VON NEUMANN-MORGENSTERN UTILITIES AND CARDINAL PREFERENCES

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FOREWORD

This Collaborative Paper is one of a series embodying the outcome of a workshop and conference on Economic Structural Change: Analytical Issues, held at IIASA in July and August 1983. The conference and workshop formed part of the continuing IIASA program on Patterns of Economic Structural Change and Industrial Adjustment.

Structural change was interpreted very broadly: the topics covered included the nature and causes of changes in different sectors of the world economy, the relationship between international markets and national economies, and issues of organization and incentives in large economic systems.

There is a general consensus that important economic structural changes are occurring in the world economy. There are, however, several alternative approaches to measuring these changes, to modeling the process, and to devising appropriate responses in terms of policy measures and institutional redesign. Other interesting questions concern the role of the international economic system in transmitting such changes, and the merits of alternative modes of economic organization in responding to structural change. All of these issues were addressed by participants in the workshop and conference, and will be the focus of the continuation of the research program's work.

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Abstract

We study the aggregation of preferences when intensities are taken into account: the aggregation of cardinal preferences and also of von Neumann-Morgenstern utilities for cases of choice under uncertainty. We show that with a finite number of choices, there exist no continuous anonymous aggregation rules that respect unanimity for such preferences or utilities. With infinitely many (discrete sets of) choices, such rules do exist and they are constructed here. However, their existence is not robust: each is a limit of rules that do not respect unanimity. Both results are for economies with a finite number of individuals.

The results are obtained by studying the global topological structure of spaces of cardinal preferences and of von Neumann-Morgenstern utilities. With a finite number of choices, these spaces are proven to be noncontractible. With infinitely many choices, on the other hand, they are proven to be contractible.

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1. Introduction

The aggregation of preferences studied in social choice theory typically describes an individual preference as a ranking among choices, i.e., in ordinal terms. In most of the literature following Arrow's and Black's pieces [1] [3], intensities of preferences are not recorded; in particular, it is not possible to express whether a choice x is preferred to another y more than to a third, z. Since most of the results in the aggregation of ordinal preferences are negative, it seems natural to inquire whether more positive results can be obtained when this property is relaxed and intensities of preferences are recorded.

A significant step in allowing the consideration of preference intensities is introduced with cardinal preferences. These preferences express precisely the notion that a choice x is preferred to another y more than to a third z. In the case of choice under uncertainty, these preferences can be shown to have the same mathematical structure as von Neumann-Morgenstern utilities, the numerical representations of preferences over lotteries: These utilities are denoted NM utilities, and are frequently used in the operations research literature in the field of decision theory since the concept was developed by von Neumann and Morgenstern (19); NM utilities are also widely used as a representation of individual behavior in game theoretic models, see Fishburn (13).

The main difference in the definitions of ordinal and cardinal preferences is the invariance they require from a numerical representation. Cardinal preferences require much weaker invariance than ordinal preferences: the representation of cardinal preference by a numerical function is invariant under (and only under) positive linear transformations. For ordinal preferences, instead, the representation must be invariant under any positive transformation.[1]

The weaker the invariance, the closer are preferences to numerical utilities, and numerical utilities have no problem of aggregation. Therefore one may expect that the task of aggregating is made easier with cardinal rather than ordinal preferences. However, this is not the case. It was shown in Chichilnisky [4] and in Chichilnisky and Heal [8] that the crucial element in our ability to aggregate preferences is the global topological structure of the space of preferences considered. In order to admit appropriate aggregation rules, these spaces must be contractible, i.e. topologically trivial. However, the topological structure of spaces of preferences may be complex even when less invariance is required. For instance, NM utilities with finite lotteries are shown here to define a <u>non-contractible space</u>, i.e. a space with a non-trivial topological structure (see Section 2).

By investigating the global topology of spaces of cardinal preferences and of NM utilities, we prove here that with <u>finitely many choices</u>, there exists no continuous anonymous social aggregation rule that respects unanimity.[2] Aggregation is impossible for cardinal preferences and for NM utilities.

With <u>infinitely many choices</u> we show instead that such aggregation rules <u>do</u> exist. However, their existence is not robust in the sense that they are the limit of rules defined on subsets of finitely many choices, which do not respect unanimity. The same results apply to von Neumann-Morgenstern utilities defined over infinitely many lotteries. A finite number of individuals is considered throughout the paper.

