ON FORECASTING URBAN AND RURAL POPULATIONS: SOME METHODOLOGICAL REFLECTIONS

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FOREWORD

Many large urban agglomerations in the developed countries are either experiencing population decline or are growing at rates lower than those of middle-sized and small settlements. This trend is in direct contrast to the one for large cities in the less developed world, which are growing rapidly. Urban contraction and decline is generating fiscal pressures and fueling interregional conflicts in the developed nations; explosive city growth in the less developed world is creating problems of urban absorption. These developments call for the reformulation of urban policies based on an improved understanding of the dynamics that have produced the current patterns.

During the period 1979-1982, the former Human Settlements and Services Area examined patterns of human settlement transformation as part of the research efforts of two tasks: the Urban Change Task and the Population, Resources, and Growth Task. This paper was written as part of that research activity. Its publication was delayed, and it is therefore being issued now a few months after the dissolution of the HSS Area.

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CONTENTS

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1. INTRODUCTION

Most attempts to forecasts the proportions of a population that will be living in urban and rural areas use a model forecasting shifts between two homogeneous states: rural and urban (Ledent, 1980; United Nations, 1980). The aim of this short paper is to attempt to estimate how such forecasts must be corrected to account for the full complexity of a multiregional system, and to allow transitions from rural to urban areas over time. Purely demographic discrete time forecasts from an initial period τ to some future period T, will be considered, and the effects of age distributions on migration and fertility rates will be neglected. Instead, attention will be focused on an issue emphasized elsewhere (Sheppard, 1980); that the dynamics of urbanization cannot be fully understood or predicted without allowing for the interdependencies between the various types of urban and rural areas; that is the full geography of population change. By introducing a methodology to correct forecasts in a way that allows for this, the importance of this full specification may be estimated.

2. THE BASE FORECAST

Under the assumption of constant migration rates, to be relaxed later, Ledent (1980) describes the following model, which seems to typify a basic urban/rural forecast:

$$
\begin{aligned}
p_r(t) &= (a_r - o_r)p_r(t-1) + o_u p_u(t-1) + p_r(t-1) \\
p_u(t) &= (a_u - o_u)p_u(t-1) + o_r p_r(t-1) + p_u(t-1)
\end{aligned}
\tag{1}
$$

where $p_r(r)$, $p_u(t)$ are the rural and urban populations at time t; o_r, o_u are the rural and urban outmigration rates; and a_r, a_u are the rural and urban rates of natural increase. For the purposes of this paper this is reformulated as:

$$
\begin{aligned}
p_r(t) &= (a_r + m_{rr})p_r(t-1) + m_{ur}p_u(t-1) \\
p_u(t) &= (a_u + m_{uu})p_u(t-1) + m_{ru}p_r(t-1)
\end{aligned}
\tag{2}
$$

where m_{ru} is the rate of migration from rural to urban areas, and m_{uu}, m_{rr} are the proportion of the population remaining respectively in the urban and rural areas. System (2) incorporates assumptions of no international migration, and births/deaths accumulated at the end of the time period. In matrix form:

$$\underline{p}'_B(t) = \underline{p}'_B(t-1) . \underset{\sim}{M}_B \tag{3}$$

where $\underline{p}'_B(t) = [p_r(t) \;\; p_u(t)]$ and

$$\underset{\sim}{M}_B = \begin{bmatrix} a_r + m_{rr} & m_{ru} \\ m_{ur} & a_u + m_{uu} \end{bmatrix}$$

Then the base forecast is:

$$\underline{p}_B'(T) = \underline{p}_B'(\tau) \cdot \underset{\sim}{M}_B^s \tag{4}$$

where $s = T - \tau$. Further, the base equilibrium forecast is given by the principal left hand eigenvector of M_B associated with the equilibrium rate of increase λ:

$$\underline{p}_B'(e) = \lambda \cdot \underline{p}_B'(e) \cdot \underset{\sim}{M}_B \tag{5}$$

