WORKING PAPER

PROBLEMS OF DECISION-MAKING WITH FUZZY INFORMATION

S.A. Orlovski

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IIASA
International Institute for Applied Systems Analysis

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ABSTRACT

A fuzzy set is a mathematical model of a collection of elements (objects) with fuzzy boundaries, which involves the possibility of gradual transition from complete belongness to nonbelongness of an element to a collection. This concept is introduced in the Fuzzy Sets Theory as the means to model mathematically fuzzy notions that are used by human beings in describing their understanding of real systems, their preferences, goals, etc. This introductory paper outlines various classes of problems of decision-making in a fuzzy environment, that is, in which information is modeled in terms of fuzzy sets and relations. The analytical approaches outlined here enable the analyst to use the information in a fuzzy form for narrowing down the scope of alternative decisions, by discarding those of them for which better alternatives can be found. A number of illustrative examples are discussed.

PREFACE

Among the important issues on the research agenda of the new IIASA project, "Impacts of Human Activities on Environmental Systems", is the one of uncertainty. In many systems and particularly in those in which human beings participate, much of the information with regard to the goals, constraints and impacts of possible human actions is often of a subjective and imprecise nature. This type of uncertainty, which differs from random uncertainty, requires special mathematical tools for its description and use in mathematical modeling and analysis. The fuzzy sets theory is an attempt to provide such tools for the utilization of subjective uncertainties in decision analysis and related mathematical modeling.

This paper provides an overview of some fundamental concepts and definitions of the fuzzy sets theory, however, its main thrust is decision-making with fuzzy information. It is hoped that it will attract IIASA's scholars to this relatively new direction in analysis and modeling that explicitly takes into account human judgement, perceptions and emotions that play such an important role at the interface among social, economic, and environmental systems.

Janusz Kindler
Project Leader
Impacts of Human Activities
on Environmental Systems

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Problems of decision-making with fuzzy information

S.A. Orlovski

1. Introduction

One of the trends in the development of applied systems analysis is the widening application of mathematical reasoning and modeling for the description and analysis of economic, social, environmental and other highly complex systems. An inherent feature of this type of system shows that, apart from objective principles, their description and behavior is largely based on subjective judgements, understanding, reasoning and even emotions of human beings.

One crucial point in the use of mathematical models for the analysis of real systems lies in the adequate description and use of information available about the system's structure and behavior. A mathematical model "accepts" the information represented in one or another mathematical form (numbers, functions, etc.). If this representation or a model of the information is not sufficiently adequate, then the results of the analysis will be misleading even using a good model of the system.

In some systems the information needed for their modeling and analysis can often be obtained by direct or indirect measurements and expressed (modeled) quantitatively. The analysis of the system can then be based on the use of traditional mathematical techniques for processing this information. But in other systems, and particularly those in which human beings participate, this kind of measurement is often not possible, and the analyst has to rely either on his own understanding of the system's structure or to use the help of experts or people who have experience in working with this system and know some of its properties, who have an understanding of the system's goals, etc. And it is important that this understanding or, in other words, the information about the system under study is of subjective nature, and its description in a natural language contains a great deal of uncertainties like "a little", "much", "to increase substantially", "high", "very effective", "a little better", etc, which do not have analogs in the traditional mathematical language. In many cases, a description of this type of information using this language makes a model too poor and an insufficiently adequate representation of the real system in question.

On the other hand, human beings modeling reality in such terms frequently find possibly not the best, but an acceptable behavior in real situations which are far too complex to be described and/or analyzed by formal methods. It is in this connection that we may say that the language of the traditional mathematics, based on the theory of sets and two-valued logics, is probably not sufficiently flexible for modeling real complex systems, since it does not have the means for describing with sufficient adequacy the concepts which are used by human beings and which have fuzzy meaning.

As a simple example, we can consider a problem of classification of objects according to their colors. Let us assume that it suffices to distinguish only between red, yellow and green objects. From the viewpoint of the traditional mathematics this classification problem consists of dividing a given collection of colored objects into three nonintersecting subsets with precisely specified boundaries between them. However, this type of classification is of little correspondence to our understanding of color. In our view, transition from red to yellow, for example, is continuous, and we accept that some objects can belong to a smaller or greater degree to different classes simultaneously, or, in other words, that the boundaries between the classes are fuzzy rather than sharp.

Thus, the further successful implementation of mathematical methods as efficient analytical tools requires among other things the elaboration of the means allowing for a more flexible use in modeling fuzzy views and understanding by human beings of the real world.

The fuzzy sets theory is a step in this direction. The concept of a fuzzy set is suggested in this theory as the means to model mathematically fuzzy notions that are used by human beings in describing their understanding of real systems, their preferences, goals, etc. A fuzzy set is a mathematical model of a collection of elements with fuzzy boundaries, and it involves the possibility of a gradual transition from complete belongness to nonbelongness of an element to this collection. And one of the directions in the development of the fuzzy sets theory lies in the elaboration of the means for processing information in the form of various types of fuzzy sets in mathematical modeling.

The innovative paper on fuzzy sets by L.Zadeh (L.A. Zadeh, 1965) appeared in 1965. The new concept attracted great attention among analysts and modelers in many fields, and the already numerous bibliography of works on various aspects of the fuzzy sets theory and its application is rapidly expanding.

Apparently, there is an implicit demand at IIASA for descriptive and analytical mathematical tools other than traditional, a demand that stems from the orientation of IIASA research towards more explicit consideration of the roles played by human actors in the evolution and coexistence of real environmental, economic, social and other systems. This paper aims basically at stimulating the interest of IIASA's researchers to this new direction in modeling and analysis of complex real systems. Apparently, it is even unnecessary to remark here that the formulations of problems and approaches to their analyses outlined in this paper are only possible ways of using mathematical means for modeling and processing information of fuzzy nature. But, on the other hand, it appears equally unnecessary to advocate for the necessity of trying to move in this direction in our research.

This paper concerns only one of the directions of application of this new approach -- problems of decision-making in a fuzzy environment. More precisely, it outlines various classes of such problems in which the information is modeled in terms of fuzzy sets and relations as introduced by L.Zadeh. To our understanding, an analysis of a decision-making process aims at screening out irrational alternatives, or, in other words, it focuses on the use of the information available to narrow the scope of alternatives by discarding those of them for which better alternatives can be found. This approach is used in all the models considered here.

The paper is organized as follows. In the subsequent Sect. 2, we introduce some preliminaries from the fuzzy sets theory needed for the further presentation. Sect. 3 presents a classification of problems which are considered in the paper. In Sect. 4, we outline an approach to problems of choice with a single fuzzy preference relation, and in Sect. 5 we consider problems in which the information about the preferences is specified in the form of multiple fuzzy preference relations. In Sect. 6, we discuss a general formulation of a fuzzy mathematical programming problem, and in Sect. 7 more specific problems with fuzzy parameters.

2. Fuzzy sets and fuzzy relations

2.1. Fuzzy sets

As has been said in the previous section, the concept of a fuzzy set is an attempt to formalize mathematically information of imprecise nature to provide for its use in mathematical modeling and analysis of real systems. Underlying this concept is the understanding that elements with some property in common and thus forming a collection may possess this common property to different degrees. With this understanding, statements like "x belongs to a given set" do not make sense since it is neccesary to indicate "how strongly" or to what degree this element belongs to the set in question.

One of the simplest ways to describe a fuzzy set mathematically is to characterize this degree of belongness by a number from, say, interval [0,1]. Let **X** be a set (in the traditional sense) of elements. In the following we consider subsets of this set.

Definition 1. *A fuzzy subset* **C** *of* **X** *is a collection of pairs* $(x, \mu_{\mathbf{C}}(x))$, *with* $x \in \mathbf{X}$ *and* $\mu_{\mathbf{C}}$ *being a function* $\mathbf{X} \to [0,1]$, *called the membership function of the fuzzy set* **C**. *A value of this function for any* $x \in \mathbf{X}$ *is refered to as the membership degree of* x *in* **C**.

Fuzzy sets of more general types can be defined (see for example, L.Zadeh, 1965).

It is noteworthy, that traditional sets constitute a subclass of a class of fuzzy sets. In fact, the membership function of a traditional set $\mathbf{B} \in \mathbf{X}$ is its characteristic function:

$$\mu_{\mathbf{B}}(x) = \begin{cases} 1, & \text{if } x \in \mathbf{B} \\ 0, & \textit{otherwise} \end{cases}$$

and in accordance with Definition 1 the traditional set **B** can be defined also as a collection of pairs $(x, \mu_{\mathbf{B}}(x))$. Therefore, a fuzzy set is a more general concept than a traditional set in that the membership function of a fuzzy set can be any function or even more generally a mapping.

For comparison consider a traditional set $\mathbf{B} = \{x \mid 0 \le x \le 2\}$ and a fuzzy set $\mathbf{C} = \{x \mid \textit{values of } x \textit{ close to } 1\}$. The membership functions of these sets are illustrated in Fig. 1. Note that the form of the membership function $\mu_{\mathbf{C}}$ of the fuzzy

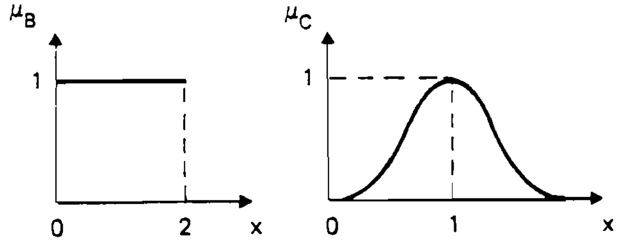

Figure 1. Membership functions of a traditional interval and of a fuzzy interval.

set **C** depends upon the meaning of the concept "close" in the context of a specific situation under analysis.

If **A** and **B** are two fuzzy subsets of **X**, then we say that **A** includes **B** ($\mathbf{B} \subseteq \mathbf{A}$), iff

$$\mu_{\mathbf{B}}(x) \leq \mu_{\mathbf{A}}(x)$$

holds for any $x \in \mathbf{X}$.

Processing of information in the form of fuzzy sets is based on **operations on fuzzy sets** which may be introduced in various ways, and the choice of a particular definition should correspond to the meaning of this operation in the context of a particular situation or problem considered. When introducing operations on fuzzy sets one should remember that the class of fuzzy sets includes sets in the traditional sense. Therefore, the operations introduced when applied to traditional sets must correspond to the usual set-theoretic operations. This, of course, does not apply to those operations which are specific only to fuzzy sets (concentration, dilution, convex hull, etc.). A discussion of operations on fuzzy sets can be found for instance in Zadeh, 1973.

In what follows, we introduce some operations in those forms which are used in this paper.