The rest of the paper is organized as follows: Section 2 gives notation and definitions; Section 3 discusses previous literature; and Section 4 gives the results.

2. <u>Notation and Definitions</u>

In the case of finite choices the <u>choice space</u> X is a finite set of points in Euclidean space

$$X = \{x_i\}, \ i = 1, \ldots, n, \ n \geq 3.$$

A preference with intensity or cardinal preference p is identified with a positive vector in R^n,

$$p \in R^{n+},\ p = (p_1, \dots p_n).$$

p_i denotes the utility value attached to the choice x_i. The total indifference preference is thus the vector with all coordinates equal. Our space of preferences contains this total indifference preference as well.

The following step is to normalize utility vectors in order to obtain a unique representation of each cardinal preference by a vector in Euclidean space. This normalization is a standard one; see, e.g., Kalai and Schmeidler [15] ; its economic content is discussed in the following section. Formally, if $p = (p_1, \dots, p_n)$ is a utility vector in R^n, p is normalized by subtracting from its coordinates the vector with all components identical to the minimum utility value

$$p \to (p_1 - m, \dots, p_n - m)$$

where $m = \min_i \{p_i\}$,

and then dividing the outcome by its maximum component M if $M \neq 0$, i.e.,

$$p \to \frac{p_1 - m}{M}, \dots, \frac{p_n - m}{M} \quad ,$$

where $M = \max_i \{p_i - m\}$. The total indifference preference is identified therefore with the vector (0, ..., 0). The fact that all normalized preferences have the same minimum and maximum utility values can be considered a weak form of interpersonal comparison.[3]

It follows therefore that with finitely many choices the space of cardinal preferences is $P = Q \cup \{0\}$, where Q is the subspace of non-zero cardinal preferences,

$$Q = \{p \epsilon R^{n+}: \sum_{i=1}^{n} p_i \leq n - 1, \; p_j = 0 \text{ and } p_k = 1 \text{ for some } k,j \; \epsilon \{1, \ldots, n\}\},$$

and {0} denotes the total indifference preference. In order to define continuity of the social choice rule, P is given the natural topology it inherits from R^n. The space P has two connected components, Q and {0}.[4]

We shall now define the space of cardinal preferences P^∞ for the case of infinitely many choices.

Assume now that the choice space X is N, the set of integers. A preference p is assumed to be a non-negative sequence of numbers, i.e., a non-negative real valued function on N. Since we are concerned with bounded sequences, without loss of generality, we may assume that

$$\sum_N p(n)\mu(n) < \infty.$$

for some finite measure μ or N given by a density function $\mu(n)$.

The space of preferences P^∞ is therefore strictly contained in the space of all bounded sequences, and this is in turn a subset (the positive cone) of a weighted ℓ_1 space.[5] Note that we could have embedded P^∞ into ℓ_∞, the space of all bounded sequences with the sup norm, $\|x\|_\infty = \sup_{i=1,2\ldots} |x_n|$. However, the space ℓ_∞ is a dense subspace of ℓ_1 with the (finite) weight $\mu(n)$. Therefore, if one defines an aggregation map ϕ for ℓ_1, one has automatically defined an aggregation map for ℓ_∞, given by the restriction of ϕ on ℓ_∞ considered as a subspace of ℓ_1. The topology induced by ℓ_1 is different than the sup norm on ℓ_∞, but since our aim is to prove an existence theorem, for some adequate topology, this procedure seems adequate. In any case, P^∞ is a strict subset of ℓ_∞ as well as of ℓ_1, and is significantly smaller than either ℓ_∞ or ℓ_1. Therefore, neither ℓ_∞ nor ℓ_1 coincide with P^∞ and thus the choice of topology is best made on the basis of mathematical adequacy. Theorem 2 shows that ℓ_1 is an adequate space; the spaces ℓ_1, or more generally ℓ_p (with $1 \leqslant p \leqslant \infty$) have been used previously in the economic literature; see e.g. Chichilnisky [6].