3. INCORPORATING MULTIREGIONAL FLOWS

Assume that the nation may be divided into many regions, of sufficiently small size that each region can be described as either rural or urban. Index these by r_i, u_i to refer to the i-th rural or urban region. If the migration rates $m_{r_i r_j}$, $m_{u_i r_j}$, $m_{u_i u_j}$, $m_{r_i u_j}$ and rates of natural increase a_{u_i}, a_{r_i} are known at time τ, then the following forecasts can be made of the population at time period T and in equilibrium:

$$\underline{p}_M'(T) = \underline{p}_M'(\tau) \cdot \underset{\sim}{M}^s \tag{6}$$

$$\underline{p}_M'(e) = \lambda_M \cdot \underline{p}_M'(e) \cdot M \tag{7}$$

where $\underline{p}_M'(t) = [p_{r_1}, p_{r_2}, \ldots, p_{r_M}, p_{u_1}, \ldots p_{u_N}]$, λ_M is the largest eigenvalue of $\underset{\sim}{M}$, and

$$\underset{\sim}{M} = \left[\begin{array}{cc|cc} a_{r_1}+m_{r_1 r_1} & m_{r_1 r_2} \cdots & \cdots & m_{r_1 u_N} \\ m_{r_2 r_1} & a_{r_2}+m_{r_2 r_2} & \ddots & m_{r_2 u_N} \\ \hline & & \ddots & \\ m_{r_2 u_N} & & a_{u_N} & +\; m_{r_2 u_N} \end{array}\right]$$

The population vector and the components of change matrix have both been partitioned into urban and rural regions.

Now define the aggregation matrix E of dimension 2 by M+N:

$$E = \left[\begin{array}{cccc|cccc} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right]$$

where the first row has M ones (corresponding to the rural regions) and the second row has N ones. Then the first corrected forecast of the urban/rural population split is:

$$\begin{aligned} \underline{p}_1'(T) &= \underline{p}_M'(T).\underset{\sim}{E}' \\ &= \underline{p}_M'(\tau).\underset{\sim}{M}^s.\underset{\sim}{E}' \end{aligned} \tag{8}$$

and the equilibrium vector is:

$$\begin{aligned} \underline{p}_1'(e) &= \underline{p}_M'(e).\underset{\sim}{E}' \\ &= \lambda_M\, \underline{p}_M'(e).\underset{\sim}{M}.\underset{\sim}{E}' \end{aligned} \tag{9}$$

It is true by definition that (see Appendix):

$$\underline{p}_B'(\tau) = \underline{p}_M'(\tau).\underset{\sim}{E}' \tag{10}$$

$$\underset{\sim}{M}_B = \underset{\sim}{E}.\underset{\sim}{M}.\underset{\sim}{E}'[\underset{\sim}{P}(\tau)]^{-1} \tag{11}$$

where $[\underset{\sim}{P}(t)] = \begin{bmatrix} p_r(\tau) & 0 \\ 0 & p_u(\tau) \end{bmatrix}$

Note in (11) that the relationship between $\underset{\sim}{M}_B$ and $\underset{\sim}{M}$ depends on the population distribution at the time the aggregate components of change were estimated. This reflects the complexities of population heterogeneity. From (11);

$$\underset{\sim}{M}_B^s = \underset{\sim}{E}.\underset{\sim}{M}.\underset{\sim}{E}'.[\underset{\sim}{P}(\tau)]^s \qquad (12)$$

The correction to be applied to the base forecast is then:

$$\Delta \underline{p}_1'(T) = \underline{p}_1'(T) - \underline{p}_B'(T) \qquad (13)$$

From equations (4), (8), and (10):

$$\Delta \underline{p}_1'(T) = \underline{p}_M'(\tau).[\underset{\sim}{M}^s.\underset{\sim}{E}' - \underset{\sim}{E}'.\underset{\sim}{M}_B^s] \qquad (14)$$

This correction must be added to the base forecast to give a modified forecast. If the correction is equal to zero, perfect aggregation is possible (Rogers, 1971). The correction is only equal to zero if either $\underset{\sim}{E}.\underset{\sim}{E}'$ equals the identity matrix (which is never true), or if there are no migrations between rural and urban areas and the urban and rural populations are initially identical (i.e., $\underset{\sim}{M}.\underset{\sim}{E}'.[\underset{\sim}{P}(\tau)].\underset{\sim}{E}.\underset{\sim}{M}=\underset{\sim}{M}^2$). The size of the correction, however, is an empirical question. It may be easily calculated given $P_{-1}(\tau)$ and $\underset{\sim}{M}$. The corrected equilibrium forecast is:

$$\Delta \underline{p}_1'(e) = \underline{p}_1'(e) - \underline{p}_B'(e) \qquad (15)$$

which is again only equal to zero under perfect aggregation, when λ of equation (5) and λ_M of equation (9) will be equal (Rogers, 1975).