1. Union (Fig. 2).

$$\mu_{\mathbf{A}\cup\mathbf{B}}(x) = \max\{\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)\}, \; x \in \mathbf{X}$$

If $\mathbf{A}_y$ is a parametric family of fuzzy sets $\mu_{\mathbf{A}_y}(x, y)$ with $y \in \mathbf{Y}$ being the parameter of the family, then the union **C** of the family is described by the membership function of the form:

$$\mu_{\mathbf{C}}(x) = \sup_{y \in \mathbf{Y}} \mu_{\mathbf{A}_y}(x, y), \; x \in \mathbf{X}$$

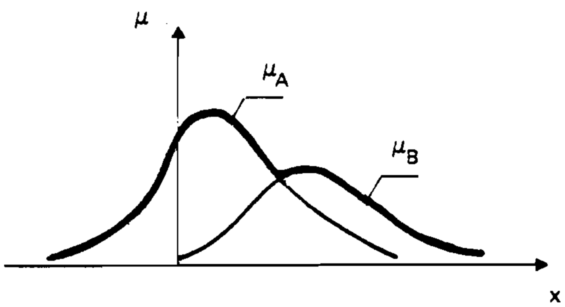

Figure 2. Union of fuzzy sets.

2. Intersection (Fig. 3).

$$\mu_{\mathbf{A}\cap\mathbf{B}}(x) = \min\{\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)\}, \; x \in \mathbf{X}$$

Similarly, for a family of fuzzy sets we have

$$\mu_{\mathbf{C}}(x) = \inf_{y \in \mathbf{Y}} \mu_{\mathbf{A}_y}(x, y), \; x \in \mathbf{X}$$

3. Complementation ($\mathbf{A}' = \mathbf{X} \setminus \mathbf{A}$) (Fig. 4).

$$\mu_{\mathbf{A}'}(x) = 1 - \mu_{\mathbf{A}}(x), \; x \in \mathbf{X}$$

It is of interest that by using this definition, we generally have $\mathbf{A} \cap \mathbf{A}' \neq \phi$, which is not the case with the traditional sets.

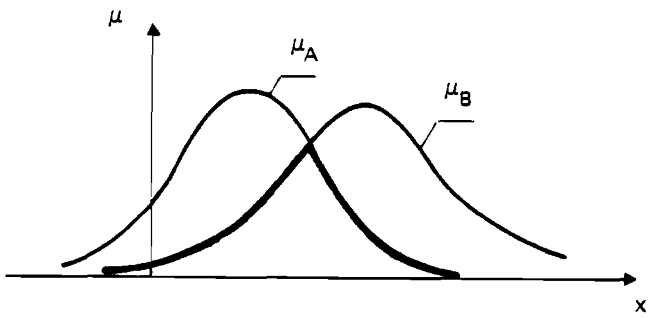

Figure 3. Intersection of fuzzy sets.

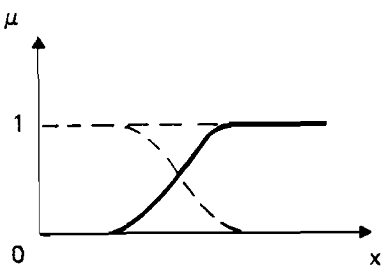

Figure 4. Complement of a fuzzy set.

For an example, consider a fuzzy set **A** = { *numbers much greater than 0*} and assume that the membership function of this set has the form shown in Fig. 4 (solid line). Then the dotted line in this figure corresponds to the membership function of the complement **A'** of **A** in the number axis. Verbally, the set **A'** can be described as a set of numbers which are not much greater than 0.

The nonempty intersection of these two sets represents a set of numbers which are and, at the same time, are not "much greater" than 0. The nonemptiness of this set reflects the fact that the concept "to be much greater" is defined fuzzily and some numbers may to certain degrees belong to both sets **A** and **A'** at the same time. In some sense, the intersection of these sets can be viewed as a fuzzy boundary between them.

4. Difference between sets

$$\mu_{\mathbf{A} \setminus \mathbf{B}} = \begin{cases} \mu_{\mathbf{A}}(x) - \mu_{\mathbf{B}}(x), & when\ \mu_{\mathbf{A}}(x) \geq \mu_{\mathbf{B}}(x) \\ 0, & otherwise \end{cases}$$

Note, that the previous definition of the complement follows from this definition.

2.2. Fuzzy relations

As will be seen from the subsequent sections of this paper, fuzzy relations represent an important concept facilitating formulation and analysis of mathematical models of decision-making problems. In problems of this type, preference relations on sets of alternatives, objects, etc., are commonly evaluated by means of consulting experts, who often do not a have fully clear idea with regard to these preferences. In such cases, fuzzy relations may serve as a more convenient, flexible, and more adequate to reality form of representation of information than traditional relations.

As is well known, a relation **R** on a set **X** can be defined as a subset of the product set **X**×**X** In accordance to this definition, to describe a relation on the set **X** means to indicate all ordered pairs $(x,y) \in \mathbf{X}$ in which x and y are connected by this relation. To indicate this connection we shall alternatively use the notation $x\mathbf{R}y$ or $(x,y) \in \mathbf{R}$.

A simple example is a relation "not smaller" on the interval [0,1]. In Fig. 5a, this relation (i.e. the set of all pairs (x,y) such that $x \geq y$) is represented by a shadowed region. As can easily be seen, the diagonal in this figure corresponds to the relation "equal".

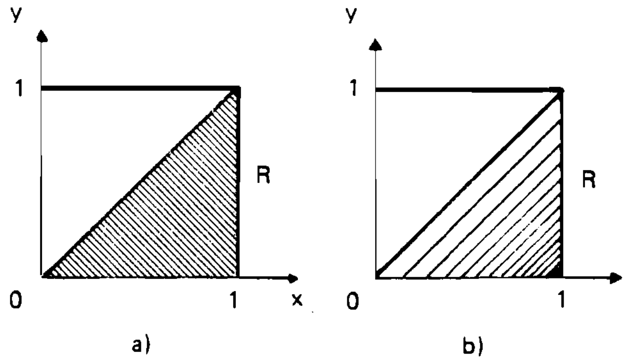

Figure 5. Relations (≥) and (>>).

In cases where the set $\mathbf{X} = \{x_1,...,x_n\}$ is finite, a relation on this set can conveniently be described with a matrix $|r_{ij}|$ with elements defined as follows:

$$r_{ij} = \begin{cases} 1, & \text{if } x_i \mathbf{R} x_j \\ 0, & otherwise \end{cases}, \quad i,j = 1,...,n.$$

The representation of a relation as a set helps to understand that, in principle, transition from traditional to fuzzy relations is similar to that in the case of sets. A description of a fuzzy relation must include not only the indication of all pairs of elements connected by this relation but also numbers from interval [0,1] reflecting degrees (or strengths) of these connections. Then we come to the following

Definition 2. *A fuzzy relation* **R** *on a set* **X** *is a fuzzy subset of the product set* **X**×**X** *with the membership function* $\mu_{\mathbf{R}} : \mathbf{X} \times \mathbf{X} \to [0,1]$ *.

It appears useful at this point to consider a simple example demonstrating the difference between traditional and fuzzy relations. Let us consider two "similar" relations on the same interval [0,1], one of which is fuzzy -- traditional

*We note here that similar to fuzzy sets, fuzzy relations of more general types can be defined (see Sect. 2.1).

relation $\mathbf{R}\,(\geq)$ and a fuzzy relation $\mathbf{R}\,(\gg)$ ("much greater"). The first of them is illustrated in Fig. 5a by the shadowed region. The diagonal of the unit square is the boundary of this region: all pairs which are beyond this boundary are not connected by the relation **R**.

The situation is more complex in the case of the fuzzy relation **R**, since the concept "much greater" is of imprecise, fuzzy nature. Trying to indicate a subset of the unit square corresponding to **R**, we find out that there are pairs (x,y) in this square which we can definitely put into the set **R** (i.e. we are certain as to $x \gg y$), and pairs that we definitely do not put into this set (i.e. we are certain as to $x \gg y$). For example, we may agree that x_1=0.9 is definitely much greater than y_1=0.001, i.e. $x_1 \gg y_1$. On the other hand, it may also be clear that for x_2=0.8 and y_2=0.6 we can definitely write $x_2 \gg y_2$. However, we may not have such certainty with respect, say, to a pair x_3=0.9; y_3=0.2. If at the same time we compare pairs x_3=0.9; y_3=0.2 and x_4=0.9; y_4=0.3 we can say that the relation $(\gg)$ is more applicable to the former of these pairs. Therefore, there exists some intermediate region of transition from pairs definitely connected by **R** to pairs to which this relation is definitely not applicable, and numbers can be assigned to pairs in this region which would reflect the degrees to which this relation is applicable to them. Therefore, we cannot find a sharp boundary for **R** in the unit square, and this fact is reflected by the gradually changing shadow density in Fig. 5b.

Similar to traditional relations, if the set **X** is finite then the membership function of a fuzzy relation on this set can be described with a matrix $|r_{ij}|$, but in this case its elements can take not only values 0 and 1 but also any other intermediate values. The value of an element $|r_{ij}|$ reflects a degree to which the relation $x_i \mathbf{R} x_j$ holds.

Since fuzzy relations may be understood as fuzzy sets, all operations on fuzzy sets like union, intersection, complementation and others are also applicable to them.

For a fuzzy relation **R** *the inverse relation* $\mathbf{R}^{-1}$ is defined as follows:

$$x\mathbf{R}^{-1}y \Rightarrow y\mathbf{R}x, \quad x,y \in \mathbf{X}$$

or using the membership functions:

$$\mu_{\mathbf{R}^{-1}}(x,y) = \mu_{\mathbf{R}}(y,x), \quad x,y \in \mathbf{X}$$

As will be seen a significant role in applied problems is played by the *composition (or product)* of relations. One of the definitions of this relation-specific operation used in this paper is as follows:

Definition 3. *The composition* **C** *of two fuzzy relations* **A** *and* **B** *is described by the membership function of the form:*

$$\mu_{\mathbf{C}}(x,y) = \sup_{z \in \mathbf{X}} \min\{\mu_{\mathbf{A}}(x,z), \mu_{\mathbf{B}}(z,y)\}.$$

2.3. Extension of fuzzy mappings and relations onto classes of fuzzy sets

In many problems a need arises for extending the domain **X** of a given mapping or a relation by including, together with elements of **X**, also fuzzy subsets of this set.

Consider for an example a set of controls **U** and a mapping $f: \mathbf{U} \to \mathbf{V}$ describing the behavior of a controlled system. An image $v = f(u)$ of a control $u \in \mathbf{U}$ is the reaction of the system to the choice of control u. If the control chosen is described fuzzily (for instance, u = {*slight increase of concentration*}) in the form of a fuzzy subset $\mu(u)$ of the set **U**, then to determine the corresponding reaction of the system we should find the image of $\mu(u)$ under the mapping f. In other words, we should have an extension of the domain of f onto the class of all fuzzy subsets of **U**. As will be seen in the sequel, similar problems of extending the domain of a fuzzy relation exist in the analysis of a general mathematical programming problem.

The way of performing this extention is called *extension principle*. This principle is of importance also because it provides for the extension of the operations for more general types of fuzzy sets. Here we introduce an extension principle based on the following definition of the image of a fuzzy set under a fuzzy mapping.

