VARIATIONAL INEQUALITIES REVISITED

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Let K be a closed convex subset of a reflexive Banach space X and A be a set-valued map from K to X^* satisfying

$$(1)\quad \begin{cases} \text{A is finitely upper-semicontinuous}^{(1)} \text{ with} \\ \text{nonempty closed convex bounded images.} \end{cases}$$

Our purpose is to solve *variational inequalities* (or generalized equations)

$$(2)\quad \begin{cases} \text{i)} & \bar{x} \in K \\ \text{ii)} & 0 \in A(\bar{x}) + N_K(\bar{x}) \end{cases}$$

where $N_K(x) := \{p \in X^* \mid \sup_{y \in K} \langle p, x-y \rangle \geq 0\}$ is the *normal cone* to K at $x \in K$, by balancing

a) *the lack of boundedness* of K, measured by its "barrier cone" b(K), defined by

$$(3)\quad b(K) := \{p \in X^* \mid \sup_{x \in K} \langle p, x \rangle < +\infty\}$$

(because the larger b(K), the lesser is K unbounded)

b) with *the degree of monotonicity* of A, measured by a nonnegative proper lower semicontinuous function β from X to $\mathbb{R} \cup \{+\infty\}$ satisfying

$$(4) \qquad \forall (x,p),\ (y,q) \in \text{graph}(A),\ \langle p-q, x-y \rangle \geq \beta(x-y) \quad .$$

We shall say that such a set-valued map A is β-*monotone*. We denote by β^* its conjugate function[2].

For instance, we can take

$$(5) \quad \begin{cases} \text{i)} & \beta(z) := 0 \text{ (and thus, } \beta^* = \psi_{\{0\}}, \text{Dom } \beta^* = \{0\})^{(3)} \\ \text{ii)} & \beta(z) := \|z\| \text{ (and thus, } \beta^* = \psi_{B_*}, \text{Dom } \beta^* = B_*)^{(4)} \\ \text{iii)} & \beta(z) := \frac{1}{\alpha}\|z\|^{\alpha} \text{ (and thus, } \beta^* = \frac{1}{\alpha_*}\|\cdot\|^{\alpha_*}, \frac{1}{\alpha}+\frac{1}{\alpha_*} = 1, \text{Dom}\beta^* = X^*) \end{cases}$$

In the following theorems, we shall measure the degree of monotonicity of G through the size of the domain of β^*: the larger Dom β^*, the more "monotone" is G.

Theorem 1. We posit assumptions (1). Assume that A is β-monotone and that

$$(6) \qquad 0 \in \text{Int } (b(K) + A(K) + \text{Dom } \beta^*) \quad .$$

Then there exists a solution $\bar{x} \in K$ to the variational inequality $0 \in A(\bar{x}) + N_K(\bar{x})$. ▲

Assumption (6) shows how the lack of boundedness of K is compensated by the degree of monotonicity of A. We point out that (6) is satisfied when one of the following instances is satisfied.

$$(7) \quad \begin{array}{ll} \text{i)} & K \text{ is bounded } (b(K) = X^*) \\ \text{ii)} & A \text{ is surjective } (A(K) = X^*) \\ \text{iii)} & A \text{ satisfies (4) with } \beta(z) := \frac{c}{\alpha}\|z\|^{\alpha},\ c > 0,\ \alpha > 1 \\ \text{iv)} & A \text{ satisfies (4) with } \beta(z) := c\|z\|,\ c > 0 \text{ and} \\ & A(K) \cap -b(K) \neq \emptyset. \end{array}$$

■

Naturally, these examples are known (See Brézis (1968), Lions (1969) and Browder (1976)). The novelty lies in the introduction of the function β as a parameter in assumption (6).

We recall that $N_K(x)$, the normal cone to K at x, is the subdifferential[3] of the indicator ψ_K. Therefore, variational inequalities are particular cases of inclusions of the form

$$(8) \qquad f \in A(\bar{x}) + \partial V(\bar{x})$$

when $V : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and A is a set-valued map from the Banach space X to X^*, which were studied by Brézis-Haraux (1976), when A is maximal monotone, for solving Hammerstein equations (see Brézis-Browder (1976)). We shall extend Theorem 1 to this case. To this end, we assume once and for all that

$$(9) \qquad \text{Dom } V \subset \text{Dom } A$$

and we observe that a necessary condition for the existence of a solution $\bar{x}$ to (8) is that

$$(10) \qquad f \in \text{Dom } V^* + A \text{ Dom } V \quad .$$

We shall prove that this condition is "almost sufficient".

Theorem 2. We posit assumptions (1). Assume moreover that A is β-monotone. Then there exists a solution $\bar{x}$ to the inclusion (8) when

$$(11) \qquad f \in \text{Int } (\text{Dom } V^* + A \text{ Dom } V + \text{Dom } \beta^*) \quad .$$

Remark. The size of Dom β^* balances the interiority condition in assumption (11), as the following corollary shows.