4. INCORPORATING URBANIZATION

When long term forecasts are made, there is an additional source of error; regions that were rural become urban as cities expand and villages evolve to towns. Assume that the change of a region from rural to urban does not alter migration rates between it and other regions. In other words, assume that urban and rural residents have the same mobility rates *ceteris paribus*. A further adjustment to population forecasts is then necessary, which may be applied either to the aggregate rural/urban, or to the disaggregate forecast.

4.1. Aggregate Adjustments

Given a two-state forecast, such as $\underline{p}_B'(T)$, this must be adjusted to account for that population that was rural at time τ but urban at time T. $\underline{p}_B'(T)$ is no longer correct simply because the original urban/rural classification is no longer valid. A simple minded adjustment can be made, however, as follows. Assume $q_{ru}(s)$ is the probability that a rural area at time τ has become urban s periods later, at time T. Then the adjusted base forecast is:

$$\underline{p}_2'(T) = \underline{p}_B'(T) \cdot \underset{\sim}{Q} \tag{16}$$

where

$$\underset{\sim}{Q} = \begin{bmatrix} q_{rr}^{(s)} & q_{ru}^{(s)} \\ q_{ur}^{(s)} & q_{uu}^{(s)} \end{bmatrix}$$

It will be assumed that $q_{ur}^{(s)}$ is zero; i.e., that urban areas never completely revert to rural areas due to the physical nature of a city. In this adjusted forecast the urban population is then simply incremented by the probability of a rural to urban transition multiplied by the forecast rural population. Clearly this assumes that the probability of a transition is not correlated with the population size of a rural area.

It remains to determine $q_{ru}^{(s)}$. We shall suppose that the two reasons for a rural-urban transition can be separately estimated. To estimate the number of regions that become urban due to autonomous urban growth, we require estimates of the 'birth rate' of new cities per annum, b (Simon, 1955; Vining, 1974), and the mean area of a new city, δ. Then the land area of rural settlement which has become urban in one time period is:

$$A_{ru}^{*}(t,t+1) = R(t).(b.\delta) \tag{17}$$

where R(t) is the total area that is rural at time t, and $A_{ru}^{*}(t,t+1)$ is the land area that changes from rural to urban due to autonomous urban growth between t and t+1. Ignoring the areal expansion of these cities:

$$A_{ru}^{*}(\tau,T) = \sum_{k=\tau}^{T-1} A_{ru}^{*}(k, k+1) \tag{18}$$

To estimate urban-rural transitions due to urban expansion, it is necessary to have the following information: the critical population density above which a region is regarded to be urban; the total area of the country that is rural; and an estimate of how population densities are changing around cities. Definition of an area as urban solely on the basis of population density is highly problematic, but in a purely demographic forecast there is no other choice. Estimation of population expansion around a city may be calculated using the following formula, which has proven tc be successful in describing intra-urban population densities (Newling 1964, 1969):

$$D(s,t) = \exp\{-\beta(t).s\}.D(0,t) \tag{19}$$

where

$$\beta(t) = \beta(0).\exp\{-\gamma t\} \tag{20}$$

$$D(0,t) = D(0,0)\exp\{\alpha t\} \tag{21}$$

Here $D(s,t)$ is the population density at distance s from the city center and time t. Combining these equations:

$$D(s,t) = D(0,0)\exp\left\{\alpha.t - \beta(0)\exp\{-\gamma t\}.s\right\} \quad (22)$$

Equation (22) must however be estimated on empirical data, and even in logarithmic form it is non-linear. The estimation equation would be:

$$\log D(s,t) = A + \alpha t - \beta\exp\{-\gamma t\} + E. \quad (23)$$

where A, α, β, γ, are constants, and E is stochastic error.

A simpler equation for describing the same spatio-temporal diffusion of population densities is (Casetti et al, 1971):

$$D(s,t) = \exp\{-\beta s + \alpha t - \gamma ts\}\, D(0,0) \quad (24)$$

This is log-linear and estimable by standard methods. Since cities of different sizes show different dynamics of population density, it would be desirable to estimate (24) separately for each major city using historical data, or for groups of cities of similar size using cross-sectional and historical data.