Definition 4. *The image* $\mathbf{B}$ *of a fuzzy subset* $\mathbf{A}$ *of a set* $\mathbf{X}$ *under a fuzzy mapping* $\mu_\varphi:\mathbf{X}\times\mathbf{Y}\to[0,1]$ *is a fuzzy subset of* $\mathbf{Y}$ *with the membership function of the form*

$$\mu_{\mathbf{B}}(y) = \sup_{x\in\mathbf{X}} \min\{\mu_{\mathbf{A}}(x),\mu_\varphi(x,y)\}$$

If μ_φ is a traditional mapping $\varphi:\mathbf{X}\to\mathbf{Y}$, or in other words,

$$\mu_\varphi(x,y) = \begin{cases} 1, & \text{if } y=\varphi(x) \\ 0, & \textit{otherwise} \end{cases}$$

then, as can easily be seen from Definition 4, we have

$$\mu_{\mathbf{B}}(y) = \sup_{x\in\varphi^{-1}(y)} \mu_{\mathbf{A}}(x)$$

which corresponds to the extension principle as introduced by L. Zadeh, 1973.

Using this principle, arithmetic operations on the number axis ($\mathbf{R}^1$) can be extended onto the class of fuzzy numbers, i.e. fuzzy subsets of this axis. For an example, operation of addition on $\mathbf{R}^1$ can be considered as the mapping $\varphi:\mathbf{R}^1\times\mathbf{R}^1\to\mathbf{R}^1$; $\varphi(r_1,r_2)=r=r_1+r_2$. Let μ_1, $\mu_2:\mathbf{R}^1\to[0,1]$ be two fuzzy numbers. The sum $\mu_\sigma=\mu_1+\mu_2$ is the image of the couple (μ_1,μ_2) under the mapping φ. Using the above extension principle, we obtain

$$\mu_\sigma(r) = \sup_{\substack{r_1,r_2\in\mathbf{R}^1 \\ r_1+r_2=r}} \min\{\mu_1(r_1),\mu_2(r_2)\}$$

In particular, if μ_1 and μ_2 represent intervals $[a_1,b_1]$ and $[a_2,b_2]$ then we have

$$[a_1,b_1] + [a_2,b_2] = [a_1+a_2,\ b_1+b_2].$$

Other operations can be extended in a similar way.

Let us now consider a problem of extending the domain of a fuzzy relation.

Let $\mathbf{Y}$ be a set with a fuzzy preference relation $\mathbf{R}$ defined on it, and denote by $\mu_{\mathbf{R}}$ the corresponding membership function $\mathbf{Y}\times\mathbf{Y}\to[0,1]$. Denote by Υ the class of all fuzzy subsets of $\mathbf{Y}$, or in other words, the class of functions $\nu:\mathbf{Y}\to[0,1]$. What is a fuzzy preference relation induced by $\mathbf{R}$ in the class Υ?. We consider this problem here basing on the extension principle introduced in Sect. 2.3.

The fuzzy relation $\mathbf{R}$ on the set $\mathbf{Y}$ can be considered as a fuzzy mapping $\mathbf{Y}\to\Upsilon$. Under this mapping, the image of any element $y^0\in\mathbf{Y}$ is a fuzzy subset of $\mathbf{Y}$ with the membership function $\mu(y^0,y)$. In fact, this function describes the fuzzy set of elements from $\mathbf{Y}$ connected with y^0 by the relation $\mathbf{R}$, i.e. such that $y^0\mathbf{R}y$.

If $\nu:\mathbf{Y}\to[0,1]$ is a fuzzy subset of $\mathbf{Y}$, then according to the extension principle the image of ν under the mapping μ is a fuzzy subset of $\mathbf{Y}$ with the membership function

$$\eta(\nu,y) = \sup_{z\in\mathbf{Y}} \min\{\nu(z),\mu(z,y)\}. \tag{2.1}$$

This function describes the extension of the original fuzzy relation $\mathbf{R}$ onto the set $\Upsilon\times\mathbf{Y}$, and a value $\eta(\nu^0,y)$ of this function is a degree to which the fuzzy set ν^0 is no less preferable than element y.

Similarly, we can obtain that a degree of the reversed preference $y\succcurlyeq\nu^0$ is as follows:

$$\eta(y,\nu^0) = \sup_{z\in\mathbf{Y}} \min\{\nu(z),\mu(y,z)\}.$$

Now we can continue this process of extension and consider the function η obtained as a fuzzy mapping $\mathbf{Y}\to\Upsilon$ with Υ being the class of fuzzy subsets of $\mathbf{Y}$. Using the extension principle we obtain for a fuzzy set $\nu^0\in\Upsilon$:

$$\eta(\nu,\nu^0) = \sup_{y\in\mathbf{Y}} \min\{\nu^0(y),\eta(\nu,y)\}. \tag{2.2}$$

Combining (2.1) and (2.2) we finally have:

$$\eta(\nu_1,\nu_2) = \sup_{y\in\mathbf{Y}} \min\{\nu_1(y),\sup_{z\in\mathbf{Y}} \min\{\nu_2(z),\mu(y,z)\} =$$

$$= \sup_{z, y \in \mathbf{Y}} \min\{\nu_1(y), \nu_2(z), \mu(y, z)\}. \qquad (2.3)$$

The value $\eta(\nu_1, \nu_2)$ is a degree to which the preference $\nu_1 \succeq \nu_2$ holds.

Similarly, we obtain that the degree of the reversed preference $\nu_2 \succeq \nu_1$ is as follows:

$$\eta(\nu_2, \nu_1) = \sup_{y, z \in \mathbf{Y}} \min\{\nu_1(y), \nu_2(z), \mu(z, y)\}.$$

If μ represents a relation in the traditional sense (unfuzzy) then it can easily be shown that (2.3) reduces to

$$\eta(\nu_1, \nu_2) = \sup_{\substack{y, z \in \mathbf{Y} \\ y \mathbf{R} z}} \min\{\nu_1(y), \nu_2(z)\}. \qquad (2.4)$$

To exemplify these results we consider $\mathbf{Y}$ to be the number axis and $\mathbf{R}$ to be the natural ordering ($\geq$) on it. Let us compare two fuzzy subsets (fuzzy numbers) ν_1 and ν_2 with the membership functions shown in Fig. 6.

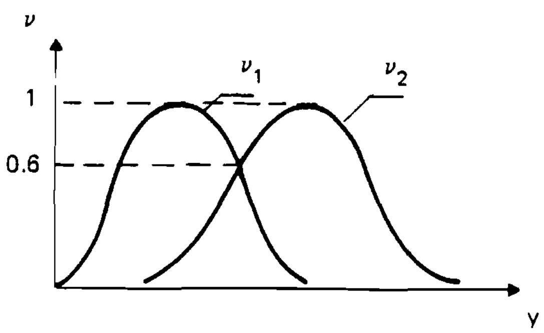

Figure 6. Comparison of two fuzzy sets.

Using (2.4) we obtain that $\eta(\nu_1, \nu_2) = 0.6$ and $\eta(\nu_2, \nu_1) = 1$, i.e. using the definition of the respective equivalence and strict preference relations:

ν_1 is equivalent (equal) to ν_2 to a degree 0.6;

ν_2 is strictly better (greater) than ν_1 to a degree 0.4.

Now we shall make some comments on the properties of the extended fuzzy relations. Consider three fuzzy subsets (intervals) of the number axis: ν_1, ν_2, ν_3 (Fig. 7).

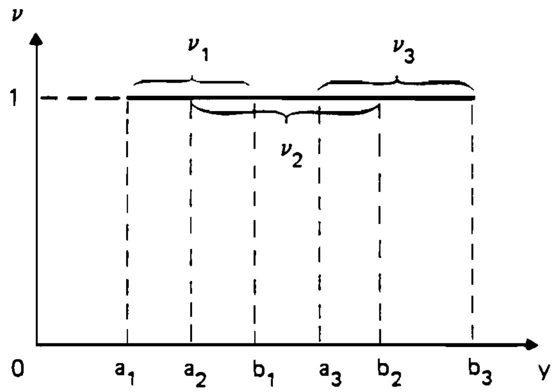

Figure 7. Membership functions of three intervals.

Using (2.4) and the definitions of the fuzzy equivalence and strict preference relations (see Sect. 4) we obtain:

$$\eta^e(\nu_1,\nu_2) = \eta^e(\nu_2,\nu_1) = 1;\ \eta^e(\nu_1,\nu_3) = 0;\ \eta^e(\nu_3,\nu_1) = 0,$$

and, therefore ν_1 and ν_2 are equivalent to a degree 1 (definitely equivalent). This may seem unnatural as the fuzzy set ν_2 is located more to the right with respect to ν_1, or in other words, ν_2 is "shifted" to the region with greater values of y.

Let us, however, give these fuzzy sets the following interpretation. Let points on the axis **Y** represent values of length and assume that ν_i, i=1,2,3 represents the result of measurement of the length of object i, width (a_i,b_i) of ν_i reflecting the precision of the measurement. It is obvious that within the given range of precision one has no justification to state that object 2 is longer than object 1 (and of course it is not reasonable to state the reverse). Thus within the given precision, objects 1 and 2 are indistinguishable from each other by their lengths, and it is this fact that is reflected by the equation $\eta^s(\nu_1,\nu_2)=1$. On the other hand, the precision in this case is sufficient to state that object 3 is longer than object 1, i.e. $\eta^s(\nu_3,\nu_1)=1$.

Concluding this section of preliminaries we should note that it contains only those notions and definitions which are necessary for going through the subsequent material in this paper. Further information on these issues can be obtained from the literature on this subject.

In the following, we outline formulations and approaches to analyses of some types of decision-making problems with fuzzy information. But before doing that we thought it useful to outline various types of such problems to help structure the further presentation in this paper.

3. Classification of decision-making problems

Two basic elements can be extracted from a description of a decision-making problem. The first is *a set of feasible choices* (decisions, alternatives, etc.) that may be described either in a fuzzy or in an unfuzzy form. The second element is *information* (fuzzy, or unfuzzy) available about the preferences between alternatives. A particular form of a decision-making problem depends largely on the form in which this information is presented.

In the general case this information has the form of fuzzy binary preference relations specified on a set of alternatives by means of their membership functions. These relations represent experts' subjective pairwise comparisons of alternatives with each other with respect to their multiple attributes. The attributes themselves may differ from each other in their relative importance, in which case still another fuzzy relation of "relative importance" may be specified on the set of the attributes. The analysis in such cases aims at determining in some sense "the best" or at least "not the worst" among the alternatives. The simplest of this type of problems of decision-making with **a single fuzzy preference relation** has been considered in Orlovski, 1978, Orlovski, 1981 and is outlined in the Sect. 4 of this paper. The more general case of **multiple fuzzy relations** has been treated in Kuz'min and Ovchinnikov, 1980 and in Orlovski, 1981, and is discussed in Sect. 5 of this paper.

In some cases preferences between alternatives may be described by a utility function. This function maps a given set of alternatives into an ordered set of estimates of alternatives and a preference relation is specified in this last set. This function, therefore, allows comparison of alternatives with each other by their estimates. If the estimates are numbers, then the decision-making problem is referred to as a mathematical programming problem.