As in the case of finitely many choices, we normalize the vector p in order to obtain a unique representation of cardinal preferences. An equivalence relation $\sim$ is defined by $p^1 \sim p^2$ if and only if $p^1 = \alpha + \beta p^2$, $\alpha \in \ell_1$, $\beta \in R^+$. A <u>preference</u> is an equivalence class of positive vectors p under the relation $\sim$. A space which is in a one to one correspondence with the space of preferences is obtained by considering all vectors with coordinates smaller than 1, with at least one coordinate zero, and with the first non-zero coordinate (if it exists) equal to 1. Therefore with many choices the space of cardinal preferences is $P^\infty = Q^\infty \cup \{0\}$, where $Q^\infty = \{f \in \ell_1^+$: for all i, $f_i \leqslant 1$; $f_j = 0$ and $f_j + 1 = 1$ for some j$\}$. P^∞ inherits the topology of ℓ_1 and is a closed subset of a Banach space. As P, P^∞ consists of exactly two connected components.

Assume now there are k agents, $k \geq 2$. With finite choices a profile of cardinal preferences is a vector $\{p^1, \ldots, p^k\} \varepsilon P^k$, the cartesian product of P with itself k-times repectively. With infinite choices a profile is a vector $\{p^1, \ldots, p^k\} \varepsilon (P^{\infty})^k$.

A rule ϕ is said to respect unanimity when $\phi(p, \ldots, p) = p$; i.e., if all voters have identical preferences over all choices, so does the social preference.[6]

A rule ϕ is anonymous when the outcome is independent of the order of the voters, i.e.,

$$\phi(p^1, \ldots, p^k) = \phi(p^{\eta 1}, \ldots, p^{\eta 1})$$

where

$$\eta: \{1, \ldots, k\} \rightarrow \{\eta_1, \ldots, \eta_k\}$$

is any permutation of the set $\{1, \ldots, k\}$.

Continuity of a rule is defined with respect to the usual product topologies of the spaces of preferences as subsets of R^{nk} or $(\ell_1)^k$ in the finite and infinite choice case, respectively.

We now discuss certain basic topological concepts used in the following.

A topological space X is contractible if there exists a continuous map $f : X \times [0, 1] \rightarrow X$ such that $f(x, 0) = x$ $\forall x$ in X, and $f(x, 1) = x_0$, for some $x_0 \varepsilon X$. Intuitively, X is contractible if it can be deformed continuously through itself, into one of its points, x_0. Clearly euclidean space and any convex set are contractible. In particular, the space of continuous real valued utility functions is a contractible space. Topologically speaking, these are all trivial spaces, since they are topologically equivalent to (i.e., continuously deformable into) points. A hollow sphere in R^n is not contractible. As we shall prove below, neither the non trival connected component the space of cardinal preferences Q,nor that of von Neumann-Morgenstern utilities, are contractible. This proves to be important for the aggregation results of this paper.

3. Relationship with Previous Work

Before proving the results, it may be useful to discuss the relationship of the spaces of preferences studied here with earlier concepts of cardinal preferences used in the literature, and also earlier results in this area.

The space of non zero cardinal preferences Q corresponds to the space of cardinal utilities as studied for instance by Kalai and Schmeidler [15]: two vectors p^1 and p^2 define the same cardinal preference when there exist a positive number β and a positive vector α such that

$$p^1 = \alpha + \beta p^2.$$

It is easy to check that our normalization of the previous section identifies each vector in Q with an equivalence class of vectors under the equivalence relation

$$p^1 \approx a + bp^2, \qquad \text{for all } a > 0,\ b > 0.$$

Consider now the case of choice under uncertainty. In this case the space of von Neumann-Morgenstern utilities, i.e., numerical representation of preferences over lotteries, corresponds precisely to our formal definition of the spaces of cardinal preferences. For further discussion see, e.g., [15] and [19].

The specification of P given here is also related to one of the forms of relaxation of the usual ordinality and comparability assumptions discussed in d'Aspremont and Gevers. Their condition CN of cardinality and non-comparability requires that if u_1 and u_2 are two utilities, then they define the same preference whenever

$$u_1\ (x,j) = \alpha_j + \beta_j(u_2\ (x,j))$$

where $j = 1,\ldots,\ell$ is the index for the voter, x denotes a choice, and where $\{\alpha_j\}$ and $\{\beta_j\}$ are positive real numbers. In our framework CN means that for each voter the vector p^1 represents the same preference as another p^2 when there exists a vector α and a positive number β such that $p^1 = \alpha + \beta p^2$. This is precisely the cardinality condition discussed above:

Let p^1 and p^2 satisfy $p^1 = \alpha + \beta p^2$. Then they yield the same element in P, since for any $p = (p_1,\ldots,p_n)$, and any $j = 1,\ldots,n$

$$\frac{\alpha + \beta p_j - \min\limits_i\ (\alpha + \beta p_i)}{\operatorname*{Max}\limits_i\ [\alpha + \beta(p_i) - \min\limits_\ell\ (\alpha + \beta p_\ell]}$$

$$= \frac{[p_j - \min\limits_i(p_i)]}{[\operatorname*{Max}\limits_i\ (p_i - \min\limits_\ell(p_\ell)]}$$

Conversely, if two utility vectors p^1 and p^2 in R^n yield the same element in the space of i-preferences P,

then $$\frac{p^1 - \min\limits_i\ (p_i^1)}{\operatorname*{Max}\limits_i\ (p_i^1 - \min\limits_\ell\ (p_\ell^1))} = \frac{p^2 - \min\limits_i\ (p_i^2)}{\operatorname*{Max}\limits_i\ (p_i^2 - \min\limits_\ell\ (p_\ell^2))}$$

which implies that

$$p^1 = \alpha + \beta p^2$$

for $\alpha = \min_i (p_i^1) \left(\frac{(\min_i (p_i^2)) \operatorname{Max}_i (p_i^1 - \min_j (p_j^1))}{\operatorname{Max}_i (p_i^2 - \min_j (p_j^2))} \right)$,

and $\beta = \frac{\operatorname{Max}_i (p_i^1 - \min_j (p_j^1)}{\operatorname{Max}_i (p_i^2 - \operatorname{Min}_j (p_j^2))}$

Therefore social choice rules that are invariant under the normalization of P correspond precisely to those satisfying condition CN.

Several authors that studied the problems involved in aggregating cardinal preferences, e.g., Sen [16] and Kalai and Schmeidler [15]. It has been shown [15] that Arrow-like paradoxes may exist even with cardinal preferences, provided Arrow-like conditions are required of the aggregation procedure: these are the somewhat controversial independence of irrelevant alternatives, Pareto and non dictatorship. Such conditions may be too strong. Also, while making the problem amenable to a combinatorial analysis, such conditions tend to leave out its intrinsic geometry. Here, instead, other conditions of the aggregation rule are studied: continuity, anonymity and respect of unanimity. These conditions admit a ready geometrical interpretation, and furthermore help to exhibit the topological nature of the problem at hand.

The following result establishes the impossibility of aggregation with finitely many choices. The space of preferences is therefore $P = Q \cup \{0\}$, as defined above.

Theorem 1

There exists no continuous aggregation rule for cardinal preferences $\phi : P^k \to P$ which respects unanimity and is anonymous. This includes cases where individual and social preferences may be indifferent among all choices.

Proof:

An aggregation rule for cardinal preferences is a map $\phi : P^k \to P$. Now, as discussed in section 2, the space P has exactly two connected components, Q and $\{0\}$ (figure 1 below illustrates the case of three choices). Therefore the product space P^k has exactly 2^k connected components.

The map ϕ is therefore a continuous function from a topological space with 2^k components into another with 2 components. It follows from continuity of ϕ that each of the connected components of P^k must be mapped by ϕ into