Given empirical estimates of β, α and γ it is then possible to estimate urbanization due to urban expansion. Define $\bar{D}$ as the critical population density defining an urban settlement pattern. If $\bar{D}$ occurs at distance $\bar{S}$ from a city center, then all areas closer to the center than $\bar{S}$ may be defined as urban by the well known property that population densities on the urban fringe decline with distance from the city center. At time period T;

$$\hat{D}_i(s,T) = \exp\{-\hat{\beta}_i.s + \hat{\alpha}_i T - \hat{\gamma}_i sT\}.D_i(0,0) \quad (25)$$

where $\hat{D}_i(s,T)$ is the predicted population density for city (or city size class) i at time T, using the estimated parameters $\hat{\beta}_i$, $\hat{\alpha}_i$, $\hat{\gamma}_i$. The distance $\bar{S}_i$ at which the critical density $\overline{D}$ is reached for city i at time T is given by solving:

$$\overline{D} = \exp\{-\hat{\beta}_i\overline{S}_i + \hat{\alpha}_i T - \hat{\gamma}_i\overline{S}_i T\}\ D(0,0) \tag{26}$$

for $\overline{S}_i$, given $\overline{D}$ and all the parameters from estimating (24):

$$\overline{S}_i = \frac{\hat{\alpha}_i T - \log\left(\overline{D}/D_i(0,0)\right)}{\hat{\beta}_i + \hat{\gamma}_i T} \tag{27}$$

The forecasting steps for urban expansion are then as follows:

[1] Estimate (24) for each city (size class)

[2] Calculate $\overline{S}_i$ for each city using equation (27)

[3] Calculate the area that has become urban around city i, A^i_{ru}, as:

$$A^i_{ru} = \pi(\overline{S}_i - s^*_i)^2 \tag{28}$$

where s^*_i was the mean radius of the city at time τ.

[4] Calculate the proportion of land that has changed from rural to urban due to city exapnsion between τ and T as

$$\sum_i A^i_{ru} / R(\tau) \tag{29}$$

4.2. Aggregate Forecast Corrected for Urban Change

The correction to be applied to the base forecast would be:

$$\begin{aligned} \Delta \underline{p}'_2(T) &= \underline{p}'_2(T) - \underline{p}'_B(T) \\ &= \underline{p}'_B(\underset{\sim}{Q} - \underset{\sim}{I}) \end{aligned} \tag{30}$$

where

$$\underset{\sim}{Q} = \begin{bmatrix} 1 - q_{ru} & q_{ru} \\ 0 & 1 \end{bmatrix}$$

and $q_{ru} = \left[A^*_{ru} + \sum_i A^i_{ru} \right] / R(\tau)$, and can only be calculated using information to estimate (17), (24), and (28). The possibility of making this correction will depend on availability of data on b, δ, D(s,T), and s^*_i, as well as specifying $\overline{D}$.

4.3. Disaggregate Adjustments

To make adjustments to the multiregional forecast $\underline{p}'_1(T)$, it is generally superior if a disaggregate adjustment is made, since this requires no new information, and is superior to simply post-multiplying $\underline{p}'_1(T)$ by $\underset{\sim}{Q}$. The steps would be as follows

[1] Estimate (24) for each city

[2] Calculate $\overline{S}_i$ for each city using equation (27)

[3] Identify the set of those rural regions that lie within distance $\overline{S}_i$ of the center of city i, $\{R_1\}$

[4] Estimate $A^*_{ru}(\tau,T)$ using (18) and calculate the estimated number of rural areas becoming urban due to autonomous city growth, $N^*_{ru}(\tau,T)$ as:

$$N^*_{ru}(\tau,T) = A^*_{ru}(\tau,T)/R_2$$

where R_2 is the mean land area of a rural region excluding those incorporated in the set $\{R_1\}$.