In the fuzzy case, utility functions may have various forms. The most general form has the function which maps the set of alternatives into a class of fuzzy subsets of a set of estimates. In other words, for every alternative this function specifies a fuzzy estimate in the form of a fuzzy set of estimates. As an illustration to this, a utility function may be thought of as a performance function of a system under control. Fuzzy values of this function are then fuzzily described reactions of the system to controls. If we want to rationally control the system, we should be able to compare fuzzy reactions with each other to decide which of them are more satisfactory. Mathematically, this problem involves the necessity of extending a generally fuzzy preference relation from elements of a set of reactions onto fuzzy subsets of this set. This type of a decision-making problem that can be referred to as **a general fuzzy mathematical programming (FMP) problem** is treated in Orlovsky, 1980, 1981, and is also discussed in Sect. 6 of this paper.

Probably more related to practical situations is another case of decision-making problems in which information about the preferences between alternatives is supplied in the form of an unfuzzily specified utility function containing parameters, the values of which are fuzzy. The set of alternatives may also be described with unfuzzily specified functions containing fuzzy parameters. This model appears to be typical for practical situations when information with regard to the values of the parameters is obtained from experts. But even in cases when these parameters are the results of measurements, they are intervals rather than just numbers. This type of a decision-making problem which we refer to as that of **mathematical programming with fuzzy parameters** can be reduced to a problem of the previous type (see Orlovski, 1981), and we discuss this in Sect. 7 of this paper.

A special class of decision-making problems are **game problems.** In these problems the results of choices of alternatives by the decision-maker depend also on the choices of other human participants having their own preferences which differ from those of the decision-maker. Clearly, the rational behavior of the decision-maker in these cases should depend on his knowledge of his opponents' preferences. If this knowledge is imprecise or fuzzy, then we deal with a fuzzy model of a game situation. This type of model is not included in the paper; an interested reader is referred to Aubin, 1982, Orlovski, 1977, and Orlovski, 1981.

4. Decision-making with a single fuzzy preference relation

In this section we consider problems in which preferences between alternatives from a given set are described by a single preference relation and we want to rationally choose alternatives from this set using this information. Clearly, it is preferable to determine the best alternatives, i.e. those which are better than all other alternatives. But such alternatives generally do not exist, and then it is rational to choose those alternatives which are not dominated, or, in other words, alternatives for which better alternatives do not exist.

A preference relation representing pairwise comparisons of alternatives with each other is usually obtained by consulting experts, who possess the knowledge of the preferences. Let us assume that as a result of such consultations a preference relation **R** has been determined. In an unfuzzy case this means that one of the following statements was ascertained with respect to each pair of alternatives (x,y):

1. *"x is not worse than y"* i.e. that $x \succcurlyeq y$, or $(x,y) \in \mathbf{R}$,
2. *"y is not worse than x"*, i.e. $y \succcurlyeq x$, or $(y,x) \in \mathbf{R}$,
3. *"x and y are not comparable"*, or $(x,y) \notin \mathbf{R}$ and $(y,x) \notin \mathbf{R}$.

In real situations experts often do not have a clear idea of the preferences between alternatives, in the sense that they find it difficult to state definitely that, for example, alternative x is better than alternative y. If, on the other hand, an expert faces the necessity of giving this type of precise judgement with regard to preferences, he will have to roughly approximate his knowledge and understanding, and the resulting model is bound to be less adequate to the real situation. In such cases the decision-maker might more readily indicate numbers from the interval [0,1] characterizing his degrees of certainty in the preferences between alternatives. As a result we obtain a fuzzy preference relation in which each pair of alternatives is assigned a degree of the preference between them.

When consulting experts, a fuzzy preference relation may arise:

1. when each expert (or some of them) is not unambiguously certain as to $x \succcurlyeq y$;
2. when different experts have different opinions as to $x \succcurlyeq y$, in which case a fraction of the number of experts having voted for $x \succcurlyeq y$ may be taken as a degree of this preference;

3. a combination of (1) and (2).

In the last two instances a method of processing the experts' estimates to obtain the desired preference relation may present a separate problem.

This more flexible form of describing preference relations allows for the introduction, to a greater degree, of subjective experts' information into a mathematical model. A problem is then how to use this form of information for rational choices of alternatives. An approach to this problem is outlined in this section.

We assume that a fuzzy nonstrict preference relation **R** with the membership function $\mu_{\mathbf{R}} : \mathbf{X}{\times}\mathbf{X} \rightarrow [0,1]$ is specified on the given (nonfuzzy) set of alternatives **X**. The value $\mu_{\mathbf{R}}(x)$ is understood as the degree to which the preference $x \succeq y$ (x is not worse than y) is true.

As is known, unfuzzy preferences are usually modeled by quasi-order relations, i.e. they are assumed to possess both reflexivity and transitivity properties. But transitivity may not be an inherent property of fuzzy preferences encountered in real life situations, and when modeling them, it appears reasonable to consider a more general class of reflexive fuzzy relations. Thus we assume that the relation $\mu_{\mathbf{R}}$ is reflexive, i.e. possesses the property

$$\mu_{\mathbf{R}}(x,x) = 1$$

for any $x \in \mathbf{X}$*.

As has been mentioned, we are concerned with the determination of nondominated (ND) alternatives and we consider a set of all ND alternatives as a solution to the problem of choice. The reason for this is that ND alternatives are either equivalent to each other, or are noncomparable with each other on the basis of the preference relation considered. Therefore, we are not in a position

* In the sequel we sometimes omit subscript **R** in the notation of this membership function.

to prefer any one of them and should consider all of them as potentially rational choices. Of course, if additional information about the preferences is provided, then we, probably, can proceed with narrowing the set of rational choices down.

In our case we have a description of the original preferences with a fuzzy relation, therefore, we can expect to obtain a fuzzy description of ND alternatives; thus we call a solution to our problem a fuzzy set of ND alternatives.

To determine this fuzzy set, we define two fuzzy relations corresponding to the given preference relation **R**: fuzzy quasi-equivalence and fuzzy strict preference relations. These relations are formally defined as follows:

1. Fuzzy quasi-equivalence relation $\mathbf{R}^e$ *:

$$\mathbf{R}^e = \mathbf{R} \cap \mathbf{R}^{-1};$$

2. Fuzzy strict preference relation $\mathbf{R}^s$:

$$\mathbf{R}^s = \mathbf{R} \setminus (\mathbf{R} \cap \mathbf{R}^{-1}) = \mathbf{R} \setminus \mathbf{R}^{-1}.$$

Using the definitions from Sect. 2 we obtain the following membership functions for both the relations introduced:

$$\mu^e(x,y) = \min\{\mu(x,y),\mu(y,x)\},$$

$$\mu^s(x,y) = \begin{cases} \mu(x,y)-\mu(y,x), & \textit{when } \mu(x,y) \geq \mu(y,x), \\ 0, & \textit{otherwise}. \end{cases}$$

For any fixed alternative $y \in \mathbf{X}$ the function $\mu(y,x)$ describes a fuzzy set of alternatives which are strictly dominated by y (strictly worse than y). Therefore, the complement of this fuzzy set, which is described in Sect. 2.1 by the membership function $1-\mu^s(y,x)$, is for any fixed y the fuzzy set of all alternatives which are not strictly dominated by y. Then the intersection of all such fuzzy sets over all $y \in \mathbf{X}$ represents the fuzzy set of those alternatives x from $\mathbf{X}$ which are strictly dominated by none of the alternatives from the set **X**. We shall

* In fact, this fuzzy relation describes fuzzy equivalence in the set $(\mathbf{X},\mu)$, but we reserve the term equivalence relation for fuzzy relations which are reflexive, symmetric and transitive.

call this set a fuzzy set μ^{ND} of nondominated (ND) alternatives of the set $(\mathbf{X},\mu)$. Thus, according to the definition of intersection (Sect. 2.1) we have

$$\mu^{ND}(x) = \inf_{y \in \mathbf{X}} [1-\mu^s(y,x)] = 1-\sup_{y \in \mathbf{X}} \mu^s(y,x).$$

The value $\mu^{ND}(x)$ represents the degree to which the alternative x is dominated by none of the alternatives from the set $\mathbf{X}$. It can be shown that if for some x: $\mu^{ND}(x) = \alpha$ then this alternative is dominated by other alternatives to a degree not higher than $1-\alpha$.

Using the above definition of μ^s it can be shown that

$$\mu^{ND}(x) = 1-\sup_{y \in \mathbf{X}}[\mu(y,x)-\mu(x,y)]. \tag{4.1}$$

It is natural to consider as rational the choices of those alternatives x of the set $(\mathbf{X},\mu)$ which have the greatest possible degrees of nondominance, or in other words, which give a value for $\mu^{ND}(x)$ that is as close as possible to the value

$$\sup_{x \in \mathbf{X}} \mu^{ND}(x) = 1-\inf_{x \in \mathbf{X}} \sup_{y \in \mathbf{X}} [\mu(y,x)-\mu(x,y)].$$

We shall call alternatives of the set

$$\mathbf{X}^{ND} = \{x \mid x \in \mathbf{X}, \mu^{ND}(x) = \sup_{z \in \mathbf{X}} \mu^{ND}(z)\}$$

maximal ND alternatives of the set $(\mathbf{X},\mu)$. If $\mu^{ND}(x) = 1$ for $x \in \mathbf{X}^{ND}$ then alternatives of $\mathbf{X}^{ND}$ will be called unfuzzy ND alternatives (UND alternatives). Evidently, $\mu^s(y,x) = 0$ for any $y \in \mathbf{X}$ if x is a UND alternative. This means that UND alternatives are dominated to a positive degree by none of the alternatives from $\mathbf{X}$. Due to this fact UND alternatives are of special interest in the context of decision-making problems since they represent unfuzzy solutions to a fuzzily posed problem.

Some sufficient conditions for the existence of UND alternatives in this type of problems can be found in Orlovski, 1981.

To illustrate the notion of a fuzzy set of nondominated alternatives we shall consider two fuzzy preference relations on a set consisting of only four elements:

Problem 1.

$\mu_1(x_i,x_j) =$

	x_1	x_2	x_3	x_4
x_1	1	0.2	0.3	0.1
x_2	0.5	1	0.2	0.6
x_3	0.1	0.6	1	0.3
x_4	0.6	0.1	0.5	1

$\mu_1^s(x_i,x_j) =$

	x_1	x_2	x_3	x_4
x_1	0	0	0.2	0
x_2	0.3	0	0	0.5
x_3	0	0.4	0	0
x_4	0.5	0	0.2	0

$\mu_1^{ND}(x_i) =$

x_1	x_2	x_3	x_4
0.5	0.6	0.8	0.5

Note that alternative x_3 has the greatest degree of nondominance (0.8) in this problem, and its choice may be suggested as rational to the DM. But, as has been mentioned, the DM may have additional information about the preferences, that due to some reasons has not been included into the problem formulation, and the DM may be willing to choose an alternative other than x_3. Therefore, suggesting only alternative x_3 in this case as the solution to the problem would be misleading.

Problem 2.

