}x_2 \quad .$$

If such a partition of A existed, then the dominant eigenvalue of A_{22}, and consequently that of A, could not be less than 1; that is, A could not be productive as assumed.

(q.e.d.)

In the light of these theorems one can see that the absence of self-serving production together with the assumption of (semi) strong self-sufficiency can be regarded as a generalization of the previous technological criteria of productivity given separately for the cases of a general and an irreducible input-output coefficient matrix. Self-sufficient production of each commodity, as indicated above, is in general by no means an unquestionable assumption. Therefore, it seems worthwhile, and not only on theoretical grounds, to turn to an examination of the dual side of the question, i.e., the economic criteria of productivity.

3. PRODUCTIVITY AND COMPLETE AUTOMATION

The concept of complete automation of a production system is very straightforward and can be viewed in many respects as the dual counterpart of the concept of self-serving production. If complete automation were impossible, this would imply that labor is indispensible in the given production system.[5] In other words, labor would be required in the production of at least one basic commodity (directly or indirectly required for final consumption). To our knowledge, the concept of complete automation was formally introduced by Morishima and Cataphores [6] in a von Neumann model framework.

DEFINITION 3: A production system, characterized by input coefficients A (an n x n nonnegative matrix) for the common

[5]Labor may be indispensible even in an economy where complete automation, as defined here, is possible. This is the case if some independent group of sectors, which can be fully automated, has a total production that, taken alone, is insufficient to meet final consumption needs.

(nonlabor) commodities and m (an 1 x n matrix) for labor power[6] can be completely automated if there is a semipositive vector x such that

$$Ax \leqq x \quad \text{and} \quad mx = 0 \quad .$$

Thus a completely automated production system is simply one capable of self-sufficient production without using any labor. Clearly, in this type of production system labor values would be conceptually invalid, but it is equally clear that such a system has never existed, except perhaps in El Dorado.

If, however, a production system could be completely automated, then it could only produce commodities that require no labor input, either directly or indirectly. This implies that, in such a case, the input coefficient matrixes could be decomposed as follows:

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

$$\begin{pmatrix} m_1 & 0 \end{pmatrix}$$

where the second group of commodities (sectors) definitely do not require labor input and A_{22} is productive.

[6]The term labor power is used here to stress the importance of Marx's distinction between the commodity bought and sold (labor power) and its service (labor). As explained by Marx, the source of exploitation is that labor power is a special commodity whose reproduction requires an amount of labor less than it can supply, and is governed not only and not so much by economic laws, as biological and social ones.

This implies some kind of structual duality between the concept of complete automation and self-serving production. In the presence of the latter, there existed a group of sectors that did not produce any net (final) output either directly or indirectly. That is, there existed the following partition of A and y (the final output vector).

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$$

and, in contrast to the complete automation case, A_{22} was assumed to be nonproductive. Thus, the two concepts are not exactly dual counterparts. The impossibility of full automation is a somewhat stronger and more subtle condition for an economy.

In the following theorem we prove that the above structural property is a necessary and sufficient condition for complete automation to be possible if A is productive.

THEOREM 3: If A is productive, then the economy can be fully automated if and only if there is a group of commodities (sectors) whose production requires no labor, either directly or indirectly.

PROOF: We prove first the sufficiency condition. Let N be the complete set of commodity indices as before and I_2 the index set of those commodities whose production does not require labor in any way. If $I_2 = N$, then m = 0 necessarily, and thus the economy can be completely automated. If $I_2 \neq N$, then we know that $m_j = 0$ for all $j \in I_2$ and we can show that $a_{ij} = 0$ for all $i \in N - I_2$ and $j \in I_2$ (otherwise some commodity in I_2 would use as an input a commodity whose production *does* require labor, and this would contradict our assumption about I_2). Since the whole input-output coefficient matrix is assumed to be productive, so is the one defined by the commodities belonging to I_2. Therefore, these commodities can be produced in amounts exceeding their use without requiring any labor input.

The necessity part of the theorem is self-evident; we confine ourselves to demonstrating the possibility of partitioning A and m as shown under Definition 3. Suppose x is a production vector of a completely automated system. Let us re-group the commodities in such a way that those produced by x (i.e., having positive components in x) are listed last. We can then partition x into $x_1 = 0$ and $x_2 > 0$. From $mx = 0$ it follows that $m_2 = 0$. Since $x \geqq Ax$, $A_{12}x_2$ must be 0, that is, $A_{12} = 0$ (where A_{12} is defined by the corresponding partitioning of matrix A).

(q.e.d.)

Next we show that a pure market economy, where complete automation is impossible, can exist only if its input-output coefficient matrix is productive.

THEOREM 4: Let an economy be characterized by nonnegative input coefficients A and m (average) prices $p_a > 0$, and (average) wage rates $w_a > 0$, which satisfy the following minimum criterion for a pure market economy

$$p_a \geqq p_a A + w_a m. \tag{4}$$

i.e., no commodity is produced at a loss. If complete automation in this economy is impossible then A is productive.