[5] Calculate an adjusted forecast as follows:

$$\underline{p}'_3(T) = \underline{p}'_M(T) . \underset{\sim}{E}^* \tag{31}$$

$$P'_{4}(T) = P'_{3}(T).\underset{\sim}{Q}^{*} \tag{32}$$

Here $\underset{\sim}{E}^{*}$ is an aggregation matrix of dimension 2 by M+N, where entries on row one equal one only for rural regions not in the set $\{R_1\}$, and entries on row two equal one only for urban regions, and rural regions in the set $\{R_1\}$. All other entries in $\underset{\sim}{E}^{*}$ are zero. $\underset{\sim}{E}^{*}$ then aggregates all urban areas, and rural areas that became urban due to city expansions into the urban sector. Equation (31) thus adjusts the forecast for city expansion. Equation (32) further adjusts the forecasts for autonomous urban growth, with $\underset{\sim}{Q}^{*}$ defined as:

$$\underset{\sim}{Q}^{*} = \begin{bmatrix} 1 - q_{ru} & q_{ru} \\ 0 & 1 \end{bmatrix}$$

with $q_{ru} = N^{*}_{ru}(\tau,T)/N^{*}_{r}$ where N^{*}_{r} is the number of rural areas excluding those in the set $\{R_1\}$.

4.4. Disaggregate Corrected Forecast for Urban Change

The correction to be applied to the base forecast is, using (14), (31), (32):

$$\Delta \underline{p}'_{4}(T) = \underline{p}'_{4}(T) - \underline{p}'_{B}(T) \tag{33}$$

$$= \underline{p}'_{M}(\tau) \left[\underset{\sim}{M}^{S}.\underset{\sim}{E}^{*}.\underset{\sim}{Q}^{*} - \underset{\sim}{E}'.\underset{\sim}{M}^{S}_{B} \right]$$

5. FORECASTING USING LARGE REGIONS

In the multiregional forecast adopted above it was assumed that each region could be uniquely described as urban or rural. In the event that this is not the case, which is likely to be relatively common because for data reasons small regions are typically inappropriate, further adjustments must be made. Let the set of regions be divided into 3 groups: Those totally urban (typically large cities), those totally rural, and those that are mixed. It is only for the last group that adjustments must be incorporated. Correspondingly, partition the vector $\underset{\sim}{p}'_M$ into three and the matrix $\underset{\sim}{M}$ into nine segments.

Assume for mixed region k the urban population at time t is $u_k(t)p_k(t)$ where $u_k(t)$ is the proportion of the population that is urban at time t. At some future point T the urban proportion in k is likely to be different. Its new value may be estimated if the rate of growth of the urban population, g_{uk}, and the rural population, g_{rk}, in region k are known at the start of the forecast. Then a reasonable estimate of $u_k(T)$ is:

$$u_k(T).\hat{p}_k(T) = u_k(\tau).p_k(\tau).(1+g_{ku})^s \qquad (34)$$

where $\hat{p}_k(T)$ is the expected population in mixed region k if current growth rates continue. If the aggregate growth rate of population in k is g_k,

$$\hat{p}_k(T) = p_k(\tau)(1+g_k)^s \qquad (35)$$

But

$$1 + g_k = (1+g_{kr}).r_k(\tau) + (1+g_{ku})u_k(\tau) \qquad (36)$$

where $r_k(\tau)$ is the proportion of the population in k that is rural at time τ. Thus, using (34), (35), (36):

$$u_k(T) = u_k(\tau).(1+g_{ku})^s/G^s \qquad (37)$$

$$G = (g_{ku}-g_{kr})u_k(\tau) + 1 + g_{kr} \quad (38)$$

$$r_k(T) = 1 - u_k(T) \quad (39)$$

The multiregional forcast with mixed regions present, at time T, is:

$$\underline{p}'_M(T) = \left[p_u(T) \ldots \quad p_k(T) \ldots \quad p_r(T) \right]$$

where the subscripts u, k and r refer to urban, mixed and rural regions. This may now be aggregated into urban/rural forecasts by:

$$\underline{p}'_M(T) = \underline{p}'_M(T) . \hat{\underset{\sim}{D}}'(T) \quad (40)$$

where

$$\hat{\underset{\sim}{D}}(T) = \left[\begin{array}{ccc|cc|cccc} 1 & 1 & \ldots & r_k(T) & \ldots & 0 & 0 & \ldots & 0 \\ & & & & & & & & \\ 0 & 0 & \ldots & u_k(T) & \ldots & 1 & 1 & \ldots & 1 \end{array} \right]$$

where $\hat{\underset{\sim}{D}}(T)$ is estimated using equations (37), (38), and (39).