Corollary 3. We posit assumptions (1).

a) If A is monotone, (i.e., $\beta=0$), then

$$(12) \qquad \text{Int } (\text{Dom } V^* + A \text{ Dom } V) \subset \text{Im } (A + \partial V) \subset \text{Dom } V^* + A \text{ Dom } V \quad .$$

b) If there exists $c > 0$ such that

$$(13) \qquad \forall (x,p),\ (y,q) \in \text{graph }(A),\ \langle p-q, x-y \rangle \geq c\|x-y\| \quad ,$$

then

$$(14) \qquad \text{Im }(A + \partial V) = \text{Dom } V^* + A \text{ Dom } V$$

c) If there exist $c > 0$ and $\alpha > 1$ such that

$$(15) \qquad \forall (x,p),\ (y,q) \in \text{graph }(A),\ \langle p-q, x-y \rangle \geq \frac{c}{\alpha} \|x-y\|^{\alpha}$$

then

$$(16) \qquad \text{Im }(A + \partial V) = \text{Dom } V^* + A \text{ Dom } V = X^* \quad .$$

■

Before proving Theorem 2, we shall characterize problem (8) by equivalent problems. For that purpose, we associate to the function V, to the map A and to an element $f \in X$ the function Φ defined on Dom V by:

$$(17) \qquad \Phi(y) := V(y) + \inf_{u \in A(y)} (V^*(f-u) - \langle f-u, y \rangle) \quad .$$

We observe that

$$(18) \qquad \forall y \in \text{Dom } V, \quad \Phi(y) \geq 0 \quad ,$$

since, for all $u \in A(y)$, $V(y) + V^*(f-u) - \langle f-u, y \rangle \geq 0$, thanks to the Fenchel inequality.

We can also characterize the set-valued map A by the function γ defined on Dom A × Dom A by

$$(19) \qquad \gamma(x,y) := \inf_{p \in A(x)} \langle p, x-y \rangle$$

Proposition 4.

Assume that the images A(x) are nonempty, closed, convex and bounded for all $x \in \mathrm{Dom}\, V$. The following problems are equivalent

(20)

i) $\exists\, \bar{x} \in \mathrm{Dom}\, V$ such that $f \in A\bar{x} + \partial V(\bar{x})$

ii) $\exists\, \bar{p} \in \mathrm{Dom}\, V$ such that $f \in \bar{p} + A\partial V^*(\bar{p})$

iii) $\exists\, \bar{x} \in \mathrm{Dom}\, V$ such that $\forall y \in \mathrm{Dom}\, V$,

$$\gamma(\bar{x},y) - \langle f,\bar{x}-y\rangle + V(\bar{x}) - V(y) \leq 0$$

iv) $\exists\, \bar{x} \in \mathrm{Dom}\, V$ such that $\Phi(\bar{x}) = 0 \quad (= \min_{y \in \mathrm{Dom}\, V} \Phi(y))$

Proof.

a) Let $\bar{x}$ be a solution to (20)i): then there exists $\bar{p} \in \partial V(\bar{x})$ such that $f-\bar{p} \in A\bar{x} \subset A\partial V^*(\bar{p})$. Conversely, let $\bar{p}$ be a solution to (20)ii). Then there exists $\bar{x} \in \partial V^*(\bar{p})$ such that $f \in \bar{p} + A\bar{x}$. Since $\bar{p} \in \partial V(\bar{x})$, then f belongs to $\partial V(\bar{x}) + A\bar{x}$.

b) Let $\bar{x}$ be a solution to (20)i). There exists $\bar{u} \in A(\bar{x})$ such that $f \in \partial V(\bar{x}) + \bar{u}$, i.e., such that $\forall y \in \mathrm{Dom}\, V$,

$$\langle \bar{u},\bar{x}-y\rangle - \langle f,\bar{x}-y\rangle + V(\bar{x}) - V(y) \leq 0 \quad .$$

By taking the infimum on $A(\bar{x})$, we deduce inequalities (20)iii).

c) Inequality (20)iii) can be written

$$\inf_{y \in \mathrm{Dom}\, V} \ \inf_{u \in A(\bar{x})} [V(x) - V(y) - \langle f-u,x-y\rangle] \leq 0 \quad .$$

Since Dom V is convex, $A(\bar{x})$ is convex weakly compact, the lop-sided minimax theorem implies that

$$\sup_{u \in A(\bar{x})} \ \sup_{y \in \mathrm{Dom}\, V} [V(\bar{x}) - V(y) - \langle f-u,\bar{x}-y\rangle]$$

$$= \inf_{u \in A(\bar{x})} [V(\bar{x}) + V^*(f-u) - \langle f-u,\bar{x}\rangle] = \Phi(\bar{x}) \quad .$$