$\mu_2(x_i,x_j) =$

	x_1	x_2	x_3	x_4
x_1	1	0.5	0	0.8
x_2	0.1	1	0.3	0.5
x_3	0.3	0.6	1	0
x_4	0.9	0.7	0.1	1

$\mu_2^s(x_i,x_j) =$

	x_1	x_2	x_3	x_4
x_1	0	0.4	0	0
x_2	0	0	0	0
x_3	0.3	0.3	0	0
x_4	0.9	0.2	0.1	0

$$\mu_2^{ND}(x_i) = \frac{x_1 \quad x_2 \quad x_3 \quad x_4}{0.8 \quad 0.6 \quad 0.9 \quad 1}$$

Note, that in this case alternative x_4 is definitely (unfuzzily) nondominated.

5. Decision-making with multiple fuzzy preference relations

5.1. Introduction

In this section we deal with the following type of problems. A set of alternatives **X** is described and each of the alternatives from this set is characterized by a number of attributes or properties $j=1,\ldots,m$. Information about pairwise comparisons of alternatives with respect to attribute j is represented in the form of a respective generally fuzzy preference relation $\mathbf{R}_j$. Therefore, we have m preference relations $\mathbf{R}_j$, $j=1,\ldots,m$ on the set **X** and the problem lies in making rational choices of alternatives from the set $(\mathbf{X}, \mathbf{R}_1, \ldots, \mathbf{R}_m)$. In some cases additional information is available about the relative importances of different attributes and, therefore, of the relations $\mathbf{R}_j$.

In the sequel, we outline three approaches to processing this type of information to determine rational choices of alternatives. The first approach is applicable to cases where there is no information about the relative importance of the relations $\mathbf{R}_j$, and it is based on the explicit definition of the effective alternatives suggested as rational choices. The second approach may be applied to problems in which the relative importance of the relations $\mathbf{R}_j$ is represented by coefficients of importance. This approach is based on the use of the weighted sum of the relations $\mathbf{R}_j$. Finally, the third approach is applicable to problems in which a relation of relative importance is defined on the set of the attributes. This approach is based on the extension principle for fuzzy relations outlined in Sect. 2.3.

5.2. First approach: Effective alternatives

In a certain sense, this approach is based on the consideration of the problem as that similar to a problem of multiobjective optimization with the difference that here, instead of utility functions, we have preference relations. To be able to speak about the rationality of choices of alternatives, first we shall indicate a way of comparing alternatives with each other using the multiple preference relations specified. To do this we assume that the relations given are not fuzzy, and then using the analogy between nonfuzzy and fuzzy relations, we shall consider the more general case of fuzzy relations.

If the relations $\mathbf{R}_j$, $j=1,\ldots,m$ are nonfuzzy, then it appears natural to consider that an alternative $x \in \mathbf{X}$ is not less preferable than an alternative $y \in \mathbf{X}$ if, and only if, x is not dominated by y with respect to any of the relations $\mathbf{R}_j$, $j=,\ldots,m$. This, in fact, implicitly contains the definition of a new preference relation $\mathbf{R}$ on the set $\mathbf{X}$

To describe this relation explicitly, we denote by $\mathbf{R}_j^I$ the indifference relation corresponding to $\mathbf{R}_j$. ($x \mathbf{R}_j^I y$ means that neither $x \mathbf{R}_j y$ nor $y \mathbf{R}_j x$ hold, or in other words, that x and y are not comparable using the relation $\mathbf{R}_j$). Then, the new relation can be defined as follows:

$$x, y \in \mathbf{X}, \; x \mathbf{R} y <=> x (\mathbf{R}_j \cup \mathbf{R}_j^I) y, \; j=1,\ldots,m \tag{5.1}$$

or

$$\mathbf{R} = \bigcap_{j=1}^{m} (\mathbf{R}_j \cup \mathbf{R}_j^I). \tag{5.2}$$

Having introduced this relation, we can define as rational the choices of those alternatives from the set $(\mathbf{X}, \mathbf{R}_1, \ldots, \mathbf{R}_m)$ which are nondominated with respect to this new relation $\mathbf{R}$, and thus consider the subset of ND alternatives of the set $(\mathbf{X}, \mathbf{R})$ as the solution to our problem with multiple relations. As can be shown (see Orlovski, 1982), every ND alternative x^o of the set $(\mathbf{X}, \mathbf{R})$ possesses the following property: *for any* $y \in \mathbf{X}$ *such that* $y(\mathrm{R}_j \cup \mathrm{R}_j^I)x^o$, $j=1,\ldots,m$ *we have* $y\mathrm{R}_j^I x^o$, $j=1,\ldots,m$. Note that this property is a generalization of the well known definition of effective alternatives to cases when $\mathrm{R}_j^I \neq \phi$ for some or all $j=1,\ldots,m$.

To apply the above reasoning to problems with fuzzy relations $\mathbf{R}_j$ we describe these relations with their membership functions and, using the operations on fuzzy sets introduced in Sect. 2, write Eq. (5.2) in the following form:

$$\mu_{\mathbf{R}}(x,y) = \min_{j=1,\ldots,m} \max\{\mu_j(x,y), \min\{1-\mu_j(x,y), 1-\mu_j(y,x)\}\}. \qquad (5.3)$$

This form together with the results from Sect. 4 can be used to determine the corresponding fuzzy subset of ND alternatives.

To illustrate this approach we present the following two examples.

Example 1 *(Nonfuzzy relations).* Let $\mathbf{X} = \{x_1, x_2, x_3\}$ and three nonfuzzy preference relations be defined on this set with the matrices of the form:

	x_1	x_2	x_3
x_1	1	1	0
x_2	1	1	1
x_3	0	0	1

	x_1	x_2	x_3
x_1	1	1	1
x_2	0	1	1
x_3	0	0	1

	x_1	x_2	x_3
x_1	1	1	0
x_2	1	1	0
x_3	1	0	1

Using (5.3) we obtain the following matrix (membership function) for the integrated relation **R**:

$\mu_{\mathbf{R}}(x_i, x_j) =$

	x_1	x_2	x_3
x_1	1	1	0
x_2	0	1	1
x_3	0	0	1

and using the definition (4.1) (Sect. 4) we have the following membership function of the subset of ND alternatives:

$$\mu_{\mathbf{R}}^{ND}(x_j) = \frac{x_1 \quad x_2 \quad x_3}{1 \quad 0 \quad 0}$$

Therefore, in this example there exists only one ND alternative x_1 and the choice of this alternative may be considered rational.

Example 2 *(Fuzzy relations).* As in the previous example $\mathbf{X} = \{x_1, x_2, x_3\}$, but here we have three fuzzy relations described with the following matrices:

	x_1	x_2	x_3
x_1	1	0.8	1
x_2	0	1	0.2
x_3	0	0.8	1

	x_1	x_2	x_3
x_1	1	0.1	0.5
x_2	0.8	1	0.8
x_3	0.5	0.3	1

	x_1	x_2	x_3
x_1	1	0	0.8
x_2	0	1	0
x_3	0.1	0	1

Using (5.3) we obtain the following matrix for the integrated relation **R**:

	x_1	x_2	x_3
x_1	1	0.2	0.5
x_2	0.2	1	0.2
x_3	0	0.3	1

and the fuzzy subset of ND alternatives (4.1) for this relation has the following form:

$$\mu_{\mathbf{R}}^{ND}(x_j) = \frac{x_1 \quad x_2 \quad x_3}{1 \quad 0.9 \quad 0.5}$$

5.3. Second approach: Weighted sum of relations

We consider a similar problem with multiple preference relations, but here it is assumed that there is additional information about the relative importance of these relations in the form of coefficients of relative importance λ_j, $j = 1, \ldots, m$, and also that the pairwise comparisons between alternatives can be made using the corresponding weighted sum of the membership functions of the original relations. Thus the membership function of the integrated preference relation on the set $\mathbf{X}$ in this case has the form:

$$\mu_{\mathbf{R}}(x,y) = \sum_{j=1}^{m} \lambda_j \mu_j(x,y), \; x,y \in \mathbf{X}, \; \lambda_j > 0, \; \sum_{j=1}^{m} \lambda_j = 1. \quad (5.4)$$

Therefore, to obtain the set of solutions to the problem with multiple relations in this case, it suffices to apply the definition (4.1) of the fuzzy subset of ND alternatives to the fuzzy relation $\mu_{\mathbf{R}}$ (5.4). Note, that the relation $\mu_{\mathbf{R}}$ may be fuzzy even in cases when all the relations μ_j are not fuzzy. Thus, this approach is generally applicable to traditional nonfuzzy relations only if these relations are considered as elements of a more general class of fuzzy relations. On the other hand, as will be illustrated in the sequel, its application to problems with nonfuzzy relations allows to obtain information that can be useful in decision-making situations.

Let us consider the application of this approach to a problem with m equally important nonfuzzy preference relations, and therefore, according to (5.4) in this case $\lambda_j = 1/m$, $j=1,\ldots,m$. Using the definition (4.1) of a fuzzy subset of ND alternatives for the set $(\mathbf{X},\mu_{\mathbf{R}})$ we obtain:

$$\mu_{\mathbf{R}}^{ND}(x) = 1 - \frac{1}{m} \sup_{x \in \mathbf{X}} \sum_{j=1}^{m} [\mu_j(y,x) - \mu_j(x,y)]. \qquad (5.5)$$

As can easily be seen, the function $\mu_{\mathbf{R}}^{ND}(x)$ has values only of the form k/m with k being a positive integer, and $k \leq m$. If for some alternative $x' \in \mathbf{X}$ we have $\mu_{\mathbf{R}}^{ND}(x') = k/m$ then according to (5.5) we obtain:

$$\sup_{y \in \mathbf{Y}} \sum_{j=1}^{m} [\mu_j(y,x') - \mu_j(x',y)] = m - k,$$

or

$$\sum_{j=1}^{m} [\mu_j(y,x') - \mu_j(x',y)] \leq m - k \qquad (5.6)$$

for any $x, y \in \mathbf{X}$

Each term of the sum in (5.6) can take only values 0,+1 and -1, and as follows from (5.6) the difference between the number of such terms with value +1 and of those with value -1 is not greater than $m-k$ for any $y \in \mathbf{X}$ Denote by $p(y,x)$ the number of those relations $\mathbf{R}_j$ with respect to which (each of them) alternative y is preferable (better than) to x, and by $q(y,x)$ the number of relations $\mathbf{R}_j$ with respect to which x is better than y. Then, if $\mu_{\mathbf{R}}^{ND}(x')=k/m$, we have:

$$p(y,x') - q(y,x') \leq m - k$$

for any $y \in \mathbf{X}$. Therefore, the function $\mu_{\mathbf{R}}^{ND}$ orders alternatives by their degrees of nondominance.For an example, if $\mu_{\mathbf{R}}^{ND}(x^\circ)=3/4$ (i.e. $m-k=1$) and some alternative y is better than x° (dominates x°) with respect to some two of four given relations ($m=4$), then with respect to at least one of the remaining relations x° is better than y.