PROOF: m must be a semipositive vector (complete automation is impossible) and w_a is positive; thus, inequality (4) implies the following semi-inequality

$$(5) \qquad p_a \geq p_a A$$

Because p_a is, by assumption, strictly positive, inequality (5) implies (see the Perron-Frobenius theorems) that the dominant eigenvalue of A is less than or equal to 1. Now, suppose the dominant eigenvalue were 1. This would imply the existence of a semi-positive vector $\hat{p}$ satisfying the equality

$$(6) \qquad \hat{p} = \hat{p}A$$

From (5) and (6) we know that p_a and $\hat{p}$ cannot be proportional to each other. Thus, there is a positive scalar α that makes p_α semipositive but not strictly positive, such that

$$p_\alpha = p_a - \alpha\hat{p}$$

and p_α also satisfies the inequality

$$(7) \qquad p_\alpha \geq p_\alpha A$$

It is also clear that the structure of equalities and strict inequalities in the inequality system (7) is the same as in (5). Let us define I_2 and I_1 in the following way

$$I_2 = \{j : p_{\alpha j} = 0\}$$

$$I_1 = N - I_2$$

and partition (7) accordingly (after suitable rearrangement of the commodity list):

(7.1) $$p_{\alpha 1} \geqq p_{\alpha 1}A_{11} + oA_{21}$$

(7.2) $$o \geqq p_{\alpha 1}A_{12} + oA_{22}$$

From (7.2) it follows that $A_{12} = 0$ and that the weak inequalities in (7.2) have in fact to be fulfilled in the form of equalities. Hence, we also know that

(8) $$p_{a2} = p_{a2}A_{22}$$

which, in turn, implies by the no-loss assumption that $m_2 = 0$, and that 1 is the dominant eigenvalue of A_{22}. Since both $A_{12} = 0$ and $m_2 = 0$, this would mean that complete automation is possible, contrary to our assumption. Thus, 1 cannot be an eigenvalue of A; or more precisely, the dominant eigenvalue of A must be less than 1, that is, A is productive.

(q.e.d.)

4. CONCEPTUAL VALIDITY OF LABOR VALUES

As mentioned earlier, one of the objectives of this paper is to show that, from some basic postulates inherent in the Marxian analysis of the capitalist mode of production, the existence and uniqueness of positive labor values can be rigorously deduced. In the previous sections we have fully prepared the ground for demonstrating this proposition, which can be seen as a conceptual justification (validation) of labor values in Marx's analysis.

Earlier contributors to this problem (Brody, Morishima, and others) have relied on assumptions of the productivity of A and the (semi)positivity of m (if only semipositivity, then with the additional assumption of the irreducibility of A). These assumptions need some justification themselves, and we will show that they are in fact even more restrictive than is actually necessary.

The assumption of the impossibility of complete automation is involved right from the outset, not only because complete automation still remains a utopian state but also because, in a fully automated economy, labor values could not of course be conceptually justified. The assumption guarantees that labor is indispensible in the production of every commodity.

But this in itself is not a sufficient condition for the productivity of the input-output coefficient matrix, which is clearly also needed here. Theorems 2 and 4 suggest that the assumption of pure market commodity production in one form or another together with that of the impossibility of complete automation will define sufficient conditions for the existence, uniqueness, and positivity of labor values. And in fact nothing is closer to the spirit of Marx's analysis than the assumption of pure market commodity production, in which no commodity is, on average, produced at a loss.

As we have indicated earlier, the most straightforward and rather weak criterion of pure market commodity production is the assumption that, at the prevailing prices and wage rates, the average cost of producing any commodity is not higher than its price. This, of course, does not exclude the possibility that temporarily or individually some producers may incur losses: we only postulate no losses *on average*. This is sufficient, together with the impossibility of full automation, to guarantee the conceptual validity (existence, uniqueness, and positivity) of labor values.

For the sake of completeness we will also show that the generalization of the technological productivity criteria introduced above could also be used in the conceptual validation of labor values. Self-serving production, as we have shown, is alien to the concept of pure market commodity production. Self-sufficient production is, however, a stronger assumption than its financial (economic) counterpart, i.e., "no losses on average", especially in view of international specialization. Thus, the argument based on the technological criterion of productivity is not as strong as the former one.

To conclude, we will complement this informal treatment with a formal theorem.

THEOREM 5 — Conceptual validity of labor values. Let an economy be characterized by a (nonnegative) input-output matrix A and a labor input coefficient vector m, and by positive prices p_a and wage rates w_a. Suppose also that this economy cannot be completely automated.

The labor values in this economy are uniquely determined and positive if either of the following additional conditions is fulfilled:

(i) on average no loss is incurred in the production of any commodity ($p_a \geqq p_a A + w_a m$);

(ii) the actual production (x_a) is at least semistrongly self-sufficient ($x_a \geq Ax_a$) and no self-serving production takes place.

PROOF: Since the conditions of Theorems 2 and 4 are met, A is a productive matrix. Thus, the labor values can be uniquely determined as $m(I - A)^{-1}$, where the L-inverse is a semipositive matrix. It remains to show that all values are positive, and we prove this indirectly. Suppose the labor value of commodity i is zero. If r_i is the i^{th} column vector of the L-inverse, we can calculate this value as mr_i. The vector r_i can be interpreted as a production vector, and in fact, as is well known, it is a self-sufficient production bundle that results in one unit of final output of commodity i. Thus, if we had $mr_i = o$, then r_i would represent completely automated production, contrary to our initial assumption.

(q.e.d.)

5. CONCLUSION

In summary, we have shown that when we analyze Marxian labor values in the framework of an open Leontief model we can with full justification assume that the input-output coefficient is productive and that labor is directly or indirectly required in the production of every commodity.

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