5.1. Corrected Forecast with Aggregate Regions

With large regions, and neglecting corrections necessitated by urban areas becoming rural areas, the correction to the basic forecast is:

$$\begin{aligned} \Delta \underline{p}'_5(T) &= \underline{p}'_5(T) - \underline{p}'_B(T) \\ &= \underline{p}'_M(\tau) \left[\underset{\sim}{M}^s . \hat{\underset{\sim}{D}}'(T) - \underset{\sim}{M}^s_B \right] \quad (41) \end{aligned}$$

When the regions are large it becomes difficult to further adjust the forecasts to account for urban expansion and new cities. These are events that typically occur within the regions since they are phenomena associated with a lower geographical scale. Thus mixed regions may become urban or the urban/rural mix in regions may shift. What is required here is a method of forecasting $u_k(T)$ that is different from the naive projection applied above. But of course forecasting $u_k(T)$ using subregional data is identical to the task of forecasting the national rural/urban population split using regional data. Thus with mixed regions the same forecasting process as outlined earlier must be repeated at a smaller scale in order to account properly for urban change.

6. CHANGING RATES OF MIGRATION

A 'gravity-type' model of urban and rural population change incorporating changing migration rates has been proposed for use by the United Nations (1980). Reformulated to conform with the notation of this paper and in discrete time:

$$\begin{aligned} p_r(t) &= \left(1+a_r-m_{ur}(t-1)\right)p_r(t-1) + m_{ur}(t-1)p_u(t-1) \\ p_u(t) &= m_{ru}(t-1)p_r(t-1) + \left(1+a_u-m_{ru}(t-1)\right)p_u(t-1) \end{aligned} \tag{42}$$

where

$$m_{ru}(t-1) = c + d.p_u(t-1)/\left[p_r(t-1)+p_u(t-1)\right] \tag{43}$$

$$m_{ur}(t-1) = e + f.p_r(t-1)/\left[p_r(t-1)+p_u(t-1)\right] \tag{44}$$

This model, however, is weak theoretically. By treating m_{rr} and m_{uu} as residuals, net of rural-urban or urban-rural migration, the implication is that there is no migration between cities or rural areas. It is more consistent to pose the existence of such migration, but to separate it in turn from non-migrants. If μ_r is the propensity of a rural migrant to migrate, then a more consistent model is:

$$p_r(t) = a_r p_r(t-1) + (1-\mu_r) p_r(t-1) + \mu_r m_{rr} p_r(t-1) + \mu_u m_{ur} p_u(t-1) \quad (45)$$

$$p_u(t) = a_u p_u(t-1) + (1-\mu_u) p_u(t-1) + \mu_r m_{ru} p_r(t-1) + \mu_u m_{uu} p_u(t-1) \quad (46)$$

Assume

$$m_{ur}(t-1) = f_{ur} p_r(t-1) / \left[f_{ur} p_r(t-1) + f_{uu} p_u(t-1) \right] \quad (47)$$

$$m_{ru}(t-1) = f_{ru} p_u(t-1) / \left[f_{ru} p_u(t-1) + f_{rr} p_r(t-1) \right] \quad (48)$$

By definition, because movers must be accounted for:

$$\mu_u p_u(t-1) = \mu_u m_{uu}(t-1) p_u(t-1) + \mu_u m_{ur}(t-1) p_u(t-1) \quad (49)$$

$$\mu_r p_r(t-1) = \mu_r m_{ru}(t-1) p_r(t-1) + \mu_r m_{rr}(t-1) p_r(t-1) \quad (50)$$

Therefore:

$$m_{uu}(t-1) = f_{uu} p_u(t-1) / \left[f_{ur} p_r(t-1) + f_{uu} p_u(t-1) \right] \quad (51)$$

$$m_{rr}(t-1) = f_{rr} p_r(t-1) / \left[f_{rr} p_r(t-1) + f_{ru} p_u(t-1) \right] \quad (52)$$

The equations (45), (46), become, in matrix form:

$$\underline{p}'_G(t) = \underline{p}'_G(t-1) \left[(\underset{\sim}{A}_B) + \underset{\sim}{I} - (\underset{\sim}{\mu}_B) + (\underset{\sim}{p}_G(t-1) \cdot (\underset{\sim}{\mu}_B) \cdot \underset{\sim}{F}'_B \right] \quad (53)$$

where

$$(\underset{\sim}{A}_B) = \begin{bmatrix} q_r & 0 \\ 0 & a_u \end{bmatrix}$$

$$(\underset{\sim}{\mu}_B) = \begin{bmatrix} \mu_r & 0 \\ 0 & \mu_u \end{bmatrix}$$

$$\left(p_G(t-1)\right) = \begin{bmatrix} p_r(t-1) & 0 \\ 0 & p_u(t-1) \end{bmatrix}$$

$$F'_B = \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix}$$

Equation (53) is close in spirit to the United Nations proposal, but is non-linear. Although it may be simplified in the two state case (Ledent, 1980), the dynamics of the multiregional case are very complex (Sheppard,1982). The general equation for forecasting T time periods ahead is very long and can only be solved recursively in the multiregional case. Thus a comparison of the multiregional case with this urban/rural case is most easily examined for just one time period.