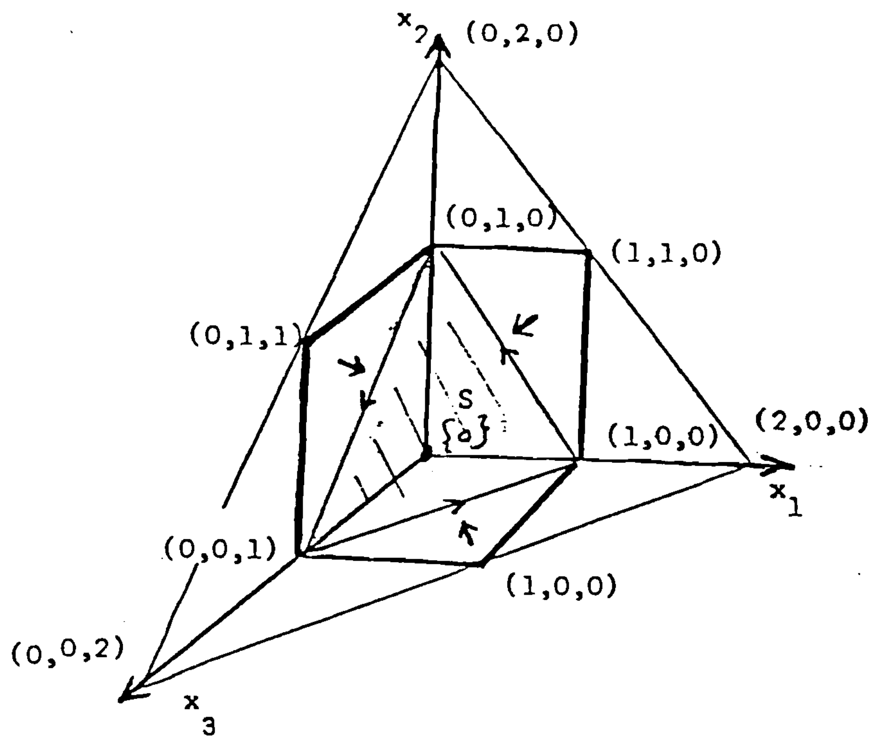

Figure 1

The space of preferences P with three choices is indicated as the union of the point {O} with the set drawn with a heavy line. The non zero connected component of P, Q, (indicated with the heavy line) is cn a one to one bicontinuous correspondence with the boundary ∂S of the simplex S in R^3, and is therefore not contractible in R^3.

one connected component of P. Consider in particular Q^k, which is the connected component of P^k consisting of all non zero cardinal preferences. Then either $\phi(Q^k) \subset Q$, or else $\phi(Q^k) \subset \{0\}$. However, by the condition of respect of unanimity $\phi(p,\ldots,p) = p$ for all p in Q, implying that $\phi(Q^k) \not\subset \{0\}$, i.e., $\phi(Q^k) \subset Q$.

Therefore, the axioms of continuity and that of respect of unanimity taken together rule out the possibility that a profile with all preferences different from the zero vector may be mapped into the zero outcome vector. Therefore, a continuous rule for cardinal preferences which respects unanimity will only assign the total indifference to a set of voters if at least one of them is totally indifferent among all choices.[7]

ϕ induces therefore a continuous map $\bar{\bar{\phi}} : Q^k \to Q$, which is also continuous, anonymous and respects unanimity. Since Q and Q^k are both connected spaces, we can now use the results in [7]. These results establish that the existence of such a map ϕ depends on certain topological invariants of the space Q. The next step of the proof consists therefore of investigating the topology of the space of non zero cardinal preferences, for any finite number of choices $n \geq 3$.

Consider the subspace R_j^{n-1} of R^{n+}, consisting of all vectors p with the j-th coordinate $p_j = 0$. The set $T_j \subset R_j^{n-1}$ defined by

$$T_j = \{p \in R_j^{n-1} : \Sigma p_i = 1\}$$

is an n-2 dimensional simplex. Now, the set $Q_j = Q \cap R_j^{n-1}$ is in a one-to-one correspondence with T_j, by the map m defined by

$$p \xrightarrow{m} \frac{p}{\sum_{i=1}^{n} p_i}, \text{ since } p \neq \{0\} \text{ for all p in Q. Since the map m is}$$

continuous, and so is its inverse, it follows that the space Q is homeomorphic to the simplex T_j.

Now, for any j, $\ell \in \{1,\ldots,n\}$ the intersection $Q_j \cap Q_\ell = T_j \cap T_\ell$. Since $Q = \bigcup_{j=1}^{n} Q_j$, and $Q_j \cap Q_\ell = T_j \cap T_\ell$ for all j and ℓ, it follows that

$\bigcup_{j=1}^{n} Q_j$ is homeomorphic to $\bigcup_{j=1}^{n} T_j$. Therefore Q is homeomorphic to $\bigcup_{j=1}^{n} T_j$.

Now, $\bigcup_{j=1}^{n} T_j$ is, in turn, homeomorphic to the boundary of an n-dimensional simplex in R^n, i.e., an n-2 dimensional sphere. It follows that Q is homeomorphic to an n-2 dimensional sphere and, therefore, in particular, Q is <u>not</u> contractible. We can now apply the results of [7], which establish that for any (para) finite CW complex X, the contractibility of X is a necessary condition for the existence of a continuous anonymous rule $\phi : X^k \to X$, which respects unanimity, for all $k \geq 2$. Since Q is a finite CW complex and is not contractible, this completes the proof.