Hence $\Phi(\overline{x}) \leq 0$.

d) Let $\overline{x} \in \text{Dom } V$ satisfy $\Phi(\overline{x}) = 0$. Since $A(\overline{x})$ is weakly compact and V is weakly lower semicontinuous, there exists $\overline{u} \in A(\overline{x})$ such that

$$\Phi(\overline{x}) := V(\overline{x}) + V^*(f-\overline{u}) - \langle f-\overline{u},\overline{x}\rangle = 0 \quad .$$

This is equivalent to saying that: $f-\overline{u} \in \partial V(\overline{x})$, i.e. that $\overline{x}$ solves (20)i).

∎

The equivalence between (20)i) and (20)iv) allows to interpret the solutions to problem (8) as a solution to a minimization problem (minimization of the functional ϕ) and provides a *variational principle*. The equivalence between (20)i) and (20)iii) allows to solve problem (8), (and, in particular, variational inequalities) by applying minimax inequalities to the function defined by

(21) $$\phi(x,y) := \gamma(x,y) - \langle f,x-y\rangle + V(x) - V(y) \quad .$$

We observe that

(22)
i) $\forall x,\ y \to \phi(x,y)$ is concave

ii) $\forall y,\ \phi(y,y) = 0 \quad ,$

that ϕ is "monotone" in the sense that

(23) $$\forall x,y \in \text{Dom }(V),\quad \phi(x,y) + \phi(y,x) \geq 0$$

and that

(24) $\forall y \in X,\ x \to \phi(x,y)$ is lower semicontinuous for the finite topology[(1)].

Therefore, if Dom V were compact, we could apply the generalization of the Ky Fan inequality (1972) due to Brézis-Nirenberg-Stampacchia (1972), which would imply the existence of a solution $\overline{x} \in \text{Dom } V$

to the inequalities (20)iii), i.e., a solution $\bar{x}$ to problem (8). When Dom V is not compact, we shall prove by approximation that assumption (11) is sufficient for the existence of a solution to inequalities (20)iii).

<u>Proof of Theorem 2</u>.

We set $K_n := \{x \in \text{Dom } V / V(x) \leq n \text{ and } \|x\| \leq n\}$. The subsets K_n are weakly compact and convex and $\text{Dom } V = \bigcup_{n=1}^{\infty} K_n$ because X is reflexive. Since K_n is weakly compact and convex, Ky Fan's inequality for monotone functions implies that, for all $n \geq 1$, there exists $x_n \in K_n$ solution to

$$\text{(25)} \qquad \forall y \in K_n, \quad \phi(x_n, y) \leq 0$$

thanks to properties (22), (23) and (24).

We shall now use assumption (11) for proving that x_n remains in a weakly compact subset of X. For that purpose, thanks to the uniform boundedness theorem, it is sufficient to prove that

$$\text{(26)} \qquad \forall p \in X^*, \ \exists \ n(p) \text{ such that } \sup_{n \geq n(p)} \langle p, x_n \rangle < +\infty \ .$$

By assumption (11), there exist $\eta > 0$, $r \in \text{Dom } \beta^*$, $q \in \text{Dom } V^*$, $y \in \text{Dom } V$, $u \in A(y)$ such that

$$\text{(27)} \qquad f + \frac{\eta p}{\|p\|} = r + q + u \ .$$

We choose n(p) to be the smallest n such that $y \in K_n$. By taking the duality product with x_n we get

$$\text{(28)} \qquad \frac{\eta}{\|p\|} \langle p, x_n \rangle = \langle r, x_n - y \rangle + \langle q, x_n \rangle + \langle u, x_n - y \rangle$$
$$- \langle f, x_n - y \rangle + \langle r+u-f, y \rangle \ .$$

We use Fenchel's inequalities $\langle r, x_n - y \rangle \leq \beta(x_n - y) + \beta^*(r)$ and $\langle q, x_n \rangle \leq V(x_n) + V^*(q)$. We obtain

$$\frac{\eta}{\|p\|}\langle p,x_n\rangle \le \langle u,x_n-y\rangle + V(x_n) - V(y) - \langle f,x_n-y\rangle$$

(29)

$$+ \beta(x_n-y) + \beta^*(r) + V^*(q) + V(y) + \langle r+u-f,y\rangle \quad .$$

Since A is β-monotone, we deduce that

$$\gamma(x_n,y) - \langle u,x_n-y\rangle = \inf_{p\in A(x_n)} \langle p-u,x_n-y\rangle \ge \beta(x_n-y) \quad .$$