6.1. The Corrected Forecast with Changing Interactions

The correction, to be applied to the basic forecast with changing interactions of equation (53) allowing for multiregional flows, is:

$$\Delta p'_6(T) = p'_6(T).E' - p'_G(T)$$

where $p'_6(T)$ is the non-linear forecast for the full multiregional system.

When a one period correction is desired:

$$\Delta p'_6(T) = p'_6(T-1).\left[H(T-1)E'-E.H_B(T-1)\right] \qquad (54)$$

where

$$H(T-1) = (A) + I - (\mu) + p_6(T-1) \;.(\mu).F'$$

$$H_B(T-1) = (A_B) + I - (\mu_B) + p_G(T-1).(\mu_B).F'_B$$

and $(\underset{\sim}{A})$, $(\underset{\sim}{\mu})$, $(\underset{\sim}{p}_6)$, $\underset{\sim}{F}$ are equivalent to matrices $(\underset{\sim}{A}_B)$, $(\underset{\sim}{\mu}_B)$, $(\underset{\sim}{p}_G)$, $\underset{\sim}{F}_B$ extended in size to represent the full multiregional system. Note that the two forecasts are assumed to be identical at time period (T-1) in equation (54):

$$\underline{p}_G'(T-1) \equiv \underline{p}_6'(T-1).\underset{\sim}{E}' \qquad (55)$$

Further corrections for rural areas that become urban areas, and for regions with mixed urban and rural populations may also be given. Thus the correction equivalent to equation (30), aggregate corrections for urban change, is:

$$\Delta\underline{p}_7'(T) = \underline{p}_6'(T).\underset{\sim}{E}.\underset{\sim}{Q} - \underline{p}_G'(T) \qquad (56)$$

The correction to the basic non-linear forecast in the presence of mixed regions is:

$$\Delta\underline{p}_8'(T) = \underline{p}_6'(T).\underset{\sim}{D}(T) - \underline{p}_G'(T) \qquad (57)$$

7. CONCLUSIONS

This paper has suggested a series of seven formulae [equations (14), (30), (33), (41), (54), (56), (57)] for correcting urban and rural forecasts that are only composed of two homogeneous and permanently defined states 'urban' and 'rural'. The corrections account for: the aggregation error in assuming that multiregional migration rates may be subsumed under only two flows (rural to urban and *vice versa*); the error induced because formerly rural areas may be urban by the end of the forecast; and regions which are partly urban and partly rural. If constant migration rates are assumed the corrections are easily computed, although those of the second type listed here have high secondary data requirements. These results should also be readily generalizable to models with full life tables. With changing migration rates, computation is cumbersome and no simple solutions are known. However, here also the corrections are in principle computable.

In this case, however, introduction of age distributions is very difficult even analytically (Gurtin and MacCamy, 1974). It is hoped that an approach such as this can at least provide a foundation for methods that will allow some rule-at-thumb adjustments of aggregate urban-rural demographic forecasts.

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APPENDIX

Equation (10) follows immediately from the definitions of $\underline{p}_B'$ and $\underline{p}_M'$. To derive (11), recall that the population from r_i that remains in some rural region during the τ-th time period is $p_{r_i}(\tau) \cdot \left[a_{r_i} + \sum_j m_{r_i r_j} \right]$ Further the following identity is true:

$$p_r(\tau)(m_{rr}+a_r) = \sum_i p_{r_i}(\tau)\left[\sum_j m_{r_i r_j} + a_{r_i}\right]$$

whence

$$m_{rr} = \left[\sum_i p_{r_i}(\tau) \sum_j m_{r_i r_j}\right] / p_r(\tau)$$

The same holds for the other three transitions, and (11) expresses this in matrix form.