Since as discussed above the space of NM utilities can be identified with P if there are finitely many choices in each state, we have therefore obtained from theorem 1:

<u>Corollary 1</u>

<u>The space of von-Neumann Morgenstern utilities with finitely many lotteries is not contractible</u>.

and

<u>Corollary 2</u>

<u>With finitely many lotteries there exist no continuous anonymous aggregation rule for von Neumann-Morgenstern utilities which respects unanimity. This includes cases where individual and social utilities may be indifferent among all lotteries</u>.

<u>Remark</u> : Even though our framework and conditions on the aggregation rule are rather different from those of Kalai and Schmeidler, our impossibility result is consistent with theirs in cases of finitely many choices.

However, this is not the case for infinitely many choices. Instead, we obtain in the next section a possibility of aggregation result for his latter case. This contrasts with the result of Kalai and Schmeidler because their impossibility result is valid also with infinitely many choices. This is because they require the axiom of independence of irrelevant alternatives, which effectively reduces the problem of aggregation with infinitely many choices to one of aggregation with finitely many choices. The difference between the two results arises from the fact that different sets of axioms are required: we do not require independence of irrelevant alternatives, but require continuity instead, and we do not require the Pareto condition, but rather a (weaker) condition of respect of unanimity. The results given below also show that the aggregation with infinitely many choices is not robust as a limiting process of aggregation on certain finite subsets of choices.

We now turn to the case of infinitely many choices. Our space of cardinal preferences is therefore P^{∞}. As before , there are a finite number of individuals , $k \geq 2$.

Theorem 2

With infinitely many choices, there exists a continuous aggregation rule $\phi : (P^{\infty})^k \to P^{\infty}$ for cardinal preferences respecting unanimity and anonymity given by a continuous deformation of a Bergsonian rule i.e., a convex addition rule However, any such rule is a limit of rules defined on arbitrarily large finite sets of choices and which do not respect unanimity; in particular they are not Pareto.

Proof:

As in theorem 1, we may consider a continuous aggregation rule

$$\phi : (P^{\infty})^k \to P^{\infty}$$

assigning to each profile of k(non-trivial) preferences in P^{∞}, an element of P^{∞}, where $P^{\infty} = Q^{\infty} \cup \{0\}$, and

$$Q^{\infty} = \{p \in \ell_1^+ : p_i \leq 1 \text{ for all } i\,,\ p_j = 0 \text{ and } p_{j+1} = 1 \text{ for some } j\,\}.$$

As in the finite dimensional case one can show that Q^{∞} is in a one-to-one bicontinuous correspondence with the boundary of a disk in ℓ_1, i.e., with an infinite dimensional sphere in ℓ_1.

Now, by corollary 5.1, p. 109 of Bessaga and Pelczynski [2] and Kuiper [14] the space Q^{∞} is homeomorphic to ℓ_1, and in particular, is contractible. This is in contrast to the finite dimensional case, where spheres are <u>not</u> contractible and indeed <u>not</u> homeomorphic to euclidean space. Let H be the homeomorphism, $H : Q^{\infty} \to \ell_1$. Since convex addition C in ℓ_1 exists and it satisfies anonymity, continuity and respect of unanimity, the composition map $\phi = H o^{-1} C o H^k$ defined by

$$(Q^{\infty})^k \xrightarrow{H^k} (\ell_1)^k \xrightarrow{C} \ell_1 \xrightarrow{H^{-1}} Q^{\infty}$$

is a continuous function $\phi : (Q^{\infty})^k \to Q^{\infty}$ satisfying anonymity and respect of unanimity. Since we can repeat this procedure for each connected component of $(P^{\infty})^k$, this proves existence. Clearly the map ϕ is a deformation of the convex addition rule C, i.e., a deformation of a Bergsonian rule.

Consider now the space of truncated sequences $T \subset Q$,

$$T = \{\{p\} : M_0 \text{ with } p_M = 0 \text{ for } M \geq M_0\}.$$

This space is dense in ℓ_1 with a (finite) measure. Consider the pointwise convergence topology of the space F of continuous functions $F = f : (\ell_1)^k \to \ell_1\}$.[8]

The sequence of restriction maps $\{\phi_d\}$, defined by the restrictions of ϕ to finite dimensional linear subspaces L_d whose dimensions define an unbounded sequence of integers $\{d\}$, converges to ϕ. Note that when restricted to any finite subspace of choices (i.e., when restricted to vectors of finite length) the map ϕ_d remains anonymous. It follows by theorem 1 that ϕ_d cannot respect unanimity on such subspace; in particular, it is not Pareto. Since the sequence of maps $\{\phi_d\}$ converges to the map ϕ, this completes the proof.