Therefore, inequality (29) becomes

$$\frac{\eta}{\|p\|}\langle p,x_n\rangle \le (\gamma(x_n,y) - \langle f,x_n-y\rangle + V(x_n) - V(y))$$

$$+ \beta^*(r) + V^*(q) + V(y) + \langle r+u-f,y\rangle.$$

Consequently, for all $n \ge n(p)$, we deduce from (25) that

(30) $$\langle p,x_n\rangle \le \frac{\|p\|}{\eta}(\beta^*(r) + V^*(q) + V(y) + \langle r+u-f,y\rangle) \quad .$$

The right-hand side is finite because $r\in \mathrm{Dom}\ \beta^*$, $q\in \mathrm{Dom}\ V^*$ and $y\in \mathrm{Dom}\ V$. Hence the sequence is bounded and thus, weakly relatively compact.

So, a subsequence of elements x_n, converges weakly to some $\bar{x}\in X$. Since V is lower semicontinuous, we deduce from the monotonicity of A and from the variational inequalities that

$$\begin{aligned} V(\bar{x}) &\le \liminf_n V(x_n) \\ &\le \liminf_n [(V(y) + \langle f,x_n-y\rangle + \gamma(y,x_n)) - \gamma(y,x_n) - \gamma(x_n,y)] \\ &\le \limsup_n [V(y) + \langle f,x_n-y\rangle + \gamma(y,x_n)] \\ &\le V(y) + \langle f,\bar{x}-y\rangle + \gamma(y,\bar{x}) \quad . \end{aligned}$$

Therefore, $\bar{x}$ belongs to Dom V and

(31) $$\forall y\in \mathrm{Dom}\ V,\quad 0 \le \phi(y,\bar{x}) \quad .$$

d) We deduce from properties (22) and (23) that

(32) $\forall z \in \mathrm{Dom}\, V, \quad \phi(\overline{x},z) \leq 0$.

Indeed, if the conclusion is false, there would exist $z \in \mathrm{Dom}\, V$ such that $0 < \phi(\overline{x},z)$ and by (24) there would exist $\overline{t} \in]0,1[$ such that

$$0 < \phi(\overline{x} + \overline{t}(\dot{z}-\overline{x}),z) \quad .$$

By taking $y = \overline{x} + \overline{t}(z-\overline{x})$, inequality (31) implies that

$$0 \leq \phi(\overline{x} + \overline{t}(z-\overline{x}),\overline{x}) \quad .$$

Hence, the concavity of ϕ with respect to the second variable yields that

(33) $0 < \phi(\overline{x} + \overline{t}(z-\overline{x}) , \overline{x} + \overline{t}(z-\overline{x}))$

a contradiction to (22)ii). Then Proposition 4 implies that the solution $\overline{x}$ of (32) is a solution to the problem (8).

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Notes

(1) The finite topology on a convex subset N of a vector space is the topology for which the maps β_K from the simplex $S^n := \{\lambda \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} \lambda_i = 1\}$ to N defined by

$$\beta_K(\lambda) := \sum_{i=1}^{n} \lambda_i x_i$$

are continuous for all finite subsets $K := \{x_1, \ldots, x_n\}$ of N. It is stronger than any vector space topology and any affine map is continuous for the finite topology (see Aubin (1979), §7.1.3). A finitely upper semicontinuous map from K to X^* is a set-valued map upper semicontinuous from K supplied with the finite topology to X^* supplied with the weak *-topology. When A is finitely upper semicontinuous, then the map $x \to \inf_{u \in A(x)} \langle u, x-y \rangle$ is lower semicontinuous on K for the finite topology (see Aubin (1979), §13.2.4).

(2) The conjugate function β^* of a function $\beta : X \to \mathbb{R} \cup \{+\infty\}$ is defined on X^* by

$$\beta^*(p) := \sup_{x \in X} [\langle p, x \rangle - \beta(x)] \quad .$$

A function β is convex and lower semicontinuous if and only if $\beta = \beta^*$. It satisfies the Fenchel inequality

$$\langle p, x \rangle \leq \beta(x) + \beta^*(p) \quad .$$

(3) The indicator of a subset K is the function ψ_K defined by $\psi_K(x) = 0$ when $x \in K$ and $\psi_K(x) = +\infty$ if not.

(4) B_* denotes the unit ball of the dual.

(5) The subdifferential of a convex function V is the subset

$$\partial V(x) := \{p \in X^* \mid \langle p, x \rangle = V(x) + V^*(p)\}$$

of gradients of the affine functions $x \to \langle p,x \rangle - V^*(p)$ below V and passing through $(x,V(x))$. When V is Gâteaux-differentiable at x, then $\partial V(x) = \{\nabla V(x)\}$. The set of points $x \in X$ for which $\partial V(x) \neq \emptyset$ is dense in Dom V.

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