In cases where the "weights" λ_j are not equal, each of the characteristics $p(y,x)$ and $q(y,x)$ will represent not the numbers of the corresponding relations, but their total relative weights (importances).

To illustrate this approach we apply it to the examples considered in the preceding section.

Example 1' *(Nonfuzzy relations).* We use here the matrices of the nonfuzzy preference relations on the set $\mathbf{X}=\{x_1,x_2,x_3\}$ from the first example in Sect. 5.2, and assume that all these relations are of equal importance, i.e. $\lambda_1=\lambda_2=\lambda_3=1/3$. Therefore, using (5.4) we obtain:

$\mu_{\mathbf{R}}(x_i,x_j) =$

	x_1	x_2	x_3
x_1	1	1	1/3
x_2	2/3	1	2/3
x_3	1/3	1	1

and the corresponding fuzzy subset of ND alternatives has the membership function

$$\mu_{\mathbf{R}}^{ND}(x_j) = \frac{x_1 \quad x_2 \quad x_3}{1 \quad 2/3 \quad 1/3}$$

Example 2' *(Fuzzy relations).* We use the matrices of the fuzzy relations on the set $\mathbf{X}=\{x_1,x_2,x_3\}$ from the second example in Sect. 5.2 and also assume that $\lambda_1=\lambda_2=\lambda_3=1/3$. Using (5.4) we obtain:

$$\mu_{\mathbf{R}}(x_i,x_j) = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0.3 & 0.77 \\ x_2 & 0.26 & 1 & 0.33 \\ x_3 & 0.2 & 0.37 & 1 \end{array}$$

and the corresponding subset of ND alternatives is described with the following membership function:

$$\mu_{\mathbf{R}}^{ND}(x_j) = \frac{x_1 \quad x_2 \quad x_3}{1 \quad 0.97 \quad 0.43}$$

5.4. Third approach: Relation of "relative importance" on the set of attributes

In this subsection we consider problems in which relative importances of the relations $\mathbf{R}_j$, $j=1,\ldots,m$ (or of the attributes) are described not with the respective coefficients λ_j, but more generally with a fuzzy relation "not less important than" on the set of the relations.

Note, that the coefficients λ_j, when specified, uniquely define the corresponding relation of importance on the set of relations. Therefore, the approach outlined in the sequel is also applicable to problems of the type considered in the preceding section. However, a tuple of such coefficients contains more information than the corresponding relation, and thus the previous approach is more applicable to problems in which such coefficients are specified.

The problem analyzed here can generally be formulated as follows. A set of alternatives $\mathbf{X}$ (or objects) is fixed together with a set $\mathbf{P}$ of attributes (or experts). For each attribute $p \in \mathbf{P}$ a fuzzy preference relation φ on the set $\mathbf{X}$ is specified, or in other words, a membership function $\varphi: \mathbf{X} \times \mathbf{X} \times \mathbf{P} \to [0,1]$ is given with a value $\varphi(x_1,x_2,p)$ understood as the degree to which x_1 is considered to be not less preferable than x_2 *with respect to the attribute* p. If $\mathbf{P}$ is a set of experts, then $\varphi(x_1,x_2,p)$ describes the preference relation on $\mathbf{X}$ obtained from expert p. Thus, the function φ describes a family of fuzzy preference relations with parameter p.

Elements of $\mathbf{P}$ generally differ in their relative importance. Let $\mu: \mathbf{P} \times \mathbf{P} \to [0,1]$ be a specified fuzzy relation of the relative importance of the attributes (experts); a value $\mu(p_1, p_2)$ is understood as a degree to which attribute (expert) p_1 is considered to be not less important than attribute (expert) p_2. The problem consists in making rational choices of alternatives from the set $\mathbf{X}$ on the basis of the above information. In what follows we outline one of possible approaches to this problem.

Denote by $\varphi^{ND}(x, p)$ the fuzzy subset of ND alternatives corresponding to $\varphi(x_1, x_2, p)$ for some fixed $p \in \mathbf{P}$. Using (4.1) we have

$$\varphi^{ND}(x, p) = 1 - \sup_{y \in \mathbf{Y}} [\varphi(y, x, p) - \varphi(x, y, p)]. \tag{5.7}$$

If the choices of alternatives were made considering only a single attribute p, then it would be natural to choose alternatives giving possible greater values of $\varphi^{ND}(x, p)$. But in our problem we should take into account all attributes $p \in \mathbf{P}$ differing in their relative importances.

Clearly, for fixed $x^0 \in \mathbf{X}$ the function $\varphi^{ND}(x^0, p)$ on the set $\mathbf{P}$ can be understood as the membership function of the fuzzy set of attributes with respect to which x^0 is a nondominated alternative. It is also clear, that if for some two alternatives $x_1, x_2 \in \mathbf{X}$ the fuzzy set of attributes (or experts) $\varphi^{ND}(x_1, p)$ "is not less important" than the fuzzy set of attributes $\varphi^{ND}(x_2, p)$, then the alternative x_1 should be considered as "no less preferable" than x_2.

Therefore, what is needed at this stage is to expand the domain of the relation $\mu(p_1, p_2)$ onto the class of fuzzy subsets of the set $\mathbf{P}$. Using the reasoning and the results from Sect. 2.3 we obtain the following preference relation on the set $\mathbf{X}$ induced by the function $\varphi^{ND}(x, p)$ and by the relation μ:

$$\eta(x_1, x_2) = \sup_{p_1, p_2 \in \mathbf{P}} \min\{\varphi^{ND}(x_1, p_1), \varphi^{ND}(x_2, p_2), \mu(p_1, p_2)\}. \tag{5.8}$$

This integrated relation can be considered as the result of combining the family of fuzzy relations $\varphi(x_1,x_2,p)$ into one relation taking into account the information about the relative importances of the attributes in the form of the relation μ. At this stage the original problem has been reduced to the problem of choice with a single preference relation, and to solve this problem we can use the approach outlined in Sect. 4.

Example 3 *(Risk analysis).* As an illustration we consider the application of this approach to a problem from a rather broad scope of problems of risk analysis. This problem in its very simplified version can be described as follows.

A regional government plans to choose a location for the construction of a liquefied gas terminal. The presence of such a terminal at any of the locations considered involves certain degrees of risk associated with great environmental damages that might occur in cases of some catastrophic events at the terminal site. Thus, the government desires to choose the location where such risk is as minimal as possible.

We assume that there are four possible locations in the region in question: L1, L2, L3 and L4. We also assume that the government invited four experts in risk analysis: E1, E2, E3 and E4 and relies on their joint opinion. However, the government values the experts' opinions differently: opinions of one expert are respected to some degree more than the opinion of another. We assume that the government describes its attitude (or respect) to the experts' opinions with the following matrix of a fuzzy relation "not less important":

	$E1$	$E2$	$E3$	$E4$
$E1$	1	0.4	0.6	0
$E2$	1	1	0.8	1
$E3$	0.2	1	1	1
$E4$	0.8	0	1	1

For instance, the element (E1,E3) of this matrix is equal to 0.6. This means that the government considers the opinion of expert E1 to be not less important than that of expert E3 to a degree 0.6. The elements (E3,E4) and (E4,E3) are both equal to 1, and this means that experts E3 and E4 are definitely (to a degree 1) equivalent from the government's viewpoint, and so on.

Each of the experts compares the alternative locations with each other in terms of potential risks associated with the construction of the terminal at these locations. The results of these comparisons are represented by matrices. If for an example, in such a matrix an element (L2,L3) is equal to 1, then to the corresponding expert's opinion, the risk of constructing the terminal at L2 is not greater than at L3. If an expert is not definite about this comparison, he may characterize its degree with a number smaller than 1.

In our case the experts' matrices (or preferences between the alternative locations) are as follows:

	$L1$	$L2$	$L3$	$L4$
$L1$	1	0.8	1	0
$L2$	0	1	0.2	1
$L3$	0	0.8	1	0
$L4$	0	0	0	1

	$L1$	$L2$	$L3$	$L4$
$L1$	1	0.1	0.5	0.3
$L2$	0.8	1	0.8	0.8
$L3$	0.5	0.3	1	0
$L4$	0.8	0	0	1

	$L1$	$L2$	$L3$	$L4$
$L1$	1	0	0.8	0
$L2$	0	1	0	0
$L3$	0.1	0	1	0.4
$L4$	1	1	1	1

	$L1$	$L2$	$L3$	$L4$
$L1$	1	1	0.9	0
$L2$	0	1	1	1
$L3$	0.4	0	1	0
$L4$	0	0	0	1

Using (5.7) we obtain the respective fuzzy subsets of ND locations:

	$L1$	$L2$	$L3$	$L4$
$\varphi^{ND}(.,E1)$	1	0.2	0	0
$\varphi^{ND}(.,E2)$	0.3	1	0.2	0.2
$\varphi^{ND}(.,E3)$	0	0	1	1
$\varphi^{ND}(.,E4)$	1	0	0	0

and then we determine a fuzzy relation η induced by the function φ^{ND} and by the relation μ:

	$L1$	$L2$	$L3$	$L4$
$L1$	1	0.4	0.4	1
$L2$	1	1	0.5	0.8
$L3$	0.5	0.5	0.5	0.5
$L4$	1	1	0.5	1

(5.8)

Using this relation we obtain the following fuzzy subset of ND locations:

$$\eta^{ND} = \frac{L1 \;\; L2 \;\; L3 \;\; L4}{1 \;\; 0.4 \;\; 0.9 \;\; 0.8} \tag{5.9}$$

Note that the relation (5.8) is not reflexive since the element (L3,L3) in its matrix is not equal to 1. In such cases (as is explained in Sect. 6), the fuzzy set (5.9) should be corrected, and the result of this correction (see Eq. (6.3)) is the fuzzy set of ND locations:

$$\tilde{\eta}^{ND} = \frac{L1 \;\; L2 \;\; L3 \;\; L4}{1 \;\; 0.4 \;\; 0.5 \;\; 0.8}$$

The greatest membership degree to this set (the greatest degree of non-dominance) has location L1, therefore, according to this approach, the choice of this location is considered rational. If more than one location have the greatest degree of nondominance, then the government can choose one of them using some additional considerations, or it can invite more experts and perform the analysis again.

6. General fuzzy mathematical programming problem

6.1. Introduction

As was mentioned in Sect. 3 of this paper, preferences in a mathematical programming problem are described by means of a utility (objective) function defined on the set of alternatives in such a way that greater values of this function correspond to more preferable alternatives. Using this function, a problem of choice among alternatives is reduced to in some sense simpler problem of choice among numbers.

The objective function represents an important part of a mathematical description of a real system. Values of this function describe effects from the choice of one or another alternative or a policy. In economic problems, for example, these values may reflect profits obtained using various means of production; in water management problems, they may have the meaning of electric power production for various water yields from a reservoir, etc. In any case, the results of the analysis depend largely upon how adequately various factors of the real system or process are reflected in the description of the goal function.