From theorem 2 we obtain immediately the analogue to corollaries 1 and 2 for von Neumann-Morgenstern utilities:

Corollary 3

With infinitely many lotteries, the space of von Neumann-Morgenstern utilities P^∞ is contractible.

and

Corollary 4

With infinitely many lotteries there exists a continuous anonymous aggregation rule for von Neumann-Morgenstern utility functions which respects unanimity. However, this rule is not robust since it is the limit of non-Pareto rules on arbitrarily large sets of lotteries.

FOOTNOTES

[1]The problem of aggregation of ordinal preferences is significantly different from that of aggregating utility functions (e.g., the classical Bergsonian social welfare function) because the aggregation of ordinal preferences must be independent of the choice of their utility representation. For instance, if u is a utility function and F is a strictly increasing numerical function, the ordinal preference associated with the utility u must be the same as that associated with the function F o u. Therefore, a rule for aggregating utilities will only induce a rule for aggregating ordinal preferences if it is invariant under any such increasing transformation of utilities. This is indeed a rather strong condition, and several possible relaxations have been studied, for instance, by Sen [16] [17], Hammond [13], Kalai and Schmeidler [15] and more recently by d'Aspremont and Gevers [10]. Sen [17] concentrates on the relaxation of the assumption of no interpersonal comparisons. d'Aspremont and Gevers discuss and characterize a wide combination of assumptions that relax both the interpersonal comparison and the ordinality assumptions. Our framework here is most closely related to axiom (CN) of d'Aspremont and Gevers, which assumes that individual utility functions are cardinal and non comparable, and to the cardinality assumptions of Sen [16] and of Kalai and Schmeidler.

[2] Respect of unanimity is strictly weaker than the Pareto condition. It requires that if all individuals agree unanimously over all choices, so does the aggregate. This condition does not imply that if one choice x is preferred to another y by all individuals, then the aggregate prefers x to y.

[3] This normalization has in particular the effect that the sum total of intensities over choices is uniformly bounded over agents; this was a suggestion of L. Gevers.

[4] A connected component of a topological space Y is a maximum connected subspace of Y. A space X is connected if it cannot be decomposed as a union $X = X_1 \cup X_2$, where $X_1 \neq \phi$, $X_2 \neq \phi$, and X_1 and X_2 are both open and closed sets. This extends the notion that any point in X can be joined to another in X by a path contained in X.

[5] (ℓ_1,μ) is the Banach space of infinite sequences of real numbers $\{x_n\}_{n=1,2,...}$ such that

$$||x||_1 = \sum_{n=1}^{\infty} |x_n| \mu(n) < \infty,$$

see [11].

[6] Note that this condition is binding only when all voters have identical preferences. It is therefore a strictly weaker condition than Pareto, since a rule ϕ satisfies the Pareto condition if whenever a choice $x \in X$ is preferred to another $y \in X$ for all preferences $p^1,\ldots,p^k$, then $\phi(p^1,\ldots,p^k)$ also prefers x to y.

[7] In addition to violating our axioms, rules that assign zero outcomes to non zero vectors are clearly undesirable for other reasons: It is easy to check that if ϕ is a continuous map from $(R^n)^3$ into R^n that maps a profile of three voters with non zero vectors into the trivial (zero) social preference, it will necessarily map the equivalent of <u>some</u> Condorcet triple into the trivial outcome $(0,\ldots,0)$. The 'Condorcet triple' we are referring to is obtained by choosing three points (xyz) in R^n, so that the three vectors giving the voters' preferences rank these choices in the orders (xyz) (zxy) and (yzx) respectively. Such aggregation would give a <u>trivial</u> (total indifference) solution to the Condorcet triple, which is clearly not an acceptable solution.

[8] The <u>pointwise</u> convergence topology τ on F is defined by the convergence rule

$$\{f^j\} \xrightarrow{\tau} f_o \text{ if } f^j(p) \to f_o(p) \;\forall\; p \text{ in } \ell_1,$$

see [11].

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