A mathematical model, if meant to be comprehendable, should not be based on the explicit consideration of too many aspects or factors of the real system under study. Two approaches are possible here. We can consider the factors not included into the model as absolutely insignificant and completely ignore them during the analysis using this model. On the other hand, using another approach, we may not explicitly introduce these "insignificant" factors into the model, but take their influence into account during the analysis, by accepting that the responses of the model to one or another choice of alternatives may be known only approximately, fuzzily. To describe this fuzziness of the model's response we can use the help of experts, who have an understanding of the roles played by these insignificant factors in the behavior of the system. Clearly, the greater the number of such factors, the larger the fuzziness of our (or of experts') description of the model. Thus, in the second approach, a complex system is described with some fuzzy goal function that to each alternative assigns a corresponding fuzzy reaction of the system.

If, for example, system's responses are described in the form of fuzzy subsets of the set of responses $\mathbf{Y}$, then the function reflecting the behavior of the system may have the form $\varphi:\mathbf{X}\times\mathbf{X}\to[0,1]$, with $\mathbf{X}$ being the set of alternatives. If $x^0 \in \mathbf{X}$ then the function $\varphi(x^0,y)$ of y is the membership function of the fuzzy response of the system to the choice of alternative x^0.

Using this type of fuzzy description of the performance function, alternatives have to be compared with each other by their respective fuzzy evaluations: those alternatives are more preferable which have more preferable fuzzy values.

In the subsequent section this problem is analyzed using the extension principle introduced in Sect. 2.3.

6.2. Fuzzy set of solutions to the general FMP problem

As has been introduced in this section, the general FMP problem is described in the following terms: a set of alternatives **X**, a set of estimates **Y**, a fuzzy goal function φ:**X**×**Y**→[0,1] and a fuzzy preference relation μ:**Y**×**Y**→[0,1]. In treating this problem here, we shall rely on the results and reasoning from Sect. 6.2 to introduce corresponding to φ and μ fuzzy preference relation η on the set **X**, and then we shall specify the fuzzy set of ND alternatives in the fuzzily ordered set (**X**,η), as has been suggested in Sect. 4 of this paper.

For every alternative $x^0 \in$ **X** the function φ gives the corresponding utility value $\varphi(x^0,y)$ in the form of a fuzzy subset of the set **Y**. Denote by η the fuzzy preference relation induced by μ on the class Υ of fuzzy subsets of **Y**. Using η we can compare fuzzy utility values of the alternatives with each other and, therefore, the alternatives themselves. In other words, as a degree of the preference between alternatives $x_1,x_2 \in$ **X** we consider the degree of the preference between their fuzzy utility values $\varphi(x_1,y)$ and $\varphi(x_2,y)$, i.e.

$$\eta(x_1,x_2) = \eta(\varphi(x_1,y),\varphi(x_2,y)).$$

Finally, using the definition (6.3) of the extended relation from Sect. 6.2 we obtain

$$\eta(x_1,x_2) = \sup_{z,y \in \mathbf{Y}} \min\{\varphi(x_1,z),\varphi(x_2,y),\mu(z,y)\}. \tag{6.1}$$

Note that for a simpler problem with an unfuzzily described goal function f ;**X**→**Y** (**Y** - number axis) definition (6.1) reduces to the traditional one:

$$x_1 \succeq x_2 <=> f(x_1) \geq f(x_2).$$

Having obtained the fuzzy relation η we reduced the original FMP problem to a problem of the type considered in Sect. 4, and the next step is the determination of the fuzzy set of ND alternatives in the set $(\mathbf{X},\eta)$. Using the definition (4.1) we have:

$$\eta^{ND}(x) = 1 - \sup_{x' \in \mathbf{X}} [\eta(x',x) - \eta(x,x')],$$

hence using (6.1) we obtain the following expression for the membership function of this set:

$$\eta^{ND}(x) = \sup_{x' \in \mathbf{X}} [\sup_{z,y \in \mathbf{Y}} \min\{\varphi(x',z),\varphi(x,y),\mu(z,y)\} - \\ - \sup_{z,y \in \mathbf{Y}} \min\{\varphi(x',z),\varphi(x,y),\mu(y,z)\}]. \tag{6.2}$$

It is important to note here that if the function $\varphi(x,y)$ is such that for some $x^0 \in \mathbf{X}$ we have

$$\sup_{y \in \mathbf{Y}} \varphi(x^0,y) = \alpha < 1,$$

then the value $\eta^{ND}(x^0)$ as in (6.2) may not reflect properly the degree of non-dominance of this alternative. As an illustration, we can consider a limit case with $\alpha=0$. In the context of the original problem this means that the utility value for x^0 is not known or is not defined (or the reaction of a system to control x^0 is not known). On the other hand, as can easily be seen using (6.1) and (6.2) for this alternative $\eta(x^0,x^0)=0$ and $\eta^{ND}(x^0)=1$, i.e., it appears to be an UND alternative, due solely to the lack of information about it. Therefore, to eliminate such "pathological" inferences, the value $\eta^{ND}(x^0)$ must be corrected by correlating it with the value α. Accordingly, we shall consider as a solution to the FMP problem not the function η^{ND} but the following corrected function:

$$\overline{\eta}^{ND}(x) = \min\{\eta^{ND}(x), \sup_{y \in \mathbf{Y}} \varphi(x,y)\},$$

or, equivalently,

$$\overline{\eta}^{ND}(x) = \min\{\eta^{ND}(x),\eta(x,x)\}. \tag{6.3}$$

If the relation η is reflexive, i.e. $\eta(x,x)=1$ for any $x \in \mathbf{X}$ then the functions η^{ND} and $\overline{\eta}^{ND}$ coincide with each other.

In a simpler and practically important FMP problem with $\mathbf{Y}$ being the number axis, equation (6.1) takes the form:

$$\eta(x_1,x_2) = \sup_{\substack{z,y \in \mathbf{Y} \\ z \geq y}} \min\{\varphi(x_1,z),\varphi(x_2,y)\}. \tag{6.4}$$

It can be shown (see Orlovski, 1981) using (6.3) and (6.4) that, in this last case to determine alternatives with nondominance degrees not smaller than α, it suffices to solve the following mathematical programming problem:

$$y \to \max_{\substack{x \in \mathbf{X} \\ y \in \mathbf{Y}}}$$
$$\varphi(x,y) \geq \alpha.$$

7. Mathematical programming problems with fuzzy parameters

7.1. Introduction

Let us consider the following simplified problem of land and water resources allocation. On the land of the total area X, crops of m types are to be planted. Denote the productivity of j-th crop by c_j and the area of land allocated to it by x_j, hence $c_j x_j$ is the yield of this crop. Denote also by w_j amount of water for the irrigation of j-th crop per unit area of land. Finally, let W be the total amount (yearly) for the irrigation and p_j benefit from the unit production of crop j. The problem lies in the determination of the maximal total profit by rationally allocating crops over the area X.

Mathematically this problem can be formulated in the following standard LP form:

Problem 1.

$$\sum_{j=1}^{m} p_j c_j x_j \to \max$$

s.t.

$$\sum_{j=1}^{m} w_j x_j \leq W, \quad \sum_{j=1}^{m} x_j \leq X,$$

$$x_j \geq 0,\ j=1,\ldots,m.$$

Now let us consider more attentively the parameters c_j, w_j, and W. Clearly, their values depend upon multiple factors which have not been included into the above formulation of the problem, for instance, the nutrient contents of the soil, soil treatment technologies, solar activity, and many others. If, trying to make the model more representative of the real system under study, we include the corresponding complex dependences into it, then the model may become cumbersome and analytically unacceptable. Moreover, it can happen (this fact being often neglected) that such attempts to increase the precision of the model will be of no practical value due to the impossibility to measure, or to measure to a sufficient accuracy, the values of the newly introduced parameters.

On the other hand, the model with some fixed values c_j, w_j, and W may still be too crude, since these values are often chosen in some arbitrary way. Apparently, at least one fact should be recognized: it is not the values of the parameters that are known, but rather sets or ranges of their possible values. A model in which parameters are evaluated not by numbers but by intervals is in greater correspondence to the reality.

In this way we come to the following more realistic version of the above formulation:

Problem 2. To determine a rational allocation pattern of land with total area X for m crops, if the benefit from an allocation $x=(x_1,\ldots,x_m)$ is described as

$$\sum_{j=1}^{m} p_j \bar{c}_j x_j$$

and the following constraints must be observed

$$\sum_{j=1}^{m} x_j \overline{w}_j \leq \overline{W},\ \sum_{j=1}^{m} x_j \leq X,$$
$$x_j \geq 0,\ j=1,\ldots,m.$$

Here $\bar{c}_j$ is a set (interval) of possible crop productivities, $\bar{w}_j$ is a set (interval) of possible values of water demand per unit area, $\bar{W}$ is a set (interval) of possible available total amounts of water for the irrigation.

Problems of this type may be called those of linear programming with set-valued coefficients. Clearly, it makes no sense in this type of formulation to speak of the maximization of the goal function (benefit in our example), since its values are intervals rather than single numbers. In this case we should first define a preference relation on the set of alternatives (land allocations in our example) that is induced by the goal function and then analyze a question of choosing rational alternatives. Problems of this type are analyzed in this section. (A simpler problem of this type with a usual linear goal function and with set-valued parameters in the constraints was analyzed by Soyster, 1973).

The next step in the direction of increasing potential adequateness of our model is the description of the parameters in the form of fuzzy sets. In this case, together with possible values of the parameters additional information is introduced into the model in the form of the membership functions. The values of these membership functions may be understood as weighing coefficients assigned by experts to these possible values. Clearly, this additional information complicates the model to a certain degree, but nevertheless the resulting model may be simpler than that involving all those multiple factors mentioned at the beginning of this section. In this way we obtain the following formulation of a fuzzy mathematical programming problem:

Problem 3. To determine a rational allocation pattern of land with total area X for m crops, if the benefit from an allocation $x = (x_1, \ldots, x_m)$ is described as

$$\sum_{j=1}^{m} p_j \bar{c}_j x_j$$

and the following constraints must be observed

$$\sum_{j=1}^{m} x_j \overline{w}_j \leq \overline{W}, \quad \sum_{j=1}^{m} x_j \leq X,$$
$$x_j \geq 0, \quad j=1,\ldots,m,$$

with values of coefficients $\bar{c}_j$, $j=1,\ldots,m$, *and* $\overline{w}_j$ *described as fuzzy sets* $\chi_j(c_j)$, $\nu_j(w_j)$.

This type of formulation, which includes the above formulations as particular cases, is discussed in the sequel.

7.2. Problem formulation and its reduction to a general FMP form

In this subsection we formulate a more general problem with fuzzy parameters and then modify this formulation to the form of the corresponding general FMP problem. This will enable us to use the results from Sect. 6 for the determination of a set of ND alternatives. At the end of this section we shall consider the application of the results obtained for Problem 3 from the previous subsection.

Denote by **X** a set of alternatives. A subset of feasible alternatives is described by an inequality of the form:

$$\psi_j(x, b_{1j}, \ldots, b_{pj}) \leq 0, \quad j=1,\ldots,n. \tag{7.1}$$

with ψ_j being functions $\mathbf{X} \times \mathbf{R}^p \to \mathbf{R}^1$, b_{ij}, $i=1,\ldots,p$; $j=1,\ldots,n$ being parameters with values described in the form of fuzzy subsets of the number axis; we denote by $\nu_{ij}(b_{ij})$ the membership functions of these subsets.

Choices of alternatives are evaluated using a function $f : \mathbf{X} \times \mathbf{R}^q \to \mathbf{R}^1$ of the form $f(x, a_1, \ldots, a_q)$ and the parameters a_i, $i=1,\ldots,q$ are also described as fuzzy subsets of the number axis with the membership functions $\chi_i(a_i)$. Note, that due to the fuzzy description of the parameters of the function f the value of any alternative $x \in \mathbf{X}$ specified by this function is a fuzzy subset of the number axis.

To complete the formulation we should describe a preference relation on the set of values of f. In our case we assume that this relation is the natural order ($\geq$) on the number axis.

To formulate the corresponding FMP problem we can directly use the extension principle to determine fuzzy values of the functions ψ_j, f for corresponding fuzzy values of their parameters. But we use here another more illustrative technique. Let us first consider fuzzy constraints (7.1) and determine the corresponding fuzzy set of feasible alternatives $\mu_{\mathbf{C}}(x)$. In doing this we shall use the following reasoning.

Consider some particular values b_{ij}^o of the parameters in (7.1). Their membership degrees in the respective fuzzy sets are $\nu_{ij}(b_{ij}^o)$. Denote by μ^o the minimum of these degrees, i.e.

$$\mu^o = \min_{\substack{i=1,\dots,p \\ j=1,\dots,n}} \nu_{ij}(b_{ij}^o)$$

If some alternative $\bar{x} \in \mathbf{X}$ satisfies the inequalities

$$\psi_j(\bar{x}, b_{1j}^o, \dots, b_{pj}^o) \le 0, \quad j=1,\dots,n,$$

then we may naturally consider this alternative as feasible to a degree not smaller than μ^o, i.e. we may consider that $\mu_{\mathbf{C}}(\bar{x}) \ge \mu^o$. This, in fact, is already sufficient for defining the fuzzy set of feasible alternatives. For convenience, we introduce the following notations:

$$\nu(B) = \min_{\substack{i=1,\dots,p \\ j=1,\dots,n}} \nu_{ij}(b_{ij}), \quad B = | b_{ij} |,$$

$$P(x) = \{ B \mid \psi_j(x, b_{1j}, \dots, b_{pj}) \le 0, \ j=1,\dots,n \}$$

Using these notations we obtain the following form of the membership function for this set:

$$\mu_{\mathbf{C}}(x) = \sup_{\mathbf{B} \in P(x)} \nu(B)$$

To each alternative $x \in \mathbf{X}$ this function assigns a degree of its feasibility taking into account the fuzzy information about the parameters.

Let us now consider the goal function f, and present it in the form of a fuzzy goal function $\varphi: \mathbf{X} \times \mathbf{R}^1 \to [0,1]$. The reasoning here is quite similar to the previous one. Let a_i^o be some particular values of the parameters; their membership degrees are given by $\chi_i(a_i^o)$. Denote by φ^o the minimum of these values, i.e.

$$\varphi^o = \min_{i=1,\ldots,q} \chi_i(a_i^o).$$

If

$$r^o = f(\bar{x}, a_1^o, \ldots, a_q^o)$$

is the corresponding value of f for some alternative $\bar{x} \in \mathbf{X}$, then it is natural to accept that this value belongs to the fuzzy evaluation of $\bar{x}$ to a degree not smaller than φ^o. Thus we can write the desired fuzzy goal function in the following form:

$$\varphi(x,r) = \sup_{a \in Q(x,r)} \chi(a).$$

where

$$\chi(a) = \min_{i=1,\ldots,q} \chi_i(a_i), \quad a = (a_1, \ldots, a_q) \in \mathbf{R}^q$$

and

$$Q(x,r) = \{a \mid a \in \mathbf{R}^q, f(x, a_1, \ldots, a_q) = r\}.$$

Thus, we finally show that the original problem with fuzzy parameters can be formulated as the following general FMP problem:

"to maximize" the fuzzy goal function

$$\varphi(x,r) = \sup_{a \in Q(x,r)} \chi(a) \tag{7.3}$$

over the fuzzy set of feasible alternatives with the membership function of the form:

$$\mu_{\mathbf{C}}(x) = \sup_{\mathbf{B} \in P(x)} \nu(B) \tag{7.4}$$

The next stage is the determination of the fuzzy set of ND alternatives for this problem.

7.3. Nondominated alternatives in problems with fuzzy parameters

We consider first a simpler problem with a fuzzy goal function (7.3) and with a nonfuzzy set of feasible alternatives, described by the inequalities

$$\psi_j(x, b_{1j}, \dots, b_{pj}) \le 0, \; j=1,\dots,n$$

with precisely known values of the parameters b_{ij}.

The function φ and the natural order ($\ge$) on the number axis induce the following fuzzy preference relation on the set $\mathbf{X}$:

$$\eta(x_1, x_2) = \sup_{\substack{z, y \in \mathbf{R}^1 \\ z \ge y}} \min\{\varphi(x_1, z), \varphi(x_2, y)\} =$$
$$= \sup_{z \ge y} \min\{ \sup_{a \in Q(x_1, z)} \chi(a), \sup_{a \in Q(x_2, y)} \chi(a)\}.$$

Denote by η^{ND} the corresponding fuzzy subset of ND alternatives of the set $(\mathbf{X}, \eta)$ and let us consider a problem of determining alternatives which have degrees of nondominance not smaller than α, i.e. alternatives for which $\eta^{ND}(x) \ge \alpha$.

It can be shown (see Orlovski, 1981) that such alternatives can be found as the solutions to the following mathematical programming problem:

$$r \to \max_{x}$$
$$\varphi(x, r) \ge \alpha$$
$$\psi_j(x, b_{1j}, \dots, b_{pj}) \le 0, \; j=1,\dots,n \qquad (7.5)$$
$$r \in \mathbf{R}^1, \; x \in \mathbf{X}$$

and also that under some conditions this problem is equivalent to the following one:

$$f(x, a_1, \dots, a_q) \to \max_{x, a_i}$$
$$\chi_i(a_i) \ge \alpha, \; a_i \in \mathbf{R}^1, \; i=1,\dots,q.$$
$$\psi_j(x, b_{1j}, \dots, b_{pj}) \le 0, \; j=1,\dots,p$$

Let us now consider another problem in which the values of the parameters a_i, $i=1,\dots,q$ are precisely known and the parameters b_{ij} are described fuzzily. In this case we have a fuzzy set of feasible alternatives described with the membership function (7.4):

$$\mu_C(x) = \sup_{B \in P(x)} \nu(B)$$

and the problem consists of the "maximization" of the function $f: \mathbf{X} \to \mathbf{R}^1$ over this fuzzy set. Therefore, alternatives in this problem should be evaluated by two functions: f – effectiveness of alternatives, and μ_C -- their degree of feasibility, and it is preferable to obtain possible greater values of both the functions.

As can be shown, UND alternatives for this problem are Pareto-maximal ones for the functions f and μ_C. To determine such alternatives it suffices, for instance, to maximize the following function:

$$L(x) = \lambda_1 f(x, a_1, \ldots, a_q) + \lambda_2 \mu_C(x)$$

with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and the corresponding problem can be formulated in the following form:

$$c \to \max_{x, b_{ij}}$$

$$\psi_j(x, b_{1j}, \ldots, b_{pj}) \leq 0, \ j = 1, \ldots, p$$

$$\lambda_1 f(x, a_1, \ldots, a_q) + \lambda_2 \nu_{ij}(b_{ij}) \geq c, \ i = 1, \ldots, q; \ j = 1, \ldots, p.$$

Any solution x^0 to this problem is a UND alternative for the original problem. By varying the coefficients λ_i we can obtain a sufficient multitude of such alternatives. Each UND alternative x^0 is characterized by two numbers: its effectiveness $f(x^0)$ and its degree of feasibility $\mu_C(x^0)$. Therefore, the choice of a particular UND alternative as a solution to the problem should be based on a trade-off between these two characteristics.

Let us now address a problem with fuzzily described parameters a_i of the goal function and b_{ij} of the constraints. As was previously shown, this problem can be formulated in the form (7.3)-(7.4). In this case the choices of alternatives should be made on the basis of two preference relations on the set $\mathbf{X}$: the fuzzy relation induced by the function $\varphi(x, r)$ (7.3):

$$\eta^1(x_1,x_2) = \sup_{\substack{y,z \in R^1 \\ y \geq z}} \min\{\varphi(x_1,y),\varphi(x_2,z)\}$$

and the unfuzzy relation induced by the function $\mu_C(x)$ (6.4):

$$\eta^2(x_1,x_2) = \begin{cases} 1, & \text{when } \mu_C(x_1) \geq \mu_C(x_2) \\ 0, & \text{otherwise} \end{cases}$$

This problem can be analyzed using one of the approaches outlined in Sect. 5 of this paper. For example, using the first approach, it can be shown that alternatives with the degree of nondominance not smaller than some fixed value α can be determined as solutions to the following problem:

$$f(x,a_1,\ldots,a_q) \to \max_{x,a_i,b_{ij}}$$
$$\psi_j(x,b_{1j},\ldots,b_{pj}) \leq 0,\ j=1,\ldots,n$$
$$\chi_i(a_i) \geq \alpha,\ i=1,\ldots,q$$
$$\nu_{ij}(b_{ij}) = 1,\ i=1,\ldots,q;\ j=1,\ldots,p$$

Using this result we can formulate the following mathematical programming problem for the determination of patterns of land allocation for Problem 3 in Sect.7.1, which are rational to a degree not smaller than α:

$$\sum_{j=1}^{m} p_j c_j x_j \to \max_{x,c_j,w_j}$$
$$\sum_{j=1}^{m} x_j w_j \leq W,\quad \sum_{j=1}^{m} x_j \leq X$$
$$\chi_j(c_j) \geq \alpha,\quad \nu_j(w_j) = 1,\quad j=1,\ldots,m.$$

Note, that despite the linear character of the original problem (see Sect. 7.1) the determination of its solution on the basis of fuzzy information requires solving a nonlinear mathematical programming problem.

8. Concluding remarks

This paper was not intended as a review of all the approaches to the use of fuzzy information in mathematical modeling, and thus it outlines only some of these approaches, which are within the author's scope of interest. The interested reader wishing to see other approaches is advised to look through issues of the International Journal of Fuzzy Sets and Systems which, together with relevant papers, contain also numerous references on the subject.

As has been mentioned in the Introduction to this paper, the use of fuzzy sets for describing information about real systems is a relatively new area and much further work is needed in order to find practically sound methods allowing to combine effectively the fuzziness of human judgements with the powerful logic and tools of mathematical analysis. Successful development in this direction may help overcome one of the essential obstacles to the application of mathematical modeling for the analysis of real systems, namely, the existing gap between the language used for mathematical models and the language used by the potential users of those models.

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